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Research article

Application of symmetry analysis and conservation laws to a fractional-order nonlinear conduction-diffusion model

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Abstract: In this paper, the Lie symmetry analysis was executed for the nonlinear fractional-order conduction-diffusion Buckmaster model (BM), which involves the Riemann-Liouville (R-L) derivative of fractional-order ‘β’. In the study of groundwater flow and oil reservoir engineering where fluid flow through porous materials is crucial, BM played an important role. The Lie point infinitesimal generators and Lie algebra were constructed for the equation. The Lie symmetries were acquired for the ordinary fractional-order BM. The power series solution and its convergence were also analyzed with the application of the implicit theorem. Noether’s theorem was employed to ensure the consistency of a system by deriving the conservation laws of its physical model.

Keywords: convection-diffusion equation; Buckmaster model; Riemann-Liouville derivatives; fractional differential equations; Erdelyi-Kober fractional operators
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1. Introduction

In recent decades, fractional-order mathematical models and their application have found huge attention in various disciplines. The reason for the popularity of the fractional-order differential equations is mainly due to the evolution of the theory of fractional calculus and their capacity to explain complex phenomena. Fractional order differential equations are a generalization of integral order differential equations and can be generalized in time and space with the help of a power-law type long memory kernel of the nonlocal operator. The fractional-order model furnishes a robust tool to explain the memories of various materials and the nature of the legacy. Altogether, these studies have a translucent visible background, which unblocks a new branch of research involving hypothetical and numerical analysis of various fractional-order dynamical systems in fluid dynamics, mechanics, biological modelling, physics, engineering, and areas of medical and other sciences. Podlubny [1], Oldham [2], Debnath [3], and Kilbas et al. [4] have elucidated the significance and practical implications of employing local generalized derivatives of fractional order in real-world phenomena.

Fractional order systems have been solved with a different group of methodologies. Wazwaz [5,6] applied the variation iterative method, Tanh method, and sine-cosine analysis to linear and nonlinear systems. Gardner equation has been solved by Lin et al. [7] with the imposition of the tanh-coth method and Iyiola et al. [8,9] have described applications of Caputo fractional derivatives in different nonlinear time-fractional homogeneous and non-homogeneous models. Jafari et al. [10] explored the numerical scheme to study the system of fractional PDEs. Gandhi et al. [11–15] studied the numerous fractional order PDEs using different techniques.

In recent times, Lie symmetry theory plays a very important role in the invariant analysis of Fractional differential equations. Olver [16] emphasized a wide range of applications of Lie group symmetries analysis to partial differential equations (PDEs). Bakkyaraj and Sahadevan [17] illustrated Lie group transformation to solve the fractional-order system. Moyo& Leach [18] presented the mathematical cancer model by symmetry analysis. The time-fractional Korteweg-de-Varies equations have been solved by Zhang [19]. Biswas et al. [20,21] organized multiple objectives like solitons, bifurcation analysis, conservation analysis, dual dispersion, and nonlinearity laws of Boussinesq equation. Bansal et al. [22] have designed optical perturbation, Lie group invariants to Fokas-Lenells equation. The symmetry reduction has been applied to clarify the soliton solution of time-fractional KdV and *K(m,n)* equations by Wang et al. [23,24]. The Harry-Dym equation with Riemann-Liouville fractional derivative has been studied by Huang et al. [25]. Garrido et al. [26] suggested Lie point symmetry along with traveling wave solution to generalized Drinfeld-Sokolov system; Bokhari et al. [27] illustrated fundamentals of symmetries to time-fractional tumour growth in the brain. Liu et al. [28] and Singla et al. [29] declared that the Lie symmetry reduction is a robust and authentic technique to solve higher-order nonlinear systems. The extensive use of Erdelyi–Kober fractional operators to help convert FPDEs into fractional ODEs has been stated by Sneddon [30]. Balsar et al. [31] attempted the sum ability of the series solution of PDEs with constant coefficients. Using the same technique, Shi et al. [32] and Razborova et al. [33] explained the additional conservation laws and exact solution to Boussinesq-Burgers system and Rosenau-KdV-RLW equation. The study of diffusion and sub-diffusion wave equations with conservation laws has been concluded by Lukashuk et al. [34]. Anco et al. [35] focused on the direct construction of conservation laws of linear and nonlinear PDEs. The nonlinear self-adjointness to the time-fractional Kompaneets equation has been studied by Gazizov et al. [36]. In addition, recently, Gandhi et al. [37] focused on invariant analysis, exact series solution and the convergence of solution by implicit theorem on fractional-order Hirota-Satsuma Coupled KdV system. A comparative study for solving Laplace fractional equation has been produced by Dubey et al. [38]. Chatibi et al. [39] have done the discrete symmetry analysis of some global and local systems. The invariant solution of generalized fractional order (2+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation has been evaluated by Chauhan et al. [40]. Gandarias et al. [41] have discussed the conservation laws and travelling wave solutions for double dispersion equations in (1+1) and (2+1) dimensions. Bruzon et al. [42] found a similarity solution of the Cooper-Shepard-Sodano equation along with the utilization of conservation analysis. The work of Edwards [45], Antim et al. [50], Hussain et al. [51,52], and Faridi et al. [53–56] is worth mentioning in this context.

