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Research article

A note on maps preserving products of matrices

Lan Lu[1] **and Yu Wang**[2,*]

[1] Department of Basic Courses, Changchun Guanghua University, Changchun 130033, China
[2] Department of Mathematics, Shanghai Normal University, Shanghai 200234, China

* **Correspondence:** Email: ywang2004@126.com.

Abstract: Let D be a division ring such that either $\mathrm{char}(D) \neq 2, 3$ or D is not a field and $\mathrm{char}(D) \neq 2$. Let $R = M_n(D)$ be the matrix ring over D, where $n > 1$. Let m, k be fixed invertible elements in R. The main purpose of the paper is to give a description of a bijective additive map $f\colon R \to R$, satisfying the identity $f(x)f(y) = m$ for every $x, y \in R$ with $xy = k$, which gives a correct version of a result due to Catalano et al. in 2019.

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1. Introduction

Let R be an associative ring. Throughout the paper we will denote by $R^{\times}$, the set of all invertible elements of R. For $x, y \in R$, we set

$$x \circ y = xy + yx.$$

A map $f\colon R \to R$ is said to be *additive* if

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in R$.

In 1999, Essannouni and Kaidi [1] obtained the following result, which generalized a well-known result due to Hua [2].

Theorem 1.1. *[1, Theorem A] Let D be a division ring with $D \neq F_2$, the field of two elements. Let $R = M_n(D)$ be the ring of $n \times n$ matrices with $n \geq 2$. Let $f\colon R \to R$ be a bijective additive map satisfying the identity*

$$f(x^{-1}) = f(x)^{-1}$$

for every $x \in R^{\times}$. Then, f is either an automorphism or an antiautomorphism.

In 2005, Chebotar et al. [3] proved that a bijective additive map f on a division ring D that satisfies

$$f(x^{-1})f(x) = f(y^{-1})f(y)$$

for all $x, y \in D^{\times}$ must have the form

$$f(x) = f(1)\varphi(x),$$

where φ is an automorphism or antiautomorphism, and $f(1)$ is a central element of D. In 2006, Lin and Wong [4] generalized this result to matrix rings.

In 2018, Catalano [5] initiated the study of maps preserving products of division rings. More precisely, she proved the following result.

Theorem 1.2. *[5, Theorem 5] Let D be a division ring with characteristic different from 2. Let Z be the center of R. With $m, k \in D^{\times}$, let $f\colon D \to D$ be a bijective additive map satisfying the identity*

$$f(x)f(y) = m$$

for every $x, y \in D^{\times}$ such that $xy = k$. Then,

$$f(x) = f(1)\varphi(x)$$

for all $x \in D$, where $\varphi\colon D \to D$ is either an automorphism or an antiautomorphism. Moreover, we have the following:

(1) If φ is an automorphism, then $f(1) \in Z$.
(2) If φ is an antiautomorphism, then $f(1) = f(k)^{-1}m$ and $f(k) \in Z$.

In 2019, Catalano et al. [6] initiated the study of maps preserving products of matrices. More precisely, they proved the following result.

Theorem 1.3. *[6, Theorem 1] Let D be a division ring with characteristic different from 2. Let $R = M_n(D)$ be the ring of $n \times n$ matrices with $n \geq 2$, and let Z be the center of R. With $m, k \in R^{\times}$, let $f\colon R \to R$ be a bijective additive map satisfying the identity*

$$f(x)f(y) = m$$

for every $x, y \in R^{\times}$ such that $xy = k$. Then,

$$f(x) = f(1)\varphi(x)$$

for all $x \in R$, where $\varphi\colon R \to R$ is either an automorphism or an antiautomorphism. Moreover, we have the following:

(1) If φ is an automorphism, then $f(1) \in Z$.
(2) If φ is an antiautomorphism, then $f(1) = f(k)^{-1}m$ and $f(k) \in Z$.

The study of maps preserving products of matrices is an active topic. For recent results on maps preserving products of matrices, we refer the reader to [7–11].

We point out that the proof of Theorem 1.3 is wrong. Indeed, they used the following identity due to Hua:

$$a - aba = (a^{-1} + (b^{-1} - a)^{-1})^{-1}, \tag{1.1}$$

where $a, b, b^{-1} - a \in R^{\times}$. By taking $a = 1$ and $b = x \in R^*$ in (1.1), they obtained that

$$\varphi(x^{-1}) = \varphi(x)^{-1}$$

for all $x \in R^{\times}$, where

$$f = f(1)\varphi.$$

By using Theorem 1.1 they got that φ is an automorphism or an antiautomorphism. However, the condition of $x^{-1} - 1 \in R^{\times}$ (equivalently, $1 - x \in R^{\times}$) should be added in the use of (1.1). Thus, they cannot obtain that

$$\varphi(x^{-1}) = \varphi(x)^{-1}$$

for all $x \in R^{\times}$, where $f = f(1)\varphi$. In fact, they can obtain that

$$\varphi(x^{-1}) = \varphi(x)^{-1}$$

for all $x \in R^{\times}$ with $1 - x \in R^{\times}$, which cannot use Theorem 1.1 to get that φ is an automorphism or an antiautomorphism.

In the present paper we shall give the following result:

Theorem 1.4. *Let D be a division ring such that either char$(D) \neq 2, 3$ or D is not a field and char$(D) \neq 2$. Let $R = M_n(D)$ be the ring of $n \times n$ matrices with $n \geq 2$, and let Z be the center of R. With $m, k \in R^{\times}$, let $f, g\colon R \to R$ be bijective additive maps satisfying the identity*

$$f(x)g(y) = m$$

for every $x, y \in R^{\times}$ such that $xy = k$. Then,

$$f(x) = f(1)\varphi(x) \quad \text{and} \quad g(x) = \varphi(xk^{-1})g(k)$$

for all $x \in R$, where $\varphi\colon R \to R$ is either an automorphism or an antiautomorphism. Moreover, we have the following:

(1) If φ is an automorphism, then $g(x) = \varphi(x)g(1)$ for all $x \in R$.
(2) If φ is an antiautomorphism, then $g(x) = f(k)^{-1}f(x)g(k)$ for all $x \in R$.

As a consequence of Theorem 1.4 we shall give the following result, which gives a correct version of Theorem 1.3.

Theorem 1.5. *Let D be a division ring such that either char$(D) \neq 2, 3$ or D is not a field and char$(D) \neq 2$. Let $R = M_n(D)$ be the ring of $n \times n$ matrices with $n \geq 2$, and let Z be the center of R. With $m, k \in R^{\times}$, let $f\colon R \to R$ be a bijective additive map satisfying the identity*

$$f(x)f(y) = m$$

for every $x, y \in R^{\times}$ such that $xy = k$. Then,

$$f(x) = f(1)\varphi(x)$$

for all $x \in R$, where $\varphi\colon R \to R$ is either an automorphism or an antiautomorphism. Moreover, we have the following:

(1) If φ is an automorphism, then $f(1) \in Z$.
(2) If φ is an antiautomorphism, then $f(1) = f(k)^{-1}m$ and $f(k) \in Z$.

We organize the paper as follows: In Section 2 we shall give the proof of Theorem 1.4. In Section 3 we shall give the proof of Theorem 1.5.

We remark that the method in the proof of Theorem 3 is different from that in the proof of Theorem 2. We believe that the method will play a certain role in the study of maps preserving products of matrices.

