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Research article

A characterization of Wolf and Schechter essential pseudospectra

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Abstract: The aim of this paper is to provide new results on the Wolf and Schechter essential pseudospectra of bounded linear operators on a Banach space. More precisely, we characterize the Wolf and Schechter essential pseudospectra by using the notion of Fredholm perturbation. Also, we state the condition under which the Wolf (respectively, Schechter) essential pseudospectrum of two different bounded linear operators coincides. Furthermore, we give some characterizations of the Wolf and Schechter essential pseudospectra of 3×3 upper triangular block operator matrices.

Keywords: Wolf essential pseudospectrum; Schechter essential pseudospectrum; Wolf essential spectrum; Schechter essential spectrum; Fredholm perturbation; block operator matrices
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1. Introduction

The concept of pseudospectra was introduced by J. M. Varah, H. Landau, L. N. Trefethen and E. B. Davies. Due in particular to L. N. Trefethen, who advanced this approach for matrices and operators. This notion appears in a number of intriguing mathematical physics issues. Likewise, many mathematicians contributed to this field (see, for instance, [8, 11, 13]).

The pseudospectrum $\sigma_\varepsilon(T)$ of a closed, densely defined linear operator T, for $\varepsilon > 0$, is determined by the following formula:

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{ \lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\},$$

where $\sigma(T)$ represents the spectrum of T.

In [8], EB. Davies has defined an equivalent definition of the pseudospectrum of a closed densely linear operator T, by

$$\sigma_\varepsilon(T) = \bigcup_{\|D\|<\varepsilon} \sigma(T + D).$$

In this paper, we are interested in giving various results on the essential pseudospectra of bounded linear operators on Banach spaces. The essential pseudospectra of block operator matrices have been studied over many years and by numerous authors (see, for example, [3, 5]). Our work focuses on Wolf and Schechter's essential pseudospectra. Theorem 3.1 introduces a characterization of the Wolf and Schechter essential pseudospectra by means of Fredholm perturbation. In Theorem 3.2, we give a relation between the essential pseudospectrum and the essential spectrum of Wolf (respectively, Schechter). Theorem 3.3 provides conditions for which the Wolf (respectively, Schechter) essential pseudospectrum of two different bounded linear operators coincides. Furthermore, we give some new characterizations of the Wolf and Schechter essential pseudospectra of 3×3 block operator matrices by investigating a new decomposition of the upper triangular block operator matrices in Theorems 3.4 and 3.5 with application of the Fredholm perturbation concept.

The structure of this paper is as follows. In Section 2, we recall some elementary results that are fundamental for our purpose. The main results are presented in Section 3.

2. Preliminaries

In this section, we collect some important definitions, notations, and preliminary results that will be needed in the sequel. Throughout this paper, X will denote a Banach space, and $\mathcal{L}(X)$ (respectively, $\mathcal{K}(X)$), the set of all bounded linear (respectively, compact) operators on X. For $T \in \mathcal{L}(X)$, we designate by $N(T)$ and $R(T)$ the null space and the range of T, respectively. The nullity, $\alpha(T)$, of T is defined as the dimension of $N(T)$, and the deficiency, $\beta(T)$, of T is defined as the codimension of $R(T)$ in X. The number $i(T) = \alpha(T) - \beta(T)$ is called the index of T.

The sets of upper semi-Fredholm and lower semi-Fredholm operators on X are respectively defined by

$$\Phi_+(X) := \{T \in \mathcal{L}(X) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed }\},$$

$$\Phi_-(X) := \{T \in \mathcal{L}(X) : \beta(T) < \infty\}.$$

The set of Fredholm operators on X is given by

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X).$$

An operator $F \in \mathcal{L}(X)$ is called a Fredholm perturbation if $T + F \in \Phi(X)$ whenever $T \in \Phi(X)$. Denote by $\mathcal{F}(X)$ the set of Fredholm perturbations; we refer to [18] for more details about these notions.

The following lemma gives the stability of the Fredholm operator under Fredholm perturbation.

Lemma 2.1. *[14, Lemma 2.1] Let $T, F \in \mathcal{L}(X)$. If $T \in \Phi(X)$ and $F \in \mathcal{F}(X)$, then $T + F \in \Phi(X)$ and $i(T + F) = i(T)$.*

Now, let us recall the notion of the pseudo-Fredholm operator.

