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Research article

Fractional dynamics and computational analysis of food chain model with disease in intermediate predator

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Abstract: In this paper, a fractional food chain system consisting of a Holling type II functional response was studied in view of a fractional derivative operator. The considered fractional derivative operator provided nonsingular as well as a nonlocal kernel which was significantly better than other derivative operators. Fractional order modeling of a model was also useful to model the behavior of real systems and in the investigation of dynamical systems. This model depicted the relationship among four types of species: prey, susceptible intermediate predators (IP), infected intermediate predators, and apex predators. One of the significant aspects of this model was the inclusion of Michaelis-Menten type or Holling type II functional response to represent the predator-prey link. A functional response depicted the rate at which the normal predator consumed the prey. The qualitative property and assumptions of the model were discussed in detail. The present work discussed the dynamics and analytical behavior of the food chain model in the context of fractional modeling. This study also examined the existence and uniqueness related analysis of solutions to the food chain system. In addition, the Ulam-Hyers stability approach was also discussed for the model. Moreover, the present work examined the numerical approach for the solution and simulation for the model with the help of graphical presentations.

Keywords: predator-prey model; fractional food chain model; fractional derivative; Ulam-Hyers stability analysis; Banach space
Mathematics Subject Classification: 26A27, 26A30, 26A33, 28A80, 35R11

1. Introduction

The linkage dynamics between predator and prey explore significant aspects in the field of ecology due to its several applications. Lotka-Volterra model [1,2] is the first mathematical framework describing the prey-predator interplay in mathematical ecology. Recently, it has been detected that stage structure models of the population present the interaction dynamics more accurately than other existing models. The recent literature regarding the development of structured models can be seen in [3–7]. In the past decade, some models discussed the impact of infectious disease on environmental ecology. These models actually show the spread of disease in populations and their transmission from susceptible to infected species. Kermack and McKendric [8] contributed toward the mathematical theory of epidemics through their model. Several researchers investigated mathematical predator-prey models with disease [9–11]. Freedman and Waltman [12] investigated three interacting predator-prey populations. Chattopadhyay et al. [13] suggested a predator-prey model. Kar et al. [14] modeled a harvested prey-predator system in 2006. In 2007, Dubey [15] studied the prey-predator model with a reserved area. In addition, the persistence and extinction of one prey and a two predator system were explored in [16].

In the past decade, several mathematical models prepared on the basis of the Beddington-DeAngelis functional response have been derived in [17–21]. Dubey et al. [22] provided the numerical treatment of the fractional food chain problem. In addition, Abdo et al. [23] investigated the three-species prey-predator (PP) model pertaining to the Mittag-Leffler kernel. Moreover, Ghanbari et al. [24] presented the numerical results of the PP system having a functional response of the Beddington-DeAngelis type. More recently, Liu et al. [25] investigated the Turing patterns of the Leslie-Gower Holling type III predator-prey model on several different networks with the help of linear stability analysis. Song et al. [26] proposed a PP model organized in multiplex networks to investigate the effect of multiplex structure on the diffusion of predator and prey, and furthermore, the influences on the formation of Turing patterns. Alsakaji et al. [27] investigated a delay differential model of one-predator two-prey system with Monod-Haldane and Holling type II functional responses. Rihan et al. [28] studied the dynamics of a fractional-order delayed model of COVID-19 with vaccination efficacy. More recently, Arif et al. [29] propounded and discussed a mathematical food chain model (FCM) involving disease in an intermediate predator. This FCM consists of four ordinary differential equations (ODEs) relating four types of species: prey, the intermediate predator (IP), susceptible infected intermediate predator (SIIP), and the apex predator. One of the significant aspects of this model is the inclusion of the Michaelis-Menten type or Holling type II functional response [30,31] to represent the PP link. A functional response depicts the rate at which the normal predator consumes the prey.

In this work, a fractional order mathematical FCM proposed by Arif et al. [29] is investigated and analyzed in view of the Atangana-Baleanu Caputo (ABC) fractional derivative operator (FDO) for the first time. This derivative operator was given by Atangana and Baleanu [32] to study the heat transfer model. This derivative operator provides a nonsingular as well as nonlocal kernel carrying the Mittag-Leffler function (MLF) [33], which is significantly better than previously established derivative operators. Atangana and Baleanu [32] have proposed two versions, the Atangana-Baleanu FDO (ABFDO) in Caputo sense (i.e., ABC derivative), which is a convolution of a local derivative of a given function with the ML function, and the ABFDO in Riemann-Liouville (RL) sense (i.e., Atangana-Baleanu Riemann (ABR) derivative), which is the derivative of a convolution of a given function that is not differentiable with the ML function. The use of a ML kernel in ABFDO is due to its natural appearance in various physical models because the MLF is a joint venture of power-law and exponential-law which induces completely the effect of memory [34]. The inspiration behind the selection of ABFDO is the nonlocal characteristic of the kernel which generates the scope of global analysis in those areas where the trends do not follow the power-law. Recently, a number of models were investigated with ABFDO which can be seen in [35–40]. This study also examines the existence and uniqueness related analysis of the solution of the model. In addition, the stability analysis for the food chain problem is also presented utilizing the Ulam-Hyers approach. In the later part of this work, the numerical solution of the model is explored along with simulations.

The rest part of the paper is subdivided as follows: Section 2 provides fundamental definitions and formulae regarding fractional integral and derivative operators. Section 3 gives a basic description of the food chain model. Section 4 discusses the qualitative property of the model. Sections 5 and 6, respectively, present the existence and uniqueness of the obtained solution. In Section 7, Ulam-Hyers stability approach is applied for FCM. In Sections 8 and 9, the numerical solution and simulation are discussed, respectively. Finally, Section 10 concludes the whole work.

2. Preliminaries

This section presents a quick review of fractional integral and derivative operators.

Definition 2.1. Let $0 \le \theta \le \theta_0 < \infty$. Then, Banach space $\Omega = E \times E \times E \times E$, where $E = C[0, \theta_0]$, with the norm

$$\|\rho\| = \|(M, N, N_1, \Theta)\| = \max_{\theta \in [0, \theta_0]} \{|M| + |N| + |N_1| + |\Theta|\},\ M, N, N_1, \Theta \in C \in [0, \theta_0]. \tag{2.1}$$

Definition 2.2. [41] Let Ω be a Banach space. The operator $\wp : \Omega \to \Omega$ is Lipschitzian if $\exists$ a constant $m > 0$ such that

$$\|\wp\,\psi_1 - \wp\,\psi_2\| \le m\|\psi_1 - \psi_2\|,\ \forall \psi_1, \psi_2 \in \Omega, \tag{2.2}$$

where m is the Lipschitz constant for $\wp$. If $m < 1$, $\wp$ is a contraction.

Definition 2.3. [32] The Caputo type fractional integral & derivative of a function $M(\theta)$ of order ω, respectively, are expressed as

$$ {}_{0}^{C}I_{\theta}^{\omega}M(\theta)=\frac{1}{\Gamma(\omega)}\int_{0}^{\theta}(\theta-\tau)^{\omega-1}M(\tau)d\tau,\ 0\le\omega<1, \tag{2.3} $$

$$ {}_{0}^{C}D_{\theta}^{\omega}M(\theta)=\frac{1}{\Gamma(k-\omega)}\int_{0}^{\theta}(\theta-\tau)^{k-\omega-1}M^{(k)}(\tau)d\tau,\ \theta>0,\ k-1\le\omega<k, \tag{2.4} $$

where $k\in Z^{+}$ and Γ is a gamma function.

Definition 2.4. [32] The AB fractional integral & derivative of $N(\theta)$ of order ω are stated as

$$ {}_{0}^{AB}I_{\theta}^{\omega}N(\theta)=\frac{1-\omega}{B(\omega)}N(\theta)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_{0}^{\theta}(\theta-\tau)^{\omega-1}N(\tau)d\tau,\ 0<\omega\le 1. \tag{2.5} $$

$$ {}_{0}^{ABC}D_{\theta}^{\omega}N(\theta)=\frac{B(\omega)}{1-\omega}\int_{0}^{\theta}E_{\omega}\left[-\frac{\omega}{1-\omega}(\theta-\tau)^{\omega}\right]N'(\tau)d\tau,\ 0<\omega\le 1. \tag{2.6} $$

Here, $E_{\omega}(\theta)$ signifies the MLF formulated as [33]:

$$ E_{\omega}(\theta)=\sum_{m=0}^{\infty}\frac{\theta^{m}}{\Gamma(\omega m+1)},\ \omega>0, \tag{2.7} $$

and $B(\omega)=1-\omega+\frac{\omega}{\Gamma(\omega)}$ denotes normalized function with $B(1)=B(0)=1$.

Definition 2.5. [32] The fractional order ABC derivative fulfills the Lipschitz criterion for two functions $M(\theta)$ and $N(\theta)$, and the inequality holds as follows:

$$ \left\| {}_{0}^{ABC}D_{\theta}^{\omega}(M(\theta)) - {}_{0}^{ABC}D_{\theta}^{\omega}(N(\theta)) \right\| \le H\left\| M(\theta)-N(\theta) \right\|. \tag{2.8} $$

Proposition 2.6. [42] The solution of

$$ {}_{0}^{ABC}D_{\theta}^{\omega}U(\theta)=V(\theta),\ \ U(0)=U_{0},\ \ \omega\in(0,1] \tag{2.9} $$

is suggested by

$$ U(\theta)=U(0)+\frac{1-\omega}{B(\omega)}V(\theta)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_{0}^{\theta}(\theta-\tau)^{\omega-1}V(\tau)d\tau. \tag{2.10} $$

3. Mathematical description of the food chain problem

In this segment, we provide the basic information about the equations and parameters of the FCM.

The mathematical structure of the FCM with three species suggested by Arif et al. [29] is represented by means of four nonlinear ODEs in this way:

$$M'(\theta)=rM(\theta)(1-M)-\alpha_0\frac{M(\theta)N(\theta)}{1+\mu M(\theta)}-\alpha_4\frac{M(\theta)N_1(\theta)}{1+\mu M(\theta)},$$

$$N'(\theta)=\alpha_1\frac{M(\theta)N(\theta)}{1+\mu X(\theta)}-\alpha_2\frac{N(\theta)\Theta(\theta)}{1+\mu N(\theta)}-c\frac{N(\theta)N_1(\theta)}{N(\theta)+N_1(\theta)}+kN_1(\theta)-d_1N(\theta),$$

$$N_1{}'(\theta)=c\frac{N(\theta)N_1(\theta)}{N(\theta)+N_1(\theta)}+\alpha_5\frac{M(\theta)N_1(\theta)}{1+\mu M(\theta)}-kN_1(\theta)-qN_1,$$

$$\Theta'(\theta)=\alpha_3\frac{N(\theta)\Theta(\theta)}{1+\mu N(\theta)}-d_2\Theta(\theta), \tag{3.1}$$

where $M'(\theta)>0$, $N'(\theta)>0$, $N_1{}'(\theta)>0$, $\Theta'(\theta)>0$. Here, $M(\theta)$, $N(\theta)$, $N_1(\theta)$ and $\Theta(\theta)$, respectively, denote functions of time representing population densities of susceptible prey (SP), susceptible IP (SIP), infected IP (IIP), and the top predator (TP), and all parameters are positive constants.

The food chain model given above describes the relation between SP, SIP, IIP, and the TP. $\alpha_0\frac{M(\theta)N(\theta)}{1+\mu M(\theta)}$ is the Michaelis-Menten type (or Holling type II) functional response and IP becomes infected with relative function $c\frac{N(\theta)N_1(\theta)}{N(\theta)+N_1(\theta)}$. Parameters are denoted as follows: μ indicates half saturation constant, rate α_0 is the per capita rate of predation of the IP, rate α_1 measures the efficiency of biomass conversion from prey to IP, rate α_2 is the per capita rate of predation of the TP, rate α_3 measures the efficiency of biomass conversion from IP to TP, rate α_4 is the per capita rate of predation of the prey, and rate α_5 measures the efficiency of biomass conversion from infected IP to TP. Furthermore, r is the intrinsic rate of growth of SP. Moreover, c measures the rate of contact between SIP and IIP while rate k represents the transformation from IIP to SIP, as this model is known as the SIS model. In this model, d_1 and d_2, respectively, stand for natural deaths of intermediate & TPs. Finally, q denotes harvesting of an IIP.

