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Research article

Applications of $q-$Ultraspherical polynomials to bi-univalent functions defined by $q-$Saigo's fractional integral operators

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Abstract: This study established upper bounds for the second and third coefficients of analytical and bi-univalent functions belonging to a family of particular classes of analytic functions utilizing $q-$Ultraspherical polynomials under $q-$Saigo's fractional integral operator. We also discussed the Fekete-Szegö family function problem. As a result of the specialization of the parameters used in our main results, numerous novel outcomes were demonstrated.

Keywords: $q-$Ultraspherical polynomials; univalent functions; bi-univalent functions; $q-$calculus; analytic functions
Mathematics Subject Classification: 30C45

1. Introduction

Quantum calculus, sometimes referred to as $q-$calculus, is a field of mathematics which expands the scope of traditional calculus to encompass the realm of quantum mechanics. $q-$calculus is a mathematical discipline that broadens the horizons of classical calculus theories and techniques by incorporating a novel parameter, denoted by q. $q-$calculus has been found to have numerous uses in different branches of mathematics and other fields. Among the most crucial and well-studied areas of $q-$calculus is the theory of $q-$orthogonal polynomials (q-op).

The theory of $q-$op began with the work of Leonard Carlitz and others in the 1940s and 1950s. Carlitz [9] introduced a new type of polynomial called q-polynomials, which are polynomials that satisfy a certain recurrence relation involving the $q-$analog of the factorial function. These polynomials were later generalized to the theory of $q-$op by Askey and Wilson [7] in the 1970s.

The $q-$op form a collection of orthogonal polynomials whose weight function is dependent on the parameter q. These polynomials have been discovered to have diverse applications in number theory, combinatorics, statistical mechanics and quantum, and other branches of mathematics and physics.

There are several types of $q-$op, including $q-$Hermite, $q-$Jacobi, $q-$Laguerre, and $q-$Ultraspherical polynomials (or $q-$Gegenbauer polynomials), among others. Each type of $q-$op has its own recurrence relation, weight function, and orthogonality properties; for a comprehensive study, see [11].

The study of $q-$op has led to the development of many important results and techniques in $q-$calculus, including the $q-$analog of the binomial theorem, $q-$difference equations, and $q-$special functions. The theory of $q-$op has also been used to study $q-$integrals and $q-$series, which are important tools in the study of $q-$calculus. In a recent development, Quesne [23] reinterpreted Jackson's $q-$exponential as a closed-form product of regular exponentials with coefficients that are already known. This finding raises important implications for the theory of $q-$op in this context and warrants further examination.

The theory of orthogonal polynomials has been extensively studied due to its numerous applications in many fields of physics and mathematics. In recent years, the use of orthogonal polynomials and their analogs has become an important tool for studying analytic functions in the complex plane, particularly bi-univalent functions.

2. Preliminaries

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{s=2}^{\infty} a_s z^s, \quad (z \in \mathbb{U}), \tag{2.1}$$

which are analytic in the disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization condition $f'(0) - 1 = 0 = f(0)$. Also, by $\mathcal{S}$, the subclass of $\mathcal{A}$ has the univalent functions in $\mathbb{U}$ of the form given in Eq (2.1).

Differential subordination of analytical functions offers powerful methods that are extremely useful in geometric function theory. The first differential subordination problem is attributed to Miller and Mocanu [14]; see [15]. Most of the field's advancements are summarized in Miller and Mocanu's book [16].

It is a commonly accepted fact that for any function $f \in \mathcal{S}$, there exists an inverse f^{-1} that is defined by a certain mathematical expression.

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(\varpi)) = \varpi \qquad (|\varpi| < r_0(f);\ r_0(f) \geq \frac{1}{4})$$

where

$$\hbar(\varpi) = f^{-1}(\varpi) = \varpi - a_2\varpi^2 + (2a_2^2 - a_3)\varpi^3 - (5a_2^3 - 5a_2a_3 + a_4)\varpi^4 + \cdots . \tag{2.2}$$

In the context of mathematics, a function is considered to be bi-univalent in the domain $\mathbb{U}$ if together the function $f(z)$ and its inverse $f^{-1}(z)$ are univalent (i.e., one-to-one) in the same domain $\mathbb{U}$.

The subclass of bi-univalent functions in the domain $\mathbb{U}$, as defined by Eq (2.1), can be denoted by the symbol Π. Several examples of functions that are in the class Π are also available

$$\frac{z}{1-z}, \qquad \log\sqrt{\frac{1+z}{1-z}}.$$

Despite its popularity, it is important to remember that the familiar Koebe function does not belong to the class Π. However, there exist other frequently used examples of functions that are defined in the domain $\mathbb{U}$, such as:

$$\frac{2z-z^2}{2} \text{ and } \frac{z}{1-z^2}$$

which are also not members of Π.

In a study by Lewin [13], the bi-univalent function class Π was explored, and it was demonstrated that $|a_2| < 1.51$. Afterward, Brannan and Clunie [8] suggested a hypothesis that $|a_2| < \sqrt{2}$. However, Netanyahu [17] subsequently provided evidence that the maximum value of $|a_2|$ among all functions in Π is equal to $4/3$.