Definition 2.1. Let $\varepsilon > 0$ and $T \in \mathcal{L}(X)$.

(1) An operator T is called pseudo-Fredholm if $T + D$ is a Fredholm operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.

(2) An operator T is called an upper (respectively, lower) pseudo semi-Fredholm if $T + D$ is an upper (respectively, lower) semi-Fredholm operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.

Denote by $\Phi^{\varepsilon}(X), \Phi^{\varepsilon}_{+}(X)$ and $\Phi^{\varepsilon}_{-}(X)$ the sets of pseudo-Fredholm operators, the upper and lower pseudo semi-Fredholm operators, respectively.

In this paper, for $S \in \mathcal{L}(X)$, our concern is mainly the following S-essential spectra of Wolf and Schechter

$$\begin{aligned} \sigma_{e4,S}(A) &:= \{\lambda \in \mathbb{C} : (\lambda S - A) \notin \Phi(X)\}, \\ \sigma_{e5,S}(A) &:= \{\lambda \in \mathbb{C} : (\lambda S - A) \notin \Phi(X) \text{ with } i(\lambda S - A) = 0\}. \end{aligned}$$

For essential pseudospectra, we are concerned with the following sets defined by

$$\begin{aligned} \sigma_{e1,\varepsilon}(T) &:= \{\lambda \in \mathbb{C} : (\lambda - T) \notin \Phi^{\varepsilon}_{+}(X)\}, \\ \sigma_{e2,\varepsilon}(T) &:= \{\lambda \in \mathbb{C} : (\lambda - T) \notin \Phi^{\varepsilon}_{-}(X)\}, \\ \sigma_{e4,\varepsilon}(T) &:= \{\lambda \in \mathbb{C} : (\lambda - T) \notin \Phi^{\varepsilon}(X)\}, \\ \sigma_{e5,\varepsilon}(T) &:= \bigcap_{K \in \mathcal{K}(X)} \sigma_{\varepsilon}(T + K). \end{aligned}$$

Note that if ε tends to 0 or $S = I$, we recover the usual definitions of Gustafson, Weidmann, Wolf, and Schechter essential spectra denoted respectively by $\sigma_{e1}(T), \sigma_{e2}(T), \sigma_{e4}(T)$ and $\sigma_{e5}(T)$ of a bounded linear operator T. Recalling that an operator T is a Riesz operator if $\sigma_{e4}(T) = \{0\}$. For more details, the reader is referred to [11].

In [11], A. Jeribi has established the following results:

Proposition 2.1. *Let $\varepsilon > 0$ and $T \in \mathcal{L}(X)$. The following conditions are equivalent:*

(1) $\lambda \in \sigma_{\varepsilon}(T)$.

(2) There exists a bounded operator D such that $\|D\| < \varepsilon$ and $\lambda \in \sigma(T + D)$.

Proposition 2.2. *Let $\varepsilon > 0$ and $T \in \mathcal{L}(X)$, then $\lambda \notin \sigma_{e5,\varepsilon}(T)$ if and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, $(T + D - \lambda) \in \Phi(X)$ and $i(T + D - \lambda) = 0$.*

3. Main results

The purpose of this section is to present our main results on the Wolf and Schechter essential pseudospectra of bounded linear operators on a Banach space. In the following theorem, we give a characterization of the Wolf and Schechter essential pseudospectra.

Theorem 3.1. *Let $\varepsilon > 0, T, D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, then there exists $S \in \mathcal{L}(X)$ with $DS \in \mathcal{F}(X)$ such that*

$$\sigma_{ei,S}(TS) \subset \sigma_{ei,\varepsilon}(T), \quad i = 4, 5.$$

Proof. (i) Let $\lambda \notin \sigma_{e4,\varepsilon}(T)$, then $(\lambda - T - D) \in \Phi(X)$ for all $\|D\| < \varepsilon$. By using the Atkinson theorem [2, Theorem 4.46, p. 161], there exists $S \in \mathcal{L}(X)$ and $K_1 \in \mathcal{K}(X)$ such that $(\lambda - T - D)S = I_X - K_1$. Since $K_1 \in \mathcal{K}(X)$, then $(I_X - K_1) \in \Phi(X)$ with zero index. So $(\lambda I - T - D)S \in \Phi(X)$, i.e., $(\lambda S - TS - DS) \in \Phi(X)$. Using the fact that $DS \in \mathcal{F}(X)$ and the stability of Fredholm operators under Fredholm perturbations, we get $(\lambda S - TS) \in \Phi(X)$. Hence $\lambda \notin \sigma_{e4,S}(TS)$.

