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Research article

On the topology $\tau_R^\diamond$ of primal topological spaces

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Abstract: The main purpose of this paper is to introduce and study two new operators $(\cdot)_R^\diamond$ and $cl_R^\diamond(\cdot)$ via primal, which is a new notion. We show that the operator $cl_R^\diamond(\cdot)$ is a Kuratowski closure operator, while the operator $(\cdot)_R^\diamond$ is not. In addition, we prove that the topology on X, shown as $\tau_R^\diamond$, obtained by means of the operator $cl_R^\diamond(\cdot)$, is finer than τ_δ, where τ_δ is the family of δ-open subsets of a space (X, τ). Moreover, we not only obtain a base for the topology $\tau_R^\diamond$ but also prove many fundamental results concerning this new structure. Furthermore, we provide many counterexamples related to our results.

Keywords: primal; primal topological space; Kuratowski closure operator; the operator $(\cdot)_R^\diamond$; the operator $cl_R^\diamond$
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1. Introduction

The desire to obtain more general results and valid solutions to many issues in topology, such as compactification, proximity space, and closure space problems, has led mathematicians to introduce some new structures. Some of these structures are grills [16], ideals [24], filters [25], and nets [28]. These classical structures are undoubtedly some of the most important objects of topology. A filter is a related idea in topology that allows for a general definition of convergence in general topological spaces. That is, filters are used to define convergence like the role of sequences in a metric space. They unify the notion of a limit across various arbitrary topological spaces. The concept of net is a generalization of the concept of sequence. Thanks to this concept, we obtain new characterizations of some concepts such as compactness in general topology. Kuratowski defined and studied the concept of ideal from filters [25]. Also, this notion has been studied by many topologists in different directions in [26, 27]. The notion of ideal comes across as the dual structure of filter. Also, the other classical structure of general topology in the literature is the notion of grill. It was introduced by Choquet [16] in 1947 and studied by many authors. Hosny [18] introduced and discussed two operators in the space $(X, \tau^{\delta}, \mathcal{G})$. Also, the author obtained a new topology $\tau_{\mathcal{G}}^{\delta}$ which is finer than $\tau_{\mathcal{G}}$ and τ_{δ}. Nasef and Azzam [30] defined and studied two operators via grill. They obtained characterization and basic properties of these operators. Also, they generalized a grill topological space via τ^{s} topology induced from these operators. Roy et al. [32–35] studied $\tau_{\mathcal{G}}$ topology and it is shown that some interesting properties and behaviours of this topology $\tau_{\mathcal{G}}$ are encountered if $\mathcal{G}$ belongs to a particular class of grills, introduced here and termed as the principal grills. Certain separation axioms and some well-known covering properties are investigated in that article. Thron [38] presented a new approach to proximity structures based on the recognition that many of the entities important in the theory are grills.

Recently, Pawlak introduced the concept of rough set in [31]. This concept has various applications in the literature, being very important in terms of this angle. One of them is topology, particularly ideals. Many mathematicians have studied the concept of rough sets via topology and ideals. Tareq et al. [7–12] have studied the concept of rough set introduced by Pawlak in terms of different angles. Güler et al. [17] generalized the notations of rough sets based on the topological space. Furthermore, they produce various topologies by using the concept of ideal, C_j-neighborhoods and P_j-neighborhoods. Hosny et al. [19–23] have also studied the concept of rough set introduced by Pawlak in terms of different angles. Mustafa et al. [29] used the system of containment neighborhoods to present new rough set models generated by topology and ideals. In addition, they discussed their fundamental characterizations and reveal the relationships among them. Tantawy et al. [37] generalized the rough set model by defining new approximation operators in more general setting of a complete atomic Boolean lattice by using an ideal.

Recently, Acharjee et al. [1] introduced a new classical structure called primal . They define the notion of primal topological space by utilizing two new operators and investigate many fundamental properties of this new structure and these two operators. Moreover, the notion of primal is the dual structure of grill. Furthermore, some new studies have been developed regarding primal topological spaces since the introduction of primal; for more details, see [2–6, 15]. Also, two new versions of primal topology have been given in the frameworks of soft setting and fuzzy setting; for more details, see [13, 14].

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (briefly, X and Y) represent topological spaces, unless otherwise stated. We denote the closure and interior of a subset A of a space X by $cl(A)$ and $int(A)$, respectively. Also, the powerset of a set X will be denoted by 2^X. The family of all open neighborhoods of a point x of X will be denoted by $O(X, x)$.

A subset A of a space X is called regular open [36] if $A = int(cl(A))$. The complement of a regular open set is called regular closed [36]. The union of all regular open subsets of X contained in A is called δ-interior of A and denoted by $\delta\text{-}int(A)$. Dually, the intersection of all regular closed subsets of X containing A of a space X is called δ-closure of A and denoted by $\delta\text{-}cl(A)$. A subset A of a space X is called δ-open [39] if $A = \delta\text{-}int(A)$. The complement of a δ-open set in a space X is called δ-closed [39].

