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Research article

A further study on an attraction-repulsion chemotaxis system with logistic source

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Abstract: This paper is concerned with the attraction-repulsion chemotaxis system (1.1) define on a bounded domain $\Omega \subset \mathbb{R}^N (N \geq 1)$ with no-flux boundary conditions. The source function f in this system is a smooth function f that satisfies $f(u) \leq a - bu^\eta$ for $u \geq 0$. It is proven that $\eta \geq 1$ is sufficient to ensure the boundedness of the solution when $r < \frac{4(N+1)}{N(N+2)}$ is in the balance case $\chi\alpha = \xi\gamma$, which improve the relevant results presented in papers such as Li and Xiang (2016), Xu and Zheng (2018), Xie and Zheng (2021), and Tang, Zheng and Li (2023).

Keywords: attraction-repulsion; boundedness; logistic source; chemotaxis
Mathematics Subject Classification: 35B35, 35K55

1. Introduction

In this paper, we study the attraction-repulsion system with a logistic source

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u(1+u)^{r-1} \nabla v) + \xi \nabla \cdot (u(u+1)^{r-1} \nabla w) + f(u), & x \in \Omega, t > 0, \\ 0 = \Delta v - \beta v + \alpha u, & x \in \Omega, t > 0, \\ 0 = \triangle w - \delta w + \gamma u, & x \in \Omega, t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary, and parameters $\chi, \xi, \alpha, \beta, \gamma, \delta, r > 0$. The logistic source $f(s)$ is smooth on $[0, \infty)$ and fulfills $f(s) \leq a - bs^\eta,\ s \geq 0$ with $a \geq 0,\ b > 0$ and $\eta \geq 1$.

In the model (1.1), u, v and w denote the cell density, the chemoattractant concentration and the chemorepellent concentration, respectively. The logistic function $f(u)$ models proliferation and death of cells. Positive parameters χ and ξ represent the chemotactic coefficients, which measure the strength of the attraction and repulsion, respectively. The mortality rates of v and w are denoted by β and δ, respectively; and parameters α and γ are the growth rates of the chemicals. The behavior of the solutions would be determined by the interaction between diffusion, attraction, repulsion and logistic sources. When $r = 1$ and $f(u) = 0$, it was proven that system (1.1) admits a global bounded solution if $n = 1$ or repulsion prevails over attraction in the sense that $\chi\alpha < \xi\gamma$ [7]. Yu et al. proved in [15] that when $n = 2$ with $\chi\alpha > \xi\gamma$ (attraction prevails over repulsion), there exists initial data such that a blow-up of solutions occurs. A source of logistic type $f(u)$ is included in (1.1) to prevent unlimited growth of the cell density, and the global boundedness of solutions to the model was established in the repulsion domination case $\chi\alpha < \xi\gamma$ with $\eta \geq 1$ [3,6] and the attraction domination case $\chi\alpha > \xi\gamma$ with $\eta > 2$ (or $\eta = 2$, b properly large) [3]. Under more interesting balance situations $\chi\alpha = \xi\gamma$ (i.e., attraction-repulsion balance), for $r = 1$, Li and Xiang [3] proved that there exists a global bounded classical solution if $\eta > \frac{\sqrt{N^2+4N}-N+2}{2}$ and $N \geq 2$. Then Xu and Zheng [14] further proved that the boundedness of the solutions to the model (1.1) can be obtained for $\eta > \frac{2N+2}{N+2}$. Xie and Zheng [12] improved the result to $\eta > \frac{2N-2}{N}$. Recently, Tang, Zheng and Li [6] considered the model (1.1) with general $r > 0$ and obtained boundedness of the solution to the model (1.1) if $\eta > \max\{r + \frac{(N-2)_+}{N}, 1\}$. More relevant results for an attraction-repulsion chemotaxis system can be found in [1–8,10–15] and the references therein.