(ii) By the same argument, we get $\sigma_{e5,S}(TS) \subset \sigma_{e5,\varepsilon}(T)$.

□

The following main theorem aims to generalize the results of [9, Theorem 2.3] by giving a relation between the Wolf (respectively, Schechter) essential pseudospectrum and the Wolf (respectively, Schechter) essential spectrum.

Theorem 3.2. *Let $\varepsilon > 0, T, D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda_0 \in \rho(T+D)$, then for $\lambda \neq \lambda_0$, we have*

$$\lambda \in \sigma_{ei,\varepsilon}(T) \quad \textit{if and only if} \quad (\lambda_0 - \lambda)^{-1} \in \sigma_{ei}((\lambda_0 - T - D)^{-1}), \quad i = 4, 5.$$

Proof. For $\lambda \in \mathbb{C}\backslash\{\lambda_0\}$ and all bounded operator D such that $\|D\| < \varepsilon$, the operator $(\lambda - T - D)$ can be written as follows:

$$(\lambda - T - D) = (\lambda_0 - \lambda)(\lambda_0 - T - D)[(\lambda_0 - \lambda)^{-1} - (\lambda_0 - T - D)^{-1}]. \tag{3.1}$$

(i) For $i = 4$, suppose that $(\lambda_0-\lambda)^{-1} \notin \sigma_{e4}((\lambda_0-T-D)^{-1})$, then $((\lambda_0-\lambda)^{-1}-(\lambda_0-T-D)^{-1}) \in \Phi(X)$. Since $(\lambda_0 - T - D) \in \Phi(X)$, the use of [18, Theorem 5.7, p. 106] implies that $(\lambda - T - D) \in \Phi(X)$, hence $\lambda \notin \sigma_{e4,\varepsilon}(T)$. Now, let $\lambda \notin \sigma_{e4,\varepsilon}(T)$, then $(\lambda - T - D) \in \Phi(X)$. Since $(\lambda_0 - T - D) \in \Phi(X)$, then using Eq (3.1) and [18, Theorem 5.13, p. 110], we get $((\lambda_0 - \lambda)^{-1} - (\lambda_0 - T - D)^{-1}) \in \Phi(X)$. Therefore $(\lambda_0 - \lambda)^{-1} \notin \sigma_{e4}((\lambda_0 - T - D)^{-1})$.

(ii) The proof is analogous for $i = 5$.

□

Now, we can state a condition under which the Wolf (respectively, Schechter) essential pseudospectrum of two different bounded linear operators coincides in the following theorem.

Theorem 3.3. *Let $\varepsilon > 0, T, L \in \mathcal{L}(X)$ and for all bounded operators, D with $\|D\| < \varepsilon$. If for some $\lambda_0 \in \rho(T+D) \cap \rho(L+D)$ such that $(\lambda_0 - T - D)^{-1} - (\lambda_0 - L - D)^{-1} \in \mathcal{F}(X)$, then*

$$\sigma_{ei,\varepsilon}(T) = \sigma_{ei,\varepsilon}(L), \quad i = 4, 5.$$

Proof. (i) Assume that $\lambda \notin \sigma_{e4,\varepsilon}(T)$. According to Theorem 3.2, this is equivalent to $(\lambda_0 - \lambda)^{-1} \notin \sigma_{e4}((\lambda_0 - T - D)^{-1})$. Since $(\lambda_0 - T - D)^{-1} - (\lambda_0 - L - D)^{-1} \in \mathcal{F}(X)$, it follows from [15, Theorem 2.1] that $(\lambda_0 - \lambda)^{-1} \notin \sigma_{e4}((\lambda_0 - L - D)^{-1})$. Therefore, by using Theorem 3.2 again, we have $\lambda \notin \sigma_{e4,\varepsilon}(L)$. This shows that $\sigma_{e4,\varepsilon}(T) = \sigma_{e4,\varepsilon}(L)$.

(ii) In the same way, we get $\sigma_{e5,\varepsilon}(T) = \sigma_{e5,\varepsilon}(L)$.