2. Preliminaries

Throughout this section, let D be a division ring and let $R = M_n(D)$ with $n > 1$. By Z we denote the center of D. We identify Z with the center of R canonically. For $A \in R$, we denote by $|A|$ the determinant of A.

We begin with the following technical result, which will be used in the proof of our main result.

Lemma 2.1. *Let D be a division ring with char$(D) \neq 2, 3$. Let $R = M_n(D)$ with $n > 1$. For any $\alpha, \beta \in D$ and $1 \leq i, j, k, l \leq n$, we set*

$$T = \alpha e_{ij} + \beta e_{kl}.$$

We claim that either there exists $\gamma \in \{1, 2, 3\}$ such that

$$\gamma + T, \gamma + T + 1 \in R^{\times}$$

or there exist $\gamma_1, \gamma_2 \in R^{\times}$ such that

$$\gamma_i + 1, \gamma_i + T, \gamma_i + T + 1, \gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^{\times}$$

for all $i = 1, 2$.

Proof. We prove the result by way of the following several cases:

Case 1. Suppose that $i = j = k = l$. We set

$$\gamma = \begin{cases} 3, & \text{if } \alpha + \beta = -1, -2; \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that $\gamma + T, \gamma + 1 + T \in R^{\times}$.

Case 2. Suppose that $i = j$, $k = l$, and $i \neq k$. We set

$$\gamma = \begin{cases} 3, & \text{if } \alpha, \beta = -1, -2; \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that $\gamma + T, \gamma + 1 + T \in R^{\times}$.

Case 3. Suppose that $i = j$ and $k \neq l$. We set

$$\gamma = \begin{cases} 3, & \text{if } \alpha = -1, -2; \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that $\gamma + T, \gamma + 1 + T \in R^{\times}$.

Case 4. Suppose that $i \neq j$ and $k = l$. We set

$$\gamma = \begin{cases} 3, & \text{if } \beta = -1, -2; \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that $\gamma + T, \gamma + 1 + T \in R^{\times}$.

Case 5. Suppose that $i \neq j$, $k \neq l$, and $(i, j) \neq (l, k)$. It is easy to check that

$$|\gamma + T| = \gamma^n$$

for all $\gamma \in D$. We set $\gamma = 1$.

It is clear that

$$\gamma + T, \gamma + T + 1 \in R^{\times}.$$

Case 6. Suppose that $i \neq j$, $k \neq l$, and $(i, j) = (l, k)$. We may assume that $i < j$. The case of $i > j$ can be discussed analogously. It is easy to check that

$$\left| \sum_{s=1}^{n} a_i e_{ii} + T \right| = a_1 \cdots a_n - a_1 \cdots a_{i-1} \alpha a_{i+1} \cdots a_{j-1} \beta a_{j+1} \cdots a_n$$

for all $a_i \in D$, $i = 1, \cdots, n$. Suppose first that $\alpha\beta \neq 1, 4$. We set $\gamma = 1$. It follows that

$$\begin{aligned} \gamma + T &= 1 + \alpha e_{ij} + \beta e_{ji}; \\ \gamma + T + 1 &= 2 + \alpha e_{ij} + \beta e_{ji}. \end{aligned}$$

This implies that

$$|\gamma + T| = 1 - \alpha\beta \neq 0$$

and

$$|\gamma + T + 1| = 2^n - 2^{n-2}\alpha\beta = 2^{n-2}(4 - \alpha\beta) \neq 0.$$

This implies that

$$\gamma + T, \gamma + T + 1 \in R^{\times}.$$

Suppose next that $\alpha\beta = 1$ or 4. We first discuss the case of $\alpha\beta = 1$. We set $\gamma = 2$. It follows that

$$\begin{aligned} \gamma + T &= 2 + \alpha e_{ij} + \beta e_{ji}; \\ \gamma + T + 1 &= 3 + \alpha e_{ij} + \beta e_{ji}. \end{aligned}$$

This implies that

$$|\gamma + T| = 2^n - 2^{n-2}\alpha\beta = 2^{n-2} \times 3 \neq 0$$

and

$$|\gamma + T + 1| = 3^n - 3^{n-2}\alpha\beta = 3^{n-2} \times 2^3 \neq 0.$$

We get that

$$\gamma + T, \gamma + T + 1 \in R^{\times}.$$

We now assume that $\alpha\beta = 4$. We set

$$\gamma_1 = \sum_{\substack{1\le s\le n\\ s\ne j}} e_{ss} + 2e_{jj};$$

$$\gamma_2 = \sum_{\substack{1\le s\le n\\ s\ne j}} 3e_{ss} + e_{jj}.$$

It is clear that $\gamma_1, \gamma_2 \in R^{\times}$. Note that

$$\gamma_1 + 1 = \sum_{\substack{1\le s\le n\\ s\ne j}} 2e_{ss} + 3e_{jj};$$

$$\gamma_2 + 1 = \sum_{\substack{1\le s\le n\\ s\ne j}} 4e_{ss} + 2e_{jj}.$$

It is clear that

$$\gamma_1 + 1, \gamma_2 + 1 \in R^{\times}.$$

Note that

$$\gamma_1 - \gamma_2 = \sum_{\substack{1\le s\le n\\ s\ne j}} (-2)e_{ss} + e_{jj};$$

$$\gamma_1 - \gamma_2 + 1 = \sum_{\substack{1\le s\le n\\ s\ne j}} (-1)e_{ss} + 2e_{jj}.$$

It is clear that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1 \in R^{\times}.$$

Note that

$$\gamma_1 + T = \sum_{\substack{1\le s\le n\\ s\ne j}} e_{ss} + 2e_{jj} + \alpha e_{ij} + \beta e_{ji};$$

$$\gamma_1 + T + 1 = \sum_{\substack{1\le s\le n\\ s\ne j}} 2e_{ss} + 3e_{jj} + \alpha e_{ij} + \beta e_{ji};$$

$$\gamma_2 + T = \sum_{\substack{1\le s\le n\\ s\ne j}} 3e_{ss} + e_{jj} + \alpha e_{ij} + \beta e_{ji};$$

$$\gamma_2 + T + 1 = \sum_{\substack{1\le s\le n\\ s\ne j}} 4e_{ss} + 2e_{jj} + \alpha e_{ij} + \beta e_{ji};$$

$$\gamma_1 - \gamma_2 + T = \sum_{\substack{1\le s\le n\\ s\ne j}} (-2)e_{ss} + e_{jj} + \alpha e_{ij} + \beta e_{ji};$$

$$\gamma_1 - \gamma_2 + T + 1 = \sum_{\substack{1\le s\le n\\ s\ne j}} (-1)e_{ss} + 2e_{jj} + \alpha e_{ij} + \beta e_{ji}.$$

It implies that

$$\begin{aligned}
|\gamma_1 + T| &= 2 - \alpha\beta = -2 \neq 0;\\
|\gamma_1 + T + 1| &= 2^{n-1} \times 3 - 2^{n-2}\alpha\beta = 2^{n-1} \neq 0;\\
|\gamma_2 + T| &= 3^{n-1} - 3^{n-2}\alpha\beta = -3^{n-2} \neq 0;\\
|\gamma_2 + T + 1| &= 4^{n-1} \times 2 - 4^{n-2}\alpha\beta = 4^{n-1} \neq 0;\\
|\gamma_1 - \gamma_2 + T| &= (-2)^{n-1} - (-2)^{n-2}\alpha\beta = (-2)^{n-2} \times (-6) \neq 0;\\
|\gamma_1 - \gamma_2 + T + 1| &= (-1)^{n-1} \times 2 - (-1)^{n-2}\alpha\beta = (-1)^{n-2} \times (-6) \neq 0.
\end{aligned}$$

This implies that

$$\gamma_i + T, \gamma_i + T + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times$$

for $i = 1, 2$. The proof of the result is complete. □

The following technical result will be used in the proof of our main result.