Askey and Ismail, in their work cited as [6], made a significant discovery of a collection of polynomials that can be considered as q-analogues of the Ultraspherical polynomials. These polynomials are referred to as $\mathfrak{G}_q^{(\aleph)}(\chi, z)$ and are essentially what they have identified.

$$\mathfrak{G}_q^{(\aleph)}(\chi, z) = \sum_{s=0}^{\infty} C_s^{(\lambda)}(\chi; q) z^s. \tag{2.3}$$

By means of the recurrence relations, Chakrabarti and colleagues, as cited in [10], made a noteworthy finding of a series of polynomials that can be understood as q-analogues of the Ultraspherical functions:

$$\begin{aligned}
&C_0^{(\aleph)}(\chi; q) = 1,\\
&C_1^{(\aleph)}(\chi; q) = [\aleph]_q C_1^1(\chi) = 2[\aleph]_q\chi,\\
&C_2^{(\aleph)}(\chi; q) = [\aleph]_{q^2} C_2^1(\chi) - \frac{1}{2}\left([\aleph]_{q^2} - [\aleph]_q^2\right) C_1^2(\chi) = 2\left([\aleph]_{q^2} + [\aleph]_q^2\right)\chi^2 - [\aleph]_{q^2}.
\end{aligned} \tag{2.4}$$

On the other hand, in 2023, Amourah et al. [2] and Alsoboh et al. [1] built several classes of analytic bi-univalent functions using $q-$Ultraspherical functions (or $q-$Gegenbauer polynomials).

In recent times, connections between bi-univalent functions and orthogonal polynomials have been the subject of research by many scholars. Some of the notable studies in this regard include ([3–5, 22, 28]). However, to the best of our knowledge, there is limited research on bi-univalent functions in the context of ultraspherical polynomials.

3. Definitions

$q-$analysis theory has found applications in numerous scientific and engineering domains. The fractional $q-$calculus is an expansion of the traditional fractional calculus. Srivastava [25] has made remarkable contributions to $q-$calculus and the fractional $q-$calculus operator. In a previous study, Purohit and Raina [21] explored the fractional $q-$calculus operator for defining various subclasses in the open disk $\mathbb{U}$. Other authors have previously issued new analytical function classes based on the $q-$calculus operator. Purohit and Raina [18–20] presented related work on open unit disk $\mathbb{U}$ and introduced new univalent classes of analytic functions. First, we employ the primary calculus definitions and notations relevant to understanding the subject of the study (all details can be found in Gasper and Rahman [11]), assuming $(0 < q < 1)$.

The q-analogue of Pochhammer symbol $(\vartheta, q)_s$ is determined by

$$(\vartheta; q)_s = (q^{\vartheta}; q)_s = \begin{cases} \prod_{\iota=0}^{s-1}(1 - \vartheta q^{\iota}), & s > 0 \\ \\ \prod_{\iota=0}^{\infty}(1 - \vartheta q^{\iota}), & s \to \infty. \end{cases} \tag{3.1}$$

Equivalently,

$$(\vartheta; q)_s = \frac{\Gamma_q(\vartheta + s)(1 - q)^s}{\Gamma_q(\vartheta)},$$

in which the definition of the function $q-$gamma is determined by

$$\Gamma_q(\vartheta) = \frac{(q, q)_\infty}{(q^{\vartheta}, q)_\infty(1 - q)^{\vartheta-1}}, \quad \vartheta \neq 0, -1, -2, \cdots . \tag{3.2}$$

Moreover, the $q-$gamma function recurrence relation is as follows:

$$\Gamma_q(\vartheta + 1) = [\vartheta]_q \Gamma_q(\vartheta), \text{ where, } [\vartheta]_q = \frac{1 - q^{\vartheta}}{1 - q}, \tag{3.3}$$

and $[\vartheta]_q$ is the $q-$analogue of ϑ.

The $q-$Binomial $(z - q\zeta)_{\gamma-1}$ is defined by

$$\begin{aligned}(z - q\zeta)_{\gamma-1} &= z^{\gamma-1} \prod_{s=0}^{\infty}\left(\frac{1 - q^s(q\zeta z^{-1})}{1 - q^{s+\gamma-1}(q\zeta z^{-1})}\right) \\ &= z^{\gamma}\,{}_1F_0\left(\begin{matrix} q^{1-\gamma} \\ - \end{matrix}; q; \frac{\zeta}{z}q^{\gamma}\right),\end{aligned}$$

where

$${}_1F_0\left(\begin{matrix}\vartheta \\ - \end{matrix}; q; z\right) = \sum_{s=0}^{\infty} \frac{(\vartheta; q)_s}{(q; q)_s} z^s = \frac{(\vartheta z; q)_\infty}{(z; q)_\infty}, \tag{3.4}$$

and

$$(a-b)_{\vartheta} = a^{\vartheta}\Big(\frac{b}{a};q\Big)_{\vartheta} = a^{\vartheta}\frac{\left(\frac{b}{a}q^{\vartheta};q\right)_{\infty}}{\left(\frac{b}{a};q\right)_{\infty}},\ (a \neq 0). \tag{3.5}$$

The Jackson's $q-$derivative [12] of a function f is defined by:

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z-qz}, & (z \in \mathbb{C}\backslash\{0\}), \\ \\ f'(0), & (z=0), \end{cases}$$

where $\lim\limits_{q\to 1}\partial_q f(z) = f'(z)$.