In this paper, we examine the nonlinear time fractional convection-diffusion equation given by

$$\partial_t^\beta v = D(v)\,v_{xx} + D^{'}(v)\,(v_x)^2 + C(v)\,v_x, \tag{1}$$

where $0<\beta<1$ and $v(x,t)$ represents the density of particles, $D(v)$ is a dispersion or diffusive term, and $C(v)$ is a conductive or convective term; both $D(v)$ and $C(v)$ are non-zero terms. We have considered a special case of conduction-dispersion phenomenon when $D(v)=4v^3$ and $C(v)=3v^2$ in Eq (1), which is known as Buckmaster Model (BM) and it is extremely effective and relevant to explore the propagation of sound, electricity, and electrodynamics in physical systems. We know that buckling is the process of uncertainty that originates in thin materials due to pressure exceeding and making the material bend out of the plane. The BM equation (1) is also meant for dynamical modeling of thin sheet fluid flows to draw buckling, suggested by Buckmaster [43].

The nonlinear convection-diffusion equations have a great contribution to the model of the evolution of thermal waves in plasma by Rosenau and Kamin [44]. Movement is caused within a fluid by the propensity of hotter or less dense material to become colder, denser material to sink under the impact of gravity, which in consequence shows in the transfer of heat is called convection. The action of distributing matter by the natural movement of particles is called diffusion. The classical nonlinear convection-diffusion equation is given by Edward [45].

Our research article is organized as some basic definitions in Section 2, Lie symmetry methodology algorithm for BM is explained in part 3, and infinitesimal generators have been deduced using symmetry reduction in Section 4, followed by reduction of FPDEs into FODEs with the utilization of Erdelyi-Kober operators in Section 5, the power series solutions of respective FODE of BM and their convergence have been studied in subsequent Sections 6 and 7, respectively, and finally conservation laws have been defined in Section 8, which impart great information about physical BM system.

2. Preliminaries

In this section, we provided an overview of the fundamental concepts surrounding fractional derivatives and integrals. Various definitions exist for fractional derivatives, including the Grunwald-Letnikov (GL), Riemann-Liouville (RL), and Caputo formulations. Each of these definitions offers unique advantages and occasional drawbacks, depending on the circumstances. However, our focus lies on utilizing the Riemann-Liouville fractional derivative to investigate symmetry reduction and attain exact solutions for Fractional Partial Differential Equations (FPDEs). Below, we present some essential definitions necessary for our exploration:

Definition 2.1. The R-L fractional partial derivative of order 'β>0' for arbitrary function $v(x,t)$ with time variable 't' is given as

$$D_t^{\beta}(u(x,t)) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \dfrac{\partial^m}{\partial t^m} \int\limits_0^t (t-\xi)^{m-\beta-1} v(\xi,x) d\xi \ ; \, for \ \ m-1<\beta<m, \, t>0, \ \ m \in N, \\ \dfrac{\partial^m u}{\partial t^m} \qquad\qquad ; \, for \ \ \beta = m. \end{cases} \quad (2)$$

Definition 2.2. The R-L integral of fractional order 'β>0'and '0<t<T' is defined as

$${}_0 J_t^{\beta} g(t,x) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\xi)^{\beta-1} g(\xi,x) d\xi; \quad (3)$$

$${}_t J_T^{\beta} g(t,x) = \frac{1}{\Gamma(\beta)} \int_t^T (t-\xi)^{\beta-1} g(\xi,x) d\xi, \quad (4)$$

Some important results associated with the above operators and used in this paper are:

$$D_t^{\beta}(t^{\alpha}) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta}; \quad (5)$$

$$J_t^{\beta}(t^{\alpha}) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}; \quad (6)$$

$$J_t^{\beta}(D_t^{\beta}(\phi(t)) = \phi(t) - \sum_{k=0}^{\lambda-1} \frac{\phi^{(k)}(0)}{k!} t^k \ \ ; \lambda-1<\beta \le \lambda; \quad (7)$$

$$D_t^{\beta}(g(t)) = D_t^n J_t^{n-\beta}(g(t)); \, n-1<\beta \le n. \quad (8)$$

3. Methodology

In this section, we give a summary of the Lie symmetry analysis for fractional partial differential equations (FPDEs) given by:

$$\partial_t^{\beta} v = F(x,t,v,v_x,v_{xx},...) \ ; \ \beta \in (0,1) \, . \quad (9)$$