We present a brief presentation of the model which may indicate the biological relevance of it. Behavior of the entire biological community is assumed to arise from the coupling of the interacting species M, N, N_1, and Θ, where the top predator Θ prey on intermediate predators N and N_1, and intermediate predators prey on M. This is the practical assumption from both mathematical and biological points of view. A specific feature of these food chain systems is that if one species dies out, all the species at higher trophic levels die out as well. It is also assumed that in the absence of the predators the prey population density grows according to a logistic curve with carrying capacity and with an intrinsic growth rate constant r $(r>0)$. The consideration of functional response provides motivation to study a food chain model under the framework of nonlinear ODEs.

Now, we replace the classical derivatives in the model (3.1) with ABC fractional derivatives ${}_{0}^{ABC}D_{\theta}^{\omega}$ of order ω to capture memory effect in the model in this way:

$$
\begin{aligned}
{}_{0}^{ABC}D_{\theta}^{\omega}M(\theta) &= rM(\theta)(1-M(\theta)) - \alpha_0 \frac{M(\theta)N(\theta)}{1+\mu M(\theta)} - \alpha_4 \frac{M(\theta)N_1(\theta)}{1+\mu M(\theta)}, \\
{}_{0}^{ABC}D_{\theta}^{\omega}N(\theta) &= \alpha_1 \frac{M(\theta)N(\theta)}{1+\mu M(\theta)} - \alpha_2 \frac{\Theta(\theta)N(\theta)}{1+\mu N(\theta)} - c\frac{N(\theta)N_1(\theta)}{N(\theta)+N_1(\theta)} + kN_1(\theta) - d_1 N(\theta), \\
{}_{0}^{ABC}D_{\theta}^{\omega}N_1(\theta) &= c\frac{N(\theta)N_1(\theta)}{N(\theta)+N_1(\theta)} + \alpha_5 \frac{M(\theta)N_1(\theta)}{1+\mu M(\theta)} - kN_1(\theta) - qN_1(\theta), \\
{}_{0}^{ABC}D_{\theta}^{\omega}\Theta(\theta) &= \alpha_3 \frac{N(\theta)\Theta(\theta)}{1+\mu N(\theta)} - d_2\Theta(\theta),
\end{aligned}
\tag{3.2}
$$

along with following conditions:

$$
M_0(\theta) = M(0), \quad N_0(\theta) = N(0), \quad N_{1,0}(\theta) = N_1(0), \quad \Theta_0(\theta) = \Theta(0). \tag{3.3}
$$

4. Nonnegative solutions: qualitative properties of the model

In this segment, we discuss qualitative properties of the nonnegative solutions of the fractional FCM (3.2).

For biological reasons, each variable in model (3.2) must be a nonnegative real-valued function. In other words, $(M, N, N_1, \Theta) \in R_+^4$, where $R_+^4 = \{u = (u_1, u_2, u_3, u_4) : u_i \geq 0, \forall i = 1, 2, 3, 4\}$. Now, we demonstrate that all the solutions of the model (3.2) with (3.3) are absolutely nonnegative.

Lemma 4.1. All solutions of the model (3.2) lie in R_+^4.

Proof. We define

$$
\varphi(l) = \{l(\theta) = 0, \forall l \in \{M, N, N_1, \Theta\} \,\&\, (M, N, N_1, \Theta) \in R_+^4\}. \tag{4.1}
$$

Then from the FCM (3.2), we attain

$$
\begin{aligned}
&{}_{0}^{ABC}D_{\theta}^{\omega}M(\theta)\Big|_{\varphi(M))} = 0, \\
&{}_{0}^{ABC}D_{\theta}^{\omega}N(\theta)\Big|_{\varphi(N))} = k\,N_1 \geq 0, \text{ since } 0 < k \leq 1 \text{ and } N_1 \geq 0, \\
&{}_{0}^{ABC}D_{\theta}^{\omega}N_1(\theta)\Big|_{\varphi(N_1))} = 0, \\
&{}_{0}^{ABC}D_{\theta}^{\omega}\Theta(\theta)\Big|_{\varphi(\Theta))} = 0.
\end{aligned}
\tag{4.2}
$$

Thus, $(M(\theta), N(\theta), N_1(\theta), \Theta(\theta)) \in R_+^4$. Hence, the lemma is proved.

Theorem 4.2. Consider the subsequent initial value problem (IVP)

$$ {}_{0}^{ABC}D_{\theta}^{\omega}U(\theta)=V(\theta,U(\theta)),\ \ U(\theta_0)=U_0,\ \ 0<\omega\le 1, \tag{4.3} $$

where ${}_{0}^{ABC}D_{\theta}^{\omega}$ signifies the ABC fractional derivative operator and $V(\theta,U(\theta)):\Re^{+}\times\Re^{m}\to\Re^{m}$ denotes a vector field. This system (4.3) definitely has a unique solution on $[0,\infty)$ if

(a) $V(\theta,U(\theta))$ and all of its partial derivatives are continuous $\forall\ U\in\Re^{m}$.

(b) $\|V(\theta,U(\theta))\|\le a_1+a_2\|U(\theta)\|$ for each $U\in R^{m}$ for $a_1,a_2>0$.

Now, it is easy to establish that the above-mentioned criteria are fulfilled by the set of equations of the model (3.2) with (3.3), and, thus, it confirms the existence of unique nonnegative solutions for the model (3.2) with (3.3).

5. Analysis of existence of the solution of the FCM

Here, we investigate the existence of a solution of arbitrary order FCM with disease in the intermediate predator. Now exerting AB integral operator of fractional order in the system of Eq (3.2) in the following way:

$$ M(\theta)-M(0)={}_{0}^{AB}I_{\theta}^{\omega}\left[r(1-M)M-\alpha_0\frac{NM}{1+M\mu}-\alpha_4\frac{M\,N_1}{1+M\mu}\right], $$

$$ N(\theta)-N(0)={}_{0}^{AB}I_{\theta}^{\omega}\left[\alpha_1\frac{M\,N}{1+\mu M}-\alpha_2\frac{N\,\Theta}{1+\mu N}-c\frac{N\,N_1}{N+N_1}+kN_1-d_1N\right], $$

$$ N_1(\theta)-N_1(0)={}_{0}^{AB}I_{\theta}^{\omega}\left[c\frac{N_1N}{N_1+N}+\alpha_5\frac{N_1M}{1+M\mu}-kN_1-qN_1\right], $$

$$ \Theta(\theta)-\Theta(0)={}_{0}^{AB}I_{\theta}^{\omega}\left[\alpha_3\frac{N\,\Theta}{1+\mu N}-d_2\Theta\right]. \tag{5.1} $$

Making use of the definition of AB fractional integral operator in the system of Eq (5.1), we acquire

$$ M(\theta)-M(0)=\frac{1-\omega}{B(\omega)}\left[rM(\theta)(1-M(\theta))-\alpha_0\frac{N(\theta)M(\theta)}{1+\mu M(\theta)}-\alpha_4\frac{M(\theta)N_1(\theta)}{1+\mu M(\theta)}\right] $$

$$ +\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\left[rM(\tau)(1-M(\tau))-\alpha_0\frac{M(\tau)N(\tau)}{1+\mu M(\tau)}-\alpha_4\frac{M(\tau)N_1(\tau)}{1+\mu M(\tau)}\right]d\tau, $$

$$ N(\theta)-N(0)=\frac{1-\omega}{B(\omega)}\left[\alpha_1\frac{M(\theta)N(\theta)}{1+\mu M(\theta)}-\alpha_2\frac{N(\theta)\Theta(\theta)}{1+\mu N(\theta)}-c\frac{N(\theta)N_1(\theta)}{N(\theta)+N_1(\theta)}+kN_1(\theta)-d_1N(\theta)\right] $$

$$ +\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\left[\alpha_1\frac{M(\tau)N(\tau)}{1+\mu M(\tau)}-\alpha_2\frac{N(\tau)\Theta(\tau)}{1+\mu N(\tau)}-c\frac{N(\tau)N_1(\tau)}{N(\tau)+N_1(\tau)}+kN_1(\tau)-d_1N(\tau)\right]d\tau, $$

$$N_1(\theta)-N_1(0)=\frac{1-\omega}{B(\omega)}\left[c\frac{N(\theta)N_1(\theta)}{N(\theta)+N_1(\theta)}+\alpha_5\frac{M(\theta)N_1(\theta)}{1+\mu X(\theta)}-kN_1(\theta)-qN_1(\theta)\right]$$

$$+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\left[c\frac{N_1(\tau)N(\tau)}{N_1(\tau)+N(\tau)}+\alpha_5\frac{N_1(\tau)M(\tau)}{1+\mu M(\tau)}-kN_1(\tau)-qN_1(\tau)\right]d\tau,$$

$$\Theta(\theta)-\Theta(0)=\frac{1-\omega}{B(\omega)}\left[\alpha_3\frac{N(\theta)\Theta(\theta)}{1+\mu N(\theta)}-d_2\Theta(\theta)\right]$$

$$+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\left[\alpha_3\frac{N(\tau)\Theta(\tau)}{1+\mu N(\tau)}-d_2\Theta(\tau)\right]d\tau. \tag{5.2}$$

For simplified presentation of the above system of Eq (5.2), we express

$$\rho_1(\theta,M(\theta))=rM(\theta)(1-M(\theta))-\alpha_0\frac{N(\theta)M(\theta)}{1+M(\theta)\mu}-\alpha_4\frac{M(\theta)N_1(\theta)}{1+\mu M(\theta)},$$

$$\rho_2(\theta,N(\theta))=\alpha_1\frac{M(\theta)N(\theta)}{1+\mu M(\theta)}-\alpha_2\frac{N(\theta)\Theta(\theta)}{1+\mu N(\theta)}-c\frac{N(\theta)N_1(\theta)}{N(\theta)+N_1(\theta)}+kN_1(\theta)-d_1N(\theta),$$

$$\rho_3(\theta,N_1(\theta))=c\frac{N(\theta)N_1(\theta)}{N(\theta)+N_1(\theta)}+\alpha_5\frac{M(\theta)N_1(\theta)}{1+\mu M(\theta)}-kN_1(\theta)-qN_1,$$

$$\rho_4(\theta,\Theta(\theta))=\alpha_3\frac{N(\theta)\Theta(\theta)}{1+\mu N(\theta)}-d_2\Theta(\theta). \tag{5.3}$$

Theorem 5.1. The kernels ρ_1, ρ_2, ρ_3, and ρ_4 fulfill the Lipschitz criterion, and contraction of the conditions $0\le\lambda_1<1$, $0\le\lambda_2<1$, $0\le\lambda_3<1$, and $0\le\lambda_4<1$ are satisfied.