The Jackson's $q-$integral [12] of a function f is defined by

$$\int_0^z f(t)d_qt = (1-q)z\sum_{s=0}^{\infty} q^s f(q^s z), \tag{3.6}$$

in which the right hand side converges.

Definition 3.1. For $0 < q < 1$, $\Re e\{\lambda\} > 0$, ϑ, δ being real or complex with $\Re e\{2-\vartheta\} > 0$ and $\Re e\{2-\vartheta+\delta\} > 0$, we define the fractional $q-$integral operator $\mathcal{F}_q^{\lambda,\vartheta,\delta} f(z)$ by

$$\mathcal{F}_q^{\lambda,\vartheta,\delta} f(z) = z^{\vartheta}\frac{\Gamma_q(2+\lambda+\delta)\Gamma_q(2-\vartheta)}{\Gamma_q(2-\vartheta+\delta)} I_{q,\delta}^{\lambda,\vartheta} f(z), \tag{3.7}$$

where

$$\begin{aligned} I_{q,\delta}^{\lambda,\vartheta} f(z) = \frac{z^{-\vartheta-1}}{\Gamma_q(\lambda)}\int_0^z \left(\frac{\zeta q}{z};q\right)_{\lambda-1}\sum_{m=0}^{\infty}\frac{(q^{\lambda+\vartheta};q)_m(q^{-\delta};q)_m}{(q;q)_m(q^{\lambda};q)_m}\times \\ \left(q^{\delta-\vartheta+1}\right)^m\left(1-\frac{t}{z}\right)_m f(q^m t)\partial_q\zeta, \end{aligned} \tag{3.8}$$

where $\left(\frac{\zeta q}{z};q\right)_{\lambda-1}$ and $\left(1-\frac{t}{z}\right)_m$ are defined in (3.1) and (3.5).

Remark 1. If $q \to 1-$, then $I_{q,\delta}^{\lambda,\vartheta} f(z)$ is reduced to the well-known Saigo's fractional integral operator studied by [26] and Srivastava and Owa [27].

In view of Jackson's integrals (3.6) and (3.8), $\mathcal{F}_q^{\lambda,\vartheta,\delta} f(z)$ is able to be expressed as

$$\begin{aligned} \mathcal{F}_q^{\lambda,\vartheta,\delta} f(z) = \frac{(1-q)^{\lambda}\Gamma_q(2+\lambda+\delta)\Gamma_q(2-\vartheta)}{\Gamma_q(2-\vartheta+\delta)}\times \\ \sum_{m=0}^{\infty}\left(\frac{(q^{\lambda+\vartheta};q)_m(q^{-\delta};q)_m}{(q;q)_m}\left(q^{\delta-\vartheta+1}\right)^m\sum_{k=0}^{\infty} q^k\frac{(q^{\lambda+m};q)_k}{(q;q)_k} f(q^{m+k}z)\right). \end{aligned} \tag{3.9}$$

Now, under fractional $q-$integral operator $\mathcal{F}_q^{\lambda,\vartheta,\delta}$, we obtain an image of the power function z^s.

Lemma 3.2. *For $\Re e(\lambda) > 0$, ϑ and δ being real or complex, if $\Re e(s+1) > 0$ and $\Re e(s-\vartheta+\delta+1) > 0$, then*

$$\mathcal{F}_q^{\lambda,\vartheta,\delta} z^s = \frac{\Gamma_q(2+\lambda+\delta)\Gamma_q(2-\vartheta)\Gamma_q(s+1)\Gamma_q(s-\vartheta+\delta+1)}{\Gamma_q(2-\vartheta+\delta)\Gamma_q(s+\lambda+\delta+1)\Gamma_q(s-\vartheta+1)} z^s. \tag{3.10}$$

Proof. Taking $f(z) = z^s$ in (3.9), we have

$$\mathcal{F}_q^{\lambda,\vartheta,\delta} z^s = z^s(1-q)^\lambda \frac{\Gamma_q(2+\lambda+\delta)\Gamma_q(2-\vartheta)}{\Gamma_q(2-\vartheta+\delta)} \times \\ \sum_{m=0}^{\infty} \frac{(q^{\lambda+\vartheta};q)_m (q^{-\delta};q)_m}{(q;q)_m} \left(q^{\delta-\vartheta+s+1}\right)^m \sum_{k=0}^{\infty} q^{k(s+1)} \frac{(q^{\lambda+m};q)_k}{(q;q)_k},$$

by using the following simplification

$$\sum_{k=0}^{\infty} q^{k(s+1)} \frac{(q^{\lambda+m};q)_k}{(q;q)_k} = \frac{(q^{\lambda+m+s+1};q)_\infty}{(q^{s+1};q)_\infty} = \frac{(q^{\lambda+m}q^{s+1};q)_\infty}{(q^{s+1};q)_\infty} = \frac{1}{(q^{s+1};q)_{\lambda+m}}.$$

