Mathematics

Research article

Analytical and numerical analyses of a viscous strain gradient problem involving type II thermoelasticity

Noelia Bazarra[1]**, José R. Fernández**[1]**, Jaime E. Muñoz-Rivera**[2,3]**, Elena Ochoa**[4] **and Ramón Quintanilla**[5,*]

[1] Departamento de Matemática Aplicada I, Universidade de Vigo, Escola de Enxeñería de Telecomunicación, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain

[2] Departamento de Matemática, Universidad del Bío-Bío, Avenida Collao 1202, Casilla 5-Concepción, Chile

[3] National Laboratory for Scientific Computation, Petrópolis, Rio de Janeiro, Brazil

[4] Departamento de Matemáticas, Facultad de Ciencias Exactas, Universidad Andres Bello, Sede Concepción, Autopista Concepción-Talcahuano 7100, Talcahuano, Chile

[5] Departament de Matemàtiques, Escuela Superior de Ingenierías Industrial, Aeroespacial y Audiovisual de Terrassa, Universitat Politécnica de Catalunya, Colom 11, 08222 Terrassa, Barcelona, Spain

* **Correspondence:** Email: ramon.quintanilla@upc.edu; Tel: +34937398162; Fax: +34937398101.

Abstract: In this paper, a thermoelastic problem involving a viscous strain gradient beam is considered from the analytical and numerical points of view. The so-called type II thermal law is used to model the heat conduction and two possible dissipation mechanisms are introduced in the mechanical part, which is considered for the first time within strain gradient theory. An existence and uniqueness result is proved by using semigroup arguments, and the exponential energy decay is obtained. The lack of differentiability for the semigroup of contractions is also shown. Then, fully discrete approximations are introduced by using the finite element method and the backward time scheme, for which a discrete stability property and a priori error estimates are proved. The linear convergence is derived under suitable additional regularity conditions. Finally, some numerical simulations are presented to demonstrate the accuracy of the approximations and the behavior of the discrete energy decay.

1. Introduction

Many studies dealing with the time decay of solutions have been developed in the last fifty years. It is common to couple several conservative equations with other dissipative equations and try to clarify the way in which the solutions of the whole system decay (when they do it). Perhaps, it corresponds to classical thermoelasticity, where a system of conservative equations is coupled with a heat equation. It is known that, in this case, there could exist (for certain geometries of the domain) undamped solutions, but, generally, we could expect the decay (polynomial) [13]. For the one-dimensional case, the decay is of exponential type. Another typical example corresponds to the thermoelastic plate with a parabolic heat equation, which leads to the coupling of a conservative equation with a dissipative one (even in dimensions greater than one), and the solutions decay in an exponential way. Moreover, they are determined by using an analytic semigroup; therefore, they exhibit a strong regularity.

However, there is a family of problems of the kind considered at the beginning for which we could not find many studies. In fact, it is common is to find a conservative system that is determined by the mechanical part of the problem and a dissipative mechanism within the thermal part of the system, but very few studies have been focused on the problem in which the mechanical mechanism is dissipative (viscosity) and the thermal one is conservative (see [14, 16, 17]), as in the case of type II thermoelastic theory (see [6–11]). In this paper, we study a problem of this type.

We will consider the problem described by a one-dimensional viscous strain gradient elastic material with type II heat conduction. It is worth remarking that, as far as we know, this is the second contribution in the literature where the viscosity is described by using fourth-order spatial derivatives. In a recent contribution [21], a similar system was considered, but it is relevant to point out that, in the present contribution, we apply a different coupling among the equations describing the system and that the type of viscosity can be different. We believe that it is important to clarify the qualitative properties to describe the decay of the solutions to this new problem. In this paper, we will focus on a problem involving a strain gradient beam with type II heat conduction (without energy dissipation), assuming that the dissipative mechanisms act on the mechanical part.

It is worth noting that we can obtain the equations by following the methods in [12,20], but, in order to ensure that they are self-contained, we can recall that, in the one-dimensional case, the evolution equations are given by

$$\rho u_{tt} = t_x - S_{xx}, \quad \rho T_0 \dot{\eta} = q_x,$$

where ρ is the mass density, u represents the displacement, t is the stress tensor, S is the double stress tensor, T_0 is the reference temperature, η is the entropy, and $q = T_0 \Phi$, where Φ is the entropy flux vector.

In our case, the constitutive equations become as follows:

$$t = \mu u_x - \beta \theta + \mu^* u_{xt},$$
$$S = \kappa_1 u_{xxt} + m \alpha_x + \kappa_1^* u_{xxt},$$
$$\rho \eta = \beta u_x + c \theta,$$
$$\Phi = m u_{xx} + \kappa_2 \alpha_x.$$

In this system, u describes the displacement and α is the thermal displacement satisfying that $\dot{\alpha} = \theta$, where θ is the temperature.

If we now introduce the constitutive equations into the evolution equations, we obtain the following linear system:

$$\rho u_{tt} = \mu u_{xx} - \kappa_1 u_{xxxx} - \beta \theta_x - m\alpha_{xxx} - \kappa_1^* u_{txxxx} + \mu^* u_{txx}, \quad \text{in } (0,\pi) \times (0,\infty), \tag{1.1}$$

$$c\alpha_{tt} = -\beta u_{tx} + m u_{xxx} + \kappa_2 \alpha_{xx}, \qquad \text{in } (0,\pi) \times (0,\infty), \tag{1.2}$$

with the following boundary conditions:

$$u(0,t) = u(\pi,t) = 0, \quad u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \quad \forall t > 0,$$
$$\alpha_x(0,t) = \alpha_x(\pi,t) = 0, \quad \forall t > 0, \tag{1.3}$$

and the initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad t > 0,$$
$$\alpha(x,0) = \alpha_0(x), \quad \alpha_t(x,0) = \alpha_1(x), \quad x \in (0,\pi), \quad t > 0. \tag{1.4}$$

The system given by (1.1) and (1.2) describes the one-dimensional strain gradient viscoelasticity with type II heat conduction. As usual, ρ is the mass density, c is the heat capacity, μ is the elastic constant, κ_1 is related to the hyperelasticity, and κ_2 is a typical constant for the type II and III theories of Green-Naghdi, but we point out that it is not the thermal conductivity; also, β and m are two coupling terms and κ_1^* and μ^* describe the dissipation. In this work, we assume that $\rho, c, \mu, \kappa_1, \kappa_2$ are positive and the signs of m and β are not definite, but $\kappa_1 \kappa_2 > m^2$ and κ_1^*, μ^* must be non-negative, where

$$\rho > 0, \quad c > 0, \quad \mu > 0, \quad \kappa_2 > 0, \quad \kappa_1 > 0, \quad \mu^* \geq 0, \quad \kappa_1^* \geq 0, \quad \kappa_1 \kappa_2 > m^2. \tag{1.5}$$

The meaning of the assumptions regarding ρ and c is clear. The assumptions regarding κ_1, κ_2 and μ can be understood with the help of the thermoelastic stability, as well as the assumption regarding m, which also guarantees that the elastic energy is positive definite. The assumptions regarding κ_1^* and μ^* are also natural to ensure the existence of dissipation.

We know that the energy of the system is defined by

$$E(t) = \frac{1}{2} \int_0^\pi \left(\rho |u_t|^2 + \kappa_1 |u_{xx}|^2 + c |\alpha_t|^2 + \kappa_2 |\alpha_x|^2 + \mu |u_x|^2 + 2m\alpha_x u_{xx} \right) dx, \tag{1.6}$$

and it satisfies

$$\frac{d}{dt} E(t) = -\kappa_1^* \int_0^\pi |u_{txx}|^2 \, dx - \mu^* \int_0^\pi |u_{tx}|^2 \, dx. \tag{1.7}$$

In this paper, we will prove the existence and uniqueness of the solutions, the exponential decay, and the lack of regularity of the solutions, which implies, in particular, that the semigroup associated with the model is not analytic or differentiable. Moreover, we will also analyze this problem from the numerical point of view. It is worth noting that we also consider the case in which $\kappa_1^* = 0$, and we obtain similar qualitative properties for the solutions. Furthermore, the numerical simulations presented in Section 6 suggest that the most efficient dissipation mechanism requires that $\kappa_1^* = 0$ but $\mu^* > 0$. This is a remarkable fact, as one could think that the case in which the dissipation is stronger ($\kappa_1^* > 0, \mu^* > 0$) corresponds to the most efficient dissipative mechanism. Nevertheless, we point out that similar unexpected results have been obtained recently (see [1]). To the best of our knowledge, there

are no previous studies that deal with the so-called type II thermal law within strain gradient theory and with dissipative mechanisms that are imposed only on the mechanical subsystem.

The paper is structured as follows. The existence and uniqueness of solutions to the problem given by (1.1)–(1.4) are derived as shown in Section 2 by using semigroup arguments. In Section 3, the exponential energy decay is proved by applying the theory of linear semigroups. We also show that the semigroup associated with the solution to the thermoelastic problem in Section 4 is not differentiable, which implies the non-analyticity property (in particular, there is a lack of regularity). Then, a fully discrete approximation is introduced in Section 5 for a variational formulation by using the finite element method and the implicit Euler scheme. The discrete stability property and a priori error estimates are shown. Finally, some numerical simulations are presented in order to demonstrate the accuracy of the approximations, and a comparison is made among the solutions to the problems defined by the different dissipation mechanisms.

2. Existence and uniqueness of solutions

In this section, we study the existence of solutions to the problem given by (1.1)–(1.4). It is important to recall that the total energy was defined in (1.6) and it satisfies (1.7).

Denoting $u_t = v, \alpha_t = \theta$, and $U = (u, v, \alpha, \theta)$, we can define the phase space $\mathcal{H}$ as follows:

$$\mathcal{H} = H^2(0, \pi) \cap H_0^1(0, \pi) \times L^2(0, \pi) \times H_*^1(0, \pi) \times L_*^2(0, \pi),$$

where

$$L_*^2(0, \pi) = \left\{ f \in L^2(0, \pi); \int_0^\pi f \, dx = 0 \right\}, \quad H_*^1 = L_*^2(0, \pi) \cap H^1(0, \pi).$$

This space encompasses an inner product which defines the following norm:

$$\|U\|_{\mathcal{H}}^2 = \int_0^\pi \left(\rho |v|^2 + \kappa_1 |u_{xx}|^2 + c |\theta|^2 + \kappa_2 |\alpha_x|^2 + \mu |u_x|^2 + 2m\alpha_x u_{xx} \right) dx.$$

The previous assumptions imply that this norm is equivalent to the typical norm for the space.

Therefore, we can construct the operator $\mathcal{A}$ for a vector $U = (u, v, \alpha, \theta)$ as follows:

$$\mathcal{A}U = \begin{pmatrix} v \\ \dfrac{1}{\rho} \left(\mu u_{xx} - \kappa_1 u_{xxxx} - \beta \theta_x - m\alpha_{xxx} - \kappa_1^* v_{xxxx} + \mu^* v_{xx} \right) \\ \theta \\ \dfrac{1}{c} \left(-\beta v_x + m u_{xxx} + \kappa_2 \alpha_{xx} \right) \end{pmatrix}. \tag{2.1}$$

Let us denote by $\mathcal{A}_0$ the operator $\mathcal{A}$ with $\kappa_1^* = 0$. It is easy to verify that both are operators over the phase space $\mathcal{H}$. Therefore, the system given by (1.1)–(1.4) can be written as

$$U_t - \mathcal{A}U = 0, \quad U(0) = U_0, \tag{2.2}$$

or

$$U_t - \mathcal{A}_0 U = 0, \quad U(0) = U_0, \tag{2.3}$$

when $\kappa_1^* > 0$ or $\kappa_1^* = 0$, respectively. Here, we have denoted $U_0 = (u_0, u_1, \alpha_0, \alpha_1)$.