In one-parameter Lie group of transformations, the infinitesimals are to be determined in such a way that the PDE (9) is invariant under the group of transformations; the entity 'ε' is a small parameter such that its square and higher powers may be neglected. The existence of such a group reduces the number of independent variables by one, which allows us to replace the PDE by an ODE and it adheres to the following condition:

$$\begin{aligned} \bar{t} &= t + \varepsilon \tau(v;x,t) + O(\varepsilon^2); \\ \bar{x} &= x + \varepsilon \xi(v;x,t) + O(\varepsilon^2); \\ \bar{v} &= v + \varepsilon \eta(v;x,t) + O(\varepsilon^2); \\ \partial_t^{\beta} \bar{v} &= \partial_t^{\beta} v + \varepsilon \eta^{\beta,t}(v;x,t) + O(\varepsilon^2); \\ \partial_t^{1} \bar{v} &= \partial_t^{1} v + \varepsilon \eta^{x}(v;x,t) + O(\varepsilon^2); \\ \partial_t^{2} \bar{v} &= \partial_t^{2} v + \varepsilon \eta^{xx}(v;x,t) + O(\varepsilon^2), \end{aligned} \quad (10)$$

where τ, ξ and η are required infinitesimals η^x, η^{xx} are extended infinitesimals and $\eta^{\beta,t}$ is extended infinitesimal of fractional parameter of order 'β' associated to Lie algebra of (9) is spanned by vector fields

$$X = \eta\frac{\partial}{\partial v} + \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} \quad with\ \tau = \left.\frac{d\bar{t}}{d\varepsilon}\right|_{\varepsilon=0},\ \xi = \left.\frac{d\bar{x}}{d\varepsilon}\right|_{\varepsilon=0},\ \text{and}\ \ \eta = \left.\frac{d\bar{v}}{d\varepsilon}\right|_{\varepsilon=0}. \tag{11}$$

Prolongation to (9) carried

$$pr^{(\beta,2)}(\partial_t^\beta v - F)\,|_{\Delta=0} = 0, \tag{12}$$

where prolongation operator is defined by

$$pr^{(\beta,2)} = X + \eta^{\beta,t}\frac{\partial}{\partial(\partial_t^\beta v)} + \eta^x\frac{\partial}{\partial v_x} + \eta^{xx}\frac{\partial}{\partial v_{xx}}. \tag{13}$$

The expressions for extended infinitesimals are given as

$$\begin{aligned}\eta^x &= D_x(\eta) - v_t D_x(\tau) - v_x D_x(\xi)\\ &= \eta_x - (\xi_x - \eta_v)v_x - v_t\tau_x - \xi_v v_x^2 - \tau_v v_t v_x,\end{aligned} \tag{14}$$

$$\begin{aligned}\eta^{xx} &= D_x(\eta^x) - v_{xt}D_x(\tau) - v_{xx}D_x(\xi)\\ &= \eta_{xx} - (\xi_{xv} - 2\eta_{xv})v_x - \tau_{xx}v_t + (\eta_{vv} - 2\xi_{xv})v_x{}^2 - 2\tau_{xv}v_x v_t - \xi_{vv}v_x{}^3\\ &- \tau_{vx}v_x{}^2 v_t - 2\tau_x v_{xt} + (\eta_v - 2\xi_x)v_{xx} - \tau_v v_{xx}v_t - 2\tau_v v_{xt}v_x - 3\xi_v v_x v_{xx}.\end{aligned} \tag{15}$$

$$D_t^\beta(\eta) = \partial_t^\beta\eta + \eta_v\,\partial_t^\beta v - v\,\partial_t^\beta\eta_v + \sum_{m=1}^{\infty}\binom{\beta}{n}\partial_t^n(\eta_v)\partial_t^{\beta-n}(v) + \mu, \tag{16}$$

where

$$\mu = \sum_{\lambda=2}^{\infty}\sum_{n=2}^{\lambda}\sum_{k=2}^{n}\sum_{r=0}^{k-1}\binom{\beta}{\lambda}\binom{\lambda}{n}\binom{k}{r}\frac{t^{\lambda-\beta}(-v)^r}{\Gamma(k+1)\Gamma(\lambda+1-\beta)}\frac{\partial^n(v^{k-r})}{\partial t^n}\frac{\partial^{\lambda-n+k}\eta}{\partial t^{\lambda-n}\partial v^k}. \tag{17}$$