Therefore,

$$\begin{aligned}
\mathcal{F}_q^{\lambda,\vartheta,\delta} z^s &= z^s(1-q)^\lambda \frac{\Gamma_q(2+\lambda+\delta)\Gamma_q(2-\vartheta)}{\Gamma_q(2-\vartheta+\delta)} \times \sum_{m=0}^{\infty} \frac{(q^{\lambda+\vartheta};q)_m (q^{-\delta};q)_m}{(q;q)_m (q^{s+1};q)_{\lambda+m}} \left(q^{\delta-\vartheta+s+1}\right)^m \\
&= z^s \frac{(1-q)^\lambda \Gamma_q(2+\lambda+\delta)\Gamma_q(2-\vartheta)}{(q^{s+1};q)_\lambda \Gamma_q(2-\vartheta+\delta)} \sum_{m=0}^{\infty} \frac{(q^{\lambda+\vartheta};q)_m (q^{-\delta};q)_m}{(q;q)_m (q^{s+1+\lambda};q)_m} \left(q^{\delta-\vartheta+s+1}\right)^m \\
&= \frac{\Gamma_q(2+\lambda+\delta)\Gamma_q(2-\vartheta)(1-q)^\lambda}{\Gamma_q(2-\vartheta+\delta)(q^{s+1};q)_\lambda} {}_2F_1\left(\begin{matrix} q^{\lambda+\vartheta}, q^{-\delta} \\ q^{s+1+\lambda} \end{matrix}; q; q^{\delta-\vartheta+s+1}\right) \\
&= \frac{\Gamma_q(2+\lambda+\delta)\Gamma_q(2-\vartheta)}{\Gamma_q(2-\vartheta+\delta)} \frac{\Gamma_q(s+\lambda+1)\Gamma_q(\delta-\vartheta+s+1)}{\Gamma_q(s+\lambda+\delta+1)\Gamma_q(s-\vartheta+1)} \frac{(1-q)^\lambda}{(q^{s+1};q)_\lambda} z^s,
\end{aligned}$$

using the Eqs (3.2) and (3.5), which yields

$$\mathcal{F}_q^{\lambda,\vartheta,\delta} z^s = \frac{\Gamma_q(2+\lambda+\delta)\Gamma_q(2-\vartheta)\Gamma_q(s+1)\Gamma_q(\delta-\vartheta+s+1)}{\Gamma_q(2-\vartheta+\delta)\Gamma_q(s+\lambda+\delta+1)\Gamma_q(s-\vartheta+1)} z^s.$$

□

For $\Re e(\lambda) > 0$, ϑ and δ being real or complex, if $\Re e(s+1) > 0$ and $\Re e(s-\vartheta+\delta+1) > 0$, we define new operator $\mathcal{F}_q^{\lambda,\vartheta,\delta} : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{F}_q^{\lambda,\vartheta,\delta} f(z) = z + \frac{\Gamma_q(2+\lambda+\delta)\Gamma_q(2-\vartheta)}{\Gamma_q(2-\vartheta+\delta)} \sum_{s=2}^{\infty} \frac{\Gamma_q(s+1)\Gamma_q(\delta-\vartheta+s+1)}{\Gamma_q(s+\lambda+\delta+1)\Gamma_q(s-\vartheta+1)} a_s z^s. \tag{3.11}$$

Remark 2. If $\lambda + \vartheta = 0$, then $\mathcal{F}_q^{\lambda,\vartheta,\delta} f(z)$ is reduced to the fractional q-differintegral operator $\Omega_q^\vartheta f(\zeta)$ introduced by Ravikumar [24], and defined by

$$\Omega_q^\vartheta f(z) = z + \sum_{s=2}^{\infty} \frac{\Gamma_q(2-\vartheta)\Gamma_q(s+1)}{\Gamma_q(s+1-\vartheta)} a_s z^s, \qquad (\vartheta \le 2,\ z \in \mathbb{U}). \tag{3.12}$$

4. Coefficient bounds of the class $\mathcal{B}_{\Pi}(\lambda, \vartheta, \delta, \mathfrak{G}_q^{(\aleph)}(\chi, z))$

Within this section, we present a novel subclass of functions that are subordinated to the $q-$Ultraspherical polynomial.

Definition 4.1. A function $f \in \Pi$ given by (2.1) is in the class $\mathcal{B}_{\Pi}(\lambda, \vartheta, \delta, \mathfrak{G}_q^{(\aleph)}(\chi, z))$ if the subsequent subordinations are met:

$$\partial_q \left(\mathcal{F}_q^{\lambda,\vartheta,\delta} f(z) \right) \prec \mathfrak{G}_q^{(\aleph)}(\chi, z) \tag{4.1}$$

and

$$\partial_q \left(\mathcal{F}_q^{\lambda,\vartheta,\delta} \hbar(\varpi) \right) \prec \mathfrak{G}_q^{(\aleph)}(\chi, \varpi), \tag{4.2}$$

where $\chi \in (\frac{1}{2}, 1]$, $\Re e(\lambda) > 0$, ϑ and δ being real or complex, $\Re e(s+1) > 0$, $\Re e(s - \vartheta + \delta + 1) > 0$, $\hbar(\varpi)$ is given by (2.2), $\mathfrak{G}_q^{(\aleph)}$ is the $q-$Ultraspherical polynomials given by (2.3), and $\prec$ stands on the subordination.