Proof. First, we initiate with the kernel ρ_1. Suppose $M(\theta)$ and $M^*(\theta)$ are two functions for the kernel ρ_1 fulfilling the conditions $\|M\|\le l_1$ and $\|M^*\|\le l_1^*$. Similarly, $N(\theta)$ and $N^*(\theta)$ are assumed to be the functions for the kernel ρ_2 satisfying the criteria $\|N\|\le l_2$ and $\|N^*\|\le l_2^*$, $N_1(\theta)$ and $N_1^*(\theta)$ are the two functions for the kernel ρ_3 satisfying the conditions $\|N_1\|\le l_3$ and $\|N_1^*\|\le l_3^*$, $\Theta(\theta)$ and $\Theta^*(\theta)$ are the two functions for the kernel ρ_4 satisfying the conditions $\|\Theta\|\le l_4$ and $\|\Theta^*\|\le l_4^*$.

$$\left\|\rho_1(\theta,M)-\rho_1(\theta,M^*)\right\|$$

$$=\left\|r(M-M^*)-r(M-M^*)(M^*+M)-\alpha_0\frac{(M-M^*)}{(1+\mu M)(1+\mu M^*)}N-\alpha_4N_1\frac{(M-M^*)}{(1+\mu M)(1+\mu M^*)}\right\|$$

$$\leq r\left\|M-M^*\right\|+r\left\|M+M^*\right\|\left\|M-M^*\right\|+\alpha_0\frac{\left\|M-M^*\right\|}{(1+\mu l_1)(1+\mu l_1^*)}\left\|N\right\|+\alpha_4\frac{\left\|M-M^*\right\|}{(1+\mu l_1)(1+\mu l_1^*)}\left\|N_1\right\|$$

$$\leq\left[r+r(l_1+l_1^*)+\frac{\alpha_0 l_2}{(1+\mu l_1)(1+\mu l_1^*)}+\frac{\alpha_4 l_3}{(1+\mu l_1)(1+\mu l_1^*)}\right]\left\|M-M^*\right\|. \tag{5.4}$$

Let

$$\lambda_1=r+r(l_1+l_1^*)+\frac{\alpha_0 l_2}{(1+\mu l_1)(1+\mu l_1^*)}+\frac{\alpha_4 l_3}{(1+\mu l_1)(1+\mu l_1^*)}. \tag{5.5}$$

Here, $M(\theta)$ signifies the bounded function, then

$$\left\|\rho_1(\theta,M)-\rho_1(\theta,M^*)\right\|\leq\lambda_1\left\|M-M^*\right\|. \tag{5.6}$$

Thus, the kernel ρ_1 satisfies the Lipschitz criterion. In addition, if $\lambda_1\in[0,1)$, then it will also be a contraction for ρ_1. Similarly,

$$\left\|\rho_2(\theta,N)-\rho_2(\theta,N^*)\right\|$$

$$=\left\|\left\{\alpha_1\frac{M\,N}{1+\mu M}-\alpha_2\frac{N\,\Theta}{1+\mu N}-c\frac{N\,N_1}{N+N_1}+kN_1-d_1N\right\}\right.$$

$$\left.-\left\{\alpha_1\frac{M\,N^*}{1+\mu M}-\alpha_2\frac{N^*\Theta}{1+\mu N^*}-c\frac{N^*N_1}{N^*+N_1}+kN_1-d_1N^*\right\}\right\|$$

$$=\left\|\alpha_1\frac{M}{1+\mu M}(N-N^*)-\alpha_2\left(\frac{N}{1+\mu N}-\frac{N^*}{1+\mu N^*}\right)\Theta-c\left(\frac{N}{N+N_1}-\frac{N^*}{N^*+N_1}\right)N_1-d_1(N-N^*)\right\|$$

$$=\left\|\alpha_1\frac{M}{1+\mu M}(N-N^*)-\alpha_2\frac{(N-N^*)\Theta}{(1+\mu N)(1+\mu N^*)}-c\frac{(N-N^*)}{(N+N_1)(N^*+N_1)}N_1^2-d_1(N-N^*)\right\|$$

$$\leq\left[\alpha_1\frac{l_1}{1+\mu l_1}+\alpha_2\frac{l_4}{(\mu l_2+1)(1+\mu l_2^*)}+c\frac{l_3^2}{(l_2+l_3)(l_2^*+l_3)}+d_1\right]\left\|N-N^*\right\|. \tag{5.7}$$

Let

$$\lambda_2=\frac{\alpha_1 l_1}{1+\mu l_1}+\frac{\alpha_2 l_4}{(1+\mu l_2)(1+\mu l_2^*)}+\frac{c\,l_3^2}{(l_2^*+l_3)(l_2+l_3)}+d_1\ . \tag{5.8}$$

Here, $N(t)$ signifies the bounded function, then

$$\left\|\rho_2(\theta,N)-\rho_2(\theta,N^*)\right\|\leq\lambda_2\left\|N-N^*\right\|. \tag{5.9}$$

Therefore, the Lipschitz criterion is fulfilled for the kernel ρ_2. Further, if $0 \le \lambda_2 < 1$, then it is also a contraction for ρ_2.

Similarly, the Lipschitz criterion is satisfied for the kernels ρ_3 and ρ_4 as follows:

$$\left\| \rho_3(\theta, N_1) - \rho_3\left(\theta, N_1^*\right) \right\| \le \lambda_3 \left\| N_1 - N_1^* \right\|, \tag{5.10}$$

$$\left\| \rho_4(\theta, \Theta) - \rho_4\left(\theta, \Theta^*\right) \right\| \le \lambda_4 \left\| \Theta - \Theta^* \right\|, \tag{5.11}$$

where

$$\lambda_3 = c \frac{l_2}{(l_2 + l_3)(l_2 + l_3^*)} + \alpha_5 \frac{l_1}{1 + \mu l_1} - (k + q), \tag{5.12}$$

$$\lambda_4 = \alpha_3 \frac{l_2}{1 + \mu l_2} - d_2 . \tag{5.13}$$

Moreover, if $0 \le \lambda_3 < 1$ and $0 \le \lambda_4 < 1$, it is also a contraction for ρ_3 and ρ_4.

In view of the system of Eq (5.3), the system of Eq (5.2) takes the following form:

$$M(\theta) = M_0 + \frac{1-\omega}{B(\omega)} \rho_1(\theta, M) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^\theta (\theta - \tau)^{\omega - 1} \rho_1(\tau, M) d\tau ,$$

$$N(\theta) = N_0 + \frac{1-\omega}{B(\omega)} \rho_2(\theta, N) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^\theta (\theta - \tau)^{\omega - 1} \rho_2(\tau, N) d\tau ,$$

$$N_1(\theta) = N_{1,0} + \frac{1-\omega}{B(\omega)} \rho_3(\theta, N_1) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^\theta (\theta - \tau)^{\omega - 1} \rho_3(\tau, N_1) d\tau ,$$

$$\Theta(\theta) = \Theta_0 + \frac{1-\omega}{B(\omega)} \rho_4(\theta, \Theta) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^\theta (\theta - \tau)^{\omega - 1} \rho_4(\tau, \Theta) d\tau . \tag{5.14}$$

Now, the subsequent recursive formulae are constructed in this way:

$$M_n(\theta) = M_0 + \frac{1-\omega}{B(\omega)} \rho_1(\theta, M_{n-1}) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^\theta (\theta - \tau)^{\omega - 1} \rho_1(\tau, M_{n-1}) d\tau ,$$

$$N_n(\theta) = N_0 + \frac{1-\omega}{B(\omega)} \rho_2(\theta, N_{n-1}) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^\theta (\theta - \tau)^{\omega - 1} \rho_2(\tau, N_{n-1}) d\tau ,$$

$$N_{1,n}(\theta) = N_{1,0} + \frac{1-\omega}{B(\omega)} \rho_3\left(\theta, N_{1,n-1}\right) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^\theta (\theta - \tau)^{\omega - 1} \rho_3\left(\tau, N_{1,n-1}\right) d\tau ,$$

$$\Theta_n(\theta) = \Theta_0 + \frac{1-\omega}{B(\omega)} \rho_4(\theta, \Theta_{n-1}) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^\theta (\theta - \tau)^{\omega - 1} \rho_4(\tau, \Theta_{n-1}) d\tau , \tag{5.15}$$

with the following conditions:

$$M_0(\theta)=M(0),\ \ N_0(\theta)=N(0),\ \ N_{1,0}(\theta)=N_1(0),\ \ \Theta_0(\theta)=\Theta(0). \tag{5.16}$$

Next, we consider difference of the successive terms as

$$\wp_{1n}=M_n-M_{n-1}$$
$$=\frac{1-\omega}{B(\omega)}\left(\rho_1(\theta,M_{n-1})-\rho_1(\theta,M_{n-2})\right)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\left(\rho_1(\tau,M_{n-1})-\rho_1(\tau,M_{n-2})\right)d\tau,$$

$$\wp_{2n}=N_n-N_{n-1}$$
$$=\frac{1-\omega}{B(\omega)}\left(\rho_2(\theta,N_{n-1})-\rho_2(\theta,N_{n-2})\right)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\left(\rho_2(\tau,N_{n-1})-\rho_2(\tau,N_{n-2})\right)d\tau,$$

$$\wp_{3n}=N_{1,n}-N_{1,n-1}$$
$$=\frac{1-\omega}{B(\omega)}\left(\rho_3(\theta,N_{1,n-1})-\rho_3(\theta,N_{1,n-2})\right)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\left(\rho_3(\tau,N_{1,n-1})-\rho_3(\tau,N_{1,n-2})\right)d\tau,$$

$$\wp_{4n}=\Theta_n-\Theta_{n-1}$$
$$=\frac{1-\omega}{B(\omega)}\left(\rho_4(\theta,\Theta_{n-1})-\rho_4(\theta,\Theta_{n-2})\right)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\left(\rho_4(\tau,\Theta_{n-1})-\rho_4(\tau,\Theta_{n-2})\right)d\tau. \tag{5.17}$$

It is worth noting that

$$M_n(\theta)=\sum_{j=0}^{n}\wp_{1j}(\theta),\ \ N_n(\theta)=\sum_{j=0}^{n}\wp_{2j}(\theta),\ \ N_{1,n}(\theta)=\sum_{j=0}^{n}\wp_{3j}(\theta),\ \ \Theta_n(\theta)=\sum_{j=0}^{n}\wp_{4j}(\theta). \tag{5.18}$$

Now, utilizing the set of Eq (5.17) along with the triangular inequality, we attain

$$\|\wp_{1n}(\theta)\|=\|M_n-M_{n-1}\|$$
$$\leq\frac{1-\omega}{B(\omega)}\|\rho_1(\theta,M_{n-1})-\rho_1(\theta,M_{n-2})\|+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\|\rho_1(\tau,M_{n-1})-\rho_1(\tau,M_{n-2})\|d\tau,$$

$$\|\wp_{2n}(\theta)\|=\|N_n-N_{n-1}\|$$
$$\leq\frac{1-\omega}{B(\omega)}\|\rho_2(\theta,N_{n-1})-\rho_2(\theta,N_{n-2})\|+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\|\rho_2(\tau,N_{n-1})-\rho_2(\tau,N_{n-2})\|d\tau,$$

$$\|\wp_{3n}(\theta)\| = \|N_{1,n}(\theta) - N_{1,n-1}(\theta)\|$$

$$\leq \frac{1-\omega}{B(\omega)}\|\rho_3(\theta, N_{1,n-1}) - \rho_3(\theta, N_{1,n-2})\| + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\|\rho_3(\tau, N_{1,n-1}) - \rho_3(\tau, N_{1,n-2})\|d\tau,$$

$$\|\wp_{4n}(\theta)\| = \|\Theta_n(\theta) - \Theta_{1,n-1}(\theta)\|$$

$$\leq \frac{1-\omega}{B(\omega)}\|\rho_4(\theta, \Theta_{n-1}) - \rho_4(\theta, \Theta_{n-2})\| + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\|\rho_4(\tau, \Theta_{n-1}) - \rho_4(\tau, \Theta_{n-2})\|d\tau. \quad (5.19)$$

It is already proved that the kernels ρ_1, ρ_2, ρ_3, and ρ_4 satisfy the Lipschitz condition, so the set of Eq (5.19) reduces to

$$\|\wp_{1n}(\theta)\| \leq \frac{1-\omega}{B(\omega)}\lambda_1\|M_{n-1} - M_{n-2}\| + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\|M_{n-1} - M_{n-2}\|\lambda_1 d\tau. \quad (5.20)$$

Consequently, we attain the following result:

$$\|\wp_{1n}(\theta)\| \leq \frac{1-\omega}{B(\omega)}\lambda_1\|\wp_{1(n-1)}(\theta)\| + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\lambda_1\|\wp_{1(n-1)}(\tau)\|d\tau. \quad (5.21)$$

Applying the same procedure, we attain other results as follows:

$$\|\wp_{2n}(\theta)\| \leq \frac{1-\omega}{B(\omega)}\lambda_2\|\wp_{2(n-1)}(\theta)\| + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\lambda_2\|\wp_{2(n-1)}(\tau)\|d\tau,$$

$$\|\wp_{3n}(\theta)\| \leq \frac{1-\omega}{B(\omega)}\lambda_3\|\wp_{3(n-1)}(\theta)\| + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\lambda_3\|\wp_{3(n-1)}(\tau)\|d\tau,$$

$$\|\wp_{4n}(\theta)\| \leq \frac{1-\omega}{B(\omega)}\lambda_4\|\wp_{4(n-1)}(\theta)\| + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\lambda_4\|\wp_{4(n-1)}(\tau)\|d\tau. \quad (5.22)$$

Taking Eqs (5.21) and (5.22) into account, we acquire the existence of the solution of the considered model. This establishes the theorem.