The corresponding domain is given by

$$
\begin{aligned}
D(\mathcal{A}) = \{U = (u, v, \alpha, \theta) \in \mathcal{H}; \; &u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \\
&v \in H^2(0, \pi) \cap H_0^1(0, \pi), \quad \theta \in H_*^1(0, \pi), \\
&\kappa_1^* v_{xx} + \kappa_1 u_{xx} + m\alpha_x \in H^2(0, \pi), \quad mu_{xxx} + \kappa_2 \alpha_{xx} \in L_*^2(0, \pi)\}.
\end{aligned}
\tag{2.4}
$$

Note that, if $\kappa_1^* = 0$, then it follows that $\kappa_1 u_{xxx} + m\alpha_{xx} = g_1 \in H^1(0, \pi)$ and $mu_{xxx} + \kappa_2 \alpha_{xx} = g_2 \in L_*^2(0, \pi)$. By condition (1.5), we obtain that $\kappa_1 \kappa_2 > m^2$; so, we can solve the two above equations for u_{xxx} and α_{xx}, concluding that $u_{xxx}, \alpha_{xx} \in L^2$. Hence, $D(\mathcal{A}_0)$ is given by

$$
\begin{aligned}
D(\mathcal{A}_0) = \{U = (u, v, \alpha, \theta) \in \mathcal{H}; \; &u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \\
&v \in H^2(0, \pi) \cap H_0^1(0, \pi), \quad u_{xxx} \in L^2(0, \pi), \quad \kappa_1 u_{xxxx} + m\alpha_{xxx} \in L^2(0, \pi), \\
&\theta \in H_*^1(0, \pi), \quad \alpha_{xx} \in L_*^2(0, \pi)\}.
\end{aligned}
\tag{2.5}
$$

Furthermore, we see that $\mathcal{A}$ and $\mathcal{A}_0$ are dissipative operators verifying that

$$
\text{Re} \, (\mathcal{A}U, U) = -\kappa_1^* \int_0^\pi |v_{xx}|^2 \, dx - \mu^* \int_0^\pi |v_x|^2 \, dx \le 0.
\tag{2.6}
$$

$$
\text{Re} \, (\mathcal{A}_0 U, U) = -\mu^* \int_0^\pi |v_x|^2 \, dx \le 0.
\tag{2.7}
$$

Using the resolvent equation $\lambda U - \mathcal{A}U = F$, we have that

$$
\kappa_1^* \int_0^\pi |v_{xx}|^2 \, dx + \mu^* \int_0^\pi |v_x|^2 \, dx = \text{Re} \, (U, F)_{\mathcal{H}},
\tag{2.8}
$$

$$
\mu^* \int_0^\pi |v_x|^2 \, dx = \text{Re} \, (U, F)_{\mathcal{H}}.
\tag{2.9}
$$

Theorem 2.1. *Let $\mathbf{A}$ be dissipative with $0 \in \varrho(\mathbf{A})$. If $\mathcal{H}$ is reflexive, then $\mathbf{A}$ is the infinitesimal generator of a semigroup of contractions.*

Proof. Since $\varrho(\mathbf{A})$ is an open set, we have that there exists $\epsilon > 0$ such that $\epsilon \in \varrho(\mathbf{A})$. This implies that any $\lambda > 0$ belongs to $\varrho(\mathbf{A})$. In particular, we find that $R(I - \mathbf{A}) = \mathcal{H}$. Using Theorem 4.6 of [18], we conclude that $\overline{D(\mathbf{A})} = \mathcal{H}$; also, by applying the Lummer-Phillips theorem [18, Theorem 1.4.3], our conclusion follows. $\qquad\square$

The method we use to show the exponential stability, as well as the the lack of differentiability, is based on the study of the resolvent system $i\lambda U - AU = F$, which, in terms of the components, is written in the following form:

$$
i\lambda u - v = f_1 \quad \text{in} \quad H^2,
\tag{2.10}
$$

$$
i\lambda \rho v - \mu u_{xx} + \kappa_1 u_{xxxx} + \beta \theta_x + m\alpha_{xxx} + \kappa_1^* v_{xxxx} - \mu^* v_{xx} = f_2 \quad \text{in} \quad L^2,
\tag{2.11}
$$

$$
i\lambda \alpha - \theta = f_3 \quad \text{in} \quad H_*^1,
\tag{2.12}
$$

$$
i\lambda c\theta + \beta v_x - mu_{xxx} - \kappa_2 \alpha_{xx} = f_4 \quad \text{in} \quad L_*^2.
\tag{2.13}
$$

Theorem 2.2. *The operators $\mathcal{A}$ and $\mathcal{A}_0$ are the infinitesimal generators of a C_0 semigroup of contractions given by $(S(t))_{t \geq 0}$ on $\mathcal{H}$. In particular, this implies the existence, uniqueness, and continuous dependence of solutions to the Cauchy problem (2.2), i.e., for any initial data $U_0 \in \mathcal{H}$, there exists only one solution U verifying that*

$$U \in C([0, \infty); \mathcal{H}).$$

Denoting by $\widetilde{\mathcal{A}}$ either the operator $\mathcal{A}$ or $\mathcal{A}_0$, we have that, if $U_0 \in D(\widetilde{\mathcal{A}})$, then the solution has the following regularity:

$$U \in C^1([0, \infty); \mathcal{H}) \cap C([0, \infty); D(\widetilde{\mathcal{A}})).$$

Proof. We will prove it only for the operator $\mathcal{A}$ because the proof for the operator $\mathcal{A}_0$ is identical, taking $\kappa_1^* = 0$. Since $\mathcal{A}$ is dissipative, it is enough to show that $0 \in \rho(\mathcal{A})$ since, by using Theorem 2.1, our result follows (see also [15]). Indeed, we show that, for any $F \in \mathcal{H}$, there exists only one $U \in D(\mathcal{A})$ verifying that $AU = F$. Let us take $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$. We must find u, v, α, and θ that satisfy (2.10)–(2.13) for $\lambda = 0$. Thus, we have that $v = f_1$ and $\theta = f_3$, and, therefore, our problem reduces finding u and α such that

$$-\mu u_{xx} + \kappa_1 u_{xxxx} + m\alpha_{xxx} = f_2 - \beta f_{3,x} - \kappa_1^* f_{1,xxxx} + \mu^* f_{1,xx} \quad \text{in} \quad (H^2 \cap H_0^1)^*, \tag{2.14}$$

$$-m u_{xxx} - \kappa_2 \alpha_{xx} = f_4 - \beta f_{1,x} \quad \text{in} \quad L_*^2. \tag{2.15}$$

We define the bilinear operator $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ over the space $V = H^2(0, \pi) \cap H_0^1(0, \pi) \times H_*^1(0, \pi)$ as follows:

$$a\left(\begin{pmatrix} u \\ \alpha \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right) = \int_0^\pi \mu u_x \phi_x + \kappa_1 u_{xx} \phi_{xx} + m\alpha_x \phi_{xx} + m u_{xx} \psi_x + \kappa \alpha_x \psi_x dx,$$

and the linear form given by

$$T\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \int_0^\pi (f_2 - \beta f_{3,x} + \mu^* f_{1,xx}) \phi - \kappa_1^* f_{1,xx} \phi_{xx} + (f_4 - \beta f_{1,x}) \psi dx.$$

It is easy to see that $T \in V^*$ and $a(\cdot, \cdot)$ is bilinear, continuous, and coercive (that is, because of conditions (1.5)). Hence, the Lax-Milgran lemma guarantees the existence and uniqueness of a solution:

$$a\left(\begin{pmatrix} u \\ \alpha \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right) = T\begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad \forall \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in V.$$

Taking $\phi \in C_0^\infty(0, \pi)$ and $\psi = 0$, we get

$$\int_0^\pi \mu u_x \phi_x + \kappa_1 u_{xx} \phi_{xx} + m\alpha_x \phi_{xx} dx = \int_0^\pi (f_2 - \beta f_{3,x} + \mu^* f_{1,xx}) \phi - \kappa_1^* f_{1,xx} \phi_{xx} dx, \quad \forall \phi \in C_0^\infty(0, \pi).$$

This variational identity implies that u satisfies (2.14) in the distributional sense. Moreover, since $\theta = f_3 \in H_*^1$, we have that $\kappa_1 u_{xxxx} + m\alpha_{xxx} + \kappa_1^* f_{1,xxxx} = \kappa_1 u_{xxxx} + m\alpha_{xxx} + \kappa_1^* v_{xxxx} \in L^2$ or $\kappa_1 u_{xx} + m\alpha_x + \kappa_1^* v_{xx} \in H^2$. Similarly, taking $\psi \in C_0^\infty(0, \pi)$ and $\phi = 0$, we get

$$\int_0^\pi m u_{xx} \psi_x + \kappa \alpha_x \psi_x dx = \int_0^\pi -\kappa_1^* f_{1,xx} \phi_{xx} + (f_4 - \beta f_{1,x}) \psi dx.$$

This variational identity ensures that α satisfies (2.15) in the distributional sense. Moreover, we also find that $-mu_{xxx} - \kappa_2\alpha_{xx} \in L^2$; so, the solution $(u, v, \alpha, \theta) \in D(\mathcal{A})$. Then, we have that $0 \in \rho(\mathcal{A})$, and, by using Theorem 2.1, our result follows. Taking $\kappa_1^* = 0$ in the above procedure, we conclude that $0 \in \rho(\mathcal{A}_0)$; so, $\mathcal{A}_0$ is also an infinitesimal generator of a C_0 semigroup of contractions. $\qquad\square$

Despite the similarities between the operators $\mathcal{A}$ and $\mathcal{A}_0$, there is a big difference from the topological point of view, as we can see in the following lemma.

Lemma 2.1. *The family of resolvent operators of $\mathcal{A}$ is not compact; however, the family of resolvent operators of $\mathcal{A}_0$ ($\mu^* = 0$) is a family of compact operators.*

Proof. First, we note that the resolvent operator $(\lambda I - \mathcal{A})^{-1} : \mathcal{H} \to \mathcal{H}$ is compact if and only if $D(\mathcal{A})$ has a compact embedding in the phase space $\mathcal{H}$. In fact, let us suppose that the sequence $F_n \in \mathcal{H}$ is bounded. So, the sequence $U_n = (\lambda I - \mathcal{A})^{-1} F_n \in D(\mathcal{A})$ is bounded in $D(\mathcal{A})$. Because of the compact embedding, we conclude that there exists a subsequence of U_n that converges strongly in $\mathcal{H}$.

Let us denote by $\mathcal{A}_*$ the operator $\mathcal{A}$ with $\mu^* = 0$. Since $D(\mathcal{A}) = D(\mathcal{A}_*)$, we conclude that the resolvent operator $(\lambda I - \mathcal{A})^{-1}$ is compact if and only if $(\lambda I - \mathcal{A}_*)^{-1}$ is also compact. Hence, in order to show the lack of compactness of the resolvent operators $(\lambda I - \mathcal{A})^{-1}$, we can suppose that $\mu^* = 0$.

We will show that there exists $\nu \in \mathbb{R}$, which is not an eigenvalue or an element of the resolvent set of $\mathcal{A}$. In fact, let us consider the following resolvent system for $\nu \in \mathbb{C}$:

$$\nu u - v = f_1 \quad \text{in} \quad H^2, \tag{2.16}$$

$$\nu\rho v - \mu u_{xx} + \kappa_1 u_{xxxx} + \beta\theta_x + m\alpha_{xxx} + \kappa_1^* v_{xxxx} = f_2 \quad \text{in} \quad L^2, \tag{2.17}$$

$$\nu\alpha - \theta = f_3 \quad \text{in} \quad H_*^1, \tag{2.18}$$

$$\nu c\theta + \beta v_x - mu_{xxx} - \kappa_2\alpha_{xx} = f_4 \quad \text{in} \quad L_*^2. \tag{2.19}$$

Note that

$$
\begin{aligned}
\kappa_1 u_{xxxx} + \kappa_1^* v_{xxxx} + m\alpha_{xxx} &= \frac{m}{\kappa_2}\left[\frac{\kappa_1\kappa_2}{m}u_{xxxx} + \frac{\kappa_1^*\kappa_2}{m}v_{xxxx} + \kappa_2\alpha_{xxx}\right] \\
&= \frac{m}{\kappa_2}\left[(\frac{\kappa_1\kappa_2}{m} + \frac{\kappa_1^*\kappa_2}{m}\nu)u_{xxxx} + \kappa_2\alpha_{xxx} - \frac{\kappa_1^*\kappa_2}{m}f_{1,xxxx}\right].
\end{aligned}
$$

Taking ν such that $\dfrac{\kappa_1\kappa_2}{m} + \dfrac{\kappa_1^*\kappa_2}{m}\nu = m$, we get

$$\kappa_1 u_{xxxx} + \kappa_1^* v_{xxxx} + m\alpha_{xxx} = \frac{m}{\kappa_2}\left[mu_{xxxx} + \kappa_2\alpha_{xxx} - \frac{\kappa_1^*\kappa_2}{m}f_{1,xxxx}\right]. \tag{2.20}$$

We will show that ν is neither an element of the resolvent set of $\mathcal{A}$ nor an eigenvalue of $\mathcal{A}$. In fact, on the contrary, let us suppose that $\nu \in \varrho(A)$; then, the solution to the resolvent system must satisfy

$$mu_{xxx} + \kappa_2\alpha_{xx} \in L^2(0, \pi) \quad \Rightarrow \quad mu_{xxxx} + \kappa_2\alpha_{xxx} \in H^{-1}(0, \pi).$$

Taking f_1 such that $f_1 \in H^2(0, \pi)$ but $f_1 \notin H^3(0, \pi)$, we get that the right hand side of (2.20) belongs to $H^{-2}(0, \pi)$, which is a contradiction because $\kappa_1 u_{xxxx} + \kappa_1^* v_{xxxx} + m\alpha_{xxx}$ must be in $L^2(0, \pi)$.