As 'η' is linear function of 'v' then $\mu\to 0$

$$\begin{aligned}\eta^{\beta,t} &= D_t^\beta(\eta) + \xi\,D_t^\beta(v_x) - D_t^\beta(\xi\,v_x) + D_t^\beta(v.D_t(\tau)) - D_t^{\beta+1}(\tau\,v) + \tau\,D_t^{\beta+1}v\\ &= \partial_t^\beta(\eta) + (\eta_v - \alpha D_t\tau)\partial_t^\beta(v) - v\partial_t^\beta(\eta_v) + \sum_{\lambda=1}^{\infty}\left[\binom{\beta}{\lambda}\partial_t^n(\eta_v) - \binom{\beta}{\lambda+1}D_t^\lambda(\xi)\partial_t^{\beta-\lambda}(v)\right]\partial_t^{\beta-\lambda}\\ &- \sum_{\lambda=1}^{\infty}\binom{\beta}{\lambda}D_t^\lambda(\xi)\partial_t^{\beta-\lambda}(v_x) + \mu.\end{aligned} \tag{18}$$

Finally, we use Eqs (13)–(18) in prolonged Eq (12), split the coefficients of v_x and v_{xx} and equate to zero; subsequently, we proceed to solve the system of fractional PDEs and ODEs derived from the process.

4. Fractional-order convection-diffusion Buckmaster model

Applying Lie symmetry method on BM (1), using Lie symmetry analysis to obtain following set of PDEs

$$\sum_{\lambda=1}^{\infty}\left[\binom{\beta}{\lambda}\partial_t^{\lambda}(\eta_v)-\binom{\beta}{\lambda+1}D_t^{\lambda}(\xi)\partial_t^{\beta-\lambda}(v)\right]=0; \tag{19}$$

$$\sum_{\lambda=1}^{\infty}\binom{\beta}{\lambda}D_t^{\lambda}(\xi)=0; \tag{20}$$

$$\tau_x=0,\ \tau_v=0:\eta_{vv}=0,\ \eta_{vx}=0\ ;\xi_v=0,\ \xi_t=0; \tag{21}$$

$$(24v)\eta+(12v^2)(\eta_v-2\xi_x+\beta\tau_t)=0; \tag{22}$$

$$(3v^2)(\xi_x-\beta\tau_t)-(6v)\eta-24v^2\eta_x-4v^3.(2\eta_{vx}-\xi_{xx})=0; \tag{23}$$

$$(4v^3)(2\xi_x-\beta\tau_t)-(12v^2)\eta=0. \tag{24}$$

In order to solve set of Eqs (19)–(24), infinitesimals in explicit form with arbitrary constants ‘p’ and ‘q’ are given by

$$\xi=px+q,\tau=\frac{-p}{\beta}t,\ \text{and}\ \eta=pv. \tag{25}$$

Infinitesimal generators are described as

$$S_1=\frac{\partial}{\partial x}\ \text{and}\ S_2=x\frac{\partial}{\partial x}-\frac{t}{\beta}\frac{\partial}{\partial t}+v\frac{\partial}{\partial v}. \tag{26}$$

The set $\{S_1, S_2\}$ forms Lie Algebra of obtained infinitesimal generators with the Lie braces operator $[X,Y]=XY-YX$ as we have explained in all above cases. The characteristic equation for S_2 is

$$\frac{dx}{x}=\frac{\beta dt}{-t}=-\frac{dv}{v}. \tag{27}$$

Solving Eq (27), we obtain similarity transformation

$$\zeta=xt^{\beta}\text{and } v=t^{-\beta}J(\zeta). \tag{28}$$

Related FODE with time fractional conduction-diffusion Buckmaster equation is

$$\frac{\partial^{\beta} v}{\partial t^{\beta}} = t^{-2\beta}[4(J(\zeta))^3 (J''(\zeta)) + 12(J(\zeta)^2)(J'(\zeta))^2 + 3(J(\zeta))^2 J'(\zeta)] \quad . \tag{29}$$

5. Applications of Erdelyi-Kober operators

Here, we illustrate the relevance of Erdelyi-Kober fractional differential and integral operators in solving FODEs. Before calculations of reduction of fractional operator $\frac{\partial^{\beta} v}{\partial t^{\beta}}$, let us define the Erdelyi-Kober operators as

$$\left(E_{\partial}^{\tau,\beta} J\right)(\zeta) = \prod_{k=0}^{n-1}\left(\tau + k - \frac{1}{\partial}\zeta \frac{d}{d\zeta}\right)(K_{\partial}^{\tau+\beta,n-\beta} J)(\zeta) \ ;$$

$$(K_{\partial}^{\tau,\beta} J)(\zeta) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_1^{\infty}(w-1)^{\beta-1} w^{-(\tau+\beta)} g(\zeta\, w^{1/\partial}) dw, \beta > 0; \\ J(\zeta) \qquad\qquad , \ \beta = 0; \end{cases}$$

$$\text{with } z>0,\ \partial > 0 \text{ and } \beta > 0; \text{ and } n = \begin{cases} [\beta]+1, & \beta \notin N; \\ \beta\ , & \text{otherwise} \end{cases} \quad . \tag{30}$$