Example 4.2. A function $f \in \Pi$ given by (2.1) is in the class $\mathcal{B}_{\Pi}(\lambda, -\lambda, \delta, \mathfrak{G}_q^{(\aleph)}(\chi, z))$ if the subsequent subordinations are met:

$$\partial_q \left(\Omega_q^{\vartheta} f(z) \right) \prec \mathfrak{G}_q^{(\aleph)}(\chi, z)$$

and

$$\partial_q \left(\Omega_q^{\vartheta} \hbar(\varpi) \right) \prec \mathfrak{G}_q^{(\aleph)}(\chi, \varpi),$$

where $\chi \in (\frac{1}{2}, 1]$, $\Re e(\lambda) > 0$, ϑ and δ being real or complex, $\Re e(s+1) > 0$, $\Re e(s - \vartheta + \delta + 1) > 0$, $\hbar(\varpi)$ is given by (2.2), and $\mathfrak{G}_q^{(\aleph)}$ is the $q-$Ultraspherical polynomials given by (2.3).

Initially, we provide the coefficient approximations for the class $\mathcal{B}_{\Pi}(\lambda, \vartheta, \delta, \mathfrak{G}_q^{(\aleph)}(\chi, z))$ described in Definition 4.1.

Theorem 4.3. *Let $f \in \Pi$ be given by (2.1) in the subclass $\mathcal{B}_{\Pi}(\lambda, \vartheta, \delta, \mathfrak{G}_q^{(\aleph)}(\chi, z))$. Then,*

$$|a_2| \leq \frac{2\,[3-\vartheta]_q[2+\lambda+\delta]_q \left|[\aleph]_q\right| \chi \sqrt{2\,[2-\vartheta]_q[3+\lambda+\delta]_q\,[\aleph]_q \chi}}{\sqrt{\begin{array}{c}[2+\delta-\vartheta]_q \left(\left(4[3]_q^2[3-\vartheta]_q[3+\delta-\vartheta]_q[2+\lambda+\delta]_q\,[\aleph]_q^2\chi^2 + [2]_q^4[2+\delta-\vartheta]_q[\aleph]_{q^2} \right.\right. \\ \left.\left. +2[2]_q^4[2+\delta-\vartheta]_q \left([\aleph]_{q^2} + [\aleph]_q^2 \right) \chi^2 \right.\right. \end{array}}},$$

and

$$\left|a_3\right| \leq \frac{4[\aleph]_q^2[2+\lambda+\delta]_q^2[3-\vartheta]_q^2\chi^2}{[2]_q^2\,[2+\delta-\vartheta]_q^2} + \frac{[3-\vartheta]_q[2-\vartheta]_q[2+\lambda+\delta]_q[3+\lambda+\delta]_q\left|[\aleph]_q\right|\chi}{[3]_q^2[3+\delta-\vartheta]_q[2+\delta-\vartheta]_q}.$$

Proof. Let $f \in \mathcal{B}_{\Sigma}(\lambda, \vartheta, \delta, \mathfrak{G}_q^{(\aleph)}(\chi, z))$. From Definition 4.1, for certain functions w, v such that $w(0) = v(0) = 0$ and $|w(z)| < 1$, $|v(\varpi)| < 1$ for all $z, \varpi \in \mathbb{U}$, after which we may write

$$\partial_q \left(\mathcal{F}_q^{\lambda,\vartheta,\delta} f(z) \right) = \mathfrak{G}_q^{(\lambda)}(\chi, w(z)) \tag{4.3}$$

and

$$\partial_q \left(\mathcal{F}_q^{\lambda,\vartheta,\delta} \hbar(\varpi) \right) = \mathfrak{G}_q^{(\lambda)}(\chi, v(\varpi)). \tag{4.4}$$

From the Eqs (4.3) and (4.4), we get that

$$\partial_q\left(\mathcal{F}_q^{\lambda,\vartheta,\delta}f(z)\right)=1+C_1^{(\lambda)}(\chi;q)c_1z+\left[C_1^{(\aleph)}(\chi;q)c_2+C_2^{(\aleph)}(\chi;q)c_1^2\right]z^2+\cdots \tag{4.5}$$

and

$$\partial_q\left(\mathcal{F}_q^{\lambda,\vartheta,\delta}\hbar(\varpi)\right)=1+C_1^{(\lambda)}(\chi;q)d_1\varpi+\left[C_1^{(\aleph)}(\chi;q)d_2+C_2^{(\aleph)}(\chi;q)d_1^2\right])\varpi^2+\cdots. \tag{4.6}$$