Finally, we will show that ν is not an eigenvalue. In fact, multiplying (2.17) by u and (2.19) by α and recalling that $f_1 = f_2 = f_3 = f_4 = 0$, we get

$$\int_0^L \nu^2\rho|u|^2 + \mu|u_x|^2 + \kappa_1|u_{xx}|^2 + \nu^2 c|\alpha|^2 + \kappa_2\,|\alpha_x|^2 + 2m\alpha_x u_{xx} + \kappa_1^*\nu|u_{xx}|^2 \, dx = 0.$$

Recalling the definition of ν, we have

$$\int_0^L \nu^2\rho|u|^2 + \mu|u_x|^2 + \frac{m^2}{\kappa_2}|u_{xx}|^2 + \nu^2 c|\alpha|^2 + \kappa_2\,|\alpha_x|^2 + 2m\alpha_x u_{xx} \, dx = 0,$$

which implies that

$$\int_0^L \nu^2\rho|u|^2 + \mu|u_x|^2 + (\frac{m}{\sqrt{\kappa_2}}u_{xx} + \sqrt{\kappa_2}\alpha_x)^2 + \nu^2 c|\alpha|^2 \, dx = 0.$$

Therefore, we conclude that $U = (u, v, \alpha, \theta)^\top = 0$; so, ν is not an eigenvalue. This implies that the family of the resolvent operators of $\mathcal{A}$ is not compact.

Finally, it is not difficult to see that $D(\mathcal{A}_0)$ has a compact embedding over the phase space $\mathcal{H}$. In fact, let us suppose that a sequence F_n is bounded in $\mathcal{H}$. Then, $(i\lambda I - \mathcal{A}_0)^{-1}F_n = U_n = (u_n, v_n, \alpha_n, \theta_n)$ is bounded in $D(\mathcal{A}_0)$. In particular, we have that $(u_n, v_n, \alpha_n, \theta_n)$ is bounded in $H^3 \times H^1 \times H^2 \times H^1 \subset \mathcal{H}$ with a compact embedding. Thus, there exists a subsequence of U_n that can be denoted in the same way, and it converges strongly in $\mathcal{H}$. Hence, the resolvent operator $(\lambda I - \mathcal{A}_0)^{-1}$ is compact. The proof is now complete. □

3. Exponential decay

In this section, we will prove that the solutions to the problems studied in the previous section decay in an exponential way. To this end, we assume that $m \neq 0$. The main tool we use is the following result according to Prüss [19].

Theorem 3.1. *Let $S(t)$ be a contraction C_0 semigroup, generated by $\mathbb{A}$ over a Hilbert space $\mathcal{H}$. Then, there exists $C, \gamma > 0$ verifying that*

$$\|S(t)\| \leq Ce^{-\gamma t} \quad \Leftrightarrow \quad i\mathbb{R} \subset \varrho(\mathbb{A}) \text{ and } \|(i\lambda I - \mathbb{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant M, \quad \forall \lambda \in \mathbb{R}. \tag{3.1}$$

Lemma 3.1. *The operator $\mathcal{A}\,(\mathcal{A}_0)$ defined by (2.1) verifies that $i\mathbb{R} \subset \varrho(\mathcal{A})\,(i\mathbb{R} \subset \varrho(\mathcal{A}_0))$.*

Proof. In the case that $\mu^* = 0$, from Lemma 2.1, the operator $\mathcal{A}_0$ has a compact family of resolvent operators. So, in order to show that $i\mathbb{R} \subset \varrho(\mathcal{A}_0)$, it is enough to prove that there are no imaginary eigenvalues. We will proceed by contradiction. Let us suppose that there exists $U \neq 0$ such that $\mathcal{A}_0 U = i\lambda U$. Since $F = 0$, the relation (2.9) yields that $v_x = 0$, and, by using (2.10), we get that $u = v = 0$. From (2.11), we get that

$$i\lambda\beta\alpha_x + m\alpha_{xxx} = 0 \quad \Rightarrow \quad \int_0^\pi i\lambda\beta|\alpha_x|^2 - m|\alpha_{xx}|^2 dx = 0.$$

Taking the real and imaginary parts in the implication above yields that $\alpha = 0$, and, by using (2.12), we get that $\theta = 0$. Therefore, we conclude that $U = 0$, which is a contradiction.

Finally, we consider the case in which $\mu^* > 0$. Since $0 \in \varrho(\mathcal{A})$, the set

$$\mathcal{N} = \{s \in \mathbb{R}^+ \,; \, (-is, is) \subset \rho(\mathcal{A})\}$$

is not empty. Setting $\sigma = \sup \mathcal{N}$, if $\sigma = +\infty$, there is nothing to prove. By contradiction, if we suppose that $\sigma < \infty$, then there exist a sequence $\lambda_n \in \mathbb{R}$ such that $\lambda_n \to \sigma < \infty$ and a sequence of unit norm vectors $\{\widetilde{F}_n\}_n$ of elements of $\mathcal{H}$ such that

$$\|\widetilde{F}_n\|_{\mathcal{H}} = 1, \text{ and } \|\underbrace{(i\lambda_n I - \mathcal{A})^{-1}\widetilde{F}_n}_{:=W_n}\|_{\mathcal{H}} \underset{n\to\infty}{\longrightarrow} +\infty.$$

Letting $U_n = \frac{W_n}{\|W_n\|_{\mathcal{H}}}$ and $F_n = \frac{\widetilde{F}_n}{\|W_n\|_{\mathcal{H}}}$, we have

$$\|U_n\|_{\mathcal{H}} = 1, \text{ and } (i\lambda_n I - \mathcal{A})U_n = F_n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } \mathcal{H}. \tag{3.2}$$

Note that the sequence $\mathcal{A}U_n$ is bounded in $\mathcal{H}$. From (2.8) and (2.10), we get

$$(u_n, v_n) \to (u, v) = (0, 0) \quad \text{strongly in} \quad H^2(0, \pi) \times H^2(0, \pi).$$

On the other hand, applying (2.11) with the sequence $(u_n, v_n, \alpha_n, \theta_n)^\top$ and multiplying the result by $\int_0^x \overline{\alpha_n} d\tau$, we find that

$$\int_0^\pi i\lambda_n \rho v_n \left(\int_0^x \overline{\alpha_n}\right) dx + \int_0^\pi \mu u_{nx}\overline{\alpha}_n + (\kappa_1 u_{nxx} + \kappa_1^* v_{nxx})\overline{\alpha}_{nx} dx$$

$$- \int_0^\pi \beta \theta_n \overline{\alpha}_n dx + \int_0^\pi m|\alpha_{nx}|^2 + \int_0^\pi \mu^* v_{nx}\overline{\alpha}_n dx \to 0. \tag{3.3}$$

Since

$$\int_0^\pi \beta \theta_n \overline{\alpha}_n dx = \int_0^\pi i\lambda_n \beta |\alpha_n|^2 dx + \int_0^\pi \beta f_{3n}\overline{\alpha}_n dx,$$

in view of the strong convergence of u_n and v_n, we conclude that

$$\int_0^\pi i\lambda_n \beta |\alpha_n|^2 dx + \int_0^\pi m|\alpha_{nx}|^2 dx \to 0. \tag{3.4}$$

From (3.3) and (3.4), it follows that U_n converges strongly to 0. This contradicts the fact that $\|U_n\|_{\mathcal{H}} = 1$ for all $n \in \mathbb{N}$.

The proof is now complete. $\qquad\square$

Theorem 3.2. *The semigroup generated by the operators $\mathcal{A}$ and $\mathcal{A}_0$ defined by* (2.1) *are exponentially stable; that is, there exist two positive constants M and ω such that*

$$\|U(t)\|_{\mathcal{H}} \le Me^{-\omega t} \|U(0)\|_{\mathcal{H}}.$$

Proof. Thanks to Theorem 3.1 and Lemma 3.1, we only need to show that

$$\left\|(i\lambda I - \mathcal{A})^{-1}\right\| \le C, \quad \forall \lambda \in \mathbb{R}. \tag{3.5}$$

Using inequality (2.9), we get

$$\int_0^\pi |v|^2\, dx \le c \int_0^\pi |v_x|^2\, dx \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \qquad \int_0^\pi |u_x|^2\, dx \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2. \qquad (3.6)$$

Multiplying (2.11) by $\overline{u}$, we have

$$\int_0^\pi i\lambda\rho v\overline{u}dx + \int_0^\pi \mu|u_x|^2 + \kappa_1|u_{xx}|^2 dx - \int_0^\pi \beta\theta_x\overline{u}dx + \int_0^\pi m\alpha_x\overline{u}_{xx} + \int_0^\pi \mu^* v_x\overline{u}_x dx = \int_0^\pi f_2\overline{u}dx.$$

Using (2.10) and (2.12), we obtain

$$-\int_0^\pi \rho|v|^2 dx + \int_0^\pi \mu|u_x|^2 + \kappa_1|u_{xx}|^2 dx + \int_0^\pi \beta\theta\overline{u}_x dx + \int_0^\pi m\alpha_x\overline{u}_{xx} + \int_0^\pi \mu^* v_x\overline{u}_x dx$$
$$= \int_0^\pi f_2\overline{u} + \rho v\overline{f_1}dx,$$

which gives

$$\kappa_1 \int_0^\pi |u_{xx}|^2 dx + m \int_0^\pi \alpha_x\overline{u}_{xx} = \mathfrak{R}_1, \qquad (3.7)$$

where

$$\mathfrak{R}_1 = \underbrace{\int_0^\pi \rho|v|^2 dx - \int_0^\pi \mu|u_x|^2 dx - \int_0^\pi \mu^* v_x\overline{u}_x dx + \int_0^\pi f_2\overline{u} + \rho v\overline{f_1}dx}_{\le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2} - \int_0^\pi \beta\theta\overline{u}_x dx.$$

Using relation (3.6), we get

$$|\mathfrak{R}_1| \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2 + \epsilon \int_0^\pi |\theta|^2 dx.$$

On the other hand, multiplying (2.19) by $\overline{u}_x$, we find that

$$\int_0^\pi (i\lambda c\theta + \beta v_x)\overline{u}_x + \int_0^\pi m|u_{xx}|^2 + \int_0^\pi \kappa_2\alpha_x\overline{u}_{xx} = \int_0^\pi f_4\overline{u}_x.$$

Then, we have

$$m \int_0^\pi |u_{xx}|^2 + \kappa_2 \int_0^\pi \alpha_x\overline{u}_{xx} = \mathfrak{R}_2, \qquad (3.8)$$

where

$$\mathfrak{R}_2 = -\int_0^\pi (i\lambda c\theta + \beta v_x)\overline{u}_x + \int_0^\pi f_4\overline{u}_x.$$

Using the above procedure, we also conclude that

$$|\mathfrak{R}_2| \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2 + \epsilon \int_0^\pi |\theta|^2 dx.$$

Solving (3.7) and (3.8) for $\int_0^\pi |u_{xx}|^2$ and $\int_0^\pi \alpha_x \overline{u}_{xx}$, we find that

$$\int_0^\pi m|u_{xx}|^2 dx \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \epsilon \int_0^\pi |\theta|^2 dx, \quad \text{and} \quad \left|\int_0^\pi \alpha_x \overline{u}_{xx} dx\right| \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \epsilon \int_0^\pi |\theta|^2 dx. \quad (3.9)$$

Multiplying (2.11) by $\int_0^x \overline{\alpha}$, we get

$$\int_0^\pi i\lambda\rho v \left(\int_0^x \overline{\alpha}\right) dx + \int_0^\pi \mu u_x \overline{\alpha} + \kappa_1 u_{xx}\overline{\alpha}_x dx - \int_0^\pi \beta\theta\overline{\alpha} dx + \int_0^\pi m|\alpha_x|^2 + \int_0^\pi \mu^* v_x \overline{\alpha} dx$$
$$= \int_0^\pi f_2 \left(\int_0^x \overline{\alpha}\right) dx.$$

Using (2.12), we obtain

$$\int_0^\pi i\lambda\rho v \left(\int_0^x \overline{\alpha}\right) dx + \int_0^\pi \mu u_x \overline{\alpha} + \kappa_1 u_{xx}\overline{\alpha}_x dx - \int_0^\pi i\lambda\beta|\alpha|^2 dx + \int_0^\pi m|\alpha_x|^2 + \int_0^\pi \mu^* v_x \overline{\alpha} dx$$
$$= \int_0^\pi f_2 \left(\int_0^x \overline{\alpha}\right) dx. \quad (3.10)$$