Theorem. Under the similarity transformations (28) for vector field X_2, the reduced FODE (29) is

$$\frac{\partial^{\beta} v}{\partial t^{\beta}} = 4(J(\zeta))^3 (J''(z) + 12(J(\zeta))^2 (J'(\zeta))^2 + 3(J(\zeta))^2 J'(\zeta).$$

Now we will solve left hand side $\frac{\partial^{\beta} u}{\partial t^{\beta}}$ with the application of E-K operator.

Riemann Liouville derivative for similarity reduction is

$$D_t^{\beta} v = D_t^{\lambda}\left(\frac{1}{\Gamma(\lambda-\beta)}\int_0^t (t-s)^{\lambda-\beta-1} s^{-\beta} J(xs^{\beta}) ds\right). \tag{31}$$

Substituting $s = \frac{t}{\gamma}$ in (31), it reduces to

$$D_t^{\beta} v = D_t^{\lambda}\left(\frac{1}{\Gamma(\lambda-\beta)}\int_1^{\infty} (t - \frac{t}{\gamma})^{\lambda-\beta-1}\left(\frac{t}{\gamma}\right)^{-\beta} J(x(t/\gamma)^{\beta}) \frac{t}{\gamma^2} dv\right),$$

$$= D_t^{\lambda}\left(\frac{t^{\lambda-2\beta}}{\Gamma(\lambda-\beta)}\int_1^{\infty}(\gamma-1)^{\lambda-\beta-1}\gamma^{-(\lambda+1-2\beta)} J(\zeta\gamma^{-\beta}) d\gamma\right). \tag{32}$$

Using Eq (31), we obtain

$$\text{if } \zeta = xt^{-\beta},\ J \in C'(0,\infty)$$

$$D_t^{\beta} v = D_t^{\lambda}\left(t^{\lambda-2\beta}\left[\left(K_{-1/\beta}^{1-\beta,\lambda-\beta} J\right)(\zeta)\right]\right), \quad t\, D_t J(\zeta) = tx(-\beta)t^{-\beta-1} D_\zeta J(\zeta) = -\beta\,\zeta\, D_\zeta J(\zeta), \tag{33}$$

$$\begin{aligned} D_t^{\beta} v &= D_t^{\lambda-1} D_t\left(t^{\lambda-2\beta}\left[\left(K_{-1/\beta}^{1-\beta,\lambda-\beta} J\right)(\zeta)\right]\right) \\ &= D_t^{\lambda-1}\left(t^{\lambda-2\beta-1}\left(\lambda - 2\beta + \beta\,\zeta\, D_\zeta\right)\left[\left(K_{-1/\beta}^{1-\beta,\lambda-\beta} J\right)(\zeta)\right]\right). \end{aligned} \tag{34}$$

Reconsider similar arguments (λ-1) times, to get

$$D_t^{\beta} v = t^{-2\beta} \prod_{j=0}^{\lambda-1}\left(1 + j - 2\beta + \beta\,\zeta\, D_\zeta\right)\left(K_{-1/\beta}^{1-\beta,\lambda-\beta} J\right)(\zeta) = t^{-2\beta}\left(P_{-1/\beta}^{1-2\beta,\beta} J\right)(\zeta). \tag{35}$$

Finally, FODE becomes

$$\left(P_{-1/\beta}^{1-2\beta,\beta} J\right)(\zeta) = [4(J(\zeta))^3 J''(\zeta) + 12(J(\zeta))^2 (J'(\zeta))^2 + 3(J(\zeta))^2 J'(\zeta)]. \tag{36}$$

6. Power series solution of BM

Now, for further solution of FODEs, we want to explore the explicit power series solution [11,12], which can be applied to solve FODE (36).