The family of all regular open [36] (resp. regular closed [36], δ-open [39], δ-closed [39]) subsets of a space X will be denoted by $RO(X)$ (resp. $RC(X)$, $\delta O(X)$, $\delta C(X)$). The family of all regular open (resp. regular closed, δ-open, δ-closed) sets of X containing a point x of X is denoted by $RO(X, x)$ (resp. $RC(X, x)$, $\delta O(X, x)$, $\delta C(X, x)$).

Now, we recall the following results concerning δ-interior and δ-closure of a set A in a topological space X.

Theorem 2.1. *[39] Let X be a topological space and $A \subseteq X$. Then the following equalities hold.*

a) $\delta\text{-}int(A) = \{x|(\exists U \in RO(X, x))(U \subseteq A)\} = \{x|(\exists U \in O(X, x))(int(cl(U)) \subseteq A)\}$,
b) $\delta\text{-}cl(A) = \{x|(\forall U \in RO(X, x))(U \cap A \neq \emptyset)\} = \{x|(\forall U \in O(X, x))(int(cl(U)) \cap A \neq \emptyset)\}$,
c) $\delta\text{-}cl(A^c) = (\delta\text{-}int(A))^c$,
d) $\delta\text{-}int(A^c) = (\delta\text{-}cl(A))^c$.

Definition 2.2. *[1] Let X be a nonempty set. A collection $\mathcal{P} \subseteq 2^X$ is called a primal on X if it satisfies the following conditions:*

a) $X \notin \mathcal{P}$,
b) if $A \in \mathcal{P}$ and $B \subseteq A$, then $B \in \mathcal{P}$,
c) if $A \cap B \in \mathcal{P}$, then $A \in \mathcal{P}$ or $B \in \mathcal{P}$.

Corollary 2.3. *[1] Let X be a nonempty set. A collection $\mathcal{P} \subseteq 2^X$ is a primal on X if and only if it satisfies the following conditions:*

a) $X \notin \mathcal{P}$,
b) if $B \notin \mathcal{P}$ and $B \subseteq A$, then $A \notin \mathcal{P}$,
c) if $A \notin \mathcal{P}$ and $B \notin \mathcal{P}$, then $A \cap B \notin \mathcal{P}$.

Definition 2.4. *[1] A topological space (X, τ) with a primal $\mathcal{P}$ on X is called a primal topological space and denoted by $(X, \tau, \mathcal{P})$.*

Definition 2.5. *[1] Let $(X, \tau, \mathcal{P})$ be a primal topological space. We consider a map $(\cdot)^\diamond : 2^X \to 2^X$ as $A^\diamond(X, \tau, \mathcal{P}) = \{x \in X|(\forall U \in O(X, x))(A^c \cup U^c \in \mathcal{P})\}$ for any subset A of X. We will use $A^\diamond(X, \tau, \mathcal{P})$ instead of $A^\diamond$ if the topology or the primal needs to be specified.*

Definition 2.6. *[1] Let $(X, \tau, \mathcal{P})$ be a primal topological space. We consider a map $cl^\diamond : 2^X \to 2^X$ as $cl^\diamond(A) = A \cup A^\diamond$, where A is any subset of X.*

Corollary 2.7. *[1] Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the function $cl^\diamond : 2^X \to 2^X$ defined by $cl^\diamond(A) = A \cup A^\diamond$, where A is any subset of X, is a Kuratowski's closure operator.*

Definition 2.8. *[1] Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the family $\tau^\diamond = \{A \subseteq X : cl^\diamond(A^c) = A^c\}$ is a topology on X induced by topology τ and primal $\mathcal{P}$.*

3. The operator $(\cdot)^\diamond_R$ and its fundamental properties

In this section, we introduce a new operator in primal topological spaces and investigate its properties. We obtain not only some fundamental properties of this new operator but also give many counterexamples related to this operator. Now, we have the following definition of the operator $(\cdot)^\diamond_R$.

Definition 3.1. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. We define a map $(\cdot)^\diamond_R : 2^X \to 2^X$ as $A^\diamond_R = \{x \in X : (\forall U \in RO(X,x))(A^c \cup U^c \in \mathcal{P})\}$ for any subset A of X. We can also use the notation $A^\diamond_R(X,\tau,\mathcal{P})$ instead of $A^\diamond_R$ to indicate the primal and the topology if necessary.*

Remark 3.2. *Let $(X,\tau,\mathcal{P})$ be a primal topological space and $A \subseteq X$. The inclusions of $A^\diamond_R \subseteq A$ or $A \subseteq A^\diamond_R$ need not always to be true, as shown by the following examples.*

Example 3.3. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset,\{b\},\{c\},\{b,c\},\{a,c\},X\}$ and $\mathcal{P} = \{\emptyset,\{b\},\{c\},\{b,c\}\}$. For the subset $A = \{b,c\}$, we get $A^\diamond_R = \emptyset$ so $A = \{b,c\} \nsubseteq \emptyset = A^\diamond_R$.*

Example 3.4. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset,X\}$ and $\mathcal{P} = \{\emptyset,\{b\},\{c\},\{b,c\}\}$. For the subset $A = \{a\}$, we get $A^\diamond_R = X$ so $A^\diamond_R = X \nsubseteq \{a\} = A$.*