Lemma 2.2. *Let D be a division ring, which is not a field. Suppose that char$(D) \neq 2$. Let $R = M_n(D)$ with $n > 1$. For any $\alpha, \beta \in D$ and $1 \leq i, j, k, l \leq n$, we set $T = \alpha e_{ij} + \beta e_{kl}$. We claim that either there exists $\gamma \in \{1, 2, 3\}$ such that*

$$\gamma + T, \gamma + T + 1 \in R^\times$$

or there exist $\gamma_1, \gamma_2 \in R^\times$ such that

$$\gamma_i + 1, \gamma_i + T, \gamma_i + T + 1, \gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times$$

for $i = 1, 2$.

Proof. In view of Lemma 2.1, we may assume that char$(D) = 3$. It is clear that D is a central division algebra over Z. It is well known that the dimension of a finite dimensional central division algebra is a perfect square (see [12, Corollary 1.37]). This implies that $\dim_Z(D) \geq 4$. We now prove the result by way of the following two cases:

Case 1. Suppose first that $\alpha, \beta, 1$ are linearly independent over Z. We get that there exists $\gamma \in D$ such that $\gamma, \alpha, \beta, 1$ are linearly independent over Z. We now discuss the following six subcases:

Subcase 1.1. Suppose that $i = j = k = l$. We set

$$\gamma_1 = \gamma \quad \text{and} \quad \gamma_2 = \gamma - \alpha.$$

It is clear that

$$\gamma_i, \gamma_i + 1, \gamma_i + (\alpha + \beta), \gamma_i + (\alpha + \beta) + 1 \in D^\times$$

for $i = 1, 2$. This implies that

$$\gamma_i, \gamma_i + 1, \gamma_i + T, \gamma_i + T + 1 \in R^\times$$

for $i = 1, 2$. Note that

$$\gamma_1 - \gamma_2 = \alpha.$$

It is clear that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + (\alpha + \beta), \gamma_1 - \gamma_2 + (\alpha + \beta) + 1 \in D^\times.$$

This implies that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times,$$

as desired.

Subcase 1.2. Suppose that $i = j$, $k = l$, and $i \neq k$. We set

$$\gamma_1 = \gamma \quad \text{and} \quad \gamma_2 = \gamma_1 - \alpha - \beta.$$

It is clear that

$$\gamma_i, \gamma_i + 1, \gamma_i + \alpha, \gamma_i + \alpha + 1, \gamma_i + \beta, \gamma_i + \beta + 1 \in D^\times$$

for $i = 1, 2$. This implies that

$$\gamma_i, \gamma_i + 1, \gamma_i + T, \gamma_i + T + 1 \in R^\times$$

for $i = 1, 2$. Note that

$$\gamma_1 - \gamma_2 = \alpha + \beta.$$

It is clear that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + \alpha, \gamma_1 - \gamma_2 + \alpha + 1, \gamma_1 - \gamma_2 + \beta, \gamma_1 - \gamma_2 + \beta + 1 \in D^\times.$$

This implies that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times,$$

as desired.

Subcase 1.3. Suppose that $i = j$ and $k \neq l$. We set

$$\gamma_1 = \gamma \quad \text{and} \quad \gamma_2 = \gamma_1 - \beta.$$

It is clear that

$$\gamma_i, \gamma_i + 1, \gamma_i + \alpha, \gamma_i + \alpha + 1 \in D^\times$$

for $i = 1, 2$. This implies that

$$\gamma_i, \gamma_i + 1, \gamma_i + T, \gamma_i + T + 1 \in R^\times$$

for $i = 1, 2$. Note that

$$\gamma_1 - \gamma_2 = \beta.$$

It is clear that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + \alpha, \gamma_1 - \gamma_2 + \alpha + 1 \in D^\times.$$

This implies that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times,$$

as desired.

Subcase 1.4. Suppose that $i \neq j$ and $k = l$. We set

$$\gamma_1 = \gamma \quad \text{and} \quad \gamma_2 = \gamma - \alpha.$$

It is clear that

$$\gamma_i, \gamma_i + 1, \gamma_i + \beta, \gamma_i + \beta + 1 \in D^\times$$

for $i = 1, 2$. This implies that

$$\gamma_i, \gamma_i + 1, \gamma_i + T, \gamma_i + T + 1 \in R^\times$$

for $i = 1, 2$. Note that $\gamma_1 - \gamma_2 = \alpha$. It is clear that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + \beta, \gamma_1 - \gamma_2 + \beta + 1 \in D^\times.$$

This implies that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times,$$

as desired.

Subcase 1.5. Suppose that $i \neq j$, $k \neq l$, and $(i, j) \neq (l, k)$. It is easy to check that

$$\left| \sum_{s=1}^{n} a_i e_{ii} + T \right| = a_1 \cdots a_n$$

for all $a_i \in D$, $i = 1, \cdots, n$. We set

$$\gamma_1 = \gamma \quad \text{and} \quad \gamma_2 = \gamma - \alpha.$$

This implies that

$$\begin{aligned} |\gamma_i| &= \gamma_i^n \neq 0; \\ |\gamma_i + 1| &= (\gamma_i + 1)^n \neq 0; \\ |\gamma_i + T| &= \gamma_i^n \neq 0; \\ |\gamma_i + T + 1| &= (\gamma_i + 1)^n \neq 0; \\ |\gamma_1 - \gamma_2| &= \alpha^n \neq 0; \\ |\gamma_1 - \gamma_2 + 1| &= (\alpha + 1)^n \neq 0; \\ |\gamma_1 - \gamma_2 + T| &= \alpha^n \neq 0; \\ |\gamma_1 - \gamma_2 + T + 1| &= (\alpha + 1)^n \neq 0 \end{aligned}$$

for $i = 1, 2$. It follows that

$$\gamma_i, \gamma_i + 1, \gamma_i + T, \gamma_i + T + 1, \gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times$$

for $i = 1, 2$, as desired.