Theorem 5.2. The fractional FCM involving Holling type II functional response expressed by the system of Eq (3.2) possesses a solution if $\exists\ \theta_0$ in this way:

$$\frac{(1-\omega)}{B(\omega)}\lambda_i + \frac{\theta_0^\omega}{B(\omega)\Gamma(\omega)}\lambda_i < 1, \quad i = 1, 2, 3, 4. \quad (5.23)$$

Proof. It is assumed that $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$ are functions of bounded nature. Now, using Eqs (5.21) and (5.22) along with the recursive algorithm, we get

$$\|\wp_{1n}(\theta)\| \le \|M(0)\| \left[\lambda_1 \frac{(1-\omega)}{B(\omega)} + \lambda_1 \frac{\theta_0^{\omega}}{B(\omega)\Gamma(\omega)} \right]^n,$$

$$\|\wp_{2n}(\theta)\| \le \|N(0)\| \left[\lambda_2 \frac{(1-\omega)}{B(\omega)} + \lambda_2 \frac{\theta_0^{\omega}}{B(\omega)\Gamma(\omega)} \right]^n,$$

$$\|\wp_{3n}(\theta)\| \le \|N_1(0)\| \left[\lambda_3 \frac{(1-\omega)}{B(\omega)} + \lambda_3 \frac{\theta_0^{\omega}}{B(\omega)\Gamma(\omega)} \right]^n,$$

$$\|\wp_{4n}(\theta)\| \le \|\Theta(0)\| \left[\lambda_4 \frac{(1-\omega)}{B(\omega)} + \lambda_4 \frac{\theta_0^{\omega}}{B(\omega)\Gamma(\omega)} \right]^n. \tag{5.24}$$

Evidently, Eq (5.18) assures about the existence and smoothness of the functions. Thus, the solution of system (3.2) exists as well as is continuous. Furthermore, to examine that the system (5.14) is a solution of the FCM model (3.2), it is presumed that

$$M(\theta) = M(0) + M_n(\theta) - F_{1n}(\theta),$$

$$N(\theta) = N(0) + N_n(\theta) - F_{2n}(\theta),$$

$$N_1(\theta) = N_1(0) + N_{1,n}(\theta) - F_{3n}(\theta),$$

$$\Theta(\theta) = \Theta(0) + \Theta_n(\theta) - F_{4n}(\theta). \tag{5.25}$$

Thus, we find

$$\|F_{1n}(\theta)\| = \left\| \frac{1-\omega}{B(\omega)} (\rho_1(\theta, M) - \rho_1(\theta, M_{n-1})) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta-\tau)^{\omega-1} (\rho_1(\tau, M) - \rho_1(\tau, M_{n-1})) d\tau \right\|,$$

$$\|F_{2n}(\theta)\| = \left\| \frac{1-\omega}{B(\omega)} (\rho_2(\theta, N) - \rho_2(\theta, N_{n-1})) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta-\tau)^{\omega-1} (\rho_2(\tau, N) - \rho_2(\tau, N_{n-1})) d\tau \right\|,$$

$$\|F_{3n}(\theta)\| = \left\| \frac{1-\omega}{B(\omega)} (\rho_3(\theta, N_1) - \rho_3(\theta, N_{1,n-1})) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta-\tau)^{\omega-1} (\rho_3(\tau, N_1) - \rho_3(\tau, N_{1,n-1})) d\tau \right\|,$$

$$\|F_{4n}(\theta)\| = \left\| \frac{1-\omega}{B(\omega)} (\rho_4(\theta, \Theta) - \rho_4(\theta, \Theta_{n-1})) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta-\tau)^{\omega-1} (\rho_4(\tau, \Theta) - \rho_4(\tau, \Theta_{n-1})) d\tau \right\|. \tag{5.26}$$

Now, we have

$$\|F_{1n}(\theta)\| \le \frac{1-\omega}{B(\omega)} \|\rho_1(\theta, M) - \rho_1(\theta, M_{n-1})\| + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta-\tau)^{\omega-1} \|\rho_1(\tau, M) - \rho_1(\tau, M_{n-1})\| d\tau$$

$$\leq \lambda_1 \frac{1-\omega}{B(\omega)}\|M - M_{n-1}\| + \lambda_1 \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)}\|M - M_{n-1}\|\theta^{\omega}. \tag{5.27}$$

Making the process recursively, we achieve

$$\|F_{1n}(\theta)\| \leq a_1\left(\frac{(1-\omega)}{B(\omega)} + \frac{\theta^{\omega}}{B(\omega)\Gamma(\omega)}\right)^{n+1} \lambda_1^{\ n+1}. \tag{5.28}$$

For $\theta = \theta_0$, we obtain

$$\|F_{1n}(\theta)\| \leq \left(\frac{(1-\omega)}{B(\omega)} + \frac{\theta_0^{\omega}}{B(\omega)\Gamma(\omega)}\right)^{n+1} \lambda_1^{\ n+1} a_1. \tag{5.29}$$

Applying the similar methodology, we further get

$$\|F_{2n}(\theta)\| \leq \left(\frac{(1-\omega)}{B(\omega)} + \frac{\theta_0^{\omega}}{B(\omega)\Gamma(\omega)}\right)^{n+1} \lambda_2^{\ n+1} a_2,$$

$$\|F_{3n}(\theta)\| \leq \left(\frac{(1-\omega)}{B(\omega)} + \frac{\theta_0^{\omega}}{B(\omega)\Gamma(\omega)}\right)^{n+1} \lambda_3^{\ n+1} a_3,$$

$$\|F_{4n}(\theta)\| \leq \left(\frac{(1-\omega)}{B(\omega)} + \frac{\theta_0^{\omega}}{B(\omega)\Gamma(\omega)}\right)^{n+1} \lambda_4^{\ n+1} a_4. \tag{5.30}$$

Now, employing the limit on inequality (5.29) as $n \to \infty$, we find $\|F_{1n}(\theta)\| \to 0$. Implementing the same procedure, we have $\|F_{in}(\theta)\| \to 0$, $i = 2,3,4$, and this establishes the theorem.

6. Uniqueness of system of solutions of the model

Here, we show the uniqueness of the solution of the fractional food chain model (3.2). We assume that $M^*(\theta)$, $N^*(\theta)$, $N_1^*(\theta)$, and $\Theta^*(\theta)$ is another set of solutions for the ABC fractional order model (3.2), then

$$M - M^* = \left(\rho_1(\theta, M) - \rho_1(\theta, M^*)\right)\frac{1-\omega}{B(\omega)} + \frac{1}{\Gamma(\omega)} \frac{\omega}{B(\omega)} \int_0^{\theta} (\theta-\tau)^{\omega-1} \left(\rho_1(\tau, M) - \rho_1(\tau, M^*)\right) d\tau,$$

$$N - N^* = \left(\rho_2(\theta, N) - \rho_2(\theta, N^*)\right)\frac{1-\omega}{B(\omega)} + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta-\tau)^{\omega-1} \left(\rho_2(\tau, N) - \rho_2(\tau, N^*)\right) d\tau,$$

$$N_1 - N_1^* = \left(\rho_3(\theta, N_1) - \rho_3(\theta, N_1^*)\right)\frac{1-\omega}{B(\omega)} + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta-\tau)^{\omega-1} \left(\rho_3(\tau, N_1) - \rho_3(\tau, N_1^*)\right) d\tau,$$

$$\Theta-\Theta^{*}=\left(\rho_4(\theta,\Theta)-\rho_4\left(\theta,\Theta^{*}\right)\right)\frac{1-\omega}{B(\omega)}+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\left(\rho_4(\tau,\Theta)-\rho_4\left(\tau,\Theta^{*}\right)\right)d\tau. \quad (6.1)$$

Taking the norm of equations of system (6.1) provides

$$\left\|M-M^{*}\right\|\leq\left\|\rho_1(\theta,M)-\rho_1\left(\theta,M^{*}\right)\right\|\frac{1-\omega}{B(\omega)}+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\left\|\rho_1(\tau,M)-\rho_1\left(\tau,M^{*}\right)\right\|d\tau,$$

$$\left\|N-N^{*}\right\|\leq\left\|\rho_2(\theta,N)-\rho_2\left(\theta,N^{*}\right)\right\|\frac{1-\omega}{B(\omega)}+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\left\|\rho_2(\tau,N)-\rho_2\left(\tau,N^{*}\right)\right\|d\tau,$$

$$\left\|N_1-N_1^{*}\right\|\leq\left\|\rho_3(\theta,N_1)-\rho_3\left(\theta,N_1^{*}\right)\right\|\frac{1-\omega}{B(\omega)}+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\left\|\rho_3(\tau,N_1)-\rho_3\left(\tau,N_1^{*}\right)\right\|d\tau,$$

$$\left\|\Theta-\Theta^{*}\right\|\leq\left\|\rho_4(\theta,\Theta)-\rho_4\left(\theta,\Theta^{*}\right)\right\|\frac{1-\omega}{B(\omega)}+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\left\|\rho_4(\tau,\Theta)-\rho_4\left(\tau,\Theta^{*}\right)\right\|d\tau. \quad (6.2)$$

Now, employing the results presented in (5.6) and (5.9)–(5.11) in the set of inequalities (6.2), we have

$$\left\|M-M^{*}\right\|\leq\frac{1-\omega}{B(\omega)}\lambda_1\left\|M(\theta)-M^{*}(\theta)\right\|+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\lambda_1\left\|M(\tau)-M^{*}(\tau)\right\|d\tau,$$

$$\left\|N-N^{*}\right\|\leq\frac{1-\omega}{B(\omega)}\lambda_2\left\|N(\theta)-N^{*}(\theta)\right\|+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\lambda_2\left\|N(\tau)-N^{*}(\tau)\right\|d\tau,$$

$$\left\|N_1-N_1^{*}\right\|\leq\frac{1-\omega}{B(\omega)}\lambda_3\left\|N_1(\theta)-N_1^{*}(\theta)\right\|+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\lambda_3\left\|N_1(\tau)-N_1^{*}(\tau)\right\|d\tau,$$

$$\left\|\Theta-\Theta^{*}\right\|\leq\frac{1-\omega}{B(\omega)}\lambda_4\left\|\Theta(\theta)-\Theta^{*}(\theta)\right\|+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\lambda_4\left\|\Theta(\tau)-\Theta^{*}(\tau)\right\|d\tau. \quad (6.3)$$

After simplification, we achieve

$$\left\|M-M^{*}\right\|\leq\frac{1-\omega}{B(\omega)}\lambda_1\left\|M-M^{*}\right\|+\frac{\theta^{\omega}}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_1\left\|M-M^{*}\right\|,$$

$$\left\|N-N^{*}\right\|\leq\frac{1-\omega}{B(\omega)}\lambda_2\left\|N-N^{*}\right\|+\frac{\theta^{\omega}}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_2\left\|N-N^{*}\right\|,$$

$$\left\|N_1-N_1^{*}\right\|\leq\frac{1-\omega}{B(\omega)}\lambda_3\left\|N_1-N_1^{*}\right\|+\frac{\theta^{\omega}}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_3\left\|N_1-N_1^{*}\right\|,$$

$$\left\|\Theta-\Theta^{*}\right\|\leq\frac{1-\omega}{B(\omega)}\lambda_4\left\|\Theta-\Theta^{*}\right\|+\frac{\theta^{\omega}}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_4\left\|\Theta-\Theta^{*}\right\|, \quad (6.4)$$

which produces

$$\left\|M(\theta)-M^*(\theta)\right\|\left(1-\frac{1-\omega}{B(\omega)}\lambda_1-\frac{\theta^\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_1\right)\leq 0,$$

$$\left\|N(\theta)-N^*(\theta)\right\|\left(1-\frac{1-\omega}{B(\omega)}\lambda_2-\frac{\theta^\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_2\right)\leq 0,$$

$$\left\|N_1(\theta)-N_1^*(\theta)\right\|\left(1-\frac{1-\omega}{B(\omega)}\lambda_3-\frac{\theta^\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_3\right)\leq 0,$$

$$\left\|\Theta(\theta)-\Theta^*(\theta)\right\|\left(1-\frac{1-\omega}{B(\omega)}\lambda_4-\frac{\theta^\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_4\right)\leq 0. \tag{6.5}$$

Theorem 6.1. The fractional food chain model (3.2) will possess the unique solution if

$$\left(1-\lambda_i\frac{1-\omega}{B(\omega)}-\frac{\theta^\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_i\right)>0,\quad i=1,2,3,4. \tag{6.6}$$

Proof. If the conditions presented in (6.6) hold, then the set of inequalities (6.5) provides

$$\left\|M(\theta)-M^*(\theta)\right\|\left(1-\frac{1-\omega}{B(\omega)}\lambda_1-\frac{\theta^\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_1\right)\leq 0,$$

$$\left\|N(\theta)-N^*(\theta)\right\|\left(1-\frac{1-\omega}{B(\omega)}\lambda_2-\frac{\theta^\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_2\right)\leq 0,$$

$$\left\|N_1(\theta)-N_1^*(\theta)\right\|\left(1-\frac{1-\omega}{B(\omega)}\lambda_3-\frac{\theta^\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_3\right)\leq 0,$$

$$\left\|\Theta(\theta)-\Theta^*(\theta)\right\|\left(1-\frac{1-\omega}{B(\omega)}\lambda_4-\frac{\theta^\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\lambda_4\right)\leq 0. \tag{6.7}$$

In view of properties of norm, the set of conditions (6.7) implies that

$$\left\|M-M^*\right\|=0,\ \left\|N-N^*\right\|=0,\ \left\|N_1-N_1^*\right\|=0,\ \left\|\Theta-\Theta^*\right\|=0. \tag{6.8}$$

Thus,

$$M=M^*,\quad N=N^*,\quad N_1=N_1^*,\quad \Theta=\Theta^*. \tag{6.9}$$

Therefore, the food chain model (3.2) has a unique solution. Hence, the theorem proved.