Theorem 3.5. *Let $(X,\tau,\mathcal{P})$ be a primal topological space and $A, B \subseteq X$. Then, the following statements hold.*

a) $A^\diamond \subseteq A^\diamond_R$,
b) If $A \in \delta C(X)$, then $A^\diamond_R \subseteq A$,
c) $\emptyset^\diamond_R = \emptyset$,
d) $A^\diamond_R \in \delta C(X)$,
e) $(A^\diamond_R)^\diamond_R \subseteq A^\diamond_R$,
f) If $A \subseteq B$, then $A^\diamond_R \subseteq B^\diamond_R$,
g) $A^\diamond_R \cup B^\diamond_R = (A \cup B)^\diamond_R$,
h) $(A \cap B)^\diamond_R \subseteq A^\diamond_R \cap B^\diamond_R$.

Proof. *a*) Obvious since every regular open set in topological spaces is open.

b) Let $A \in \delta C(X)$ and $x \notin A$. We will show that $x \notin A^\diamond_R$.

$$\left.\begin{array}{r} A \in \delta C(X) \Rightarrow A^c \in \delta O(X) \Rightarrow (\exists \mathcal{A} \subseteq RO(X))\,(A^c = \bigcup \mathcal{A}) \\ x \notin A \Rightarrow x \in A^c \end{array}\right\} \Rightarrow (\exists B \in RO(X))(x \in B \subseteq A^c)$$
$$\Rightarrow (B \in RO(X,x))(B^c \cup A^c \supseteq (A^c)^c \cup A^c = A \cup A^c = X \notin \mathcal{P})$$
$$\Rightarrow (B \in RO(X,x))(B^c \cup A^c \notin \mathcal{P})$$
$$\Rightarrow x \notin A^\diamond_R.$$

c) $\emptyset^\diamond_R = \{x \in X : (\forall U \in RO(X,x))(U^c \cup \emptyset^c = U^c \cup X = X \in \mathcal{P})\} = \emptyset$.

d) We will show that $A^\diamond_R = \delta\text{-}cl(A^\diamond_R)$. We already have $A^\diamond_R \subseteq \delta\text{-}cl(A^\diamond_R) \ldots (1)$.
Conversely, now let $x \in \delta\text{-}cl(A^\diamond_R)$ and $U \in RO(X,x)$.

$$\left.\begin{array}{r} U \in RO(X,x) \\ x \in \delta\text{-}cl(A^\diamond_R) \end{array}\right\} \Rightarrow U \cap A^\diamond_R \neq \emptyset \Rightarrow (\exists y \in X)(y \in U \cap A^\diamond_R) \Rightarrow (\exists y \in X)(y \in U)(y \in A^\diamond_R)$$
$$\Rightarrow (U \in RO(X,y))(y \in A^\diamond_R)$$
$$\Rightarrow U^c \cup A^c \in \mathcal{P}.$$

Then, we have $x \in A^\diamond_R$. Thus, $\delta\text{-}cl(A^\diamond_R) \subseteq A^\diamond_R \ldots (2)$
$(1),(2) \Rightarrow \delta\text{-}cl(A^\diamond_R) = A^\diamond_R \Rightarrow A^\diamond_R \in \delta C(X)$.

e) Let $A \subseteq X$.

$$A \subseteq X \overset{(d)}{\Rightarrow} A^\diamond_R \in \delta C(X) \overset{(b)}{\Rightarrow} (A^\diamond_R)^\diamond_R \subseteq A^\diamond_R.$$

f) Let $A \subseteq B$ and $x \in A^\diamond_R$. We will show that $x \in B^\diamond_R$.

$$\left.\begin{array}{r} x \in A_R^\diamond \Rightarrow (\forall U \in RO(X,x))(U^c \cup A^c \in \mathcal{P}) \\ A \subseteq B \Rightarrow B^c \subseteq A^c \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\forall U \in RO(X,x))(U^c \cup B^c \subseteq U^c \cup A^c \in \mathcal{P}) \\ \mathcal{P} \text{ is a primal on } X \end{array}\right\} \Rightarrow (\forall U \in RO(X,x))(U^c \cup B^c \in \mathcal{P}) \Rightarrow x \in B_R^\diamond.$$

g) Let $A, B \subseteq X$.

$$\left.\begin{array}{r} A, B \subseteq X \Rightarrow A \subseteq A \cup B \Rightarrow A_R^\diamond \subseteq (A \cup B)_R^\diamond \\ A, B \subseteq X \Rightarrow B \subseteq A \cup B \Rightarrow B_R^\diamond \subseteq (A \cup B)_R^\diamond \end{array}\right\} \Rightarrow A_R^\diamond \cup B_R^\diamond \subseteq (A \cup B)_R^\diamond \ldots (1)$$

Conversely, let $x \notin A_R^\diamond \cup B_R^\diamond$.

$$\left.\begin{array}{r} x \notin A_R^\diamond \cup B_R^\diamond \Rightarrow (x \notin A_R^\diamond)(x \notin B_R^\diamond) \Rightarrow (\exists U, V \in RO(X,x))(U^c \cup A^c \notin \mathcal{P})(V^c \cup B^c \notin \mathcal{P}) \\ W := U \cap V \end{array}\right\} \Rightarrow$$

$\Rightarrow (W \in RO(X,x))(W^c \cup A^c \notin \mathcal{P})(W^c \cup B^c \notin \mathcal{P})$
$\Rightarrow (W \in RO(X,x))(W^c \cup (A \cup B)^c = (W^c \cup A^c) \cap (W^c \cup B^c) \notin \mathcal{P})$
$\Rightarrow x \notin (A \cup B)_R^\diamond$.