Subcase 1.6. Suppose that $i \neq j$, $k \neq l$, and $(i, j) = (l, k)$. We may assume that $i < j$. The case of $i > j$ can be discussed analogously. It is easy to check that

$$\left| \sum_{s=1}^{n} a_i e_{ii} + T \right| = a_1 \cdots a_n - a_1 \cdots a_{i-1} \alpha a_{i+1} \cdots a_{j-1} \beta a_{j+1} \cdots a_n$$

for all $a_i \in D$, $i = 1, \cdots, n$. We set

$$\begin{aligned} \gamma_1 &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} (\gamma + \alpha) e_{ss} + (\gamma + \beta + 2) e_{jj} + 2\alpha e_{ij}; \\ \gamma_2 &= \gamma + 2\alpha e_{ij}. \end{aligned}$$

It is clear that

$$\begin{aligned}|\gamma_1| &= (\gamma+\alpha)^{j-1}(\gamma+\beta+2)(\gamma+\alpha)^{n-j}\neq 0;\\ |\gamma_1+1| &= (\gamma+\alpha+1)^{j-1}(\gamma+\beta)(\gamma+\alpha+1)^{n-j}\neq 0;\\ |\gamma_2| &= \gamma^n\neq 0;\\ |\gamma_2+1| &= (\gamma+1)^n\neq 0.\end{aligned}$$

This implies that

$$\gamma_1,\gamma_1+1,\gamma_2,\gamma_2+1\in R^\times.$$

Since $\mathrm{char}(D)=3$ we get that

$$\begin{aligned}\gamma_1+T &= \sum_{\substack{1\le s\le n\\ s\neq j}}(\gamma+\alpha)e_{ss}+(\gamma+\beta+2)e_{jj}+\beta e_{ji};\\ \gamma_1+T+1 &= \sum_{\substack{1\le s\le n\\ s\neq j}}(\gamma+\alpha+1)e_{ss}+(\gamma+\beta)e_{jj}+\beta e_{ji};\\ \gamma_2+T &= \gamma+\beta e_{ji};\\ \gamma_2+T+1 &= \gamma+1+\beta e_{ji}.\end{aligned}$$

It is easy to check that

$$\begin{aligned}|\gamma_1+T| &= (\gamma+\alpha)^{j-1}(\gamma+\beta+2)(\gamma+\alpha)^{n-j}\neq 0;\\ |\gamma_1+T+1| &= (\gamma+\alpha+1)^{j-1}(\gamma+\beta)(\gamma+\alpha+1)^{n-j}\neq 0;\\ |\gamma_2+T| &= \gamma^n\neq 0;\\ |\gamma_2+T+1| &= (\gamma+1)^n\neq 0.\end{aligned}$$

This implies that

$$\gamma_i+T,\gamma_i+T+1\in R^\times$$

for $i=1,2$. Note that

$$\begin{aligned}\gamma_1-\gamma_2 &= \sum_{\substack{1\le s\le n\\ s\neq j}}\alpha e_{ss}+(\beta+2)e_{jj};\\ \gamma_1-\gamma_2+1 &= \sum_{\substack{1\le s\le n\\ s\neq j}}(\alpha+1)e_{ss}+\beta e_{jj}.\end{aligned}$$

We get that

$$\begin{aligned}|\gamma_1-\gamma_2| &= \alpha^{j-1}(\beta+2)\alpha^{n-j}\neq 0;\\ |\gamma_1-\gamma_2+1| &= (\alpha+1)^{j-1}\beta(\alpha+1)^{n-j}\neq 0.\end{aligned}$$

This implies that

$$\gamma_1+\gamma_2,\gamma_1+\gamma_2+1\in R^\times.$$

Note that

$$\gamma_1 - \gamma_2 + T = \sum_{\substack{1\le s\le n\\ s\ne j}} \alpha e_{ss} + (\beta+2)e_{jj} + \alpha e_{ij} + \beta e_{ji};$$
$$\gamma_1 - \gamma_2 + T + 1 = \sum_{\substack{1\le s\le n\\ s\ne j}} (\alpha+1) e_{ss} + \beta e_{jj} + \alpha e_{ij} + \beta e_{ji}.$$

We get that

$$\begin{aligned}|\gamma_1 - \gamma_2 + T| &= \alpha^{j-1}(\beta+2)\alpha^{n-j} - \alpha^{j-1}\beta\alpha^{n-j}\\ &= 2\alpha^{n-1}\\ &\ne 0\end{aligned}$$

and

$$\begin{aligned}|\gamma_1 - \gamma_2 + T + 1| &= (\alpha+1)^{j-1}\beta(\alpha+1)^{n-j} - (\alpha+1)^{i-1}\alpha(\alpha+1)^{j-i-1}\beta(\alpha+1)^{n-j}\\ &= (\alpha+1)^{j-1}\beta(\alpha+1)^{n-j} - (\alpha+1)^{j-2}\alpha\beta(\alpha+1)^{n-j}\\ &= (\alpha+1)^{j-2}(\alpha+1-\alpha)\beta(\alpha+1)^{n-j}\\ &= (\alpha+1)^{j-2}\beta(\alpha+1)^{n-j}\\ &\ne 0.\end{aligned}$$

This implies that

$$\gamma_1 + \gamma_2 + T, \gamma_1 + \gamma_2 + T + 1 \in R^{\times},$$

as desired.

Case 2. Suppose next that $\alpha, \beta, 1$ are linearly dependent over Z. Note that $\dim_Z D \ge 4$. We get that there exists $\gamma_1' \in D$ such that $\gamma_1' \notin L(1, \alpha, \beta)$, where $L(1, \alpha, \beta)$ is a subspace of D generalized by $1, \alpha, \beta$. It is clear that

$$dim_Z(L(1, \alpha, \beta, \gamma_1')) \le 3,$$

where $L(1, \alpha, \beta, \gamma_1)$ is a subspace of D generalized by $1, \alpha, \beta, \gamma_1'$. We get that there exists $\gamma_2' \in D$ such that $\gamma_2' \notin L(1, \alpha, \beta, \gamma_1')$. We now discuss the following six subcases.

Subcase 2.1. Suppose that $i = j = k = l$. We set $\gamma_1 = \gamma_1'$ and $\gamma_2 = \gamma_2'$. It is clear that

$$\gamma_i, \gamma_i + 1, \gamma_i + (\alpha+\beta), \gamma_i + (\alpha+\beta) + 1 \in D^{\times}$$

for $i = 1, 2$. This implies that

$$\gamma_i + T, \gamma_i + T + 1 \in R^{\times}$$

for $i = 1, 2$. Note that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + (\alpha+\beta), \gamma_1 - \gamma_2 + (\alpha+\beta) + 1 \in D^{\times}.$$

This implies that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^{\times},$$

as desired.

Subcase 2.2. Suppose that $i = j$, $k = l$, and $i \neq k$. We set $\gamma_1 = \gamma_1'$ and $\gamma_2 = \gamma_2'$. It is clear that

$$\gamma_i, \gamma_i + 1, \gamma_i + \alpha, \gamma_i + \alpha + 1, \gamma_i + \beta, \gamma_i + \beta + 1 \in D^\times$$

for $i = 1, 2$. This implies that

$$\gamma_i, \gamma_i + 1, \gamma_i + T, \gamma_i + T + 1 \in R^\times$$

for $i = 1, 2$. Note that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + \alpha, \gamma_1 - \gamma_2 + \alpha + 1, \gamma_1 - \gamma_2 + \beta, \gamma_1 - \gamma_2 + \beta + 1 \in D^\times.$$

This implies that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times,$$

as desired.

Subcase 2.3. Suppose that $i = j$ and $k \neq l$. We set $\gamma_1 = \gamma_1'$ and $\gamma_2 = \gamma_2'$. It is clear that

$$\gamma_i, \gamma_i + 1, \gamma_i + \alpha, \gamma_i + \alpha + 1 \in D^\times$$

for $i = 1, 2$. This implies that

$$\gamma_i, \gamma_i + 1, \gamma_i + T, \gamma_i + T + 1 \in R^\times$$

for $i = 1, 2$. Note that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + \alpha, \gamma_1 - \gamma_2 + \alpha + 1 \in D^\times.$$

This implies that

$$\gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times,$$

as desired.