That is

$$|w(z)|=\left|c_1z+c_2z^2+c_3z^3+\cdots\right|<1,\qquad (z\in\mathbb{U})$$

and

$$|v(\varpi)|=\left|d_1\varpi+d_2\varpi^2+d_3\varpi^3+\cdots\right|<1,\quad (\varpi\in\mathbb{U}),$$

then

$$|c_s|\leq 1 \text{ and } |d_s|\leq 1 \text{ for all } s\in\mathbb{N}. \tag{4.7}$$

Thus, from comparing the Eqs (4.5) and (4.6), we have

$$\frac{[2]_q^2\,[2+\delta-\vartheta]_q}{[2+\lambda+\delta]_q[3-\vartheta]_q}a_2=C_1^{(\aleph)}(\chi;q)c_1, \tag{4.8}$$

$$\frac{[3]_q^2[3+\delta-\vartheta]_q[2+\delta-\vartheta]_q}{[3+\lambda+\delta]_q[2+\lambda+\delta]_q[3-\vartheta]_q[2-\vartheta]_q}a_3=C_1^{(\aleph)}(\chi;q)c_2+C_2^{(\aleph)}(\chi;q)c_1^2, \tag{4.9}$$

$$-\frac{[2]_q^2\,[2+\delta-\vartheta]_q}{[2+\lambda+\delta]_q[3-\vartheta]_q}a_2=C_1^{(\aleph)}(\chi;q)d_1, \tag{4.10}$$

and

$$\frac{[3]_q^2[3+\delta-\vartheta]_q[2+\delta-\vartheta]_q}{[3+\lambda+\delta]_q[2+\lambda+\delta]_q[3-\vartheta]_q[2-\vartheta]_q}\left(2a_2^2-a_3\right)=C_1^{(\aleph)}(\chi;q)d_2+C_2^{(\aleph)}(\chi;q)d_1^2. \tag{4.11}$$

It follows from (4.8) and (4.10) that

$$c_1=-d_1, \tag{4.12}$$

and

$$2\left(\frac{[2]_q^2\,[2+\delta-\vartheta]_q}{[2+\lambda+\delta]_q[3-\vartheta]_q}\right)^2a_2^2=\left[C_1^{(\aleph)}(\chi;q)\right]^2\left(c_1^2+d_1^2\right). \tag{4.13}$$

If we add (4.9) and (4.11), we get

$$\frac{2[3]_q^2[3+\delta-\vartheta]_q[2+\delta-\vartheta]_q}{[3+\lambda+\delta]_q[2+\lambda+\delta]_q[3-\vartheta]_q[2-\vartheta]_q}a_2^2=C_1^{(\lambda)}(x;q)\,(c_2+d_2)+C_2^{(\lambda)}(x;q)\left(c_1^2+d_1^2\right). \tag{4.14}$$

By replacing the value of $\left(c_1^2+d_1^2\right)$ from (4.13) in (4.14), we get that

$$\begin{aligned}\left(\frac{[3]_q^2[3+\delta-\vartheta]_q}{[3+\lambda+\delta]_q[2-\vartheta]_q}-\frac{C_2^{(\aleph)}(\chi;q)}{\left[C_1^{(\aleph)}(\chi;q)\right]^2}\left(\frac{[2]_q^4\,[2+\delta-\vartheta]_q}{[2+\lambda+\delta]_q[3-\vartheta]_q}\right)\right)a_2^2&\\ =\frac{[2+\lambda+\delta]_q[3-\vartheta]_q}{2[2+\delta-\vartheta]_q}C_1^{(\aleph)}(\chi;q)\,(c_2+d_2)\,,&\end{aligned} \tag{4.15}$$

or the equivalent to

$$a_2^2 = \frac{[2-\vartheta]_q[3-\vartheta]_q^2[2+\lambda+\delta]_q^2[3+\lambda+\delta]_q\left[C_1^{(\aleph)}(\chi;q)\right]^3(c_2+d_2)}{2[2+\delta-\vartheta]_q\left([3]_q^2[3-\vartheta]_q[3+\delta-\vartheta]_q[2+\lambda+\delta]_q\left[C_1^{(\aleph)}(\chi;q)\right]^2-[2]_q^4[2+\delta-\vartheta]_qC_2^{(\aleph)}(\chi;q)\right)}. \tag{4.16}$$

Moreover, from computations using (4.6), (4.7), and (4.15), we have

$$|a_2| \le \frac{2\,[3-\vartheta]_q[2+\lambda+\delta]_q\left|[\aleph]_q\right|\chi\sqrt{2\,[2-\vartheta]_q[3+\lambda+\delta]_q\,[\aleph]_q\chi}}{\sqrt{\begin{array}{c}[2+\delta-\vartheta]_q\left(\left(4[3]_q^2[3-\vartheta]_q[3+\delta-\vartheta]_q[2+\lambda+\delta]_q\,[\aleph]_q^2\chi^2+[2]_q^4[2+\delta-\vartheta]_q[\aleph]_{q^2}\right.\right.\\ \left.\left.+2[2]_q^4[2+\delta-\vartheta]_q\left([\aleph]_{q^2}+[\aleph]_q^2\right)\chi^2\right)\right)\end{array}}}.$$