7. Ulam-Hyers stability analysis for the model

The Ulam-Hyers stability approach [43,44] has been used for problems with fractional derivatives [45–48]. Now, we apply this approach to discuss stability for model (3.2) by virtue of nonlinear functional analysis.

Definition 7.1. System (3.2) with (3.3) possesses Ulam-Hyers stability if there exist $\sigma = \max(\sigma_1, \sigma_2, \sigma_3, \sigma_4) > 0$ and $\aleph = \max(\aleph_1, \aleph_2, \aleph_3, \aleph_4) > 0$, for each $\tilde{M},\ \tilde{N},\ \tilde{N}_1, \tilde{\Theta} \in E \times E \times E \times E$, having the subsequent inequalities

$$\left| {}^{ABC}_{0}D^{\omega}_{\theta}\tilde{M}(\theta) - \rho_1\left(\theta, \tilde{M}\right) \right| \le \aleph_1,$$

$$\left| {}^{ABC}_{0}D^{\omega}_{\theta}\tilde{N}(\theta) - \rho_1\left(\theta, \tilde{N}\right) \right| \le \aleph_2,$$

$$\left| {}^{ABC}_{0}D^{\omega}_{\theta}\tilde{N}_1(\theta) - \rho_1\left(\theta, \tilde{N}_1\right) \right| \le \aleph_3,$$

$$\left| {}^{ABC}_{0}D^{\omega}_{\theta}\tilde{\Theta}(\theta) - \rho_1\left(\theta, \tilde{\Theta}\right) \right| \le \aleph_4, \tag{7.1}$$

then there exists $\left(M, N, N_1, \tilde{\Theta}\right) \in E \times E \times E \times E$ fulfilling system (3.2) with conditions given by

$$M(0) = \tilde{M}(0),\quad N(0) = \tilde{N}(0),\quad N_1(0) = \tilde{N}_1(0),\quad \Theta(0) = \tilde{\Theta}(0) \tag{7.2}$$

such that

$$\left\| \left(\tilde{M},\ \tilde{N},\ \tilde{N}_1, \tilde{\Theta}\right) - \left(M, N, N_1, \Theta\right) \right\|_{\Omega} \le \sigma \aleph. \tag{7.3}$$

Remark 7.2. Consider the small perturbations $h_1, h_2, h_3, h_4 \in C[0, \theta_0]$ that depend only on the solutions such that $h_1(0) = 0,\ h_2(0) = 0,\ h_3(0) = 0,\ h_4(0) = 0$, with the following properties:

(1)

$$|h_1(\theta)| \le \aleph_1,\ |h_2(\theta)| \le \aleph_2,\ |h_3(\theta)| \le \aleph_3,\ |h_4(\theta)| \le \aleph_4, \text{ for } \theta \in [0, \theta_0],\ \aleph_i > 0,\ i = 1, 2, 3, 4. \tag{7.4}$$

(2) Furthermore, one has

$${}^{ABC}_{0}D^{\omega}_{\theta}\tilde{M}(\theta) = \rho_1\left(\theta, \tilde{M}\right) + h_1(\theta),\ \theta \in [0, \theta_0],$$

$${}^{ABC}_{0}D^{\omega}_{\theta}\tilde{N}(\theta) = \rho_1\left(\theta, \tilde{N}\right) + h_2(\theta),\ \theta \in [0, \theta_0],$$

$${}^{ABC}_{0}D^{\omega}_{\theta}\tilde{N}_1(\theta) = \rho_1\left(\theta, \tilde{N}_1\right) + h_3(\theta),\ \theta \in [0, \theta_0],$$

$${}^{ABC}_{0}D^{\omega}_{\theta}\tilde{\Theta}(\theta) = \rho_1\left(\theta, \tilde{\Theta}\right) + h_4(\theta),\ \theta \in [0, \theta_0]. \tag{7.5}$$

Now,

$$\left\|\tilde{M}(\theta)-M(\theta)\right\|\le\sigma_1\aleph_1,$$

$$\left\|\tilde{N}(\theta)-N(\theta)\right\|\le\sigma_2\aleph_2,$$

$$\left\|\tilde{N}_1(\theta)-N_1(\theta)\right\|\le\sigma_3\aleph_3,$$

$$\left\|\tilde{\Theta}(\theta)-\Theta(\theta)\right\|\le\sigma_4\aleph_4. \tag{7.6}$$

Lemma 7.3. The solution of perturbed problems

$$\begin{cases} {}^{ABC}_{0}D^{\omega}_{\theta}\tilde{M}(\theta)=\rho_1\left(\theta,\tilde{M}\right)+h_1(\theta), \\ \tilde{M}(0)=\tilde{M}_0, \end{cases} \tag{7.7}$$

$$\begin{cases} {}^{ABC}_{0}D^{\omega}_{\theta}\tilde{N}(\theta)=\rho_1\left(\theta,\tilde{N}\right)+h_2(\theta), \\ \tilde{N}(0)=\tilde{N}_0, \end{cases} \tag{7.8}$$

$$\begin{cases} {}^{ABC}_{0}D^{\omega}_{\theta}\tilde{N}_1(\theta)=\rho_1\left(\theta,\tilde{N}_1\right)+h_3(\theta), \\ \tilde{N}_1(0)=\tilde{N}_{1,0}, \end{cases} \tag{7.9}$$

$$\begin{cases} {}^{ABC}_{0}D^{\omega}_{\theta}\tilde{\Theta}(\theta)=\rho_1\left(\theta,\tilde{\Theta}\right)+h_4(\theta), \\ \tilde{\Theta}(0)=\tilde{\Theta}_0 \end{cases} \tag{7.10}$$

fulfills the relations

$$\left|\tilde{M}_{h_1}(\theta)-\tilde{M}(\theta)\right|\le m\aleph_1,$$

$$\left|\tilde{N}_{h_2}(\theta)-\tilde{N}(\theta)\right|\le m\aleph_2,$$

$$\left|\tilde{N}_{1,h_3}(\theta)-\tilde{N}_1(\theta)\right|\le m\aleph_3,$$

$$\left|\tilde{\Theta}_{h_4}(\theta)-\tilde{\Theta}(\theta)\right|\le m\aleph_4, \tag{7.11}$$

where $\tilde{M}_{h_1}(\theta)$, $\tilde{N}_{h_2}(\theta)$, $\tilde{N}_{1,h_3}(\theta)$, $\tilde{\Theta}_{h_4}(\theta)$ are solutions of Eqs (7.7)–(7.10), respectively. Here, $\tilde{M}$, $\tilde{N}$, $\tilde{N}_1$, $\tilde{\Theta}$ satisfy the set of conditions (7.1) and $m=\dfrac{\Gamma(\omega)-\Gamma(\omega+1)+\theta_0^{\omega}}{B(\omega)\Gamma(\omega)}$.

Proof. As suggested by Remark 7.2 and Lemma 7.3, the solutions of Eqs (7.7)–(7.10) are, respectively, given by

$$\tilde{M}_{h_1}(\theta)=\tilde{M}_0+\frac{1-\omega}{B(\omega)}\rho_1\left(\theta,\tilde{M}\right)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}\rho_1\left(\tau,\tilde{M}\right)d\tau$$

$$+\frac{1-\omega}{B(\omega)}h_1(\theta)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^{\theta}(\theta-\tau)^{\omega-1}h_1(\tau)d\tau,$$

$$\tilde{N}_{h_2}(\theta)=\tilde{N}_0+\frac{1-\omega}{B(\omega)}\rho_2(\theta,\tilde{N})+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\rho_2(\tau,\tilde{N})d\tau$$

$$+\frac{1-\omega}{B(\omega)}h_2(\theta)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}h_2(\tau)d\tau,$$

$$\tilde{N}_{1,h_3}(\theta)=\tilde{N}_{1,0}+\frac{1-\omega}{B(\omega)}\rho_3(\theta,\tilde{N}_1)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\rho_3(\tau,\tilde{N}_1)d\tau$$

$$+\frac{1-\omega}{B(\omega)}h_3(\theta)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}h_3(\tau)d\tau,$$

$$\tilde{\Theta}_{h_4}(\theta)=\tilde{\Theta}_0+\frac{1-\omega}{B(\omega)}\rho_4(\theta,\tilde{\Theta})+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\rho_4(\tau,\tilde{\Theta})d\tau$$

$$+\frac{1-\omega}{B(\omega)}h_4(\theta)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}h_4(\tau)d\tau. \tag{7.12}$$

Also, we find

$$\tilde{M}(\theta)=\tilde{M}_0+\frac{1-\omega}{B(\omega)}\rho_1(\theta,\tilde{M})+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\rho_1(\tau,\tilde{M})d\tau,$$

$$\tilde{N}(\theta)=\tilde{N}_0+\frac{1-\omega}{B(\omega)}\rho_2(\theta,\tilde{N})+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\rho_2(\tau,\tilde{N})d\tau,$$

$$\tilde{N}_1(\theta)=\tilde{N}_{1,0}+\frac{1-\omega}{B(\omega)}\rho_3(\theta,\tilde{N}_1)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\rho_3(\tau,\tilde{N}_1)d\tau,$$

$$\tilde{\Theta}(\theta)=\tilde{\Theta}_0+\frac{1-\omega}{B(\omega)}\rho_4(\theta,\tilde{\Theta})+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\rho_4(\tau,\tilde{\Theta})d\tau. \tag{7.13}$$

It follows from the Remark 7.2 that

$$\left|\tilde{M}_{h_1}(\theta)-\tilde{M}(\theta)\right|\le\frac{1-\omega}{B(\omega)}h_1(\theta)+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\left|h_1(\tau)\right|d\tau$$

$$\le\left(\frac{\Gamma(\omega)-\Gamma(\omega+1)+\theta_0^\omega}{B(\omega)\Gamma(\omega)}\right)\aleph_1\le m\aleph_1. \tag{7.14}$$

Similarly,

$$\left|\tilde{N}_{h_2}(\theta)-\tilde{N}(\theta)\right|\le h_2(\theta)\frac{1-\omega}{B(\omega)}+\frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta(\theta-\tau)^{\omega-1}\left|h_2(\tau)\right|d\tau$$

$$\leq \left(\frac{\Gamma(\omega) - \Gamma(\omega+1) + \theta_0^{\omega}}{B(\omega)\Gamma(\omega)} \right) \aleph_2 \leq m \aleph_2 . \tag{7.15}$$

$$\left| \tilde{N}_{1,h_3}(\theta) - \tilde{N}_1(\theta) \right| \leq \frac{1-\omega}{B(\omega)} h_3(\theta) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta - \tau)^{\omega-1} \left| h_3(\tau) \right| d\tau$$

$$\leq \left(\frac{\Gamma(\omega) - \Gamma(\omega+1) + \theta_0^{\omega}}{B(\omega)\Gamma(\omega)} \right) \aleph_3 \leq m \aleph_3 . \tag{7.16}$$

Similarly,

$$\left| \tilde{\Theta}_{h_4}(\theta) - \tilde{\Theta}(\theta) \right| \leq m \aleph_4 . \tag{7.17}$$

Hence, the lemma is proved.