Subcase 2.4. Suppose that $i \neq j$ and $k = l$. We set $\gamma_1 = \gamma_1'$ and $\gamma_2 = \gamma_2'$. It is clear that

$$\gamma_i, \gamma_i + 1, \gamma_i + \beta, \gamma_i + \beta + 1 \in D^\times$$

for $i = 1, 2$. This implies that

$$\gamma_i, \gamma_i + 1, \gamma_i + T, \gamma_i + T + 1 \in R^\times$$

for $i = 1, 2$. Note that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + \beta, \gamma_1 - \gamma_2 + \beta + 1 \in D^\times.$$

This implies that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times,$$

as desired.

Subcase 2.5. Suppose that $i \neq j$, $k \neq l$ and $(i, j) \neq (l, k)$. We set $\gamma_1 = \gamma_1'$ and $\gamma_2 = \gamma_2'$. It is easy to check that

$$\left| \sum_{s=1}^{n} a_i e_{ii} + T \right| = a_1 \cdots a_n$$

for all $a_i \in D$, $i = 1, \cdots, n$. It is clear that

$$\gamma_i, \gamma_i + 1, \gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1 \in D^\times$$

for $i = 1, 2$. We get that

$$\gamma_i, \gamma_i + 1, \gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1 \in D^\times$$

for $i = 1, 2$. Note that

$$\begin{gathered} |\gamma_i + T| = \gamma_i^n \neq 0; \\ |\gamma_i + T + 1| = (\gamma_i + 1)^n \neq 0; \\ |\gamma_1 - \gamma_2 + T| = (\gamma_1 - \gamma_2)^n \neq 0; \\ |\gamma_1 - \gamma_2 + T + 1| = (\gamma_1 - \gamma_2 + 1)^n \neq 0 \end{gathered}$$

for $i = 1, 2$. It follows that

$$\gamma_i + T, \gamma_i + T + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^\times$$

for $i = 1, 2$, as desired.

Subcase 2.6. Suppose that $i \neq j$, $k \neq l$, and $(i, j) = (l, k)$. We may assume that $i < j$. The case of $i > j$ can be discussed analogously. It is easy to check that

$$\left|\sum_{s=1}^{n} a_i e_{ii} + T\right| = a_1 \cdots a_n - a_1 \cdots a_{i-1} \alpha a_{i+1} \cdots a_{j-1} \beta a_{j+1} \cdots a_n$$

for all $a_i \in D$, $i = 1, \cdots, n$. Suppose first that $\alpha\beta \in Z$. We set

$$\begin{aligned} \gamma_1 &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} (\gamma_1' + 1) e_{ss} + \gamma_1' e_{jj} + 2\alpha e_{ij}; \\ \gamma_2 &= \sum_{\substack{1 \leq s \leq n \\ s \neq j}} \gamma_1' e_{ss} + \gamma_2' e_{jj} + 2\alpha e_{ij}. \end{aligned}$$

It is clear that

$$\begin{gathered} |\gamma_1| = (\gamma_1' + 1)^{j-1} \gamma_1' (\gamma_1' + 1)^{n-j} \neq 0; \\ |\gamma_1 + 1| = (\gamma_1' + 2)^{j-1} (\gamma_1' + 1)(\gamma_1' + 2)^{n-j} \neq 0; \\ |\gamma_2| = (\gamma_1')^{j-1} \gamma_2' (\gamma_1')^{n-j} \neq 0; \\ |\gamma_2 + 1| = (\gamma_1' + 1)^{j-1} (\gamma_2' + 1)(\gamma_1' + 1)^{n-j} \neq 0. \end{gathered}$$

This implies that

$$\gamma_1, \gamma_1 + 1, \gamma_2, \gamma_2 + 1 \in R^\times.$$

Since char$(D) = 3$ we get that

$$\begin{aligned}
\gamma_1 + T &= \sum_{\substack{1\le s\le n\\ s\ne j}} (\gamma_1' + 1)e_{ss} + \gamma_1' e_{jj} + \beta e_{ji};\\
\gamma_1 + T + 1 &= \sum_{\substack{1\le s\le n\\ s\ne j}} (\gamma_1' + 2)e_{ss} + (\gamma_1' + 1)e_{jj} + \beta e_{ji};\\
\gamma_2 + T &= \sum_{\substack{1\le s\le n\\ s\ne j}} \gamma_1' e_{ss} + \gamma_2' e_{jj} + \beta e_{ij};\\
\gamma_2 + T + 1 &= \sum_{\substack{1\le s\le n\\ s\ne j}} (\gamma_1' + 1)e_{ss} + (\gamma_2' + 1)e_{jj} + \beta e_{ij}.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
|\gamma_1 + T| &= (\gamma_1' + 1)^{j-1}\gamma_1'(\gamma_1' + 1)^{n-j} \ne 0;\\
|\gamma_1 + T + 1| &= (\gamma_1' + 2)^{j-1}(\gamma_1' + 1)(\gamma_1' + 2)^{n-j} \ne 0;\\
|\gamma_2 + T| &= (\gamma_1')^{j-1}\gamma_2'(\gamma_1')^{n-j} \ne 0;\\
|\gamma_2 + T + 1| &= (\gamma_1' + 1)^{j-1}(\gamma_2' + 1)(\gamma_1' + 1)^{n-j} \ne 0.
\end{aligned}$$

This implies that

$$\gamma_i + T, \gamma_i + T + 1 \in R^{\times}$$

for $i = 1, 2$. Note that

$$\begin{aligned}
\gamma_1 - \gamma_2 &= \sum_{\substack{1\le s\le n\\ s\ne j}} e_{ss} + (\gamma_1' - \gamma_2')e_{jj};\\
\gamma_1 - \gamma_2 + 1 &= \sum_{\substack{1\le s\le n\\ s\ne j}} 2e_{ss} + (\gamma_1' - \gamma_2' + 1)e_{jj}.
\end{aligned}$$

We get that

$$\begin{aligned}
|\gamma_1 - \gamma_2| &= \gamma_1' - \gamma_2' \ne 0;\\
|\gamma_1 - \gamma_2 + 1| &= 2^{n-1}(\gamma_1' - \gamma_2' + 1) \ne 0.
\end{aligned}$$

This implies that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1 \in R^{\times}.$$

Note that

$$\begin{aligned}
\gamma_1 - \gamma_2 + T &= \sum_{\substack{1\le s\le n\\ s\ne j}} e_{ss} + (\gamma_1' - \gamma_2')e_{jj} + \alpha e_{ij} + \beta e_{ji};\\
\gamma_1 - \gamma_2 + T + 1 &= \sum_{\substack{1\le s\le n\\ s\ne j}} 2e_{ss} + (\gamma_1' - \gamma_2' + 1)e_{jj} + \alpha e_{ij} + \beta e_{ji}.
\end{aligned}$$

Since $\alpha\beta \in Z$, we get that

$$|\gamma_1 - \gamma_2 + T| = \gamma_1' - \gamma_2' - \alpha\beta \neq 0$$

and

$$\begin{aligned} |\gamma_1 - \gamma_2 + T + 1| &= 2^{n-1}(\gamma_1' - \gamma_2' + 1) - 2^{n-2}\alpha\beta \\ &= 2^{n-2}(2\gamma_1' - 2\gamma_2' + 2 - \alpha\beta) \\ &\neq 0. \end{aligned}$$

This implies that

$$\gamma_1 + \gamma_2 + T, \gamma_1 + \gamma_2 + T + 1 \in R^{\times},$$

as desired.