Additionally, if we subtract (4.11) from (4.9), we obtain

$$\frac{2[3]_q^2[3+\delta-\vartheta]_q[2+\delta-\vartheta]_q}{[3-\vartheta]_q[2-\vartheta]_q[2+\lambda+\delta]_q[3+\lambda+\delta]_q}\left(a_3-a_2^2\right)=C_1^{(\aleph)}(\chi;q)\,(c_2-d_2)+C_2^{(\aleph)}(\chi;q)\left(c_1^2-d_1^2\right). \tag{4.17}$$

Then, in view of (2.4) and (4.13), Eq (4.17) becomes

$$\begin{aligned}a_3 = {} & \frac{[2+\lambda+\delta]_q^2[3-\vartheta]_q^2}{2[2]_q^2\,[2+\delta-\vartheta]_q^2}\left[C_1^{(\aleph)}(\chi;q)\right]^2\left(c_1^2+d_1^2\right)\\ & +\frac{[3-\vartheta]_q[2-\vartheta]_q[2+\lambda+\delta]_q[3+\lambda+\delta]_q}{2[3]_q^2[3+\delta-\vartheta]_q[2+\delta-\vartheta]_q}C_1^{(\aleph)}(\chi;q)\,(c_2-d_2)\,.\end{aligned}$$

Thus, applying (2.4), we conclude that

$$\left|a_3\right| \le \frac{4[\aleph]_q^2[2+\lambda+\delta]_q^2[3-\vartheta]_q^2\chi^2}{[2]_q^2\,[2+\delta-\vartheta]_q^2}+\frac{[3-\vartheta]_q[2-\vartheta]_q[2+\lambda+\delta]_q[3+\lambda+\delta]_q\left|[\aleph]_q\right|\chi}{[3]_q^2[3+\delta-\vartheta]_q[2+\delta-\vartheta]_q}.$$

This concludes the theorem's proof. □

5. The Fekete and Szegö problem

In 1933, Fekete-Szegö established a bound for the functional $\eta a_2^2 - a_3$ for a univalent function f [**?**]. Since that time, the challenge of finding the optimal bounds for this function over any compact set of functions $f \in \mathcal{A}$ with a complex η has commonly been referred to as the classical Fekete-Szegö problem. In this part, we will examine this problem for functions in the subclass $\mathcal{B}\Pi(\lambda,\vartheta,\delta,\mathfrak{G}_q^{(\aleph)}(\chi,z))$, which is motivated by Zaprawa's outcome as described in [29].

Theorem 5.1. *Let $f \in \Pi$ as is in (2.1) be in the subclass $\mathcal{B}_\Pi(\lambda,\vartheta,\delta,\mathfrak{G}_q^{(\aleph)}(\chi,z))$. Then,*

$$\left|a_3-Fa_2^2\right| \le \begin{cases}\frac{2\left|[\aleph]_q\right|[2-\vartheta]_q[3-\vartheta]_q[2+\lambda+\delta]_q[3+\lambda+\delta]_q}{2[3]_q^2[3+\delta-\vartheta]_q[2+\delta-\vartheta]_q}\chi, & |1-F|\le \mathsf{K},\\ \\ \frac{2\left|[\aleph]_q\right|[2-\vartheta]_q[3-\vartheta]_q[2+\lambda+\delta]_q[3+\lambda+\delta]_q}{[2+\delta-\vartheta]_q}\chi\left|\mathfrak{L}(F)\right|, & |1-F|\ge \mathsf{K},\end{cases}$$

where

$$\mathfrak{L}(F)=\frac{(1-F)[3-\vartheta]_q[2+\lambda+\delta]_q\left[C_1^{(\aleph)}(\chi;q)\right]^2}{[3]_q^2[3-\vartheta]_q[3+\delta-\vartheta]_q[2+\lambda+\delta]_q\left[C_1^{(\aleph)}(\chi;q)\right]^2-[2]_q^4[2+\delta-\vartheta]_qC_2^{(\aleph)}(\chi;q)},$$

and

$$\mathsf{K}=\left|1-\frac{[2]_q^4[2+\delta-\vartheta]_qC_2^{(\aleph)}(\chi;q)}{[3]_q^2[3+\delta-\vartheta]_q[3-\vartheta]_q[2+\lambda+\delta]_q\left[C_1^{(\aleph)}(\chi;q)\right]^2}\right|.$$

Proof. From (4.15) and (4.17),

$$\begin{aligned}a_3-Fa_2^2&=\frac{[3-\vartheta]_q[2-\vartheta]_q[2+\lambda+\delta]_q[3+\lambda+\delta]_q}{2[3]_q^2[3+\delta-\vartheta]_q[2+\delta-\vartheta]_q}C_1^{(\aleph)}(\chi;q)\,(c_2-d_2)+\\&\frac{(1-F)[2-\vartheta]_q[3-\vartheta]_q^2[2+\lambda+\delta]_q^2[3+\lambda+\delta]_q\left[C_1^{(\aleph)}(\chi;q)\right]^3(c_2+d_2)}{2[2+\delta-\vartheta]_q\left([3]_q^2[3-\vartheta]_q[3+\delta-\vartheta]_q[2+\lambda+\delta]_q\left[C_1^{(\aleph)}(\chi;q)\right]^2-[2]_q^4[2+\delta-\vartheta]_qC_2^{(\aleph)}(\chi;q)\right)}\\&=\left\{\frac{[2-\vartheta]_q[3-\vartheta]_q[2+\lambda+\delta]_q[3+\lambda+\delta]_q}{2[2+\delta-\vartheta]_q}C_1^{(\aleph)}(\chi;q)\right\}\times\\&\qquad\left[\left(\mathfrak{L}(F)+\frac{1}{[3]_q^2[3+\delta-\vartheta]_q}\right)c_2+\left(\mathfrak{L}(F)-\frac{1}{[3]_q^2[3+\delta-\vartheta]_q}\right)d_2\right].\end{aligned}$$