Then, we have $(A \cup B)_R^\diamond \subseteq A_R^\diamond \cup B_R^\diamond \ldots (2)$

$(1), (2) \Rightarrow (A \cup B)_R^\diamond = A_R^\diamond \cup B_R^\diamond$.

h) This follows from (f). □

Remark 3.6. *The equality in Theorem 4.1 (h) need not to be true, as seen in the following example.*

Example 3.7. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ *and* $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. *Now, if* $A = \{b\}$ *and* $B = \{c\}$, *then, we have* $A_R^\diamond \cap B_R^\diamond = \{b\} \cap \{b, c\} = \{b\} \neq \emptyset = \emptyset_R^\diamond = (A \cap B)_R^\diamond$.

Theorem 3.8. *Let* $(X, \tau, \mathcal{P})$ *be a primal topological space and* $A, B \subseteq X$. *If* $A \in \delta O(X)$, *then* $A \cap B_R^\diamond \subseteq (A \cap B)_R^\diamond$.

Proof. Let $A \in \delta O(X)$ and $x \in A \cap B_R^\diamond$.

$$\left.\begin{array}{r} x \in A \cap B_R^\diamond \Rightarrow (x \in A)(x \in B_R^\diamond) \Rightarrow (x \in A)(\forall U \in RO(X,x))(U^c \cup B^c \in \mathcal{P}) \\ A \in \delta O(X) \Rightarrow (\exists \mathcal{A} \subseteq RO(X))\,(A = \bigcup \mathcal{A}) \Rightarrow (\exists V \in RO(X))(V \subseteq A) \end{array}\right\} \Rightarrow$$

$\Rightarrow (\forall U \in RO(X,x))(U \cap V \in RO(X,x))(U^c \cup (A \cap B)^c = (U \cap A)^c \cup B^c \subseteq (U \cap V)^c \cup B^c \in \mathcal{P})$
$\Rightarrow (\forall U \in RO(X,x))(U^c \cup (A \cap B)^c \in \mathcal{P})$
$\Rightarrow x \in (A \cap B)_R^\diamond$. □

Theorem 3.9. *Let* $(X, \tau, \mathcal{P})$ *be a primal topological space. Then, the following statements are equivalent:*

a) $X_R^\diamond = X$;
b) $RC(X) \setminus \{X\} \subseteq \mathcal{P}$;
c) $A \subseteq A_R^\diamond$ for all regular open subsets A of X.

Proof. $(a) \Rightarrow (b)$: Let $X_R^\diamond = X$.

$$\begin{array}{rcl} X_R^\diamond = X & \Rightarrow & (\forall x \in X)(x \in X_R^\diamond) \\ & \Rightarrow & (\forall x \in X)(\forall U \in RO(X,x))(U^c \cup X^c = U^c \in \mathcal{P}) \\ & \Rightarrow & (\forall V \in RC(X) \setminus \{X\})(V \in \mathcal{P}) \\ & \Rightarrow & RC(X) \setminus \{X\} \subseteq \mathcal{P}. \end{array}$$

$(b) \Rightarrow (a)$: Let $x \in X$ and $U \in RO(X, x)$.

$$\left.\begin{array}{r} U \in RO(X,x) \Rightarrow U^c \in RC(X) \setminus \{X\} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow U^c \cup X^c = U^c \cup \emptyset = U^c \in \mathcal{P}.$$

Then, we have $x \in X_R^\diamond$. Thus, $X \subseteq X_R^\diamond \subseteq X$ and so $X_R^\diamond = X$.

$(b) \Rightarrow (c)$: Let $A \in RO(X)$.

$$\left.\begin{array}{r} A \in RO(X) \overset{\text{Theorem 3.8}}{\Rightarrow} A \cap X_R^\diamond \subseteq (A \cap X)_R^\diamond = A_R^\diamond \\ RC(X) \setminus \{X\} \subseteq \mathcal{P} \Rightarrow X_R^\diamond = X \end{array}\right\} \Rightarrow A \subseteq A_R^\diamond.$$

$(c) \Rightarrow (b)$: Let $A \in RC(X) \setminus \{X\}$.

$$\left.\begin{array}{r} A \in RC(X) \setminus \{X\} \Rightarrow A^c \in RO(X) \setminus \{\emptyset\} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow A^c \subseteq (A^c)_R^\diamond \Rightarrow (\forall x \in A^c)(x \in (A^c)_R^\diamond).$$

$$\left.\begin{array}{r} \Rightarrow (\forall x \in A^c)(\forall U \in RO(X,x))(U^c \cup (A^c)^c = U^c \cup A \in \mathcal{P}) \\ A \subseteq U^c \cup A \end{array}\right\} \Rightarrow A \in \mathcal{P}.$$