It can be seen from the existing research results that the behavior of solutions depends on the effects of logistic sources (i.e., the exponent η) on the model (1.1) under the balance situation $\chi\alpha = \xi\gamma$ (i.e., attraction-repulsion balance). A natural question is to determine the optimal restriction of the exponent η, which guarantees the global boundedness of the solution. The main objective of this short paper is to do some further study and give an optimal condition on the logistic source for small r in the balance case $\chi\alpha = \xi\gamma$. Our main results are stated as follows:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with a smooth boundary. Suppose that the nonnegative initial data $u_0(x) \in C(\overline{\Omega})$. If $\chi\alpha = \xi\gamma, \eta \geq 1$ and $r < \frac{4(N+1)}{N(N+2)}$, then the solution of the model (1.1) is globally bounded in the sense that $\|u\|_\infty \leq C$, where C is a positive constant independent of t.*

Remark 1.1. *For the model (1.1) with $r = 1$ in $\mathbb{R}^N (N = 1, 2, 3)$, Theorem 1.1 implies that $\eta \geq 1$ is an optimal condition on the logistic source in the balance case $\chi\alpha = \xi\gamma$ and our results remove the restriction on η in [3,6,12,14] for $N = 1, 2, 3$.*

2. Preliminaries

In this section, we first show the local well-posedness and then give two necessary estimates and the Gagliardo-Nirenberg inequality that will be used in the proof of Theorem 1.1.

Lemma 2.1. *([14], Lemma 2.1) Suppose that $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary. Then, for any nonnegative $u_0 \in C(\overline{\Omega})$, there exist nonnegative functions $u, v, w \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max}))$ with $T_{max} \in (0, \infty]$ classically solving (1.1) in $\Omega \times (0, T_{max})$. Moreover, if $T_{max} < \infty$, then*

$$\lim_{t \to T_{max}} \|u(., t)\|_{L^\infty(\Omega)} = \infty.$$

Lemma 2.2. *([8], Lemma 2.2) Let (u, v, w) be a solution to the model (1.1). Then $\|w\|_{L^1} = \frac{\gamma}{\delta}\|u\|_{L^1}$ and for any $\kappa > 0,\ \theta > 1$, there is $c_0 = c_0(\kappa, \theta) > 0$ such that*

$$\int_\Omega w^\theta \le \kappa \int_\Omega u^\theta + c_0 \quad for\ all\ \ t > 0. \tag{2.1}$$

The next lemma directly results from an integration of the first equation in the model (1.1).

Lemma 2.3. *The solution of (1.1) satisfies $\int_\Omega u(\cdot, t) \le C$ for all $t \ge 0$ with $C > 0$.*

Lemma 2.4. *([12], Lemma 2.1) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary. Suppose $p \in (0, q)$ and $\phi \in W^{1,2}(\Omega) \bigcap L^q(\Omega)$. Then there exists a positive constant C_{GN} depending on Ω, p and q such that*

$$\|\phi\|_{L^q(\Omega)} \le C_{GN}(\|\nabla\phi\|^k_{L^2(\Omega)}\|\phi\|^{1-k}_{L^p(\Omega)} + \|\phi\|_{L^p(\Omega)}),$$