□

Now, we consider the following 3×3 block operator matrices defined on X^3 by $T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix}$, $D = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix}$ where $T_{ij}, D_k \in \mathcal{L}(X)$ and $\|D\| = \max\limits_{k=1,2,3} \|D_k\| < \varepsilon$, (i.e., $\|D_k\| < \varepsilon$, for all $k = 1, 2, 3$).

In the following main theorem, we characterize the Wolf essential pseudospectrum of the 3×3 block operator matrix T.

Theorem 3.4. *Let $\varepsilon > 0$. If for all bounded operator D_3 such that $\|D_3\| < \varepsilon$, $(T_{33} + D_3)$ is a Reisz operator and $T_{12} \in \mathcal{F}(X)$, then*

$$\sigma_{e4,\varepsilon}(T)\backslash\{0\} = [\sigma_{e4,\varepsilon}(T_{11}) \cup \sigma_{e4,\varepsilon}(T_{22})]\backslash\{0\}.$$

Proof. Let $\lambda \notin [\sigma_{e4,\varepsilon}(T_{11}) \cup \sigma_{e4,\varepsilon}(T_{22})]\backslash\{0\}$, then $(\lambda - T_{11} - D_1) \in \Phi(X)$ and $(\lambda - T_{22} - D_2) \in \Phi(X)$ for all $\|D_k\| < \varepsilon, k = 1, 2$. We have

$$\lambda - T - D = \begin{pmatrix} \lambda - T_{11} - D_1 & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} - D_2 & -T_{23} \\ 0 & 0 & \lambda - T_{33} - D_3 \end{pmatrix} = A_1 \times A_2 \times A_3 + B,$$

where $A_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \lambda - T_{33} - D_3 \end{pmatrix}$, $A_2 = \begin{pmatrix} I & 0 & -T_{13} \\ 0 & I & -T_{23} \\ 0 & 0 & I \end{pmatrix}$, $A_3 = \begin{pmatrix} \lambda - T_{11} - D_1 & 0 & 0 \\ 0 & \lambda - T_{22} - D_2 & 0 \\ 0 & 0 & I \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -T_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since block operator matrices A_1, A_2 and A_3 are Fredholm, then $A_1 \times A_2 \times A_3$ is Fredholm. So, $(\lambda - T - D) \in \Phi(X)$ by using the stability of the Fredholm operator by Fredholm perturbation B. Hence $\lambda \notin \sigma_{e4,\varepsilon}(T)\backslash\{0\}$. For the inverse inclusion, let $\lambda \notin \sigma_{e4,\varepsilon}(T)\backslash\{0\}$. Then

$$\lambda - T - D = \begin{pmatrix} \lambda - T_{11} - D_1 & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} - D_2 & -T_{23} \\ 0 & 0 & \lambda - T_{33} - D_3 \end{pmatrix} \in \Phi(X).$$

By using the stability of the Fredholm operator by Fredholm perturbation $-B$, we get

$$\begin{pmatrix} \lambda - T_{11} - D_1 & 0 & -T_{13} \\ 0 & \lambda - T_{22} - D_2 & -T_{23} \\ 0 & 0 & \lambda - T_{33} - D_3 \end{pmatrix} \in \Phi(X).$$

Since $A_1 \times A_2 \in \Phi(X)$, then using [18, Theorem 5.13, p. 110], we get

$$A_3 = \begin{pmatrix} \lambda - T_{11} - D_1 & 0 & 0 \\ 0 & \lambda - T_{22} - D_2 & 0 \\ 0 & 0 & I \end{pmatrix} \in \Phi(X).$$

Consequently, $(\lambda - T_{11} - D_1) \in \Phi(X)$ and $(\lambda - T_{22} - D_2) \in \Phi(X)$ for all $\|D_k\| < \varepsilon, k = 1, 2$. Hence $\lambda \notin [\sigma_{e4,\varepsilon}(T_{11}) \cup \sigma_{e4,\varepsilon}(T_{22})]\backslash\{0\}$. □

Remark 3.1. If for all bounded operator D_1 such that $\|D_1\| < \varepsilon$, $(T_{11} + D_1)$ is a Reisz operator and $T_{23} \in \mathcal{F}(X)$, then

$$\sigma_{e4,\varepsilon}(T)\backslash\{0\} = [\sigma_{e4,\varepsilon}(T_{22}) \cup \sigma_{e4,\varepsilon}(T_{33})]\backslash\{0\}.$$