Therefore, we have

$$-\int_0^\pi i\lambda\beta|\alpha|^2 dx + \int_0^\pi m|\alpha_x|^2 = \Re_3, \quad (3.11)$$

where

$$\Re_3 = -\int_0^\pi i\lambda\rho v \left(\int_0^x \overline{\alpha}\right) dx - \int_0^\pi \mu u_x \overline{\alpha} + \kappa_1 u_{xx}\overline{\alpha}_x dx - \int_0^\pi \mu^* v_x \overline{\alpha} dx + \int_0^\pi f_2 \left(\int_0^x \overline{\alpha}\right) dx.$$

Keeping in mind (3.6) and (2.12), we conclude that

$$|\Re_3| \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2 + \epsilon \int_0^\pi |\theta|^2 dx. \quad (3.12)$$

Hence, taking the real and imaginary parts in (3.11), we get

$$\int_0^\pi |\lambda||\alpha|^2 dx + \int_0^\pi |\alpha_x|^2 \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2 + \epsilon \int_0^\pi |\theta|^2 dx. \quad (3.13)$$

Multiplying (2.13) by $\overline{\alpha}$ and performing an integration by parts, we have

$$\int_0^\pi i\lambda c\theta\overline{\alpha} - \beta v\overline{\alpha}_x + m u_{xx}\overline{\alpha}_x + \kappa_2|\alpha_x|^2 dx = \int_0^\pi f_4 \overline{\alpha} dx.$$

Using (2.12), we find that

$$\int_0^\pi c|\theta|^2 = \int_0^\pi -c\theta f_3 - \beta v\overline{\alpha}_x + m u_{xx}\overline{\alpha}_x + \kappa_2|\alpha_x|^2 dx - \int_0^\pi f_4 \overline{\alpha} dx, \quad (3.14)$$

and, taking into account (3.9) and (3.12), we get

$$\int_0^\pi |\theta|^2 \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2 + \epsilon \int_0^\pi |\theta|^2 dx.$$

Therefore, we obtain

$$\|U\|_{\mathcal{H}}^2 \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2 + \epsilon\|U\|_{\mathcal{H}}^2,$$

which implies that $\|U\|_{\mathcal{H}} \le c\|F\|_{\mathcal{H}}$. Therefore, the exponential stability follows.

Finally, in the case in which $\mu^* > 0$, relation (2.8) implies that

$$\int_0^\pi |v|^2 \, dx \le c \int_0^\pi |v_x|^2 + |v_{xx}|^2 \, dx \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \qquad \int_0^\pi |u_{xx}|^2 \, dx \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2. \quad (3.15)$$

Multiplying (2.11) by $\int_0^x \overline{\alpha}$, and by using the same procedure as that for inequality (3.10), we get

$$\int_0^\pi i\lambda\rho v \left(\int_0^x \overline{\alpha}\right) dx + \int_0^\pi \mu u_x \overline{\alpha} + (\kappa_1 u_{xx} + \kappa_1^* v_{xx})\overline{\alpha}_x dx - \int_0^\pi i\lambda\beta|\alpha|^2 dx + \int_0^\pi m|\alpha_x|^2 + \int_0^\pi \mu^* v_x \overline{\alpha} dx$$

$$= \int_0^\pi f_2 \left(\int_0^x \overline{\alpha}\right) dx.$$

Then, we have

$$-\int_0^\pi i\lambda\beta|\alpha|^2 dx + \int_0^\pi m|\alpha_x|^2 = \Re_4, \qquad (3.16)$$

where

$$\Re_4 = -\int_0^\pi i\lambda\rho v \left(\int_0^x \overline{\alpha}\right) dx - \int_0^\pi \mu u_x \overline{\alpha} + (\kappa_1 u_{xx} + \kappa_1^* v_{xx})\overline{\alpha}_x dx - \int_0^\pi \mu^* v_x \overline{\alpha} dx + \int_0^\pi f_2 \left(\int_0^x \overline{\alpha}\right) dx.$$

Using (3.15) and (2.12), we conclude that

$$|\Re_4| \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2 + \epsilon \int_0^\pi |\theta|^2 dx. \qquad (3.17)$$

Hence, taking the real and imaginary parts in (3.12), we get

$$\int_0^\pi |\lambda||\alpha|^2 dx + \int_0^\pi |\alpha_x|^2 \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2 + \epsilon \int_0^\pi |\theta|^2 dx. \qquad (3.18)$$

Finally, using (3.14), we obtain

$$\int_0^\pi |\theta|^2 \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2 + \epsilon \int_0^\pi |\theta|^2 dx.$$

Therefore, we conclude that

$$\|U\|_{\mathcal{H}}^2 \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|_{\mathcal{H}}^2 + \epsilon\|U\|_{\mathcal{H}}^2.$$

It implies that $\|U\|_{\mathcal{H}} \le c\|F\|_{\mathcal{H}}$; so, the proof is now complete.

Remark 3.1. *We have proved the existence and the exponential stability of the solutions to the problem given by* (1.1)–(1.4). *In this case, we have assumed homogeneous Neumann boundary conditions on the thermal displacement. It is clear that, using the standard modifications, we can obtain the same kind of results when we consider homogeneous Dirichlet boundary conditions on the thermal displacement. In fact, in Sections 5 and 6, we obtain similar results when these boundary conditions are imposed, but from a numerical point of view.*

4. Lack of differentiability of the semigroup

In this section, we will show that the semigroup associated with the system given by (1.1)–(1.4) is not differentiable [18] (and not immediately differentiable [5]). To see this, we recall the following results.

Theorem 4.1. *Let $S = (S(t))_{t \geq 0}$ be an immediately differentiable semigroup on the Banach space X; then, $S(t)$ is an immediately norm-continuous semigroup (see [5], Definition 4.17, page 112).*

Proof. If $S(t)$ is immediately differentiable, then $S(t)$ is immediately differentiable with the uniform norm of $\mathcal{L}(X)$ for any $t > 0$. This implies that the semigroup is immediately norm-continuous. □

Theorem 4.2. *If $\mathcal{A}$ is the generator of an immediately norm-continuous exponentially stable semigroup, then*

$$\lim_{\lambda \to \pm \infty} \|(i\lambda I - \mathcal{A})^{-1}\| = 0.$$

Proof. See [5], Corollary 4.19, page 114. □

Theorem 4.3. *The semigroup $S = (S(t))_{t \geq 0}$ defined by the system given by (1.1)–(1.4) is not differentiable.*

Proof. From Theorems 4.1 and 4.2, it is enough to show that there exist a sequence $(\lambda_n)_n$ of real numbers and a bounded sequence $(F_n)_n$ in $\mathcal{H}$ with $\lim_{n \to \infty} \lambda_n = \infty$ such that

$$\lim_{n \to \infty} \left\| (i\lambda_n I - \mathcal{A})^{-1} \right\| > 0.$$

In fact, for each $n \in \mathbb{N}$, we consider $F_n = (0, 0, 0, \sqrt{\frac{2}{\pi}} \cos(nx))$, which is bounded in $\mathcal{H}$. Let $(u_n, v_n, \alpha_n, \theta_n) \in D(\mathcal{A})$ be the unique solution to the following resolvent equation:

$$(i\lambda_n I - \mathcal{A})U_n = F_n. \tag{4.1}$$

Now, let us fix any constant $\mu^* > 0$. We consider two cases:

Case 1. $\kappa_1^* \neq 0$. In this case, system (4.1) can be broken down into its components as follows:

$$i\lambda_n u_n - v_n = 0 \text{ in } H^2(0, \pi) \cap H_0^1(0, \pi), \tag{4.2}$$

$$i\lambda_n \rho v_n - (-\mu u_{nxx} + \kappa_1 u_{nxxxx} + \beta \theta_{nx} + m\alpha_{nxxx} + \kappa_1^* v_{nxxxx} - \mu^* v_{nxx}) = 0 \text{ in } L^2(0, \pi), \tag{4.3}$$

$$i\lambda_n \alpha_n - \theta_n = 0 \text{ in } H_*^1(0, \pi), \tag{4.4}$$

$$i\lambda_n c\theta_n + (\beta v_{nx} - m u_{nxxx} - \kappa_2 \alpha_{nxx}) = \sqrt{\frac{2}{\pi}} \cos(nx) \text{ in } L_*^2(0, \pi). \tag{4.5}$$

Given by boundary conditions described by (1.3), the solutions to the above system have the following forms:

$$u_n = A_n \sin(nx), \quad \alpha_n = B_n \cos(nx), \quad v_n = i\lambda_n A_n \sin(nx), \quad \theta_n = i\lambda_n B_n \cos(nx). \tag{4.6}$$

Substituting this into the previous system, we get that

$$i\lambda_n\rho(i\lambda_n A_n \sin(nx)) - \mu(-n^2 A_n \sin(nx)) + \kappa_1(n^4 A_n \sin(nx)) + \beta(-i\lambda_n n B_n \sin(nx))$$
$$+ m(n^3 B_n \sin(nx)) + \kappa_1^*(i\lambda_n n^4 A_n \sin(nx)) + \mu^*(i\lambda_n n^2 A_n \sin(nx)) = 0,$$

$$i\lambda_n c(i\lambda_n B_n \cos(nx)) + \beta(i\lambda_n n A_n \cos(nx)) + m(n^3 A_n \cos(nx)) + \kappa_2(n^2 B_n \cos(nx)) = \sqrt{\frac{2}{\pi}} \cos(nx).$$

Then, the following two equations must be satisfied:

$$(-\rho\lambda_n^2 + \mu n^2 + \kappa_1 n^4 + i\lambda_n \kappa_1^* n^4 + i\lambda_n \mu^* n^2)A_n + (mn^3 - i\lambda_n\beta n)B_n = 0,$$
$$(i\lambda_n\beta n + mn^3)A_n + (\kappa_2 n^2 - \lambda_n^2 c)B_n = \sqrt{\frac{2}{\pi}}.$$

Then, from (4.7) and (4.8), we have

$$A_n = -\frac{\sqrt{\dfrac{2}{\pi}}\left(mn^3 - i\lambda_n\beta n\right)}{(\kappa_2 n^2 - \lambda_n^2 c)\left(-\rho\lambda_n^2 + \mu n^2 + \kappa_1 n^4 + i\lambda_n\kappa_1^* n^4 + i\lambda_n\mu^* n^2\right) - (m^2 n^6 + \lambda_n^2\beta^2 n^2)},$$

$$B_n = \frac{\sqrt{\dfrac{2}{\pi}}\left(-\rho\lambda_n^2 + \mu n^2 + \kappa_1 n^4 + i\lambda_n\kappa_1^* n^4 + i\lambda_n\mu^* n^2\right)}{(\kappa_2 n^2 - \lambda_n^2 c)\left(-\rho\lambda_n^2 + \mu n^2 + \kappa_1 n^4 + i\lambda_n\kappa_1^* n^4 + i\lambda_n\mu^* n^2\right) - (m^2 n^6 + \lambda_n^2\beta^2 n^2)}.$$

We let $\kappa_2 n^2 - \lambda_n^2 c = 0$. Note that our choice is correct because

$$\lambda_n^2 = \frac{\kappa_2}{c}n^2 > 0 \quad \Rightarrow \quad \lambda_n = \pm\gamma n,$$

given the conditions on κ_2 and c, with $\gamma = \sqrt{\frac{\kappa_2}{c}}$. Under the above conditions, we have

$$B_n = \frac{\sqrt{\dfrac{2}{\pi}}\left(-\rho\lambda_n^2 + \mu n^2 + \kappa_1 n^4 + i\lambda_n\kappa_1^* n^4 + i\lambda_n\mu^* n^2\right)}{-m^2 n^6 - \lambda_n^2\beta^2 n^2}.$$

Hence, we find that the $L^2(0,\pi)$-norm of θ_n is given by

$$\|\theta_n\|_{L^2}^2 = \lambda_n^2 |B_n|^2 \to \sqrt{\frac{2}{\pi}}\frac{\kappa_1^{*2}\gamma^4}{m^4} > 0.$$

Hence, we conclude that

$$\|U_n\|_{\mathcal{H}}^2 \geq c\,\|\theta_n\|_{L^2(\Omega)}^2 \to \sqrt{\frac{2}{\pi}}\frac{\kappa_1^{*2}\gamma^4}{m^4} > 0.$$

Since $U_n = (i\lambda_n I - \mathcal{A})^{-1} F_n$, our conclusion follows.