where $k \in (0, 1)$ satisfying

$$k = \frac{\frac{N}{p} - \frac{N}{q}}{1 - \frac{N}{2} + \frac{N}{p}}.$$

3. Proof of Theorem 1.1

Proof. Testing the first equation of (1.1) by $u^{p-1}(p > 1)$ and integrating by part over Ω, we have that there exists $C_1 > 0$ such that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \\
&\le \chi(p-1)\int_\Omega u^{p-1}(1+u)^{r-1}\nabla u \cdot \nabla v \\
&\quad - \xi(p-1)\int_\Omega u^{p-1}(1+u)^{r-1}\nabla u \cdot \nabla w + a\int_\Omega u^{p-1} - b\int_\Omega u^{p+\eta-1} \\
&\le -\chi(p-1)\int_\Omega \left[\int_0^u s^{p-1}(1+s)^{r-1}ds\right]\Delta v \\
&\quad + \xi(p-1)\int_\Omega \left[\int_0^u s^{p-1}(1+s)^{r-1}ds\right]\Delta w + a\int_\Omega u^{p-1} - b\int_\Omega u^{p+\eta-1} \\
&\le (\chi\alpha - \xi\gamma)(p-1)\int_\Omega \left[\int_0^u s^{p-1}(1+s)^{r-1}ds\right]u \\
&\quad + \delta\xi(p-1)\int_\Omega [\int_0^u s^{p-1}(1+s)^{r-1}ds]w + a\int_\Omega u^{p-1} - b\int_\Omega u^{p+\eta-1} \\
&\le \frac{C_1\delta\xi(p-1)}{p+r-1}\int_\Omega (u^{p+r-1}+1)w + a\int_\Omega u^{p-1} - b\int_\Omega u^{p+\eta-1}.
\end{aligned} \tag{3.1}$$

When $r < \frac{2}{N}$, for any $\epsilon > 0$, by Young's inequality,

$$\frac{\delta\xi(p-1)}{p+r-1}\int_\Omega (u^{p+r-1}+1)w \le \frac{\epsilon}{2}\int_\Omega u^{p+r} + C_2\int_\Omega w^{p+r} + C_3$$

with $C_2, C_3 > 0$. By (2.1) with $\kappa = \frac{\epsilon}{2C_2}$,we have

$$\frac{\delta\xi(p-1)}{p+r-1}\int_\Omega (u^{p+r-1}+1)w \le \epsilon\int_\Omega u^{p+r} + C_4 \tag{3.2}$$

with $C_4 > 0$. By the Gagliardo-Nirenberg inequality and Lemma 2.3, we discover that there exist $C_5, C_6 > 0$ such that

$$\begin{aligned}\int_\Omega u^{p+r} &= \|u^{\frac{p}{2}}\|^{\frac{2(p+r)}{p}}_{L^{\frac{2(p+r)}{p}}(\Omega)}\\ &\le C_5\|\nabla u^{\frac{p}{2}}\|^{\frac{2(p+r)}{p}\cdot k_1}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{\frac{2(p+r)}{p}\cdot(1-k_1)}_{L^{\frac{2}{p}}(\Omega)} + C_5\|u^{\frac{p}{2}}\|^{\frac{2(p+r)}{p}}_{L^{\frac{2}{p}}(\Omega)}\\ &\le C_6\|\nabla u^{\frac{p}{2}}\|^{\frac{2(p+r)}{p}\cdot k_1}_{L^2(\Omega)} + C_6\end{aligned}$$

for all $t > 0$, where

$$k_1 = \frac{\frac{Np}{2} - \frac{Np}{2(p+r)}}{1 - \frac{N}{2} + \frac{Np}{2}} \in (0,1).$$

We know from $r < \frac{2}{N}$ that $\frac{2(p+r)}{p}\cdot k_1 < 2$. Consequently

$$\int_\Omega u^{p+r} \le C_7\|\nabla u^{\frac{p}{2}}\|^2_{L^2(\Omega)} + C_7 \tag{3.3}$$

for all $t > 0$ with $C_7 > 0$. We use Young's inequality such that

$$a\int_\Omega u^{p-1} \le b\int_\Omega u^{p+\eta-1} + C_8 \ \text{ and } \ \int_\Omega u^p \le \epsilon\int_\Omega u^{p+r} + C_9 \tag{3.4}$$

with $C_8, C_9 > 0$. Finally, letting $\epsilon = \frac{2(p-1)}{C_7 p^2}$ and combining (3.1)–(3.4), we can obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le C_{10} \ \text{ for all } \ t > 0$$

with $C_{10} > 0$, thus, by an ODE comparison argument, we have that $\int_\Omega u^p < C(p)$.