Suppose next that $\alpha\beta \notin Z$. We set

$$\begin{aligned} \gamma_1 &= \sum_{\substack{1\leq s\leq n \\ s\neq j}} (\gamma_1' + 1)e_{ss} + \alpha\beta e_{jj} + 2\alpha e_{ij}; \\ \gamma_2 &= \sum_{\substack{1\leq s\leq n \\ s\neq j}} \gamma_1' e_{ss} + e_{jj} + 2\alpha e_{ij}. \end{aligned}$$

It is clear that

$$\begin{aligned} |\gamma_1| &= (\gamma_1' + 1)^{j-1}\alpha\beta(\gamma_1' + 1)^{n-j} \neq 0; \\ |\gamma_1 + 1| &= (\gamma_1' + 2)^{j-1}(\alpha\beta + 1)(\gamma_1' + 2)^{n-j} \neq 0; \\ |\gamma_2| &= (\gamma_1')^{n-1} \neq 0; \\ |\gamma_2 + 1| &= 2(\gamma_1' + 1)^{n-1} \neq 0. \end{aligned}$$

This implies that

$$\gamma_1, \gamma_1 + 1, \gamma_2, \gamma_2 + 1 \in R^{\times}.$$

Since $\mathrm{char}(D) = 3$ we get that

$$\begin{aligned} \gamma_1 + T &= \sum_{\substack{1\leq s\leq n \\ s\neq j}} (\gamma_1' + 1)e_{ss} + \alpha\beta e_{jj} + \beta e_{ji}; \\ \gamma_1 + T + 1 &= \sum_{\substack{1\leq s\leq n \\ s\neq j}} (\gamma_1' + 2)e_{ss} + (\alpha\beta + 1)e_{jj} + \beta e_{ji}; \\ \gamma_2 + T &= \sum_{\substack{1\leq s\leq n \\ s\neq j}} \gamma_1' e_{ss} + e_{jj} + \beta e_{ij}; \\ \gamma_2 + T + 1 &= \sum_{\substack{1\leq s\leq n \\ s\neq j}} (\gamma_1' + 1)e_{ss} + 2e_{jj} + \beta e_{ij}. \end{aligned}$$

It is easy to check that

$$\begin{aligned} |\gamma_1 + T| &= (\gamma_1' + 1)^{j-1}\alpha\beta(\gamma_1' + 1)^{n-j} \neq 0; \\ |\gamma_1 + T + 1| &= (\gamma_1' + 2)^{j-1}(\alpha\beta + 1)(\gamma_1' + 2)^{n-j} \neq 0; \\ |\gamma_2 + T| &= (\gamma_1')^{n-1} \neq 0; \\ |\gamma_2 + T + 1| &= 2(\gamma_1' + 1)^{n-1} \neq 0. \end{aligned}$$

This implies that

$$\gamma_i + T, \gamma_i + T + 1 \in R^{\times}$$

for $i = 1, 2$. Note that

$$\gamma_1 - \gamma_2 = \sum_{\substack{1 \le s \le n \\ s \ne j}} e_{ss} + (\alpha\beta - 1)e_{jj};$$

$$\gamma_1 - \gamma_2 + 1 = \sum_{\substack{1 \le s \le n \\ s \ne j}} 2e_{ss} + \alpha\beta e_{jj}.$$

Since $\alpha\beta \notin Z$ we get that

$$|\gamma_1 - \gamma_2| = \alpha\beta - 1 \ne 0;$$

$$|\gamma_1 - \gamma_2 + 1| = 2^{n-1}\alpha\beta \ne 0.$$

This implies that

$$\gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1 \in R^{\times}.$$

Note that

$$\gamma_1 - \gamma_2 + T = \sum_{\substack{1 \le s \le n \\ s \ne j}} e_{ss} + (\alpha\beta - 1)e_{jj} + \alpha e_{ij} + \beta e_{ji};$$

$$\gamma_1 - \gamma_2 + T + 1 = \sum_{\substack{1 \le s \le n \\ s \ne j}} 2e_{ss} + \alpha\beta e_{jj} + \alpha e_{ij} + \beta e_{ji}.$$

Since $\alpha\beta \notin Z$, we get that

$$|\gamma_1 - \gamma_2 + T| = \alpha\beta - 1 - \alpha\beta = -1 \ne 0$$

and

$$|\gamma_1 - \gamma_2 + T + 1| = 2^{n-1}\alpha\beta - 2^{n-2}\alpha\beta = 2^{n-2}\alpha\beta \ne 0.$$

This implies that

$$\gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^{\times},$$

as desired. The proof of the result is complete. □

3. The proof of Theorem 1.4

The following result will be used in the proof of our main result, which is of some independent interests.

Proposition 3.1. *Let D be a division ring such that either char$(D) \ne 2, 3$ or D is not a field and char$(D) \ne 2$. Let $R = M_n(D)$, where $n > 1$. Let φ: $R \to R$ be a bijective additive map such that $\varphi(1) = 1$ and*

$$\varphi(x^2) = \varphi(x)^2$$

for all $x \in R^{\times}$ with $x + 1 \in R^{\times}$. Then, φ is either an automorphism or an antiautomorphism.

Proof. We first claim that φ is a Jordan automorphism by way of the following three steps:

Step 1. We claim that

$$\varphi((\alpha e_{ij} + \beta e_{kl})^2) = \varphi(\alpha e_{ij} + \beta e_{kl})^2 \tag{3.1}$$

for all $1 \leq i, j, k, l \leq n$ and $\alpha, \beta \in D$. In particular, we have

$$\varphi((\alpha e_{ij})^2) = \varphi(\alpha e_{ij})^2 \tag{3.2}$$

for all $1 \leq i, j \leq n$ and $\alpha \in D$. We set

$$T = \alpha e_{ij} + \beta e_{kl}.$$

In view of both Lemmas 2.1 and 2.2 we note that either there exists $\gamma \in \{1, 2, 3\}$ such that

$$\gamma + T, \gamma + T + 1 \in R^{\times}$$

or there exist $\gamma_1, \gamma_2 \in R^{\times}$ such that

$$\gamma_i + 1, \gamma_i + T, \gamma_i + T + 1, \gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^{\times}$$

for all $i = 1, 2$. Suppose first that there exists $\gamma \in \{1, 2, 3\}$ such that

$$\gamma + T, \gamma + T + 1 \in R^{\times}.$$

By our hypothesis we have

$$\varphi((\gamma + T)^2) = \varphi(\gamma + T)^2.$$

Note that

$$\varphi(\gamma) = \gamma \quad \text{and} \quad \varphi(\gamma T) = \gamma\varphi(T).$$

Expanding the last relation we get

$$\gamma^2 + 2\gamma\varphi(T) + \varphi(T^2) = \gamma^2 + 2\gamma\varphi(T) + \varphi(T)^2,$$

which implies that $\varphi(T^2) = \varphi(T)^2$, as desired.