Set power series

$$J(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n, \tag{37}$$

substituting (37) in (36), it yields

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\Gamma(2-\beta-n\beta)}{\Gamma(3-2\beta-n\beta)}.a_n.\zeta^n &= 4\sum_{n=0}^{\infty}\sum_{k=o}^{n}\sum_{i=0}^{k}\sum_{j=0}^{i}(n+2-k).(n+1-k)\,a_i\, a_{k-i}\, a_{i-j} a_{n+2-k}.\zeta^n \\ &+ 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)^2.a_i a_{k-i} a^2{}_{n+1-k}.\zeta^n + 3\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1).a_i a_{k-i} a_{n+k+1}\,\zeta^n. \end{aligned} \tag{38}$$

Put n=0 in (24) and comparing coefficients of ζ^n, we get

$$a_2 = -\frac{1}{8a^3{}_0}\left(\frac{(1-2\beta)\Gamma(2-\beta)}{\Gamma(3-2\beta)} a_0 - 12a_0{}^2.a_1{}^2 - 3a_0{}^2 a_1\right),\ a_0 \text{ and } a_1 \neq 0.$$

$$a_{n+2} = \frac{1}{(n+1)(n+2)(4a_n{}^3)}\left(\begin{array}{l} \dfrac{(1-2\beta+n\beta)\,\Gamma(2-(n+1)\beta)}{\Gamma(3-(n+2)\beta)} a_n - 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(i+1-k)(i+1)a_i a_{k-i} a^2{}_{n+1-k} \\ -3\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(i+1-k).a_i a_{k-i} a_{n+1-k} \end{array}\right).$$

As

$$J(\zeta)=a_0+a_1(\zeta)+a_2(\zeta)^2+\sum_{n=1}^{\infty}a_{n+2}(\zeta)^{n+2},$$

$$v(x,t)=a_0+a_1xt^{-\alpha/3}+a_2x^2t^{-2\alpha/3}+\sum_{n=1}^{\infty}a_{n+2}x^{n+2}t^{-(n+2)\alpha/3}. \quad (39)$$

Hence, we found the exact power series solution (39).

Now, we are expecting the convergence of solution of BM, so a_{n+2} in Eq (38) taken as

$$|a_{n+2}|\le\frac{1}{|d|}\left(\begin{array}{l}\left|\frac{(1-2\beta+n\beta)\Gamma(2-(n+2)\beta)}{\Gamma(3-(2+n)\beta)}\right||a_n|+12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}|a_i||a_{k-i}||a^2{}_{n+1-k}|\\+3\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}|a_i||a_{n+1-k}||a_{k-i}|\end{array}\right). \quad (40)$$

We can find $\left|\frac{(1-2\beta+n\beta)\Gamma(2-(n-1)\beta)}{\Gamma(2-2\beta+n\beta)}\right|<1$, for large arbitrary value of *n*.

$$|a_{n+2}|\le M\left(|a_n|+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}|a_i||a^2{}_{n+1-k}||a_{k-i}|+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}|a_i||a_{k-i}||a_{n+1-k}|\right); \quad (41)$$

where $M=\text{greatest}\left\{\frac{1}{|d|},\frac{12}{|d|},\frac{3}{|d|}\right\}$.

Introduce another majorant series

$$G(\zeta)=\sum_{n=0}^{\infty}c_n\zeta^n;c_i=|a_i|,i=0,1,2,3...,\quad (42)$$

where

$$c_{n+3}=M\left(c_n+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c^2{}_{n+1-k}.+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n+1-k}\right). \quad (43)$$

It can be observed that $|a_n|\le c_n, n=0,1,2$ ....

Further, the series function G (ζ) has non-negative convergence radius and it presents

$$G(\zeta)=c_0+c_1\zeta+c_2\zeta^2+M\sum_{n=1}^{\infty}\left(c_n+\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n+1-k}{}^2+\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n-k+1}\right)\zeta^{n+2}. \quad (44)$$

Now, the implicit function system is defined with the variable ζ.

$$I(\zeta,G)=G-c_0-c_1\zeta-c_2\zeta^2-M\sum_{n=1}^{\infty}\left(c_n+\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n+1-k}{}^2+\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n-k+1}\right)\zeta^{n+2}. \tag{45}$$

As I (ζ, G) is regular in vicinity of (0, c_0) and $I(0,c_0)=0$ with $\frac{\partial}{\partial G}I(0,c_0)\neq 0$, by implicit function theorem explained in Rudin [46]. We observed that G (ζ) is regular in the vicinity of the point (0, c_0) and have real positive radius; the series solution (39) converges in the vicinity of the point (0, c_0).

7. Construction of conservation laws for BM

In the physical and mathematical vision, conservation laws play a key role in the analysis of time-fractional PDEs. To obtain the conservation laws of convection-diffusion BM, we are generalizing the Noether's theorem suggested by Ibragimov [47,48] and Bourdin et al. [49]. The applications of conservation laws in FPDEs are almost the same as the application of these laws in classical order PDEs. These conservation laws can be extended from PDEs to FDEs. Let us define a conserved vector for BM (2), where λ^t and λ^x are components of vector