Case 2. $\kappa_1^* = 0$. In this case, we are solving the system given by (4.2)–(4.5), and by using the solutions detailed in (4.6), we get

$$i\lambda_n \rho(i\lambda_n A_n \sin(nx)) - \mu(-n^2 A_n \sin(nx)) + \kappa_1(n^4 A_n \sin(nx)) + \beta(-i\lambda_n n B_n \sin(nx))$$
$$+ m(n^3 B_n \sin(nx)) - \mu^*(-i\lambda_n n^2 A_n \sin(nx)) = 0,$$

$$i\lambda_n c(i\lambda_n B_n \cos(nx)) + \beta(i\lambda_n n A_n \cos(nx)) + m(n^3 A_n \cos(nx)) + \kappa_2(n^2 B_n \cos(nx)) = \sqrt{\frac{2}{\pi}} \cos(nx).$$

Then, the following two equations must be satisfied:

$$(-\rho\lambda_n^2 + \mu n^2 + \kappa_1 n^4 + i\lambda_n \mu^* n^2)A_n + (mn^3 - i\lambda_n \beta n)B_n = 0, \tag{4.7}$$

$$(i\lambda_n \beta n + mn^3)A_n + (\kappa_2 n^2 - \lambda_n^2 c)B_n = \sqrt{\frac{2}{\pi}}. \tag{4.8}$$

Then, from (4.7) and (4.8), we get

$$A_n = -\frac{\sqrt{\frac{2}{\pi}}\left(mn^3 - i\lambda_n \beta n\right)}{(\kappa_2 n^2 - \lambda_n^2 c)\left(-\rho\lambda_n^2 + \mu n^2 + \kappa_1 n^4 + i\lambda_n \mu^* n^2\right) - (m^2 n^6 + \lambda_n^2 \beta^2 n^2)},$$

$$B_n = \frac{\sqrt{\frac{2}{\pi}}\left(-\rho\lambda_n^2 + \mu n^2 + \kappa_1 n^4 + i\lambda_n \mu^* n^2\right)}{(\kappa_2 n^2 - \lambda_n^2 c)\left(-\rho\lambda_n^2 + \mu n^2 + \kappa_1 n^4 + i\lambda_n \mu^* n^2\right) - (m^2 n^6 + \lambda_n^2 \beta^2 n^2)}.$$

We let $\kappa_2 n^2 - \lambda_n^2 c = \frac{m^2}{\kappa_1}n^2$, which is correct because

$$\kappa_2 n^2 - \lambda_n^2 c = \frac{m^2}{\kappa_1}n^2 \quad \Rightarrow \quad \lambda_n^2 = \frac{\kappa_1 \kappa_2 - m^2}{c\kappa_1}n^2 > 0 \quad \Rightarrow \quad \lambda_n = \pm\gamma n,$$

given the conditions on κ_1, κ_2, and m, with $\gamma = \sqrt{\frac{\kappa_1\kappa_2 - m^2}{c\kappa_1}}$. Under the above conditions, we have

$$B_n = \frac{\sqrt{\frac{2}{\pi}}\left(-\rho\lambda_n^2 + \mu n^2 + \kappa_1 n^4 + i\lambda_n \mu^* n^2\right)}{-\rho\frac{m^2}{\kappa_1}n^2\lambda_n^2 + \mu\frac{m^2}{\kappa_1}n^4 + i\lambda_n \mu^*\frac{m^2}{\kappa_1}n^4 - \lambda_n^2\beta^2 n^2}.$$

Hence, we find that the $L^2(0, \pi)$-norm of θ_n is given by

$$\|\theta_n\|_{L^2}^2 = \lambda_n^2 |B_n|^2 \to \sqrt{\frac{2}{\pi}}\frac{\gamma^2 \kappa_1^4}{|\mu^*|^2 m^4} > 0.$$

Therefore, we find that

$$\|U_n\|_{\mathcal{H}}^2 \geq c\,\|\theta_n\|_{L^2(\Omega)}^2 \to \sqrt{\frac{2}{\pi}}\frac{\gamma^2 \kappa_1^4}{|\mu^*|^2 m^4} > 0.$$

So, our conclusion follows.

Remark 4.1. *The above result implies that the semigroup defined by system given by (1.1)–(1.4) is not differentiable and, hence, not analytic. In particular, it implies that the solution does not have the smoothing effect property.*

5. A fully discrete approximation: a priori error estimates

In this section, we will provide an a priori error analysis of the thermoelastic problems described in the previous sections. However, for the sake of generality, we will assume that the length of the beam is $\ell > 0$ instead of π, and we will study its deformation over a finite time interval $[0, T]$, with $T > 0$. We recall the assumptions required for the constitutive coefficients of (1.5). Now, we shall derive the weak form of the thermomechanical problem defined by system given by (1.1) and (1.2) with the initial conditions presented in (1.4), as well as the following boundary conditions:

$$u(0, t) = u(\ell, t) = 0, \quad u_x(0, t) = u_x(\ell, t) = 0, \quad \alpha(0, t) = \alpha(\ell, t) = 0, \quad \forall t \in [0, T]. \tag{5.1}$$

Thus, let us denote $Y = L^2(0, \ell)$, $E = H_0^1(0, \ell)$, and $V = H_0^2(0, \ell)$, and denote by $(\cdot, \cdot)$ the inner product of $L^2(0, \ell)$.

Therefore, by applying integration by parts to the constitutive equations (1.1) and (1.2), and by taking into account the new boundary conditions of (5.1), we obtain the following variational formulation written in terms of the velocity $v = u_t$ and the temperature $\theta = \alpha_t$.

Find the velocity field $v : [0, T] \to V$ and the temperature $\theta : [0, T] \to E$ such that $v(0) = u_1$ and $\theta(0) = \alpha_1$, and, for all $w \in V$ and $r \in E$,

$$\rho(v_t(t), w) + \mu(u_x(t), w_x) + \mu^*(v_x(t), w_x) + \kappa_1(u_{xx}(t), w_{xx})$$
$$+\kappa_1^*(v_{xx}(t), w_{xx}) + m(\alpha_x(t), w_{xx}) + \beta(\theta_x(t), m) = 0, \tag{5.2}$$
$$c(\theta_t(t), r) + \kappa_2(\alpha_x(t), r_x) + \beta(v_x(t), r) + m(u_{xx}(t), r) = 0, \tag{5.3}$$

where the displacement field u and the thermal displacement α are recovered via the following relations:

$$u(t) = \int_0^t v(s)\, ds + u_0, \quad \alpha(t) = \int_0^t \theta(s)\, ds + \alpha_0. \tag{5.4}$$

In what follows, we provide an a priori error analysis of a fully discrete scheme that can be obtained by using the classical finite element method and the implicit Euler scheme.

Now, we introduce the approximation of problem given by (5.2)–(5.4). This is done first from the spatial point of view. So, we assume that the interval $[0, \ell]$ is split into M subintervals $a_0 = 0 < a_1 < \ldots < a_M = \ell$ of uniform length $h = a_{i+1} - a_i = \ell/M$. Therefore, we approximate the variational spaces E and V by the finite element spaces $E^h \subset E$ and $V^h \subset V$, defined as follows:

$$E^h = \{r^h \in C([0, \ell]) \cap E \; ; \; r^h_{|[a_i, a_{i+1}]} \in P_1([a_i, a_{i+1}]) \; i = 0, \ldots, M-1\}, \tag{5.5}$$
$$V^h = \{w^h \in C^1([0, \ell]) \cap V \; ; \; w^h_{|[a_i, a_{i+1}]} \in P_3([a_i, a_{i+1}]) \; i = 0, \ldots, M-1\}. \tag{5.6}$$

In these definitions, we have represented by $P_q([a_i, a_{i+1}])$ the space of polynomials of degree less or equal to q in the subinterval $[a_i, a_{i+1}]$. Moreover, let us define the discrete initial conditions u^{0h}, v^{0h}, α^{0h} and θ^{0h} as follows:

$$u^{0h} = \mathcal{P}_2^h u_0, \quad v^{0h} = \mathcal{P}_2^h u_1, \quad \alpha^{0h} = \mathcal{P}_1^h \alpha_0, \quad \theta^{0h} = \mathcal{P}_1^h \alpha_1. \tag{5.7}$$

Here, $\mathcal{P}_1^h$ and $\mathcal{P}_2^h$ denote the respective projection operators over the finite element spaces E^h and V^h (see [3]).

Now, we discretize the time derivatives. Therefore, we use a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, with time step size $k = T/N$ and nodes $t_n = nk$ for $n = 0, 1, \ldots, N$. As usual, for a continuous function $z(t)$, we denote $z_n = z(t_n)$ and, given a sequence $\{z_n\}_{n=0}^N$, let $\delta z_n = (z_n - z_{n-1})/k$ be its divided differences.

By using the well-known implicit Euler scheme, a fully discrete approximation of the problem given by (5.2)–(5.4) can be obtained as follows.

Find the discrete velocity $\{v_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete temperature $\{\theta_n^{hk}\}_{n=0}^N \subset E^h$ such that $v_0^{hk} = v^{0h}$, $\theta_0^{hk} = \theta^{0h}$, and, for all $w^h \in V^h$, $r^h \in E^h$, and $n = 1, \ldots, N$,

$$
\rho(\delta v_n^{hk}, w^h) + \mu((u_n^{hk})_x, w_x^h) + \mu^*((v_n^{hk})_x, w_x^h) + \kappa_1((u_n^{hk})_{xx}, w_{xx}^h)
$$
$$
+ \kappa_1^*((v_n^{hk})_{xx}, w_{xx}^h) + m((\alpha_n^{hk})_x, w_x^h) + \beta((\theta_n^{hk})_x, w^h) = 0, \tag{5.8}
$$
$$
c(\delta \theta_n^{hk}, r^h) + \kappa_2((\alpha_n^{hk})_x, r_x^h) + \beta((v_n^{hk})_x, r^h) + m((u_n^{hk})_{xx}, r_x^h) = 0, \tag{5.9}
$$

where the discrete displacements denoted by u_n^{hk} and the discrete thermal displacement denoted by α_n^{hk} are now recovered from the following relations:

$$
u_n^{hk} = k \sum_{j=1}^n v_j^{hk} + u^{0h}, \quad \alpha_n^{hk} = k \sum_{j=1}^n \theta_j^{hk} + \alpha^{0h}. \tag{5.10}
$$

We note that it is easy to prove that the above fully discrete problem admits a unique solution. It is obtained by using the classical Lax-Milgram lemma and the conditions defined in (1.5).

Now, we will confirm the discrete stability property, which is summarized as follows.

Lemma 5.1. *Under conditions defined in (1.5), it follows that the discrete solution to the problem given by (5.8)–(5.10) $\{u_n^{hk}, v_n^{hk}, \alpha_n^{hk}, \theta_n^{hk}\}_{n=0}^N$ satisfies the following estimates:*

$$
\|v_n^{hk}\|^2 + \|u_n^{hk}\|_V^2 + \|\theta_n^{hk}\|^2 + \|\alpha_n^{hk}\|_E^2 \le C,
$$

where the positive constant C is independent of the discretization parameters h and k, and, from now on, let $\|\cdot\|_{\mathcal{H}}$ be the typical norm in the Hilbert space $\mathcal{H}$.