Suppose next that there exist $\gamma_1, \gamma_2 \in R^{\times}$ such that

$$\gamma_i + 1, \gamma_i + T, \gamma_i + T + 1, \gamma_1 - \gamma_2, \gamma_1 - \gamma_2 + 1, \gamma_1 - \gamma_2 + T, \gamma_1 - \gamma_2 + T + 1 \in R^{\times}$$

for all $i = 1, 2$. By our hypothesis we have that

$$\varphi(\gamma_1^2) = \varphi(\gamma_1)^2; \tag{3.3}$$

$$\varphi((\gamma_1 + T)^2) = \varphi(\gamma_1 + T)^2. \tag{3.4}$$

Expanding (3.4) we get

$$\varphi(\gamma_1^2) + \varphi(\gamma_1 \circ T) + \varphi(T^2) = \varphi(\gamma_1)^2 + \varphi(\gamma_1) \circ \varphi(T) + \varphi(T)^2. \tag{3.5}$$

Using (3.3) we get from (3.5) that

$$\varphi(\gamma_1 \circ T) + \varphi(T^2) = \varphi(\gamma_1) \circ \varphi(T) + \varphi(T)^2. \tag{3.6}$$

By our hypothesis we have that

$$\varphi(\gamma_2^2) = \varphi(\gamma_2)^2; \tag{3.7}$$
$$\varphi((\gamma_2 + T)^2) = \varphi(\gamma_2 + T)^2. \tag{3.8}$$

Expanding (3.8) we get

$$\varphi(\gamma_2^2) + \varphi(\gamma_2 \circ T) + \varphi(T^2) = \varphi(\gamma_2)^2 + \varphi(\gamma_2) \circ \varphi(T) + \varphi(T)^2. \tag{3.9}$$

Using (3.7) we get from (3.9) that

$$\varphi(\gamma_2 \circ T) + \varphi(T^2) = \varphi(\gamma_2) \circ \varphi(T) + \varphi(T)^2. \tag{3.10}$$

By our hypothesis we have that

$$\varphi((\gamma_1 - \gamma_2)^2) = \varphi((\gamma_1 - \gamma_2)^2; \tag{3.11}$$
$$\varphi(((\gamma_1 - \gamma_2 + T)^2) = \varphi(\gamma_1 - \gamma_2 + T)^2. \tag{3.12}$$

Expanding (3.12) we get

$$\varphi((\gamma_1 - \gamma_2)^2) + \varphi((\gamma_1 - \gamma_2) \circ T) + \varphi(T^2) = \varphi(\gamma_1 - \gamma_2)^2 + \varphi(\gamma_1 - \gamma_2) \circ \varphi(T) + \varphi(T)^2. \tag{3.13}$$

Using (3.11) we get from (3.13) that

$$\varphi((\gamma_1 - \gamma_2) \circ T) + \varphi(T^2) = \varphi(\gamma_1 - \gamma_2) \circ \varphi(T) + \varphi(T)^2. \tag{3.14}$$

Subtracting (3.6) from (3.10) we get

$$\varphi((\gamma_1 - \gamma_2) \circ T) = \varphi(\gamma_1 - \gamma_2) \circ \varphi(T). \tag{3.15}$$

It follows from both (3.14) and (3.15) that

$$\varphi(T^2) = \varphi(T)^2,$$

as desired.

Step 2. We claim that

$$\varphi(\alpha e_{ij} \circ \beta e_{kl}) = \varphi(\alpha e_{ij}) \circ \varphi(\beta e_{ij}) \tag{3.16}$$

for all $\alpha, \beta \in D$ and $1 \le i, j, k, l \le n$.

On one hand, we get from (3.2) that

$$\begin{aligned}\varphi((\alpha e_{ij} + \beta e_{kl})^2) &= \varphi((\alpha e_{ij})^2 + \alpha e_{ij} \circ \beta e_{kl} + (\beta e_{kl})^2) \\ &= \varphi((\alpha e_{ij})^2) + \varphi(\alpha e_{ij} \circ \beta e_{kl}) + \varphi((\beta e_{kl})^2) \\ &= \varphi(\alpha e_{ij})^2 + \varphi(\alpha e_{ij} \circ \beta e_{kl}) + \varphi(\beta e_{kl})^2.\end{aligned} \tag{3.17}$$

On the other hand, we get from (3.1) that

$$\begin{aligned}\varphi((\alpha e_{ij} + \beta e_{kl})^2) &= \varphi(\alpha e_{ij} + \beta e_{kl})^2 \\ &= (\varphi(\alpha e_{ij}) + \varphi(\beta e_{kl}))^2 \\ &= \varphi(\alpha e_{ij})^2 + \varphi(\alpha e_{ij}) \circ \varphi(\beta e_{kl}) + \varphi(\beta e_{kl})^2.\end{aligned} \tag{3.18}$$

Combining (3.17) with (3.18) we get

$$\varphi(\alpha e_{ij} \circ \beta e_{kl}) = \varphi(\alpha e_{ij}) \circ \varphi(\beta e_{ij})$$

for all $\alpha, \beta \in D$ and $1 \leq i, j, k, l \leq n$, as desired.

Step 3. We claim that $\varphi(x^2) = \varphi(x)^2$ for all $x \in R$.

For any

$$x = \sum_{1 \leq i,j \leq n} \alpha_{ij} e_{ij} \in R,$$

we get from both (3.2) and (3.16) that

$$\begin{aligned}
\varphi(x^2) &= \varphi\left(\left(\sum_{1\leq i,j\leq n} \alpha_{ij} e_{ij}\right)^2\right) \\
&= \varphi\left(\sum_{1\leq i,j\leq n} (\alpha_{ij} e_{ij})^2 + \sum_{\substack{1\leq i,j,k,l\leq n \\ (i,j)<(k,l)}} \alpha_{ij} e_{ij} \circ \alpha_{kl} e_{kl}\right) \\
&= \sum_{1\leq i,j\leq n} \varphi((\alpha_{ij} e_{ij})^2) + \sum_{\substack{1\leq i,j,k,l\leq n \\ (i,j)<(k,l)}} \varphi(\alpha_{ij} e_{ij} \circ \alpha_{kl} e_{kl}) \\
&= \sum_{1\leq i,j\leq n} \varphi(\alpha_{ij} e_{ij})^2 + \sum_{\substack{1\leq i,j,k,l\leq n \\ (i,j)<(k,l)}} \varphi(\alpha_{ij} e_{ij}) \circ \varphi(\alpha_{kl} e_{kl}) \\
&= \left(\sum_{1\leq i,j\leq n} \varphi(\alpha_{ij} e_{ij})\right)^2 \\
&= \varphi\left(\sum_{1\leq i,j\leq n} \alpha_{ij} e_{ij}\right)^2 \\
&= \varphi(x)^2.
\end{aligned}$$

In view of Step 3, we get that φ is a Jordan automorphism. Since char$(D) \neq 2$, we get from [13, Theorem 1] that φ is an automorphism or antiautomorphism. This proves the result. □

The following simple result will be used in the proof of our main result:

Lemma 3.1. *Let D be a division ring such that either char$(D) \neq 2, 3$ or D is not a field and char$(D) \neq 2$. Let $R = M_n(D)$, where $n > 1$. Let $f\colon R \to R$ be an additive map such that $f(x) = 0$ for all $x \in R^\times$ with $x + 1 \in R^\times$. Then $f = 0$.*

Proof. For any $\alpha \in D$ and $1 \leq i, j \leq n$, In view of both Lemmas 2.1 and 2.2 we get that there exists $\gamma \in R^\times$ such that

$$\begin{aligned}
&\gamma + 1 \in R^\times; \\
&\gamma + \alpha e_{ij} \in R^\times; \\
&\gamma + \alpha e_{ij} + 1 \in R^\times.
\end{aligned}$$

By our hypothesis we have

$$f(\gamma) = 0 \quad \text{and} \quad f(\gamma + \alpha e_{ij}) = 0.$$

Since f is additive, we get that $f(\alpha e_{ij}) = 0$. For any

$$x = \sum_{1 \leq i,j \leq n} \alpha_{ij} e_{ij} \in R,$$

we get that $f(x) = 0$, as desired. □

We are in a position to give the proof of our main result.