When $\frac{2}{N} \le r < \frac{4(N+1)}{N(N+2)}$, we divide the first term on the right side of (3.1) into two terms by Young's inequality

$$\frac{C_1\delta\xi(p-1)}{p+r-1}\int_\Omega (u^{p+r-1}+1)w \le \lambda_1(\epsilon)\int_\Omega w^{\frac{Np+2}{N(1-r)+2}} + \epsilon_1\int_\Omega u^{\frac{Np+2}{N}} + C_{11} \tag{3.5}$$

for all $t > 0$ with $\lambda_1(\epsilon_1) > 0$, where the positive constant ϵ_1 is to be determined later. For the term $\lambda_1(\epsilon_1)\int_\Omega w^{\frac{Np+2}{N(1-r)+2}}$ in (3.5), testing $w^{\frac{N(p+r-1)}{N(1-r)+2}}$ on both sides of the third equation of the model (1.1) and using integrating by parts, we can derive

$$\frac{4N(p+r-1)[N(1-r)+2]}{[Np+2]^2}\|\nabla w^{\frac{Np+2}{2[N(1-r)+2]}}\|^2_{L^2(\Omega)} + \delta\int_\Omega w^{\frac{Np+2}{N(1-r)+2}} = \gamma\int_\Omega uw^{\frac{N(p+r-1)}{N(1-r)+2}} \tag{3.6}$$

for all $t > 0$. Then we divide the right side of (3.6) into two terms by the Young inequality again

$$\gamma \int_\Omega u w^{\frac{N(p+r-1)}{N(1-r)+2}} \le \epsilon_2 \int_\Omega u^{\frac{Np+2}{N}} + \lambda_2(\epsilon_2) \int_\Omega w^{\frac{N(p+r-1)}{N(1-r)+2}\cdot\frac{Np+2}{N(p-1)+2}} \tag{3.7}$$

for all $t > 0$ with $\lambda_2(\epsilon_2) > 0$, where $\epsilon_2 \le \frac{\delta\epsilon_1}{\lambda_1(\epsilon_1)}$. For the second term on the right side of (3.7), we utilize the Gagliardo-Nirenberg inequality to see that

$$\begin{aligned}
&\lambda_2(\epsilon_2) \int_\Omega w^{\frac{N(p+r-1)}{N(1-r)+2}\cdot\frac{Np+2}{N(p-1)+2}} \\
&= \lambda_2(\epsilon_2) \| w^{\frac{Np+2}{2[N(1-r)+2]}} \|^{\frac{2N(p+r-1)}{N(p-1)+2}}_{L^{\frac{2N(p+r-1)}{N(p-1)+2}}(\Omega)} \\
&\le C_{12} \Bigg(\| \nabla w^{\frac{Np+2}{2[N(1-r)+2]}} \|^{\frac{2N(p+r-1)}{N(p-1)+2}\cdot k_2}_{L^2(\Omega)} \| w^{\frac{Np+2}{2[N(1-r)+2]}} \|^{\frac{2N(p+r-1)}{N(p-1)+2}\cdot(1-k_2)}_{L^{\frac{2[N(1-r)+2]}{Np+2}}(\Omega)} \\
&\quad + \| w^{\frac{Np+2}{2[N(1-r)+2]}} \|^{\frac{2N(p+r-1)}{N(p-1)+2}}_{L^{\frac{2[N(1-r)+2]}{Np+2}}(\Omega)} \Bigg)
\end{aligned} \tag{3.8}$$

for all $t > 0$ with $C_{12} > 0$, where

$$k_2 = \frac{\frac{N[Np+2]}{2[N(1-r)+2]} - \frac{[N(p-1)+2]}{2(p+r-1)}}{1 - \frac{N}{2} + \frac{N(Np+2)}{2[N(1-r)+2]}} \in (0, 1).$$