Theorem 7.4. Under the assumptions of Theorems 5.1 and 5.2 and conditions (5.6), (5.9)–(5.11), the system (3.2) with (3.3) will possess Ulam-Hyers stability in Ω.

Proof. Let $\tilde{M}, \tilde{N}, \tilde{N}_1, \tilde{\Theta} \in E$ be the solutions of inequalities (7.1) and the functions $M, N, N_1, \Theta \in E$ be unique solutions of Eq (3.2) with the conditions (7.2). Now, due to the set of conditions (7.2), we obtain

$$M_0 = \tilde{M}_0, \quad N_0 = \tilde{N}_0, \quad N_{1,0} = \tilde{N}_{1,0}, \quad \Theta_0 = \tilde{\Theta}_0 . \tag{7.18}$$

That is,

$$M(\theta) = M_0 + \frac{1-\omega}{B(\omega)} \rho_1(\theta, M) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta - \tau)^{\omega-1} \rho_1(\tau, M) d\tau ,$$

$$N(\theta) = N_0 + \frac{1-\omega}{B(\omega)} \rho_2(\theta, N) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta - \tau)^{\omega-1} \rho_2(\tau, N) d\tau ,$$

$$N_1(\theta) = N_{1,0} + \frac{1-\omega}{B(\omega)} \rho_3(\theta, N_1) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta - \tau)^{\omega-1} \rho_3(\tau, N_1) d\tau ,$$

$$\Theta(\theta) = \Theta_0 + \frac{1-\omega}{B(\omega)} \rho_4(\theta, \Theta) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta - \tau)^{\omega-1} \rho_4(\tau, \Theta) d\tau . \tag{7.19}$$

Hence, the set of Eq (7.19) transforms to

$$M = \tilde{M}_0 + \frac{1-\omega}{B(\omega)} \rho_1(\theta, M) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta - \tau)^{\omega-1} \rho_1(\tau, M) d\tau ,$$

$$N = \tilde{N}_0 + \frac{1-\omega}{B(\omega)} \rho_2(\theta, N) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta - \tau)^{\omega-1} \rho_2(\tau, N) d\tau ,$$

$$N_1 = \tilde{N}_{1,0} + \frac{1-\omega}{B(\omega)}\rho_3(\theta, N_1) + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta (\theta-\tau)^{\omega-1}\rho_3(\tau, N_1)d\tau,$$

$$\Theta = \tilde{\Theta}_0 + \frac{1-\omega}{B(\omega)}\rho_4(\theta, M) + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta (\theta-\tau)^{\omega-1}\rho_4(\tau, \Theta)d\tau. \tag{7.20}$$

Now, making use of inequality (5.6) and Lemma 7.3, we get

$$\left|\tilde{M}(\theta) - M(\theta)\right| \le \left|\tilde{M}(\theta) - \tilde{M}_{h_1}(\theta)\right| + \left|\tilde{M}_{h_1}(\theta) - M(\theta)\right|$$

$$\le m\aleph_1 + \frac{1-\omega}{B(\omega)}\left|\rho_1(\theta, \tilde{M}) - \rho_1(\theta, M)\right| + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta (\theta-\tau)^{\omega-1}\left|\rho_1(\tau, \tilde{M}) - \rho_1(\tau, M)\right|d\tau + m\aleph_1$$

$$\le 2m\aleph_1 + \left(\frac{1-\omega}{B(\omega)} + \frac{\theta_0^\omega}{B(\omega)\Gamma(\omega)}\right)\lambda_1\left\|\tilde{M} - M\right\|,$$

which yields that

$$\left\|\tilde{M} - M\right\|_T \le \frac{2m\aleph_1}{1-\varphi_1}, \tag{7.21}$$

where

$$\varphi_1 = \left(\frac{1-\omega}{B(\omega)} + \frac{\theta_0^\omega}{B(\omega)\Gamma(\omega)}\right)\lambda_1 < 1. \tag{7.22}$$

For $\sigma_1 = \frac{2m}{1-\varphi_1}$, we obtain

$$\left\|\tilde{M} - M\right\|_E \le \sigma_1\aleph_1. \tag{7.23}$$

Similarly, by condition (5.9) and Lemma 7.3, we obtain

$$\left|\tilde{N} - N\right| \le \left|\tilde{N} - \tilde{N}_{h_2}\right| + \left|\tilde{N}_{h_2} - N\right|$$

$$\le m\aleph_2 + \frac{1-\omega}{B(\omega)}\left|\rho_2(\theta, \tilde{N}) - \rho_2(\theta, N)\right| + \frac{\omega}{B(\omega)}\frac{1}{\Gamma(\omega)}\int_0^\theta (\theta-\tau)^{\omega-1}\left|\rho_2(\tau, \tilde{N}) - \rho_1(\tau, N)\right|d\tau + m\aleph_2$$

$$\le 2m\aleph_2 + \left(\frac{1-\omega}{B(\omega)} + \frac{\theta_0^\omega}{B(\omega)\Gamma(\omega)}\right)\lambda_2\left\|\tilde{N} - N\right\|,$$

which implies that

$$\left\| \tilde{N} - N \right\|_T \le \frac{2m\aleph_2}{1-\varphi_2}, \tag{7.24}$$

where

$$\varphi_2 = \left(\frac{1-\omega}{B(\omega)} + \frac{\theta_0^{\omega}}{B(\omega)\Gamma(\omega)} \right) \lambda_2 < 1. \tag{7.25}$$

For $\sigma_2 = \dfrac{2m}{1-\varphi_2}$, we obtain

$$\left\| \tilde{N} - N \right\|_E \le \sigma_2 \aleph_2. \tag{7.26}$$

Similarly, we conclude that

$$\left\| \tilde{N}_1 - N_1 \right\|_E \le \sigma_3 \aleph_3, \quad \left\| \tilde{\Theta} - \Theta \right\|_E \le \sigma_4 \aleph_4, \tag{7.27}$$

where

$$\sigma_i = \frac{2m}{1-\varphi_i}, \quad i = 3, 4. \tag{7.28}$$

Thus, for some $\aleph, \sigma > 0$,

$$\left\| \left(\tilde{M}, \tilde{N}, \tilde{N}_1, \tilde{\Theta} \right) - \left(M, N, N_1, \Theta \right) \right\|_{\Omega} \le \sigma \aleph. \tag{7.29}$$

Therefore, the Ulam-Hyers stability of the model (3.2) with (3.3) is established.

8. Numerical solution of the ABC type FCM model

Here, a numerical approach for the solution of the model is discussed. For this, we consider an IVP with the ABC fractional derivative as follows:

$$^{ABC}_{\ \ 0}D^{\omega}_{\theta} U(\theta) = V(\theta, U(\theta)). \tag{8.1}$$

Employing the AB fractional integral operator on Eq (8.1), we get

$$U(\theta) - U(0) = \frac{1-\omega}{B(\omega)} V(\theta, U(\theta)) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta - \xi)^{\omega-1} V(\xi, U(\xi)) d\xi. \tag{8.2}$$

Taking $\theta = \theta_n = n\hbar$ in Eq (8.2), we obtain

$$U(\theta_n) = U(0) + \frac{1-\omega}{B(\omega)} V(\theta_n, U(\theta_n)) + \frac{\omega}{B(\omega)} \frac{1}{\Gamma(\omega)} \sum_{i=0}^{n-1} \int_{\theta_i}^{\theta_{i+1}} V(\xi, U(\xi)) (\theta_n - \xi)^{\omega-1} d\xi. \tag{8.3}$$

Now, the linear Lagrange interpolation of $V(t,U(t))$ provides

$$V(\theta,U(\theta)) \approx V(\theta_{i+1},U_{i+1}) + \frac{\theta-\theta_{i+1}}{\hbar}\left(V(\theta_{i+1},U_{i+1}) - V(\theta_i,U_i)\right), \quad \theta \in [\theta_i,\theta_{i+1}], \tag{8.4}$$

where $U_i = U(\theta_i)$.

By using Eq (8.4) in Eq (8.3), the estimated solution of Eq (8.1) is obtained as

$$U_n = U_0 + \frac{\omega\hbar^{\omega}}{B(\omega)}\left(\omega_n V(\theta_0,U_0) + \sum_{i=1}^{n}\theta_{n-i} V(\theta_i,U_i)\right), \tag{8.5}$$

where

$$\omega_n = \frac{(n-1)^{\omega+1} - (n-\omega-1)n^{\omega}}{\Gamma(\omega+2)}, \tag{8.6}$$

$$\theta_{\aleph} = \begin{cases} \dfrac{1}{\Gamma(\omega+2)} + \dfrac{1-\omega}{\omega\hbar^{\omega}}, & \aleph = 0, \\ \dfrac{(\aleph-1)^{\omega+1} - 2\aleph^{\omega+1} + (\aleph+1)^{\omega+1}}{\Gamma(\omega+2)}, & \aleph = 1,2,3,\ldots,n-1. \end{cases} \tag{8.7}$$

Using the numerical method (8.5)–(8.7), the solution of the model (3.2) is generated recursively in this way:

$$M_n = M_0 + \frac{\omega\hbar^{\omega}}{B(\omega)}\left(\begin{array}{l} \omega_n\left(r M_0(1-M_0) - \alpha_0 \dfrac{M_0 N_0}{1+\mu M_0} - \alpha_4 \dfrac{M_0 N_{1,0}}{1+\mu M_0(\theta)}\right) \\ +\displaystyle\sum_{i=1}^{n}\theta_{n-i}\left(r M_i(\theta)(1-M_i(\theta)) - \alpha_0 \dfrac{M_i(\theta) N_i(\theta)}{\mu M_i(\theta)+1} - \alpha_4 \dfrac{M_i N_{1,i}}{1+\mu M_i}\right) \end{array}\right),$$

$$N_n = N_0 + \frac{\omega\hbar^{\omega}}{B(\omega)}\left(\begin{array}{l} \omega_n\left(\alpha_1 \dfrac{M_0(\theta)N_0(\theta)}{1+\mu M_0(\theta)} - \alpha_2 \dfrac{N_0(\theta)\Theta_0(\theta)}{1+\mu N_0(\theta)} - c\dfrac{N_0(\theta)N_{1,0}(\theta)}{N_0(\theta)+N_{1,0}(\theta)} + kN_{1,0}(\theta) - d_1 N_0(\theta)\right) \\ +\displaystyle\sum_{i=1}^{n}\theta_{n-i}\left(\alpha_1 \dfrac{M_i(\theta)N_i(\theta)}{1+\mu M_i(\theta)} - \alpha_2 \dfrac{N_i(\theta)\Theta_i(\theta)}{1+\mu N_i(\theta)} - c\dfrac{N_i(\theta)N_{1,i}(\theta)}{N_i(\theta)+N_{1,i}(\theta)} + kN_{1,i}(\theta) - d_1 N_i(\theta)\right) \end{array}\right),$$

$$N_{1,n} = N_{1,0} + \frac{\omega\hbar^{\omega}}{B(\omega)}\left(\begin{array}{l} \omega_n\left(c\dfrac{N_0(\theta)N_{1,0}(\theta)}{N_0(\theta)+N_{1,0}(\theta)} + \alpha_5 \dfrac{M_0(\theta)N_{1,0}(\theta)}{1+\mu M_0(\theta)} - kN_{1,0}(\theta) - qN_{1,0}\right) \\ +\displaystyle\sum_{i=1}^{n}\theta_{n-i}\left(c\dfrac{N_i N_{1,i}}{N_i+N_{1,i}} + \alpha_5 \dfrac{M_i N_{1,i}}{1+\mu M_i} - kN_{1,i} - qN_{1,i}\right) \end{array}\right),$$

$$\Theta_n = \Theta_0 + \frac{\omega\hbar^{\omega}}{B(\omega)}\left(\begin{array}{l}\omega_n\left(\alpha_3\dfrac{N_0(\theta)\Theta_0(\theta)}{1+\mu N_0(\theta)} - d_2\Theta_0(\theta)\right)\\ +\sum_{i=1}^{n}\theta_{n-i}\left(\alpha_3\dfrac{N_i(\theta)\Theta_i(\theta)}{1+\mu N_i(\theta)} - d_2\Theta_i(\theta)\right)\end{array}\right). \tag{8.8}$$

The solution of the model (3.2) is achieved by means of the above obtained iterative numerical schemes (8.8).