So, we have $RC(X) \setminus \{X\} \subseteq \mathcal{P}$. □

Theorem 3.10. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A \subseteq X$. If $A_R^\diamond \neq \emptyset$, then, $A^c \in \mathcal{P}$.*

Proof. Let $A_R^\diamond \neq \emptyset$.

$$\left.\begin{array}{r} A_R^\diamond \neq \emptyset \Rightarrow (\exists x \in X)(x \in A_R^\diamond) \Rightarrow (\forall U \in RO(X,x))(A^c \subseteq U^c \cup A^c \in \mathcal{P}) \\ \mathcal{P} \text{ is a primal on } X \end{array}\right\} \Rightarrow A^c \in \mathcal{P}.$$ □

Corollary 3.11. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A \subseteq X$. If $A^c \notin \mathcal{P}$, then $A_R^\diamond = \emptyset$.*

Theorem 3.12. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A, B \subseteq X$. Then, $A_R^\diamond \setminus B_R^\diamond = (A \setminus B)_R^\diamond \setminus B_R^\diamond$.*

Proof. Let $A, B \subseteq X$.

$$\begin{array}{rcl} A, B \subseteq X & \Rightarrow & A = (A \setminus B) \cup (A \cap B) \\ & \Rightarrow & A_R^\diamond = [(A \setminus B) \cup (A \cap B)]_R^\diamond \\ & \Rightarrow & A_R^\diamond = (A \setminus B)_R^\diamond \cup (A \cap B)_R^\diamond \subseteq (A \setminus B)_R^\diamond \cup B_R^\diamond \\ & \Rightarrow & A_R^\diamond \setminus B_R^\diamond \subseteq (A \setminus B)_R^\diamond \\ & \Rightarrow & A_R^\diamond \setminus B_R^\diamond \subseteq (A \setminus B)_R^\diamond \setminus B_R^\diamond \ldots (1) \\ A, B \subseteq X & \Rightarrow & A \setminus B \subseteq A \\ & \Rightarrow & (A \setminus B)_R^\diamond \subseteq A_R^\diamond \\ & \Rightarrow & (A \setminus B)_R^\diamond \setminus B_R^\diamond \subseteq A_R^\diamond \setminus B_R^\diamond \ldots (2) \end{array}$$

$(1), (2) \Rightarrow A_R^\diamond \setminus B_R^\diamond = (A \setminus B)_R^\diamond \setminus B_R^\diamond$. □

Theorem 3.13. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A, B \subseteq X$. If $B^c \notin \mathcal{P}$, then $(A \cup B)_R^\diamond = A_R^\diamond = (A \setminus B)_R^\diamond$.*

Proof. Let $A, B \subseteq X$.

$$\left.\begin{array}{r} A, B \subseteq X \overset{\text{Theorem 3.12}}{\Rightarrow} A_R^\diamond \setminus B_R^\diamond = (A \setminus B)_R^\diamond \setminus B_R^\diamond \\ B^c \notin \mathcal{P} \overset{\text{Corollary 3.11}}{\Rightarrow} B_R^\diamond = \emptyset \end{array}\right\} \Rightarrow A_R^\diamond = (A \setminus B)_R^\diamond \ldots (1)$$

$$\left.\begin{array}{r} A, B \subseteq X \overset{\text{Theorem 4.1}}{\Rightarrow} (A \cup B)_R^\diamond = A_R^\diamond \cup B_R^\diamond \\ B^c \notin \mathcal{P} \overset{\text{Corollary 3.11}}{\Rightarrow} B_R^\diamond = \emptyset \end{array}\right\} \Rightarrow (A \cup B)_R^\diamond = A_R^\diamond \ldots (2)$$

$(1), (2) \Rightarrow (A \cup B)_R^\diamond = A_R^\diamond = (A \setminus B)_R^\diamond$. □

4. The operator $cl^{\diamond}_{R}$ and $\tau^{\diamond}_{R}$ topology

In this section, we define and investigate a new operator using the operator $(\cdot)^{\diamond}_{R}$ and investigate its properties. The operator defined in this section turns out to be a Kuratowski closure operator whereas the operator $(\cdot)^{\diamond}_{R}$ is not. Using this operator, we obtain a new topology, denoted by $\tau^{\diamond}_{R}$, which is finer than τ_{δ}. Also, we get a base for this new topology.