$$\lambda=(\lambda^t,\lambda^x), \tag{46}$$

which satisfy the continuity or conservation equation given by

$$D_t(\lambda^t)+D_x(\lambda^x)\,|_{\Delta=0}=0\ . \tag{47}$$

A formal Lagrangian form with 'u' as a new independent variable described as

$$\ell=u\left[\partial_t^{\beta}v-4v^3v_{xx}-12v^2v_x{}^2-3v^2v_x\right], \tag{48}$$

where $\delta/\delta v$ is Euler-Lagrangian operator, is defined as

$$\frac{\delta}{\delta v}=\frac{\partial}{\partial v}+(D_t^{\beta})^*\frac{\partial}{\partial(D_t^{\beta}v)}+\sum_{k=1}^{\infty}(-1)^kD_{i_1}D_{i_2}...D_{i_k}\frac{\partial}{\partial v_{i_1},_{i_2},_{i_3}\cdots_{i_k}}, \tag{49}$$

where $(D_t^{\beta})^*$ is adjoint of R-L fractional differential operator (D_t^{β}).

Adjoint equation of (1), is given by

$$(D_t^{\beta})^*=(-1)^n\,{}_tJ_T^{n-\beta}(D_t^n)={}_t^cD_T^{\beta}\ ;\ \Delta^*=\frac{\delta\ell}{\delta v}=0, \tag{50}$$

where ${}_tJ_T^{n-\beta}$ is right-handed fractional integral of order *(n-β)* and ${}_t^cD_T^{\beta}$ is Caputo right-handed derivative operator of fractional order β.

The idea of the physical property of self-adjointness for establishing these laws has been discussed in [36] and this concept can also be applied and expanded to fractional PDEs. The time fractional convection-diffusion equation will be self-adjoint if the adjoint Eq (50) is well pleased for the obtained solution of model (1).

For further discussion, the basic Noether expression is defined as

$$\overline{X} + D_t(\tau) + D_x(\xi) = W\frac{\delta}{\delta v} + D_t(N^t) + D_x(N^x), \tag{51}$$

where N^t and N^x are noether operators. As N^x in (1) does not have the non-integer or fractional derivatives with variable 'x', so the general expression is

$$N^x = \xi\ell + W\left(\frac{\partial}{\partial v_x} - D_x\frac{\partial}{\partial v_{xx}}\right) + D_x(W)\left(\frac{\partial}{\partial v_{xx}}\right), \tag{52}$$

and N^t involves fractional derivative, so this can be expressed by RL derivatives as

$$N^t = \ell\tau + \sum_{j=0}^{n-1}(-1)^j D_t^j \frac{\partial}{\partial(D_t^\beta v)} D_t^{\beta-1-j}(W) - (-1)^n I\left(W,\ D_t^n \frac{\partial}{\partial(D_t^\beta v)}\right), \tag{53}$$

In Eq (51), $\overline{X}$ represents the prolongation of symmetry reduction with characteristics of the vector field $W=\eta-\tau v_t-\xi v_x$, and operator I in (53) is described as

$$I(g,f) = \frac{1}{\Gamma(n-\beta)}\int_0^t\int_t^T \frac{g(\tau,x)f(\mu,x)}{(\mu-\tau)}d\mu d\tau. \tag{54}$$

Applying Lagrangian operator 'ℓ' on (51) for any vector X of (1), we obtained

$$\overline{X}\ell + D_t(\tau)\ell + D_x(\xi)\ell\,|_{\Delta=0} = 0 \ \text{ also } \frac{\delta\ell}{\delta v} = 0. \tag{55}$$

Thus, we obtained the conservation law of (1)

$$D_t(N^t\ell) + D_x(N^x\ell) = 0. \tag{56}$$

The components λ^t and λ^x of conserved vector fields in (30) can be expressed by

$$\begin{aligned} \lambda^t &= N^t\ell = \tau\ell + \sum_{j=0}^{m-1}(-1)^j D_t^j \frac{\partial\ell}{\partial(D_t^\beta v)} D_t^{\beta-1-j}(W) - (-1)^m I\left(W, D_t^m \frac{\partial\ell}{\partial(D_t^\beta v)}\right), \\ \lambda^x &= N^x\ell = \xi\ell + W\left(\frac{\partial\ell}{\partial v_x} - D_x\frac{\partial\ell}{\partial v_{xx}}\right) + D_x(W)\left(\frac{\partial\ell}{\partial v_{xx}}\right). \end{aligned} \tag{57}$$

The adjoint equation for (1) is found as

$$\begin{aligned} \Delta^* &= D_t^{\beta^*}(v) - 12uv^2v_{xx} - 24uvv_x^2 - 6uvv_x + D_x(24uv^2v_x) + D_x(3uv^2) - D_x^2(4uv^3) \\ &= D_t^{\beta^*}(v) + 3u_xv^2 - 4u_{xx}v^3 \\ &= 0 \end{aligned} . \tag{58}$$