9. Numerical simulation

In this part, the above obtained numerical iterative schemes represented by Eqs (8.6)–(8.8) are utilized to perform the simulations for the fractional food chain model (3.2). The following values for the parameters of the discussed model are considered for the simulation purpose [29]: $r = 0.6$, $\alpha_0 = 0.44$, $\mu = 0.406$, $\alpha_4 = 0.479$, $\alpha_1 = 0.309$, $\alpha_2 = 0.45$, $c = 0.202$, $k = 0.095$, $d_1 = 0.126$, $d_2 = 0.07$, $\alpha_5 = 0.292$, $q = 0.5$, $\alpha_3 = 0.35$, and a time step size $\hbar = 1.0\times 10^{-3}$.

The graphs presented here show the behavior of solutions for different values of some parameters. The numerical solution of the model is plotted through Figures 1–10 for various initial values and fractional order ω. Figure 1 shows the impact of distinct values of fractional order ω on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$. Figure 2 presents the effect of various values of μ on the nature of solutions M, N, N_1, and Θ. Figure 3 describes the behavior of solutions M, N, N_1, and Θ for distinct values of r. Figures 4 and 5, respectively, depict the dynamics of solutions M, N, N_1 and Θ for distinct values of α_0 and α_1. Figures 6 and 7, respectively, show the impact of distinct values of α_2 and α_3 on the dynamics of solutions M, N, N_1 and Θ. Figures 8 and 9 elucidate the nature of solutions M, N, N_1, and Θ for various values of d_1 and d_2. Finally, Figure 10 explains the impact of various values of k on the dynamics of solutions M, N, N_1, and Θ. It is easy to examine that the achieved numerical results are compatible with the conceptual conjectures in the foregoing sections.

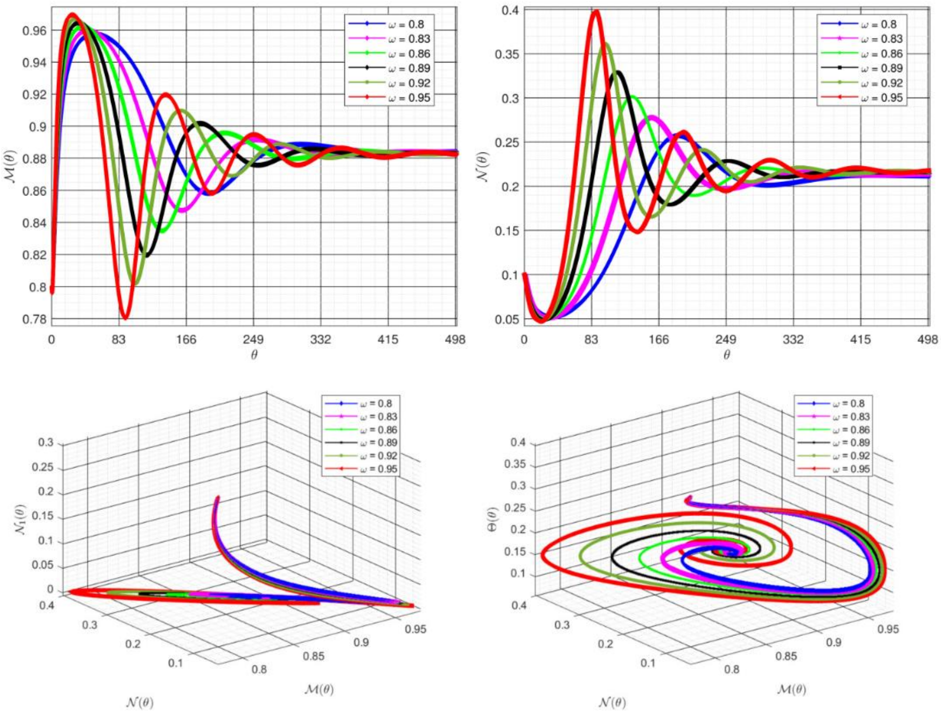

Figure 1. Impact of distinct values of fractional order ω on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$.

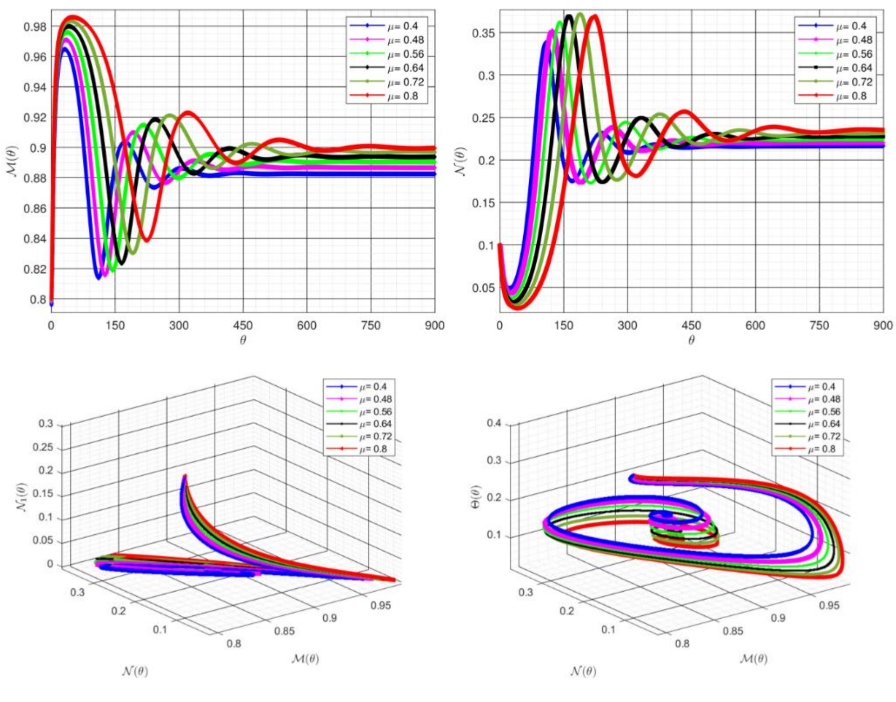

Figure 2. Effect of various values of μ on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$.

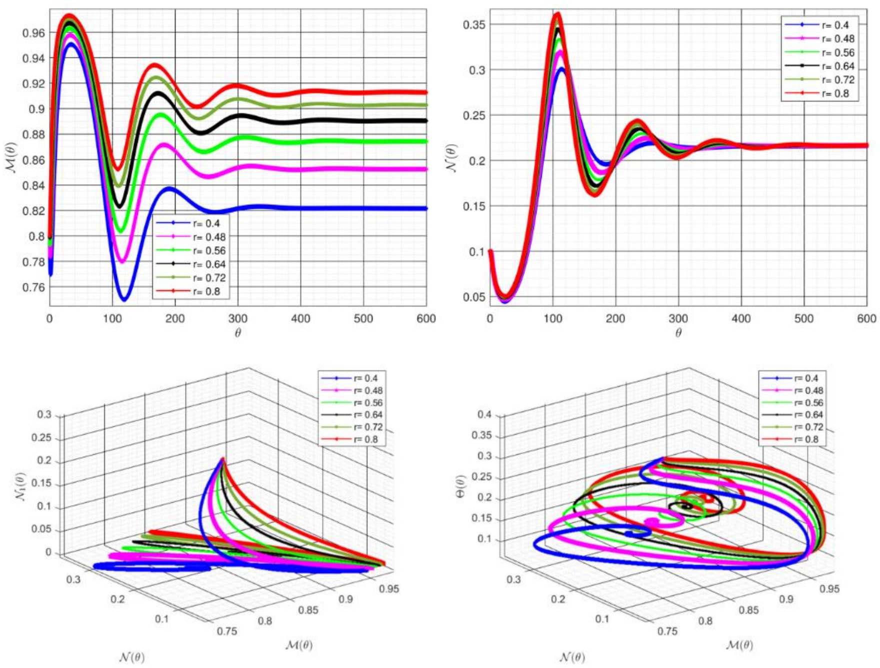

Figure 3. Effect of distinct values of r on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$.

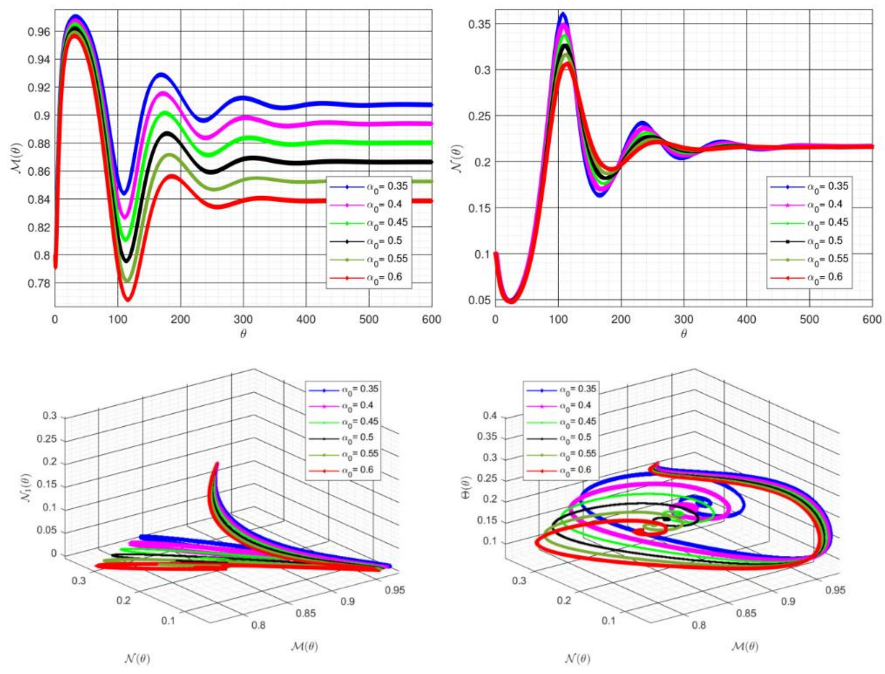

Figure 4. Impact of distinct values of α_0 on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$.

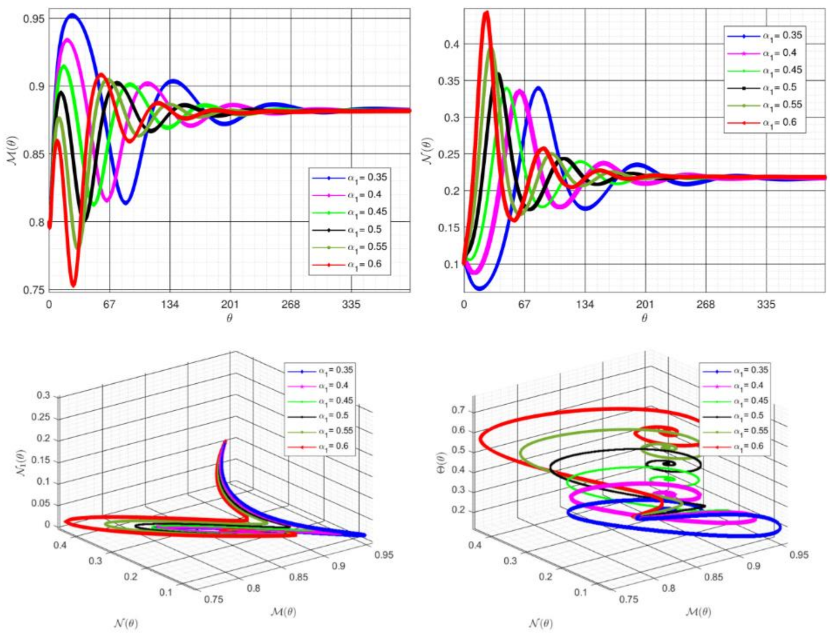

Figure 5. Impact of various values of α_1 on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$.