Proof. Taking $w^h = v_n^{hk}$ as a test function in discrete variational equation (5.8), we find that

$$
\rho(\delta v_n^{hk}, v_n^{hk}) + \mu((u_n^{hk})_x, (v_n^{hk})_x) + \mu^*((v_n^{hk})_x, (v_n^{hk})_x) + \kappa_1((u_n^{hk})_{xx}, (v_n^{hk})_{xx})
$$
$$
+ \kappa_1^*((v_n^{hk})_{xx}, (v_n^{hk})_{xx}) + m((\alpha_n^{hk})_x, (v_n^{hk})_{xx}) + \beta((\theta_n^{hk})_x, v_n^{hk}) = 0.
$$

Now, keeping in mind that (given the conditions of (1.5))

$$
\rho(\delta v_n^{hk}, v_n^{hk}) \ge \frac{\rho}{2k}\left\{\|v_n^{hk}\|^2 - \|v_{n-1}^{hk}\|^2\right\},
$$
$$
\mu((u_n^{hk})_x, (v_n^{hk})_x) \ge \frac{\mu}{2k}\left\{\|(u_n^{hk})_x\|^2 - \|(u_{n-1}^{hk})_x\|^2\right\},
$$
$$
\kappa_1((u_n^{hk})_{xx}, (v_n^{hk})_{xx}) = \frac{\kappa_1}{2k}\left\{\|(u_n^{hk})_{xx}\|^2 - \|(u_{n-1}^{hk})_{xx}\|^2 + \|(u_n^{hk} - u_{n-1}^{hk})_{xx}\|^2\right\},
$$
$$
\beta((\theta_n^{hk})_x, v_n^{hk}) = -\beta(\theta_n^{hk}, (v_n^{hk})_x),
$$
$$
\mu^*((v_n^{hk})_x, (v_n^{hk})_x) \ge 0, \quad \kappa_1^*((v_n^{hk})_{xx}, (v_n^{hk})_{xx}) \ge 0,
$$

we obtain

$$\frac{\rho}{2k}\left\{\|v_n^{hk}\|^2 - \|v_{n-1}^{hk}\|^2\right\} + \frac{\mu}{2k}\left\{\|(u_n^{hk})_x\|^2 - \|(u_{n-1}^{hk})_x\|^2\right\} + m((\alpha_n^{hk})_x, (v_n^{hk})_{xx})$$
$$+ \frac{\kappa_1}{2k}\left\{\|(u_n^{hk})_{xx}\|^2 - \|(u_{n-1}^{hk})_{xx}\|^2 + \|(u_n^{hk} - u_{n-1}^{hk})_{xx}\|^2\right\} - \beta(\theta_n^{hk}, (v_n^{hk})_x) \le 0.$$

Proceeding in a similar manner as with the discrete variational equation (5.9), for a test function $r^h = \theta_n^{hk}$, we obtain

$$\frac{c}{2k}\left\{\|\theta_n^{hk}\|^2 - \|\theta_{n-1}^{hk}\|^2\right\} + \frac{\kappa_2}{2k}\left\{\|(\alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1}^{hk})_x\|^2 + \|(\alpha_n^{hk} - \alpha_{n-1}^{hk})_x\|^2\right\}$$
$$+ m((u_n^{hk})_{xx}, (\theta_n^{hk})_x) + \beta((v_n^{hk})_x, \theta_n^{hk}) \le 0.$$

Combining the previous estimates, it follows that

$$\frac{\rho}{2k}\left\{\|v_n^{hk}\|^2 - \|v_{n-1}^{hk}\|^2\right\} + \frac{\mu}{2k}\left\{\|(u_n^{hk})_x\|^2 - \|(u_n^{hk})_x\|^2\right\} + m((\alpha_n^{hk})_x, (v_n^{hk})_{xx})$$
$$+ \frac{\kappa_1}{2k}\left\{\|(u_n^{hk})_{xx}\|^2 - \|(u_{n-1}^{hk})_{xx}\|^2 + \|(u_n^{hk} - u_{n-1}^{hk})_{xx}\|^2\right\} + \frac{c}{2k}\left\{\|\theta_n^{hk}\|^2 - \|\theta_{n-1}^{hk}\|^2\right\}$$
$$+ \frac{\kappa_2}{2k}\left\{\|(\alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1}^{hk})_x\|^2 + \|(\alpha_n^{hk} - \alpha_{n-1}^{hk})_x\|^2\right\} + m((u_n^{hk})_{xx}, (\theta_n^{hk})_x) \le 0.$$

Observing that

$$m((\alpha_n^{hk})_x, (v_n^{hk})_{xx}) + m((u_n^{hk})_{xx}, (\alpha_n^{hk})_x)$$
$$= \frac{m}{k}\left\{((\alpha_n^{hk})_x, (u_n^{hk})_{xx}) - ((\alpha_{n-1}^{hk})_x, (u_{n-1}^{hk})_{xx}) + ((\alpha_n^{hk} - \alpha_{n-1}^{hk})_x, (u_n^{hk} - u_{n-1}^{hk})_{xx})\right\},$$

and taking into account that, under the condition that $\kappa_1\kappa_2 > m^2$, we have

$$\kappa_1\|(u_n^{hk} - u_{n-1}^{hk})_{xx}\|^2 + \kappa_2\|(\alpha_n^{hk} - \alpha_{n-1}^{hk})_x\|^2 + 2m((\alpha_n^{hk} - \alpha_{n-1}^{hk})_x, (u_n^{hk} - u_{n-1}^{hk})_{xx}) \ge 0.$$

Taking the above estimates, multiplying them by k, and summing up to n, we find that

$$\rho\|v_n^{hk}\|^2 + \mu\|(u_n^{hk})_x\|^2 + c\|\theta_n^{hk}\|^2 + \kappa_1\|(u_n^{hk})_{xx}\|^2 + \kappa_2\|(\alpha_n^{hk})_x\|^2 + 2m((\alpha_n^{hk})_x, (u_n^{hk})_{xx})$$
$$\le C\left(\|v^{0h}\|^2 + \|u^{0h}\|_V^2 + \|\theta^{0h}\|^2 + \|\alpha^{0h}\|_E^2\right),$$

where C is a positive constant which does not depend on h or k.

By again applying the condition that $\kappa_1\kappa_2 > m^2$, we have verified the desired stability property. □

In the rest of the section, we will derive some estimates on the numerical errors $v_n - v_n^{hk}$ and $\theta_n - \theta_n^{hk}$. First, we will obtain the estimates for the velocity field. So, taking (5.2) with a test function $w = w^h \in V^h$ at time $t = t_n$, and subtracting it by discrete variational equation (5.8), it follows that, for all $w^h \in V^h$,

$$\rho(v_t(t_n) - \delta v_n^{hk}, v_n - v_n^{hk}) + \mu((u_n - u_n^{hk})_x, (v_n - v_n^{hk})_x) + \mu^*((v_n - v_n^{hk})_x, (v_n - v_n^{hk})_x)$$
$$+ \kappa_1((u_n - u_n^{hk})_{xx}, (v_n - v_n^{hk})_{xx}) + \kappa_1^*((v_n - v_n^{hk})_{xx}, (v_n - v_n^{hk})_{xx}) + m((\alpha_n - \alpha_n^{hk})_x, (v_n - v_n^{hk})_{xx})$$
$$+ \beta((\theta_n - \theta_n^{hk})_x, v_n - v_n^{hk})$$
$$= \rho(v_t(t_n) - \delta v_n^{hk}, v_n - w^h) + \mu((u_n - u_n^{hk})_x, (v_n - w^h)_x) + \mu^*((v_n - v_n^{hk})_x, (v_n - w^h)_x)$$

$$+ \kappa_1((u_n - u_n^{hk})_{xx}, (v_n - w^h)_{xx}) + \kappa_1^*((v_n - v_n^{hk})_{xx}, (v_n - w^h)_{xx}) + m((\alpha_n - \alpha_n^{hk})_x, (v_n - w^h)_{xx})$$
$$+ \beta((\theta_n - \theta_n^{hk})_x, v_n - w^h).$$

Keeping in mind that

$$\rho(v_t(t_n) - \delta v_n^{hk}, v_n - v_n^{hk}) = \rho(v_t(t_n) - \delta v_n, v_n - v_n^{hk}) + \rho(\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}),$$

$$\rho(\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}) \geq \frac{\rho}{2k}\left\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\right\},$$

$$\mu^*((v_n - v_n^{hk})_x, (v_n - v_n^{hk})_x) \geq 0,$$

$$\kappa_1^*((v_n - v_n^{hk})_{xx}, (v_n - v_n^{hk})_{xx}) \geq 0,$$

$$\mu((u_n - u_n^{hk})_x, (\delta u_n - \delta u_n^{hk})_x) \geq \frac{\mu}{2k}\left\{\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\right\},$$

$$\kappa_1((u_n - u_n^{hk})_{xx}, (\delta u_n - \delta u_n^{hk})_{xx})$$
$$= \frac{\kappa_1}{2k}\left\{\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 + \|(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))_{xx}\|^2\right\},$$

$$\beta((\theta_n - \theta_n^{hk})_x, v_n - v_n^{hk}) = -\beta(\theta_n - \theta_n^{hk}, (v_n - v_n^{hk})_x),$$

$$\beta((\theta_n - \theta_n^{hk})_x, v_n - w^h) = -\beta(\theta_n - \theta_n^{hk}, (v_n - w^h)_x),$$

and by applying Cauchy's inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$ for $a, b, \varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, it follows that, for all $w^h \in V^h$,

$$\frac{\rho}{2k}\left\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\right\} + \frac{\mu}{2k}\left\{\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\right\}$$
$$+ \frac{\kappa_1}{2k}\left\{\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 + \|(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))_{xx}\|^2\right\}$$
$$+ m((\alpha_n - \alpha_n^{hk})_x, (\delta u_n - \delta u_n^{hk})_{xx}) - \beta(\theta_n - \theta_n^{hk}, (v_n - v_n^{hk})_x)$$
$$\leq C\Big(\|v_t(t_n) - \delta v_n\|^2 + \|u_t(t_n) - \delta u_n\|_V^2 + \|v_n - w^h\|_V^2 + \|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2$$
$$+ \|(u_n - u_n^{hk})_{xx}\|^2 + \|(\alpha_n - \alpha_n^{hk})_x\|^2 + \|\theta_n - \theta_n^{hk}\|^2 + (\delta u_n - \delta u_n^{hk}, v_n - w^h)\Big).$$

Proceeding in a similar manner, we obtain the following estimates for the temperature field for all $r^h \in E^h$:

$$\frac{c}{2k}\left\{\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\right\} + m((u_n - u_n^{hk})_{xx}, (\delta\alpha_n - \delta\alpha_n^{hk})_x) + \beta((v_n - v_n^{hk})_x, \theta_n - \theta_n^{hk})$$
$$+ \frac{\kappa_2}{2k}\left\{\|(\alpha_n - \alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1} - \alpha_{n-1}^{hk})_x\|^2 + \|(\alpha_n - \alpha_n^{hk} - (\alpha_{n-1} - \alpha_{n-1}^{hk}))_x\|^2\right\}$$
$$\leq C\Big(\|\theta_t(t_n) - \delta\theta_n\|^2 + \|\alpha_t(t_n) - \delta\alpha_n\|_E^2 + \|\theta_n - r^h\|_E^2 + \|\theta_n - \theta_n^{hk}\|^2 + \|(\alpha_n - \alpha_n^{hk})_x\|^2$$
$$+ \|(u_n - u_n^{hk})_{xx}\|^2 + \|v_n - v_n^{hk}\|^2 + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - r^h)\Big).$$

After easy algebraic manipulations, we have

$$m((\alpha_n - \alpha_n^{hk})_x, (\delta u_n - \delta u_n^{hk})_{xx}) + m((u_n - u_n^{hk})_{xx}, (\delta\alpha_n - \delta\alpha_n^{hk})_x)$$
$$= \frac{m}{k}\Big\{((\alpha_n - \alpha_n^{hk})_x, (u_n - u_n^{hk})_{xx}) - ((\alpha_{n-1} - \alpha_{n-1}^{hk})_x, (u_{n-1} - u_{n-1}^{hk})_{xx})$$
$$+ ((\alpha_n - \alpha_n^{hk} - (\alpha_{n-1} - \alpha_{n-1}^{hk}))_x, (u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))_{xx})\Big\}.$$

Observing that, thanks again to the condition that $\kappa_1 \kappa_2 > m^2$, we find that

$$\kappa_1 \|(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))_{xx}\|^2 + \kappa_2 \|(\alpha_n - \alpha_n^{hk} - (\alpha_{n-1} - \alpha_{n-1}^{hk}))_x\|^2$$
$$+ 2m((\alpha_n - \alpha_n^{hk} - (\alpha_{n-1} - \alpha_{n-1}^{hk}))_x, (u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))_{xx}) \geq 0,$$

and, summing up to n the previous estimates on the velocity and temperatures, it follows that

$$\rho \|v_n - v_n^{hk}\|^2 + \mu \|(u_n - u_n^{hk})_x\|^2 + \kappa_1 \|(u_n - u_n^{hk})_{xx}\|^2 + c\|\theta_n - \theta_n^{hk}\|^2 + \kappa_2 \|(\alpha_n - \alpha_n^{hk})_x\|^2$$
$$+ 2m((\alpha_n - \alpha_n^{hk})_x, (u_n - u_n^{hk})_{xx})$$

$$\leq Ck \sum_{j=1}^{n} \Big(\|v_t(t_j) - \delta v_j\|^2 + \|u_t(t_j) - \delta u_j\|_V^2 + \|v_j - w_j^h\|_V^2 + \|v_j - v_j^{hk}\|^2 + \|(u_j - u_j^{hk})_x\|^2$$
$$+ \|(u_j - u_j^{hk})_{xx}\|^2 + \|(\alpha_j - \alpha_j^{hk})_x\|^2 + \|\theta_j - \theta_j^{hk}\|^2 + \|\theta_t(t_j) - \delta\theta_j\|^2 + \|\alpha_t(t_j) - \delta\alpha_j\|_E^2$$
$$+ (\delta u_j - \delta u_j^{hk}, v_j - w_j^h) + \|\theta_j - r_j^h\|_E^2 + (\delta\theta_j - \delta\theta_j^{hk}, \theta_j - r_j^h) \Big) + C \Big(\|u_1 - v^{0h}\|^2 + \|u_0 - u^{0h}\|_V^2$$
$$+ \|\alpha_1 - \theta^{0h}\|^2 + \|\alpha_0 - \alpha^{0h}\|_E^2 \Big), \quad \forall \{w_j^h\}_{j=1}^n \subset V^h, \quad \{r_j^h\}_{j=1}^n \subset E^h.$$

Now, taking into account that

$$\kappa_1 \|(u_n - u_n^{hk})_{xx}\|^2 + \kappa_2 \|(\alpha_n - \alpha_n^{hk})_x\|^2 + 2m((\alpha_n - \alpha_n^{hk})_x, (u_n - u_n^{hk})_{xx})$$
$$\geq C(\|(u_n - u_n^{hk})_{xx}\|^2 + \|(\alpha_n - \alpha_n^{hk})_x\|^2),$$

where we have again applied the conditions of (1.5) as well as the estimates

$$k \sum_{j=1}^{n} (\delta u_j - \delta u_j^{hk}, v_j - w_j^h)$$

$$= (v_n - v_n^{hk}, v_n - w_n^h) + (v^{0h} - u_1, v_1 - w_1^h) + \sum_{j=1}^{n-1} (v_j - v_j^{hk}, v_j - w_j^h - (v_{j+1} - w_{j+1}^h)),$$

with a similar expression for the temperature, after an application of a discrete version of Gronwall's lemma (see [4]), we obtain the following a priori error estimate result.