In view of (2.4), we conclude that

$$\left|a_3-Fa_2^2\right|\leq\begin{cases}\frac{[2-\vartheta]_q[3-\vartheta]_q[2+\lambda+\delta]_q[3+\lambda+\delta]_q}{2[3]_q^2[3+\delta-\vartheta]_q[2+\delta-\vartheta]_q}\left|C_1^{(\aleph)}(\chi;q)\right|, & |\mathfrak{L}(F)|\leq\frac{1}{[3]_q^2[3+\delta-\vartheta]_q},\\ \\ \frac{[2-\vartheta]_q[3-\vartheta]_q[2+\lambda+\delta]_q[3+\lambda+\delta]_q}{[2+\delta-\vartheta]_q}\left|C_1^{(\lambda)}(x;q)\right||\mathfrak{L}(F)|, & |\mathfrak{L}(F)|\geq\frac{1}{[3]_q^2[3+\delta-\vartheta]_q}.\end{cases}$$

This concludes the theorem's proof. □

6. Corollaries

Corollary 6.1. *Let $f\in\Pi$ be given by (2.1) in the class $\mathcal{B}_\Pi(\lambda,-\lambda,\delta,\mathfrak{G}_q^{(\aleph)}(\chi,z))$. Then*

$$|a_2|\leq\frac{2\,[3+\lambda]_q\left|[\aleph]_q\right|\chi\sqrt{2\,[2-\vartheta]_q[3+\lambda+\delta]_q\,[\aleph]_q\chi}}{\sqrt{4[3]_q^2[3+\lambda]_q[3+\delta+\lambda]_q\,[\aleph]_q^2\chi^2+2[2]_q^4\left([\aleph]_{q^2}+[\aleph]_q^2\right)\chi^2+[2]_q^4[\aleph]_{q^2}}},$$

$$\left|a_3\right|\leq\frac{4[\aleph]_q^2[3+\lambda]_q^2\chi^2}{[2]_q^2}+\frac{[3-\vartheta]_q[2-\vartheta]_q\left|[\aleph]_q\right|\chi}{[3]_q^2},$$

and

$$\left|a_3-Fa_2^2\right|\leq\begin{cases}\frac{[2+\lambda]_q[3+\lambda]_q}{[3]_q^2}\left|[\aleph]_q\right|\chi, & |1-F|\leq\mathrm{K},\\ \\ 2\left|[\aleph]_q\right|[2+\lambda]_q[3+\lambda]_q[3+\lambda+\delta]_q\chi\,|\mathfrak{L}(F)|, & |1-F|\geq\mathrm{K},\end{cases}$$

where

$$\mathfrak{L}(F) = \frac{(1-F)[3+\lambda]_q \left[C_1^{(\aleph)}(\chi;q)\right]^2}{[3]_q^2[3+\lambda]_q[3+\delta+\lambda]_q \left[C_1^{(\aleph)}(\chi;q)\right]^2 - [2]_q^4 C_2^{(\aleph)}(\chi;q)},$$

and

$$\mathrm{K} = \left|1 - \frac{[2]_q^4 C_2^{(\aleph)}(\chi;q)}{[3]_q^2[3+\delta+\lambda]_q[3+\lambda]_q \left[C_1^{(\aleph)}(\chi;q)\right]^2}\right|.$$

7. Conclusions

In this study, we have investigated the coefficient issues associated with a new subclass $\mathcal{B}_\Sigma(\lambda, \vartheta, \delta, \mathfrak{G}^{(\aleph)}q(\chi, z))$ of bi-univalent functions within the unit disk $\mathbb{U}$. These bi-univalent functions are defined in Definition 4.1. We have established estimations for the coefficients $|a_2|$ and $|a_3|$, furthermore to the Fekete-Szegö problem for this novel subclass of the function. By specializing the parameters in our fundamental findings, we have demonstrated numerous new findings as in Corollary 6.1. However, it remains an unsolved problem to obtain approximations regarding the boundaries of $|a_s|$ for $s \geq 4$; $s \in \mathbb{N}$ for the introduced class.

Author contributions

Tariq Al-hawar: Conceptualization, Validation, Formal analysis, Investigation, Supervision; Ala Amourah: Methodology, Formal analysis, Investigation; Abdullah Alsoboh: Methodology; Osama Ogilat: Methodology, Validation; Irianto Harny: Writing-original draft, Writing-review; Maslina Darus: Writing-original draft, Writing-review. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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