Similarly, we can prove this equality by using the decomposition given in [6]

$$\lambda - T - D = \begin{pmatrix} \lambda - T_{11} - D_1 & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} - D_2 & -T_{23} \\ 0 & 0 & \lambda - T_{33} - D_3 \end{pmatrix} = A_1 \times A_2 \times A_3 \times A_4 + B,$$

where $A_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & \lambda - T_{22} - D_2 & 0 \\ 0 & 0 & \lambda - T_{33} - D_3 \end{pmatrix}$, $A_2 = \begin{pmatrix} I & -T_{12} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$, $A_3 = \begin{pmatrix} I & 0 & -T_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$,
$A_4 = \begin{pmatrix} \lambda - T_{11} - D_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_{23} \\ 0 & 0 & 0 \end{pmatrix}$.

An immediate consequence of Theorem 3.4 is the following remark concerning the Schechter essential pseudospectrum of the upper block operator matrix T.

Remark 3.2. Under the same hypotheses as Theorem 3.4, we can easily prove the following inclusion:

$$\sigma_{e5,\varepsilon}(T)\backslash\{0\} \subset [\sigma_{e5,\varepsilon}(T_{11}) \cup \sigma_{e5,\varepsilon}(T_{22})]\backslash\{0\}.$$

Remark 3.3. Assume that the hypotheses of Remark 3.1 hold, then

$$\sigma_{e5,\varepsilon}(T)\backslash\{0\} \subset [\sigma_{e5,\varepsilon}(T_{22}) \cup \sigma_{e5,\varepsilon}(T_{33})]\backslash\{0\}.$$

Similarly, we can prove this statement by using the same decomposition defined in Remark 3.1.

Finally, we conclude this paper by giving another characterization of the Schechter essential pseudospectrum of the block operator matrix T.

Theorem 3.5. *Let $\varepsilon > 0$ and T be the block operator matrix defined above, then we have*

$$\sigma_{e1,\varepsilon}(T_{11}) \cup \sigma_{e2,\varepsilon}(T_{33}) \subset \sigma_{e5,\varepsilon}(T) \subset \bigcup_{k=1}^{3} \sigma_{e5,\varepsilon}(T_{kk}).$$

Proof. Let $\lambda \notin \sigma_{e5,\varepsilon}(T)$, then $(\lambda - T - D) \in \Phi(X)$ with $i(\lambda - T - D) = 0$ for all $\|D\| < \varepsilon$. So, the block operator matrix

$$\lambda - T - D = \begin{pmatrix} \lambda - T_{11} - D_1 & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} - D_2 & -T_{23} \\ 0 & 0 & \lambda - T_{33} - D_3 \end{pmatrix} \in \Phi(X).$$

By using [1, Proposition 2.1, p. 1190], we get $(\lambda - T_{11} - D_1) \in \Phi_+(X)$ and $(\lambda - T_{22} - D_2) \in \Phi_-(X)$. We deduce that $\lambda \notin \sigma_{e1,\varepsilon}(T_{11}) \cup \sigma_{e2,\varepsilon}(T_{22})$. For the second inclusion, suppose that $\lambda \notin \bigcup_{k=1}^{3} \sigma_{e5,\varepsilon}(T_{kk})$, then $(\lambda - T_{kk} - D_k) \in \Phi(X)$ with $i(\lambda - T_{kk} - D_k) = 0$ for all $\|D_k\| < \varepsilon, k = 1, 2, 3$. By using [16, Lemma 1.33, p. 10], the triangular block operator matrix

$$\begin{pmatrix} \lambda - T_{11} - D_1 & -T_{12} & -T_{13} \\ 0 & \lambda - T_{22} - D_2 & -T_{23} \\ 0 & 0 & \lambda - T_{33} - D_3 \end{pmatrix} \in \Phi(X)$$

is Fredholm with a zero index. This means that $(\lambda - T - D) \in \Phi(X)$ with $i(\lambda - T - D) = 0$ for all $\|D\| < \varepsilon$, which implies that $\lambda \notin \sigma_{e5,\varepsilon}(T)$. □

4. Conclusions

In this work, we have introduced some new characterizations of the Wolf and Schechter essential pseudospectra of 3×3 block operator matrices by investigating a new decomposition of the upper triangular block operator matrices with application of the Fredholm perturbation notion.

Author contributions

S. Smail investigated the results of this paper and prepared the manuscript; C. Belabbaci provided critical feedback, reviewed, and approved the final version of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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