Theorem 5.1. *Under the conditions of (1.5), let us denote by (u, v, α, θ) the solution to the problem given by (5.2)–(5.4), and denote by $(u^{hk}, v^{hk}, \alpha^{hk}, \theta^{hk})$ the solution to the problem given by (5.8)–(5.10); then, we have the following a priori error estimates for all $w^h = \{w_j^h\}_{j=0}^N \subset V^h$: $r^h = \{r_j^h\}_{j=0}^N \subset E^h$:*

$$\max_{0 \leq n \leq N} \Big\{ \|v_n - v_n^{hk}\|^2 + \|u_n - u_n^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|^2 + \|\alpha_n - \alpha_n^{hk}\|_E^2 \Big\}$$

$$\leq Ck \sum_{j=1}^{N} \Big(\|v_t(t_j) - \delta v_j\|^2 + \|u_t(t_j) - \delta u_j\|_V^2 + \|v_j - w_j^h\|_V^2 + \|\theta_t(t_j) - \delta\theta_j\|^2$$

$$+ \|\alpha_t(t_j) - \delta\alpha_j\|_E^2 + \|\theta_j - r_j^h\|_E^2 \Big) + \frac{C}{k} \sum_{j=1}^{N-1} \Big(\|v_j - w_j^h - (v_{j+1} - w_{j+1}^h)\|^2 + \|\theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h)\|^2 \Big)$$

$$+ C \Big(\|u_1 - v^{0h}\|^2 + \|u_0 - u^{0h}\|_V^2 + \|\alpha_1 - \theta^{0h}\|^2 + \|\alpha_0 - \alpha^{0h}\|^2 \Big) + C \max_{0 \leq n \leq N} \Big(\|v_n - w_n^h\|^2 + \|\theta_n - r_n^h\|^2 \Big),$$

where C is a positive constant which does not depend on the discretization parameters h and k, and we recall that $\| \cdot \|_{\mathcal{H}}$ denotes the norm of the Hilbert space $\mathcal{H}$.

From the above result, we can derive the convergence order of the approximations for the discrete problem given by (5.8)–(5.10). As an example, if we assume the following additional regularity:

$$u \in H^3(0, T; Y) \cap H^2(0, T; V) \cap C^1([0, T]; H^3(0, \ell)),$$
$$\alpha \in H^3(0, T; Y) \cap H^2(0, T; E) \cap C^1([0, T]; H^2(0, \ell)),$$

(5.11)

we conclude that the convergence of the algorithm is linear. It can be proved by applying some well-known results on the approximation through the use of finite elements (see, e.g., [2]) and some estimates already used in [4]. That is, it follows that there exists a positive constant $C > 0$ such that

$$\max_{0 \leq n \leq N} \left\{ \|v_n - v_n^{hk}\| + \|u_n - u_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\| + \|\alpha_n - \alpha_n^{hk}\|_E \right\} \leq C(h + k).$$

6. Numerical examples

In this final section, the numerical scheme for solving the problem given by (5.8)–(5.10) is described and some numerical simulations are presented to show the convergence of the approximations and the behavior of the discrete energy.

First, we note that, given the solutions u_{n-1}^{hk} and α_{n-1}^{hk} at time t_{n-1}, the discrete velocity v_n^{hk} and the discrete temperature θ_n^{hk} are obtained by solving the following linear system for all $w^h \in V^h$, $r^h \in E^h$:

$$\frac{\rho}{k}(v_n^{hk}, w^h) + \mu k((v_n^{hk})_x, w_x^h) + \mu^* k((v_n^{hk})_x, w_x^h) + \kappa_1 k((v_n^{hk})_{xx}, w_{xx}^h) + \kappa_1^* k((v_n^{hk})_{xx}, w_{xx}^h)$$

$$= \frac{\rho}{k}(v_{n-1}^{hk}, w^h) - \mu((u_{n-1}^{hk})_x, w_x^h) - \kappa_1((u_{n-1}^{hk})_{xx}, w_{xx}^h) - m((\alpha_n^{hk})_x, w_{xx}^h) - \beta((\theta_n^{hk})_x, w^h),$$

$$\frac{c}{k}(\theta_n^{hk}, r^h) + \kappa_2 k((\theta_n^{hk})_x, r_x^h) = \frac{c}{k}(\theta_{n-1}^{hk}, r^h) - \kappa_2((\alpha_{n-1}^{hk})_x, r_x^h) - m((u_n^{hk})_{xx}, r_x^h) - \beta((v_n^{hk})_x, r^h).$$

This numerical scheme was implemented on a 3.2 GHz PC by using MATLAB, and a typical run (using parameters $h = k = 0.001$) took about 0.52s of CPU time.

As an example, in order to show the accuracy of the approximations and the behavior of the solution, we solved problem given by (1.1), (1.2), (1.4), and (5.1) with the following data:

$$\ell = 1, \ T = 1, \ \rho = 1, \ \mu = 2, \ \kappa_1 = 2, \ \kappa_2 = 3, \ \beta = 2, \ m = 2, \ c = 1, \ \mu^* = 3, \ \kappa_1^* = 1.$$

By applying the following initial conditions for all $x \in (0, 1)$:

$$u_0(x) = u_1(x) = x^3(x - 1)^3, \quad \alpha_0(x) = \alpha_1(x) = x(x - 1),$$

and by considering homogeneous Dirichlet boundary conditions and the following (artificial) supply terms for all $(x, t) \in (0, 1) \times (0, 1)$:

$$F_1(x, t) = e^t(x^6 - 3x^5 - 147x^4 + 299x^3 + 900x^2 - 1046x + 214),$$
$$F_2(x, t) = e^t(12x^5 - 30x^4 - 216x^3 + 355x^2 - 145x + 6),$$

the exact solution to this problem can be obtained, and it has the following form for $(x, t) \in [0, 1] \times [0, 1]$:

$$u(x, t) = v(x, t) = e^t x^3(x - 1)^3, \quad \alpha(x, t) = \theta(x, t) = e^t x(x - 1).$$

We note that the analysis performed in the previous section and the numerical scheme presented above should be modified to include these supply terms, but the corresponding changes are really minor.

Thus, taking some discretization parameters h and k,

$$\max_{0 \le n \le N} \left\{ \|v_n - v_n^{hk}\| + \|u_n - u_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\| + \|\alpha_n - \alpha_n^{hk}\|_E \right\}$$

was used to obtain the estimated approximation errors as are shown in Table 1. Moreover, the evolution of the error according to the parameter $h + k$ was determined and plotted in Figure 1. We notice that the convergence of the algorithm can be clearly observed, and that the linear convergence, as stated in the previous section, is achieved.

Table 1. Example 1: Numerical errors for some values of h and k.

| $h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^2$	2.275072	2.285012	2.293769	2.297389	2.299363	2.300604	2.301028
$1/2^3$	1.267086	1.275154	1.283056	1.286605	1.288630	1.289941	1.290397
$1/2^4$	0.651628	0.654311	0.657076	0.658292	0.658962	0.659393	0.659544
$1/2^5$	0.328939	0.329831	0.330947	0.331538	0.331917	0.332192	0.332295
$1/2^6$	0.165232	0.165479	0.165942	0.166213	0.166392	0.166524	0.166574
$1/2^7$	0.083020	0.082934	0.083090	0.083211	0.083295	0.083359	0.083384
$1/2^8$	0.042009	0.041629	0.041598	0.041639	0.041674	0.041703	0.041715
$1/2^9$	0.021810	0.021053	0.020851	0.020840	0.020848	0.020859	0.020864
$1/2^{10}$	0.012152	0.010921	0.010504	0.010443	0.010436	0.010433	0.010435
$1/2^{11}$	0.007798	0.006015	0.005269	0.005289	0.005265	0.005256	0.005250
$1/2^{12}$	0.005869	0.004363	0.004072	0.003086	0.001821	0.001807	0.001805
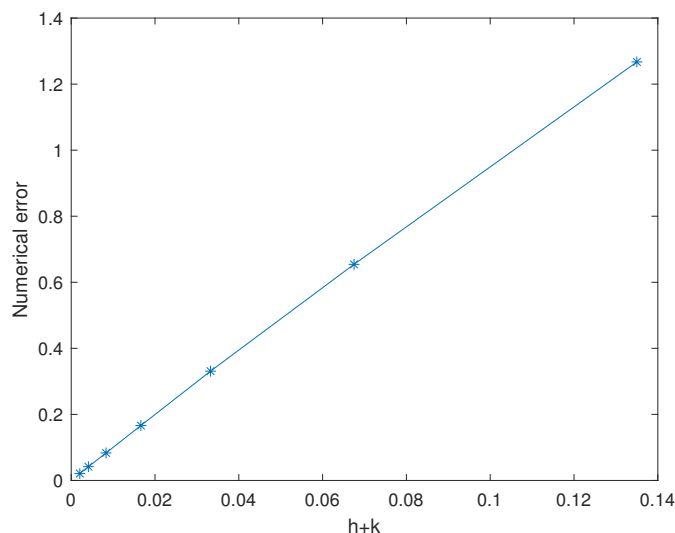
Figure 1. Asymptotic constant error.

Now, we are going to compare the energy decay of the two problems corresponding to the dissipation mechanisms studied in this work. So, let us denote as "Problem 1" the problem resulting

from applying $\mu^* = 1$ and $\kappa_1^* = 0$ (that is, with a dissipation mechanism of second order), denote as "Problem 2" the problem associated with applying $\kappa_1^* = 1$ and $\mu^* = 0$ (i.e., a dissipation mechanism of fourth order), and, finally, denote as "Problem 3" the problem associated with applying $\kappa_1^* = 1$ and $\mu^* = 1$ (that is, with the previous two dissipation mechanisms).