Then we can choose p sufficiently large such that

$$\frac{2N(p+r-1)}{N(p-1)+2} \cdot k_2 = \frac{\frac{N^2(p+r-1)(Np+2)}{[N(p-1)+2][N(1-r)+2]} - N}{1 - \frac{N}{2} + \frac{N[Np+2]}{2[N(1-r)+2]}} < 2$$

since $r < \frac{4(N+1)}{N(N+2)}$. Substituting it into (3.8) and applying Lemmas 2.2 and 2.3, the Young inequality yields

$$\lambda_2(\epsilon_2) \int_\Omega w^{\frac{N(p+r-1)}{N(1-r)+2}\cdot\frac{Np+2}{N(p-1)+2}} \le \frac{4N(p+r-1)[N(1-r)+2]}{[Np+2]^2} \| \nabla w^{\frac{Np+2}{2[N(1-r)+2]}} \|^2_{L^2(\Omega)} + C_{13} \tag{3.9}$$

for all $t > 0$ with $C_{13} > 0$. Combining (3.9) with (3.6) and (3.7), we obtain

$$\delta \int_\Omega w^{\frac{Np+2}{N(1-r)+2}} \le \epsilon_2 \int_\Omega u^{\frac{Np+2}{N}} + C_{14} \tag{3.10}$$

for all $t > 0$ with $C_{14} > 0$. Then, combining (3.1) with (3.5) and (3.10) implies

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt} \int_\Omega u^p + \frac{4(p-1)}{p^2} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 \\
&\qquad \le 2\epsilon_1 \int_\Omega u^{\frac{Np+2}{N}} + a \int_\Omega u^{p-1} - b \int_\Omega u^{p+\eta-1} + C_{15}
\end{aligned} \tag{3.11}$$

with $C_{15} > 0$ since $\epsilon_2 \le \frac{\delta\epsilon_1}{\lambda_1(\epsilon_1)}$. Through the Gagliardo-Nirenberg inequality, we discover that

$$\begin{aligned}
&\int_\Omega u^{\frac{Np+2}{N}} = \| u^{\frac{p}{2}} \|^{\frac{2[Np+2]}{Np}}_{L^{\frac{2[Np+2]}{Np}}(\Omega)} \\
&\le C_{16} \| \nabla u^{\frac{p}{2}} \|^{\frac{2[Np+2]}{Np}\cdot k_3}_{L^2(\Omega)} \| u^{\frac{p}{2}} \|^{\frac{2[Np+2]}{Np}\cdot(1-k_3)}_{L^{\frac{2}{p}}(\Omega)} + C_{16} \| u^{\frac{p}{2}} \|^{\frac{2[N(p+2]}{Np}}_{L^{\frac{2}{p}}(\Omega)}
\end{aligned} \tag{3.12}$$

for all $t > 0$ with $C_{16} > 0$, where

$$k_3 = \frac{\frac{Np}{2} - \frac{N \cdot Np}{2[Np+2]}}{1 - \frac{N}{2} + \frac{Np}{2}} = \frac{Np}{Np+2} \in (0,1),$$

then, by Lemma 2.3 and (3.12), we can choose ϵ_1 small enough such that

$$2\epsilon_1 \int_\Omega u^{\frac{Np+2}{N}} \leq \frac{4(p-1)}{p^2} \|\nabla u^{\frac{p}{2}}\|^2_{L^2(\Omega)} + C_{17} \tag{3.13}$$

with $C_{17} > 0$. Finally, we substitute (3.13) into (3.11) to conclude that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \leq a\int_\Omega u^{p-1} - b\int_\Omega u^{p+\eta-1} + C_{18} \text{ for all } t > 0$$

with $C_{18} > 0$, thus, by an ODE comparison argument, we have that $\int_\Omega u^p < C(p)$.

Next, using the standard Moser-type iteration in [9], we easily have $C_{19} > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C_{19} \text{ for all } t > 0.$$

The proof of Theorem 1.1 is completed. □

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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