Definition 4.1. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. We define the operator $cl^{\diamond}_{R} : 2^X \to 2^X$ by $cl^{\diamond}_{R}(A) = A \cup A^{\diamond}_{R}$ for all $A \in 2^X$.*

Theorem 4.2. *Let $(X,\tau,\mathcal{P})$ be a primal topological space and $A, B \subseteq X$. Then, the following statements hold:*

a) $cl^{\diamond}_{R}(\emptyset) = \emptyset$,
b) $cl^{\diamond}_{R}(X) = X$,
c) $A \subseteq cl^{\diamond}(A) \subseteq cl^{\diamond}_{R}(A)$,
d) If $A \subseteq B$, then $cl^{\diamond}_{R}(A) \subseteq cl^{\diamond}_{R}(B)$,
e) $cl^{\diamond}_{R}(A \cup B) = cl^{\diamond}_{R}(A) \cup cl^{\diamond}_{R}(B)$,
f) $cl^{\diamond}_{R}(cl^{\diamond}_{R}(A)) = cl^{\diamond}_{R}(A)$.

Proof. *a*) Since $\emptyset^{\diamond}_{R} = \emptyset$, we have $cl^{\diamond}_{R}(\emptyset) = \emptyset \cup \emptyset^{\diamond}_{R} = \emptyset$.

b) Since $X^{\diamond}_{R} \subseteq X$, we have $cl^{\diamond}_{R}(X) = X \cup X^{\diamond}_{R} = X$.

c) Since $cl^{\diamond}_{R}(A) = A \cup A^{\diamond}_{R}$, we have $A \subseteq cl^{\diamond}_{R}(A)$. Also, since $A^{\diamond} \subseteq A^{\diamond}_{R}$, we have $cl^{\diamond}(A) \subseteq cl^{\diamond}_{R}(A)$.

d) Let $A \subseteq B$,

$$A \subseteq B \Rightarrow A^{\diamond}_{R} \subseteq B^{\diamond}_{R} \Rightarrow A \cup A^{\diamond}_{R} \subseteq B \cup B^{\diamond}_{R} \Rightarrow cl^{\diamond}_{R}(A) \subseteq cl^{\diamond}_{R}(B).$$

e) Let $A, B \subseteq X$,

$$\begin{aligned} cl^{\diamond}_{R}(A \cup B) &= (A \cup B) \cup (A \cup B)^{\diamond}_{R} \\ &= (A \cup B) \cup (A^{\diamond}_{R} \cup B^{\diamond}_{R}) \\ &= (A \cup A^{\diamond}_{R}) \cup (B \cup B^{\diamond}_{R}) \\ &= cl^{\diamond}_{R}(A) \cup cl^{\diamond}_{R}(B). \end{aligned}$$

f) Let $A \subseteq X$. It is obvious from (c) that $cl^{\diamond}_{R}(A) \subseteq cl^{\diamond}_{R}(cl^{\diamond}_{R}(A)) \ldots (1)$

$$\left.\begin{array}{r} cl^{\diamond}_{R}(cl^{\diamond}_{R}(A)) = cl^{\diamond}_{R}(A) \cup \left(cl^{\diamond}_{R}(A)\right)^{\diamond}_{R} = cl^{\diamond}_{R}(A) \cup (A \cup A^{\diamond}_{R})^{\diamond}_{R} = cl^{\diamond}_{R}(A) \cup A^{\diamond}_{R} \cup (A^{\diamond}_{R})^{\diamond}_{R} \\ A \subseteq X \Rightarrow A^{\diamond}_{R} \in \delta C(X) \Rightarrow (A^{\diamond}_{R})^{\diamond}_{R} \subseteq A^{\diamond}_{R} \end{array}\right\} \Rightarrow$$

$$\Rightarrow cl^{\diamond}_{R}(cl^{\diamond}_{R}(A)) \subseteq cl^{\diamond}_{R}(A) \cup A^{\diamond}_{R} \cup A^{\diamond}_{R} = cl^{\diamond}_{R}(A) \ldots (2)$$

$(1), (2) \Rightarrow cl^{\diamond}_{R}(cl^{\diamond}_{R}(A)) = cl^{\diamond}_{R}(A)$. □

Corollary 4.3. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. Then, the operator $cl^{\diamond}_{R} : 2^X \to 2^X$ defined by $cl^{\diamond}_{R}(A) = A \cup A^{\diamond}_{R}$, where A is any subset of X, is a Kuratowski closure operator.*

Definition 4.4. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. Then, the family $\tau^{\diamond}_{R} = \{A \subseteq X : cl^{\diamond}_{R}(A^c) = A^c\}$ is a topology on X induced by topology τ and primal $\mathcal{P}$. We can also use the notation $\tau^{\diamond}_{R(\mathcal{P})}$ instead of $\tau^{\diamond}_{R}$ to indicate the primal if necessary.*

Theorem 4.5. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the following statements hold:*

a) $\tau_\delta \subseteq \tau^\diamond_R$, where τ_δ is the family of all δ-open sets in a topological space (X, τ).
b) $\tau^\diamond_R \subseteq \tau^\diamond$.