In these simulations, we have assumed that there are no supply terms, and we have applied the following data:

$$T = 120, \quad \rho = 1, \quad \mu = 2, \quad \kappa_1 = 2, \quad \kappa_2 = 3, \quad \beta = 2, \quad m = 2, \quad c = 1.$$

By applying the initial conditions for all $x \in (0, 1)$:

$$u_0(x) = u_1(x) = 0, \quad \alpha_0(x) = \alpha_1(x) = x(x - 1),$$

and, by taking the discretization parameters $h = k = 0.001$, the evolution in time of the discrete energy of the two problems given by

$$E_n^{hk} = \rho\|v_n^{hk}\|^2 + \mu\|u_n^{hk}\|_E^2 + \kappa_1\|u_n^{hk}\|_V^2 + c\|\theta_n^{hk}\|^2 + \kappa_2\|\alpha_n^{hk}\|_E^2$$

was obtained as in Figure 2 (in both natural and semi-log scales) for the three cases described above. As can be seen, the three discrete energies converge to zero and an exponential decay seems to be achieved. Moreover, as expected, the decay is faster for the second-order mechanism (Problem 1); meanwhile, the case with the two dissipation mechanisms together has a worse rate decay, which is similar to that of the fourth-order dissipation mechanism. This is an amazing result.
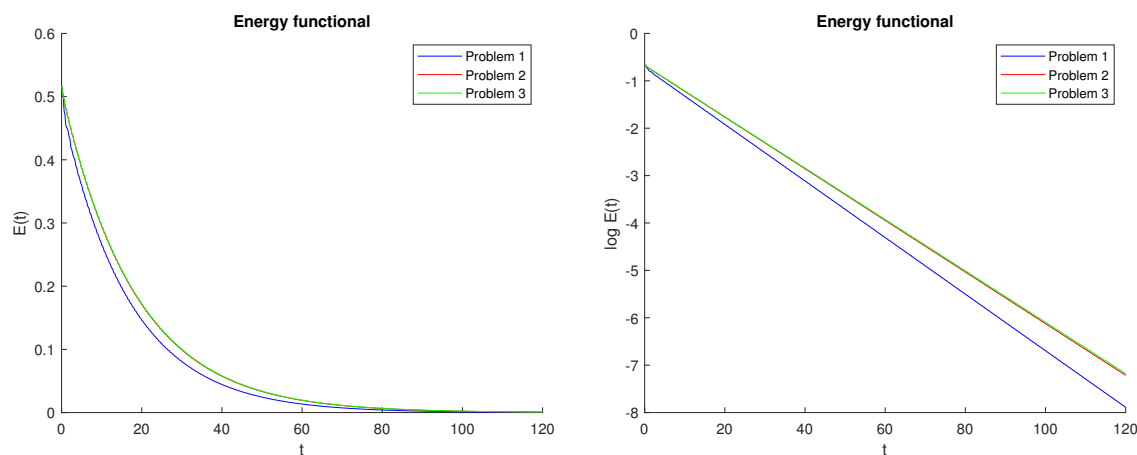

Figure 2. Evolution in time of the discrete energy for the three problems considered ((a) natural and (b) semi-log scales).

Finally, our aim will be to numerically demonstrate the lack of differentiability of the solutions to the problem given by (1.1)–(1.4). Hence, following the convergences described by (4.2)–(4.5), we will solve the following coupled discrete linear system:

$$(i\lambda_n u_n^h - v_n^h, w^h) = 0, \quad \forall w^h \in V^h,$$
$$-(i\lambda_n v_n^h, \xi^h) + \mu((u_n^h)_x, \xi_x^h) + \kappa_1((u_n^h)_{xx}, \xi_{xx}^h) + \beta((\theta_n^h)_x, \xi^h) + m((\alpha_n^h)_x, \xi_{xx}^h) + \kappa_1^*((v_n^h)_{xx}, \xi_{xx}^h)$$
$$+\mu^*((v_n^h)_x, \xi_x^h) = 0, \quad \forall \xi^h \in V^h,$$
$$(i\lambda_n \alpha_n^h - \theta_n^h, r^h) = 0, \quad \forall r^h \in E^h,$$
$$(i\lambda_n c\theta_n^h, \zeta^h) + \kappa_2((\alpha_n^h)_x, \zeta_x^h) + \beta((v_n^h)_x, \zeta^h) + m((u_n^h)_{xx}, \zeta_x^h) = (F_n, \zeta^h), \quad \forall \zeta^h \in E^h,$$

where the function $F_n(x) = \sqrt{\dfrac{2}{\pi}} \cos nx$ for all $x \in [0, \pi]$. We note that, in this case, the problem is static; so, the approximation is done only in space. Moreover, the subscript n represents the discrete solution obtained for each value λ_n.

In this example, we have used the following data:

$$\ell = \pi, \quad \rho = 1, \quad \mu = 2, \quad \kappa_1 = 2, \quad \kappa_2 = 3, \quad \beta = 2, \quad m = 2, \quad c = 1, \quad \mu^* = 3, \quad \kappa_1^* = 1.$$

Our aim here will be to show that the discrete solution to the above system $U_n^h = (u_n^h, v_n^h, \alpha_n^h, \theta_n^h)$ has a norm defined by

$$\|U_n^h\|^2 = \int_0^\pi \left(\rho|v_n^h|^2 + \kappa_1|(u_n^h)_{xx}|^2 + c|\theta_n^h|^2 + \kappa_2|(\alpha_n^h)_x|^2 + \mu|(u_n^h)_x|^2 + 2m(\alpha_n^h)_x(u_n^h)_{xx} \right) dx,$$

which is greater than a positive constant when $\lambda_n \to \infty$.

Therefore, taking as a discretization parameter $h = \pi/1000$, in Figure 3, we show the evolution of the norm $\|U_n^h\|^2$ when λ_n increases to 10^5. As we can see on the left-hand side, the norm of these discrete solutions seems to stabilize near 0.663; this can be clearly checked if we zoom in near value 10^6 (see the right-hand side). Thus, proceeding as in the proofs in Section 4, we could conclude that the semigroup is not differentiable.
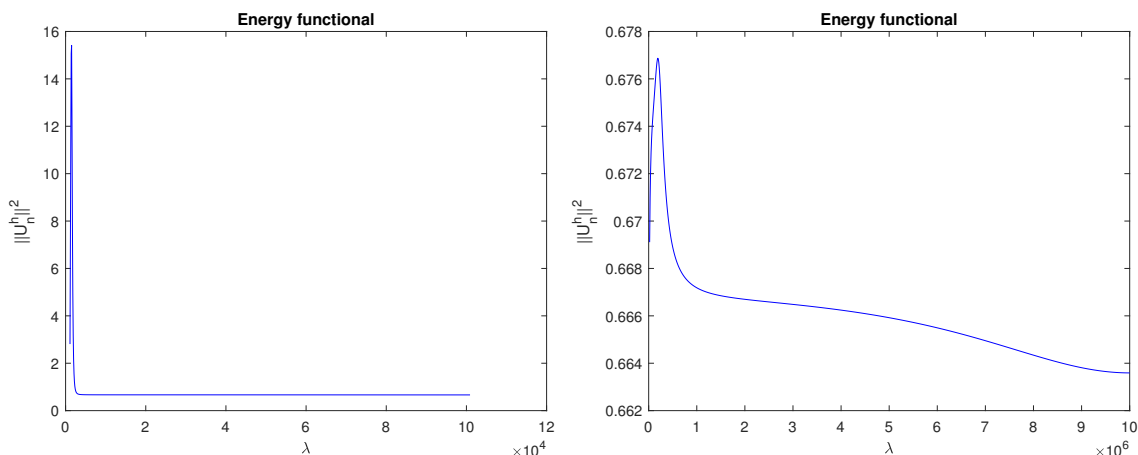

Figure 3. Evolution of the norm $\|U_n^h\|^2$ when λ_n increases (to 10^5 (left-hand side) and zoom in near value 10^6 (right)).

7. Conclusions

In this work, we studied a one-dimensional viscous strain gradient problem with several dissipation mechanisms from the analytical and numerical points of view. It is remarkable that we have assumed mechanical dissipation of strain gradient type under the condition of a conservative heat conduction.

From the analytical point of view, we have proved that the problem has a unique solution, and that the energy decays in an exponential way. Moreover, we have also shown the lack of differentiability of the semigroup.

From the numerical point of view, we have introduced a fully discrete approximation of a weak form of the above thermal problem, and we have proved the existence of a discrete stability property and a main a priori error estimate result. Through the numerical example, we have demonstrated the linear convergence and analyzed the convergence rate according to the number of dissipation mechanisms considered. We have found that the energy decays faster when the second-order dissipation is used. Finally, we have shown numerically that the discrete solutions have a norm which is greater than a positive value for any value λ_n introduced into the problem; so, we have concluded that the semigroup is not differentiable.

Author contributions

J. E. Muñoz-Rivera, E. Ochoa and R. Quintanilla: Conceptualization, Methodology; J. R. Fernández, J. E. Muñoz-Rivera, E. Ochoa and R. Quintanilla: Formal analysis; J. R. Fernández, E. Ochoa and R. Quintanilla: Writing-original draft; B. Noelia and J. R. Fernández: Writing-review & editing, Visualization. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Conflict of interest

Ramón Quintanilla is an editorial board member and was not involved in the editorial review or the decision to publish this article. The authors declare that they have no conflict of interest.

References

1. J. Baldonedo, J. R. Fernández, A. Magaña, R. Quintanilla, Decay for strain gradient porous elastic waves, *Z. Angew. Math. Phys.*, **74** (2023), 35. https://doi.org/10.1007/s00033-022-01930-6

2. P. G. Ciarlet, Basic error estimates for elliptic problems, In: *Handbook of numerical analysis*, Amsterdam: North-Holland, **2** (1991), 17–351. https://doi.org/10.1016/S1570-8659(05)80039-0

3. P. Clement, Approximation by finite element functions using local regularization, *R.A.I.R.O. Anal. Numer.*, **9** (1975), 77–84. https://doi.org/10.1051/m2an/197509R200771

4. M. Campo, J. R. Fernández, K. L. Kuttler, M. Shillor, J. M. Viaño, Numerical analysis and simulations of a dynamic frictionless contact problem with damage, *Comput. Methods Appl. Mech. Eng.*, **196** (2006), 476–488. https://doi.org/10.1016/j.cma.2006.05.006

5. K. J. Engel, R. Nagel, *One-parameter semigroups for linear evolution equations*, New York: Springer, 2000. https://doi.org/10.1007/b97696

6. A. E. Green, P. M. Naghdi, A re-examination of the basic postulates of themomechanics, *Proc. R. Soc. Lond. A*, **432** (1991), 171–194. https://doi.org/10.1098/rspa.1991.0012

7. A. E. Green, P. M. Naghdi, On undamped heat waves in an elastic solid, *J. Thermal Stresses*, **15** (1992), 253–264. https://doi.org/10.1080/01495739208946136

8. A. E. Green, P. M. Naghdi, Thermoelasticity without energy dissipation, *J. Elasticity*, **31** (1993), 189–208. https://doi.org/10.1007/BF00044969

9. A. E. Green, P. M. Naghdi, A unified procedure for construction of theories of deformable media: I. Classical continuum physics, *Proc. Roy. Soc. London A*, **448** (1995), 335–356. https://doi.org/10.1098/rspa.1995.0020

10. A. E. Green, P. M. Naghdi, A unified procedure for construction of theories of deformable media: II. Generalized continua, *Proc. Roy. Soc. London A*, **448** (1995), 357–377. https://doi.org/10.1098/rspa.1995.0021

11. A. E. Green, P. M. Naghdi, A unified procedure for construction of theories of deformable media: III. Mixtures of interacting continua, *Proc. Roy. Soc. London A*, **448** (1995), 379–388. https://doi.org/10.1098/rspa.1995.0022

12. D. Ieşan, R. Quintanilla, Strain gradient theory of chiral Cosserat thermoelasticity without energy dissipation, *J. Math. Anal. Appl.*, **437** (2016), 1219–1235. https://doi.org/10.1016/j.jmaa.2016.01.058

13. S. Jiang, R. Racke, *Evolution equations in thermoelasticity*, Chapman & Hall/CRC, 2000. https://doi.org/10.1201/9781482285789

14. M. C. Leseduarte, A. Magaña, R. Quintanilla, On the time decay of solutions in porous-thermo-elasticity of type II, *Discrete Contin. Dyn. Syst. Ser. B*, **13** (2010), 375–391.

15. Z. Y. Liu, S. M. Zheng, *Semigroups associated with dissipative systems*, Chapman & Hall/CRC, 1999.

16. A. Magaña, R. Quintanilla, Decay of quasi-static porous-thermo-elastic waves, *Z. Angew. Math. Phys.*, **72** (2021), 125. https://doi.org/10.1007/s00033-021-01557-z

17. A. Miranville, R. Quintanilla, Exponential decay in one-dimensional type II thermoviscoelasticity with voids, *J. Comput. Appl. Math.*, **368** (2020), 112573. https://doi.org/10.1016/j.cam.2019.112573

18. A. Pazy, *Semigroup of linear operators and applications to partial diferential equations*, New York: Springer, 1983. https://doi.org/10.1007/978-1-4612-5561-1

19. J. Prüss, On the spectrum of C_0-semigroups, *Trans. Amer. Math. Soc.*, **284** (1984), 847–857.

20. R. Quintanilla, Thermoelasticity without energy dissipation of nonsimple materials, *Z. Angew. Math. Mech.*, **83** (2003), 172–180. https://doi.org/10.1002/zamm.200310017

21. J. M. Rivera, E. O. Ochoa, R. Quintanilla, Time decay of viscoelastic plates with type II heat conduction, *J. Math. Anal. Appl.*, **528** (2023), 127592. https://doi.org/10.1016/j.jmaa.2023.127592