Proof of Theorem 1.4. For $x \in R^{\times}$ with $x + 1 \in R^{\times}$, we note that

$$(x^{-1} - (x+1)^{-1})^{-1} = x(x+1).$$

We set

$$y = x(x+1).$$

Since

$$y(y^{-1}k) = k,$$

we have that

$$f(y)g(y^{-1}k) = m.$$

It follows that

$$\begin{aligned} m &= f(y)g(y^{-1}k) \\ &= f(x(x+1))g((x^{-1} - (x+1)^{-1})k) \\ &= (f(x^2) + f(x))(g(x^{-1}k) - g((x+1)^{-1}k)) \\ &= f(x^2)g(x^{-1}k) - f(x^2)g((x+1)^{-1}k) + f(x)g(x^{-1}k) - f(x)g((x+1)^{-1}k). \end{aligned} \tag{3.19}$$

Note that

$$f(x)g(x^{-1}k) = m.$$

It follows from (3.19) that

$$0 = f(x^2)g(x^{-1}k) - f(x^2)g((x+1)^{-1}k) - f(x)g((x+1)^{-1}k). \tag{3.20}$$

For any $z \in R^{\times}$, we note that $z(z^{-1}k) = k$. This implies that

$$f(z)g(z^{-1}k) = m$$

and so

$$g(z^{-1}k) = f(z)^{-1}m.$$

We get from (3.20) that

$$0 = f(x^2)f(x)^{-1}m - f(x^2)f(x+1)^{-1}m - f(x)f(x+1)^{-1}m. \tag{3.21}$$

Multiplying (3.21) by $m^{-1}f(x+1)$ on the right hand side, we get

$$\begin{aligned}0 &= f(x^2)f(x)^{-1}f(x+1) - f(x^2) - f(x)\\ &= f(x^2)f(x)^{-1}f(x) + f(x^2)f(x)^{-1}f(1) - f(x^2) - f(x)\\ &= f(x^2) + f(x^2)f(x)^{-1}f(1) - f(x^2) - f(x)\\ &= f(x^2)f(x)^{-1}f(1) - f(x),\end{aligned}$$

which implies that

$$f(x^2) = f(x)f(1)^{-1}f(x) \tag{3.22}$$

for all $x \in R^\times$ with $x+1 \in R^\times$. It follows from (3.22) that

$$f(1)^{-1}f(x^2) = f(1)^{-1}f(x)f(1)^{-1}f(x) \tag{3.23}$$

for all $x \in R^\times$ with $x+1 \in R^\times$. We define

$$\varphi(x) = f(1)^{-1}f(x)$$

for all $x \in R$. Then,

$$f(x) = f(1)\varphi(x)$$

for all $x \in R$. It is clear that $\varphi(1) = 1$. The additivity of f immediately yields the additivity of φ. It follows from (3.23) that

$$\varphi(x^2) = \varphi(x)^2$$

for all $x \in R^\times$ with $x+1 \in R^\times$. In view of Proposition 3.1, we can conclude that φ is an automorphism or antiautomorphism.

For any $x \in R^\times$, since

$$(kx^{-1})x = k$$

we get that

$$f(kx^{-1})g(x) = m.$$

This implies that

$$\begin{aligned}g(x) &= f(kx^{-1})^{-1}m\\ &= (f(1)\varphi(kx^{-1}))^{-1}m\\ &= \varphi(kx^{-1})^{-1}f(1)^{-1}f(1)g(k)\\ &= \varphi(kx^{-1})^{-1}g(k)\\ &= \varphi(xk^{-1})g(k)\end{aligned}$$

for all $x \in R^\times$. In view of Lemma 3.1, we get that

$$g(x) = \varphi(xk^{-1})g(k) \tag{3.24}$$

for all $x \in R$. Suppose first that φ is an automorphism. We get from (3.24) that

$$\begin{aligned} g(x) &= \varphi(xk^{-1})g(k) \\ &= \varphi(x)\varphi(k^{-1})g(k) \\ &= \varphi(x)\varphi(k)^{-1}g(k) \\ &= \varphi(x)(f(1)^{-1}f(k))^{-1}g(k) \\ &= \varphi(x)f(k)^{-1}f(1)g(k) \\ &= \varphi(x)f(k)^{-1}m \\ &= \varphi(x)f(k)^{-1}f(k)g(1) \\ &= \varphi(x)g(1) \end{aligned}$$

for all $x \in R$. In particular, if $f = g$, we have that

$$f(1)\varphi(x) = \varphi(x)f(1)$$

for all $x \in R$. This implies that $f(1) \in Z$. Suppose next that φ is an antiautomorphism. We get from (3.24) that

$$\begin{aligned} g(x) &= \varphi(xk^{-1})g(k) \\ &= \varphi(k^{-1})\varphi(x)g(k) \\ &= \varphi(k)^{-1}\varphi(x)g(k) \\ &= (f(1)^{-1}f(k))^{-1}f(1)^{-1}f(x)g(k) \\ &= f(k)^{-1}f(x)g(k) \end{aligned}$$

for all $x \in R$. In particular, if $f = g$, we get that

$$f(x) = f(k)^{-1}f(x)f(k)$$

for all $x \in R$. This implies that

$$f(k)f(x) = f(x)f(k)$$

for all $x \in R$. Since f is a bijective map we obtain that $f(k) \in Z$. Note that $f(k)f(1) = m$, and so $f(1) = f(k)^{-1}m$. The proof of the result is complete. □

4. The proof of Theorem 1.5

As a consequence of Theorem 1.4 we give the proof of Theorem 1.5 as follows:

Proof of Theorem 1.5. In view of Theorem 1.4, we have that

$$f(x) = f(1)\varphi(x)$$

for all $x \in R$, where $\varphi\colon R \to R$ is either an automorphism or an antiautomorphism. Moreover, we have the following:

(1) If φ is an automorphism, then $f(x) = \varphi(x)f(1)$ for all $x \in R$.

(2) If φ is an antiautomorphism, then $f(x) = f(k)^{-1}f(x)f(k)$ for all $x \in R$.

Suppose first that φ is an automorphism. Since

$$f(x) = \varphi(x)f(1)$$

for all $x \in R$, we get that

$$f(1)\varphi(x) = f(x) = \varphi(x)f(1)$$

for all $x \in R$. Since φ is an automorphism, we get that $f(1) \in Z$, as desired.

Suppose next that φ is an antiautomorphism. Since

$$f(x) = f(k)^{-1}f(x)f(k)$$

for all $x \in R$, we get that

$$f(k)f(x) = f(x)f(k)$$

for all $x \in R$. Recall that f is a bijective map. We get from the last relation that $f(k) \in Z$. Since $k1 = k$, we have that

$$f(k)f(1) = m.$$

This implies that

$$f(1) = f(k)^{-1}m,$$

as desired. The proof of the result is complete. □

5. Conclusions

We give a complete description of maps preserving products of matrices over a division D such that either char(D) $\neq 2, 3$ or D is not a field and char(D) $\neq 2$, which gives a correct version of Theorem 1.3. The future study of this field is to give a complete description of maps preserving products of matrices over a division D with char(D) $\neq 2$.

Author contributions

Lan Lu: writing the draft of the manuscript. Yu Wang: correcting some errors in the proof of some results and writing the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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