Proof. *a*) Let $A \in \tau_\delta$. We will show that $A \in \tau^\diamond_R$,

$$\left.\begin{array}{r} A \in \tau_\delta \Rightarrow A^c \in \delta C(X) \Rightarrow (A^c)^\diamond_R \subseteq A^c \Rightarrow A^c \cup (A^c)^\diamond_R = A^c \\ cl^\diamond_R(A^c) = A^c \cup (A^c)^\diamond_R \end{array}\right\} \Rightarrow cl^\diamond_R(A^c) = A^c \Rightarrow A \in \tau^\diamond_R.$$

b) Let $A \in \tau^\diamond_R$. We will show that $A \in \tau^\diamond$,

$$\begin{array}{rcl} A \in \tau^\diamond_R & \Rightarrow & cl^\diamond_R(A^c) = A^c \\ & \Rightarrow & A^c \cup (A^c)^\diamond_R = A^c \\ & \overset{\text{Theorem 4.1}}{\Rightarrow} & (A^c)^\diamond \subseteq (A^c)^\diamond_R \subseteq A^c \\ & \Rightarrow & (A^c)^\diamond \cup A^c = A^c \\ & \Rightarrow & cl^\diamond(A^c) = A^c \\ & \Rightarrow & A \in \tau^\diamond. \quad \square \end{array}$$

Corollary 4.6. *From Theorem 4.5, we have the following diagram of implications:*

$$\begin{array}{ccccc} & & \tau^\diamond\text{-}open & & \\ & \nearrow & & \nwarrow & \\ \tau^\diamond_R\text{-}open & & & & \tau\text{-}open \\ & \nwarrow & & \nearrow & \\ & & \tau_\delta\text{-}open & & \end{array}$$

Remark 4.7. *The converses of the implications given in the above diagram need not to be true, as shown by the following examples. Also, the notions of $\tau^\diamond_R$-open and τ-open are independent of each other.*

Example 4.8. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$ and let $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Simple calculations show that $\tau_\delta = \{\emptyset, X\}$, $\tau^\diamond_R = \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$, and $\tau^\diamond = 2^X$.*

1) The set $\{c\}$ is $\tau^\diamond_R$-open but not τ_δ-open.
2) The set $\{a\}$ is $\tau^\diamond$-open but not $\tau^\diamond_R$-open.
3) The set $\{c\}$ is $\tau^\diamond_R$-open but not τ-open.
4) The set $\{b\}$ is τ-open but not $\tau^\diamond_R$-open.

Theorem 4.9. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A \subseteq X$. Then, the following statements hold:*

a) $A \in \tau^\diamond_R$ if and only if for all x in A, there exists a regular open set U containing x such that $U^c \cup A \notin \mathcal{P}$.
b) if $A \notin \mathcal{P}$, then, $A \in \tau^\diamond_R$.

Proof. *a*) Let $A \subseteq X$,

$$\begin{array}{rcl} A \in \tau_R^\diamond & \Leftrightarrow & cl_R^\diamond(A^c) = A^c \\ & \Leftrightarrow & A^c \cup (A^c)_R^\diamond = A^c \\ & \Leftrightarrow & (A^c)_R^\diamond \subseteq A^c \\ & \Leftrightarrow & A \subseteq ((A^c)_R^\diamond)^c \\ & \Leftrightarrow & (\forall x \in A)(x \notin (A^c)_R^\diamond) \\ & \Leftrightarrow & (\forall x \in A)(\exists U \in RO(X,x))(U^c \cup (A^c)^c = U^c \cup A \notin \mathcal{P}). \end{array}$$

b) Let $A \notin \mathcal{P}$ and $x \in A$. We will make use of (a),

$$\left.\begin{array}{r} (U := X)(x \in A) \Rightarrow (U \in RO(X,x))(A = U^c \cup A) \\ A \notin \mathcal{P} \end{array}\right\} \Rightarrow U^c \cup A \notin \mathcal{P}.$$

Thus, we have $A \in \tau_R^\diamond$ from (a). □

Remark 4.10. *The converse of Theorem 4.9(b) need not to be true, as shown by the following example.*

Example 4.11. *Let $X = \{a,b,c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$. Simple calculations show that $\tau_R^\diamond = 2^X$. It is obvious that the set $\{b\}$ belongs to both $\tau_R^\diamond$ and $\mathcal{P}$.*

Theorem 4.12. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the following statements hold:*

a) if $\mathcal{P} = \emptyset$, then $\tau_R^\diamond = 2^X$,
b) if $\mathcal{P} = 2^X \setminus \{X\}$, then $\tau_\delta = \tau_R^\diamond$.

Proof. *a*) We have $\tau_R^\diamond \subseteq 2^X \ldots (1)$. Now, let $A \in 2^X$. We need to show that $A \in \tau_R^\diamond$.

$$\left.\begin{array}{r} cl_R^\diamond(A^c) = A^c \cup (A^c)_R^\diamond \\ \mathcal{P} = \emptyset \Rightarrow (A^c)_R^\diamond = \emptyset \end{array}\right\} \Rightarrow cl_R^\diamond(A^c) = A^c \Rightarrow A \in \tau_R^\diamond.$$

So, we have $2^X \subseteq \tau_R^\diamond \ldots (2)$

$(1), (2) \Rightarrow \tau_R^\diamond = 2^X$.

b) We have $\tau_\delta \subseteq \tau_R^\diamond$ by Theorem 4.5. Let us now show $\tau_R^\diamond \subseteq \tau_\delta$. Let $A \in \tau_R^\diamond$.