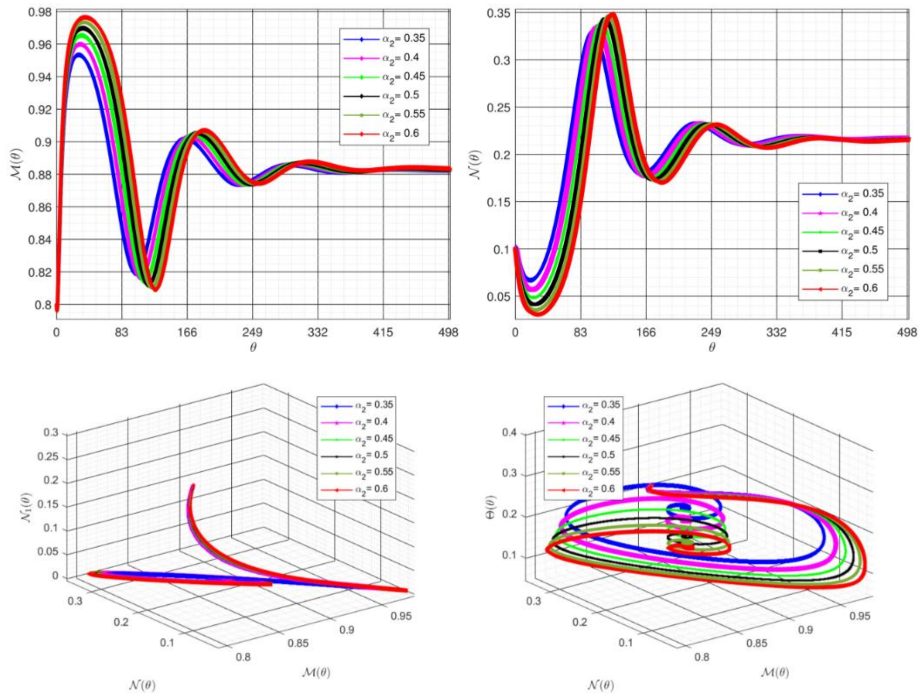

Figure 6. Impact of distinct values of α_2 on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$.

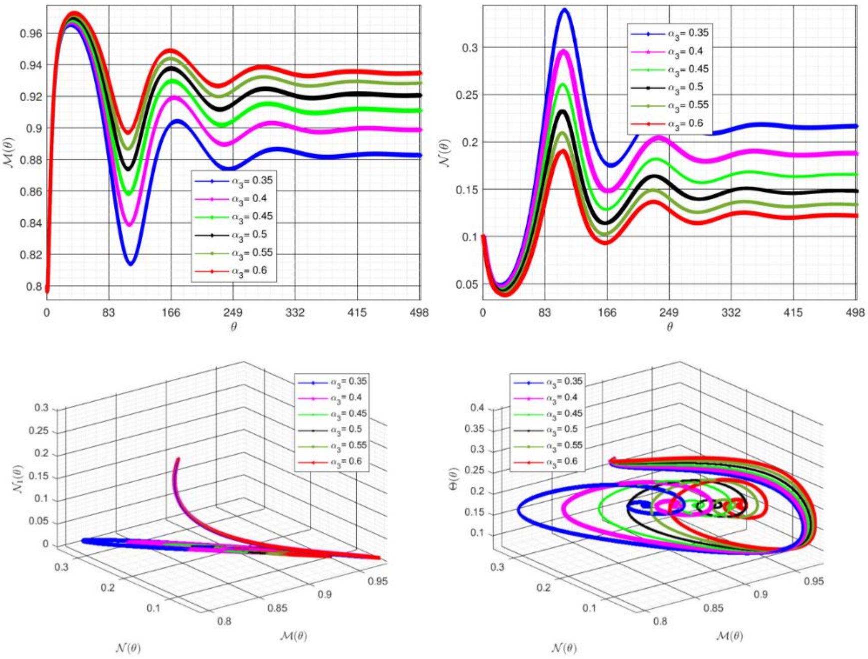

Figure 7. Effect of distinct values of α_3 on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$.

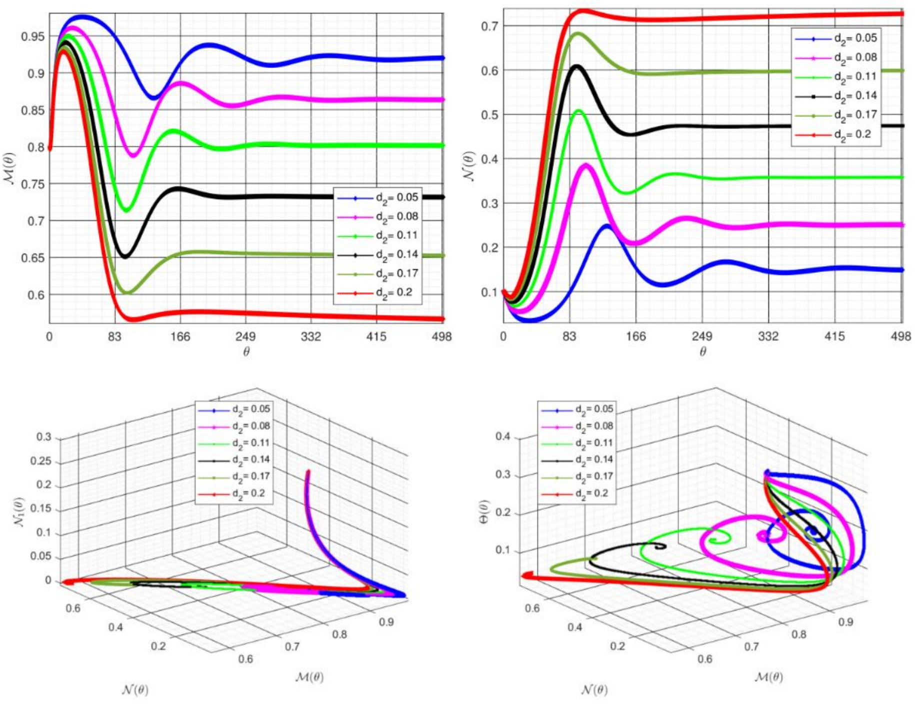

Figure 8. Impact of various values of d_1 on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$.

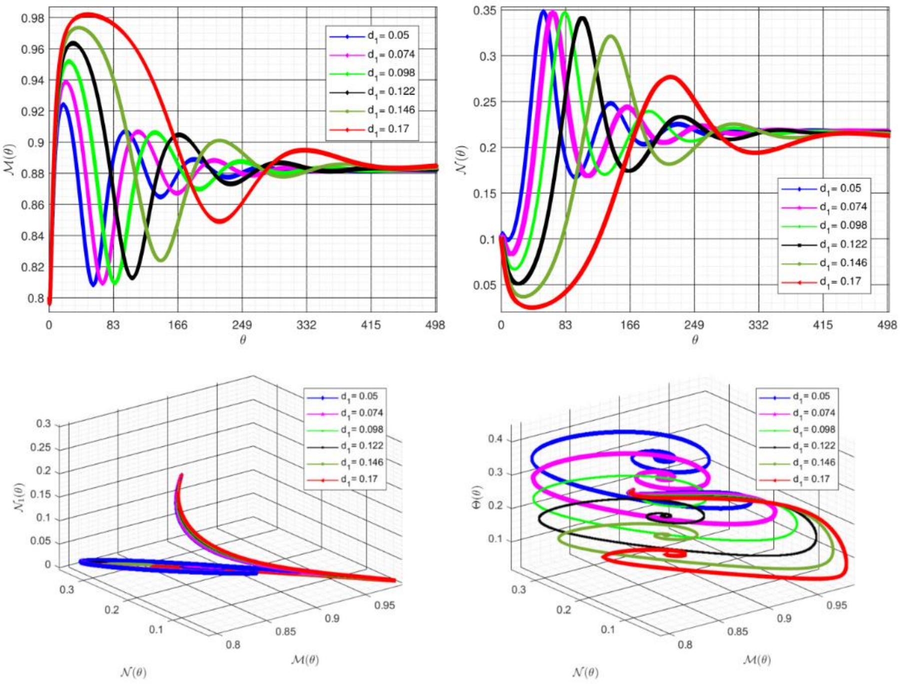

Figure 9. Effect of distinct values of d_2 on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$.

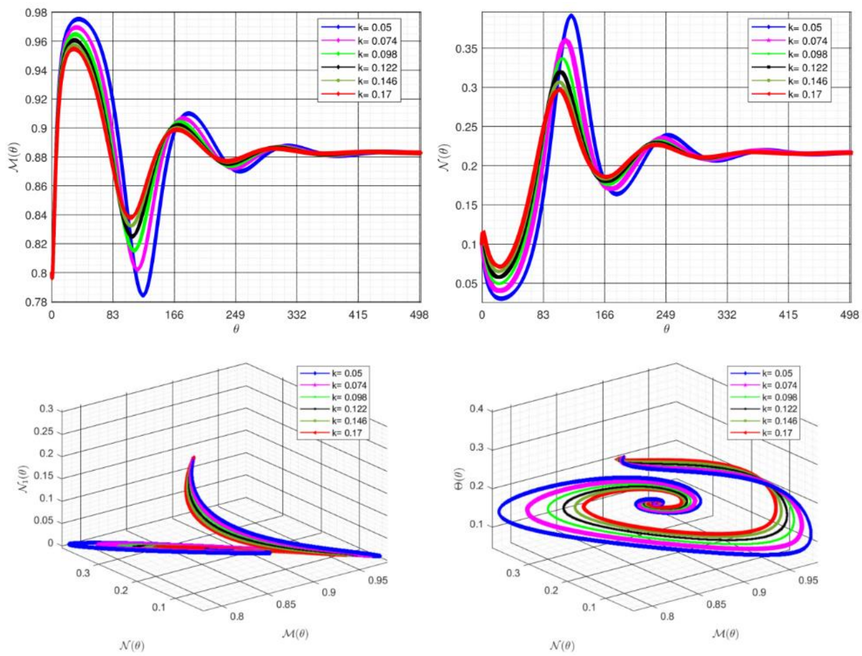

Figure 10. Impact of various values of k on the dynamics of solutions $M(\theta)$, $N(\theta)$, $N_1(\theta)$, and $\Theta(\theta)$.

10. Concluding remarks and observations

This paper examines the fractional order FCM having Holling type II functional response. The ABC fractional derivative operator providing Mittag-Leffler kernel is used for this purpose. The best feature of this derivative operator is that it provides scope for inclusion of memory-related properties very efficiently along with utilization of the information. This FDO possesses all the features of the Caputo-Fabrizio derivative with extra features of nonsingular and nonlocal character of a kernel. The main reason behind the use of FDO is the better description of memory characteristics which can be possible through the inclusion of nonlocal features. The use of an ML kernel in ABFDO is due to its natural appearance in various physical and biological models because the ML function is a joint venture of power-law and exponential-law which induces the memory effect completely. The inspiration behind the selection of the AB derivative is the nonlocal characteristic of the kernel which generates the scope of global analysis in those areas where the trends do not follow the power-law. Fractional order modeling also enhances the accuracy of the analysis and provides an extended degree of freedom in the model. However, the most important attribute of the fractional order modeling is to provide an excellent tool for the description of memory and hereditary characteristics which are generally ignored by systems of integer-order derivatives. In addition, fractional order derivatives are also useful to model the behavior of real systems and in the investigation of dynamical systems. The numerical approach is implemented to solve the model to get an approximate solution. The existence and uniqueness related analysis for the model are also presented. In addition, the model is also discussed regarding the Ulam-Hyers stability approach. Moreover, the graphical presentations for numerical solutions of the model confirm the authenticity of the numerical scheme utilized in this paper. It is clear that the achieved numerical results for the model correspond well with the conceptual findings. As a future research scope of the work, the presented numerical approach and stability analysis can also be applied to other physical and biological models.

Author contributions

Jagdev Singh and Devendra Kumar: Conceptualization; Behzad Ghanbari and Kottakkaran Sooppy Nisar: Methodology; Jagdev Singh, Behzad Ghanbari, Ved Prakash Dubey and Kottakkaran Sooppy Nisar: Software; Jagdev Singh, Ved Prakash Dubey and Devendra Kumar: Investigation; Jagdev Singh, Behzad Ghanbari and Kottakkaran Sooppy Nisar: Validation; Jagdev Singh, Behzad Ghanbari, Ved Prakash Dubey, Devendra Kumar and Kottakkaran Sooppy Nisar: Writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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