If adjoint Eq (58) is satisfied for all solutions of (1), is said to be nonlinear self-adjoint. It shows

$$D_t^{\beta^*}(v)+3u_xv^2-4u_{xx}v^3=\lambda\left[\partial_t^{\beta}v-4v^3v_{xx}-12v^2{v_x}^2-3v^2v_x\right] \tag{59}$$

Substituting $u=\psi(t,x)=\omega(t)\rho(x)\neq 0$ represents the nonlinear self adjointness of (1). By using the above (59), we obtain

$$D_t^{\beta^*}(\omega(t))={}_t^cD_t^{\beta^*}(\omega(t))=0,$$
$$3\rho_x(x)v^2-4\rho_{xx}(x)v^3=0.$$

Which implies that $v=\psi(t,x)=C$, C is arbitrary constant.

Thus, the Lagrangian operator for (1) is $\ell=C\left[\partial_t^{\beta}v-4v^3v_{xx}-12v^2{v_x}^2-3v^2v_x\right]$. Now, we proceed with the calculation of conservation laws of BM using (57).

Case1. For $0\leq\beta<1, S_1=\frac{\partial}{\partial x}$, the Lie characteristic is $W_1=-v_x$, so the components of conserved vectors are as follows:

$$\begin{aligned}\lambda^x&=\xi\ell+W_1\left(-D_x\frac{\partial}{\partial v_{xx}}+\frac{\partial}{\partial v_x}\right)\ell+D_x(W_1)\left(\frac{\partial\ell}{\partial v_{xx}}\right)\\&=(24c.v^2v_x+3c.v^2-4cD_x(v^3))v_x-D_x(v_x)(-4cv^3)\\&=c.v^2(3v_x+12{v_x}^2+4vv_{xx}),\end{aligned} \tag{60}$$

$$\begin{aligned}\lambda^t&=c.D_t^{\beta-1}(-v_x)+I(-v_x,0)\\&=-c.D_t^{\beta-1}(v_x).\end{aligned} \tag{61}$$

Case2. For $S_2=x\frac{\partial}{\partial x}-\frac{t}{\beta}\frac{\partial}{\partial t}+v\frac{\partial}{\partial v}$ the Lie characteristic is $W_2=v-\frac{t}{\beta}v_t+xv_x$, so the components of conserved vectors are as follows:

$$\begin{aligned}\lambda^x&=\xi\ell+W_2\left(-D_x\frac{\partial}{\partial v_{xx}}+\frac{\partial}{\partial v_x}\right)\ell+D_x(W_2)\left(\frac{\partial\ell}{\partial v_{xx}}\right)\\&=(v+\frac{t}{\beta}v_t-xv_x)(-24cv^2v_x-3cv^2-D_x(-4cv^3))-4cv^3D_x(u+\frac{t}{\beta}v_t-xv_x)\\&=-3cv^2(4v_x+1)(v+\frac{t}{\beta}v_t-xv_x)-4cv^3(v_x-\frac{t}{\beta}v_{tx}-v_x-xv_{xx}),\end{aligned} \tag{62}$$

$$\lambda^t = \sum_{k=0}^{n-1} (-1)^k D_t^k \frac{\partial \ell}{\partial (D_t^\beta v)} D_t^{\beta-1-k}(W_2) - (-1)^n I\left(W_2\,,\ D_t^n \frac{\partial \ell}{\partial (D_t^\beta v)} \right)$$
$$= c\, D_t^{\beta-1}(v + \frac{t}{\beta} v_t - x v_x) + I(v + \frac{t}{\beta} v_t - x v_x, 0)$$
$$= c\, D_t^{\beta-1}(v + \frac{t}{\beta} v_t - x v_x) \tag{63}$$

8. Conclusions

In the article, we have utilized the symmetry reduction to fractional-order convection-diffusion Buckmaster model. The Lie point infinitesimal generators and Lie algebra have been constructed. Also, Erdelyi-Kober operators are used to transforming the fractional-PDE into fractional-ODE. Finally, the power series solutions of the model are obtained with their convergence the implicit function theorem. To construct the conservation laws of the model, Ibragimov's method and Noether's theorem have been used. The study of the obtained solutions with conservation laws is supposed to be very useful in the future for various stream of physical and applied sciences.

Author contributions

A. Tomar: Conceptualization, Supervision, Formal analysis, Writing-original draft, Writing review editing; H. Kumar: Conceptualization, Software, Visualization, Writing original draft; M. Ali: Formal analysis, Writing-review, editing; H. Gandhi: Formal analysis, Methodology, Writing-original draft, Visualization; D. Singh: Formal analysis, Supervision, Formatting, Writing review; G. Pathak: Software, Formal analysis, Visualization. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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