$$\left.\begin{array}{r} A \in \tau_R^\diamond \overset{\text{Theorem 4.9}}{\Rightarrow} (\forall x \in A)(\exists U \in RO(X,x))(U^c \cup A \notin \mathcal{P}) \\ \mathcal{P} = 2^X \setminus \{X\} \end{array}\right\} \Rightarrow$$

$$\begin{array}{l} \Rightarrow (\forall x \in A)(\exists U \in RO(X,x))(U^c \cup A = X) \\ \Rightarrow (\forall x \in A)(\exists U \in RO(X,x))(U \cap A^c = \emptyset) \\ \Rightarrow (\forall x \in A)(x \notin \delta\text{-}cl(A^c) = (\delta\text{-}int(A))^c) \\ \Rightarrow (\forall x \in A)(x \in \delta\text{-}int(A)) \\ \Rightarrow A = \delta\text{-}int(A) \\ \Rightarrow A \in \tau_\delta. \end{array}$$

□

Remark 4.13. *The converses of (a) and (b) in Theorem 4.12 need not to be true, as shown by the following examples.*

Example 4.14. *Let $X = \{a,b,c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$. Simple calculations show that $\tau_R^\diamond = 2^X$, but $\mathcal{P} \neq \emptyset$.*

Example 4.15. *Let $X = \{a,b,c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $\mathcal{P} = 2^X \setminus \{X, \{a,b\}\}$. Simple calculations show that $\tau = \tau_\delta = \tau_R^\diamond$, but $\mathcal{P} \neq 2^X \setminus \{X\}$.*

Theorem 4.16. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. Then, the family $\mathcal{B} = \{T \cap P | (T \in RO(X))(P \notin \mathcal{P})\}$ is a base for the topology $\tau_R^\diamond$ on X.*

Proof. Let $B \in \mathcal{B}$,

$$\left.\begin{array}{r} B \in \mathcal{B} \Rightarrow (\exists T \in RO(X))(\exists P \notin \mathcal{P})(B = T \cap P) \\ RO(X) \subseteq \tau_\delta \subseteq \tau_R^\diamond \end{array}\right\} \overset{\text{Theorem 4.9}}{\Rightarrow} (T, P \in \tau_R^\diamond)(B = T \cap P)$$

$\Rightarrow B \in \tau_R^\diamond$.

Thus, we have $\mathcal{B} \subseteq \tau_R^\diamond$. Now, let $A \in \tau_R^\diamond$ and $x \in A$. We need to find $B \in \mathcal{B}$ such that $x \in B \subseteq A$.

$$\left.\begin{array}{r} x \in A \in \tau_R^\diamond \overset{\text{Theorem 4.9}}{\Rightarrow} (\exists U \in RO(X,x))(U^c \cup A \notin \mathcal{P}) \\ B := U \cap (U^c \cup A) \end{array}\right\} \Rightarrow (B \in \mathcal{B})(x \in B \subseteq A).$$

Hence, $\mathcal{B}$ is a base for the topology $\tau_R^\diamond$ on X. □

Theorem 4.17. *Let $(X,\tau,\mathcal{P})$ and $(X,\tau,\mathcal{Q})$ be two primal topological spaces. If $\mathcal{P} \subseteq \mathcal{Q}$, then $\tau^\diamond_{R(\mathcal{Q})} \subseteq \tau^\diamond_{R(\mathcal{P})}$.*

Proof. Let $A \in \tau^\diamond_{R(\mathcal{Q})}$,

$$\left.\begin{array}{r} A \in \tau^\diamond_{R(\mathcal{Q})} \Rightarrow (\forall x \in A)(\exists U \in RO(X,x))(U^c \cup A \notin \mathcal{Q}) \\ \mathcal{P} \subseteq \mathcal{Q} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\forall x \in A)(\exists U \in RO(X,x))(U^c \cup A \notin \mathcal{P})$

$\Rightarrow A \in \tau^\diamond_{R(\mathcal{P})}$. □

5. Conclusions and future work

In this article, we defined two new operators, denoted by $(\cdot)_R^\diamond$ and $cl_R^\diamond(\cdot)$, via the notion of primal and investigated their properties. The second one turns out to be a Kuratowski closure operator, whereas the first is not. Hence, we obtained a new topology $\tau_R^\diamond$, which is finer than τ_δ. On the other hand, we have shown that the notions of $\tau_R^\diamond$-open and τ-open are independent. Moreover, we constructed a basis for this new topology $\tau_R^\diamond$ and proved several fundamental results. Furthermore, we proved some relationships between this new topology and the other topologies that existed in the literature and we also gave several examples. We hope that this paper will stimulate further research on primals and rough sets as ideals.

In future work, we will study rough approximations based on different topologies via primals. Also, we will generate new topologies from other types of neighborhoods and primals.

Author contributions

Murad ÖZKOÇ: conceptualization, formal analysis, investigation, methodology, project administration, visualization, writing-review and editing. Büşra KÖSTEL: conceptualization, formal analysis, investigation, methodology, visualization, writing-original draft preparation.

All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Authors do not have any conflict of interest with any other person or organization.

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