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Research article

Infinite series involving harmonic numbers and reciprocal of binomial coefficients

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Abstract: Yamamoto's integral was the integral associated with 2-posets, which was first introduced by Yamamoto. In this paper, we obtained the values of infinite series involving harmonic numbers and reciprocal of binomial coefficients by using some techniques of Yamamoto's integral. We determine the value of infinite series of the form:

$$\sum_{m_1,\ldots,m_n,\ell_1,\ldots,\ell_k\geq 1} \frac{H_{m_1}^{(a_1)}\cdots H_{m_n}^{(a_n)}}{m_1^{b_1}\cdots m_n^{b_n}\ell_1^{c_1}\cdots\ell_k^{c_k}\binom{m_1+\cdots+m_n+\ell_1+\cdots+\ell_k}{\ell_k}},$$

in terms of a finite sum of multiple zeta values, for positive integers $a_1,\ldots,a_n,b_1,\ldots,b_n,c_1,\ldots,c_k$.

Keywords: multiple zeta values; harmonic numbers; binomial coefficients; Yamamoto's integral; 2-poset
Mathematics Subject Classification: 11M32, 05A10

1. Introduction

Given a r-tuple $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_r)$ of positive integers with $\alpha_r \geq 2$, a multiple zeta value $\zeta(\boldsymbol{\alpha})$ (MZV) is defined to be [6, 8, 9]

$$\zeta(\boldsymbol{\alpha}) = \sum_{1\leq k_1<k_2<\cdots<k_r} \frac{1}{k_1^{\alpha_1}k_2^{\alpha_2}\cdots k_r^{\alpha_r}}.$$

We let $\{a\}^k$ be k repetitions of a such that $\zeta(\{1\}^3,3) = \zeta(1,1,1,3)$ and $\zeta(2,\{3\}^2,5) = \zeta(2,3,3,5)$.

The generalized harmonic numbers are defined by

$$H_0^{(s)} = 0 \quad \text{and} \quad H_n^{(s)} = \sum_{j=1}^{n} \frac{1}{j^s},$$

where s and n are positive integers. In particular, $H_n^{(1)} = H_n$ is the classical harmonic number. The famous formula

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$$

was discovered by Euler. Series incorporating harmonic numbers find application in various mathematical disciplines and related fields [1, 2, 5]. It is well-known that binomial coefficients play a crucial role in various subjects, such as combinatorics, graph theory, number theory, and probability.

Sofo [16, 17] discovered closed-form representations for sums involving harmonic numbers with reciprocal binomial coefficients of various forms:

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{\binom{n+k}{k}}, \quad \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n\binom{n+k}{k}}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{H_n^2}{\binom{n+k}{k}}.$$

In the literature, numerous papers explore infinite sums involving reciprocals of binomial and harmonic numbers [1, 13, 15, 18]. Recently, Chu [7] investigated double series of the following forms:

$$\sum_{n,m\geq 1} \frac{H_n}{nm\binom{m+n}{m}} = 2\zeta(3), \quad \text{and} \quad \sum_{n,m\geq 1} \frac{H_{n+m}}{nm\binom{m+n}{m}} = 3\zeta(3).$$

Additionally, the first author [4] developed q-analogues of such series. It prompts us to explore this general type of series further. This paper focuses on determining the value of an infinite series given by the expression:

$$\sum_{m_1,\ldots,m_n,\ell_1,\ldots,\ell_k\geq 1} \frac{H_{m_1}^{(a_1)} \cdots H_{m_n}^{(a_n)}}{m_1^{b_1} \cdots m_n^{b_n} \ell_1^{c_1} \cdots \ell_k^{c_k} \binom{m_1+\cdots+m_n+\ell_1+\cdots+\ell_k}{\ell_k}}. \tag{1.1}$$

This series is evaluated as a finite sum of MZVs, with the parameters being positive integers $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_k$. For example,

$$\begin{aligned}\sum_{m,n\geq 1} \frac{H_m H_n}{m^a n^b \binom{m+n}{m}} &= \zeta(1, a, \{1\}^{b-2}, 3) + \zeta(b+1, \{1\}^{a-2}, 3) \\ &\quad + \zeta(a+1, \{1\}^{b-2}, 3) + \zeta(a+1, \{1\}^{b-1}, 2),\end{aligned}$$

where a, b are positive integers. It is noted that for our convenience, we denote $\zeta(a_1, \ldots, a_n, \{1\}^{-1}, c)$ as $\zeta(a_1, \ldots, a_n + c - 1)$. Therefore, we have

$$\sum_{m,n\geq 1} \frac{H_m H_n}{mn\binom{m+n}{m}} = 3\zeta(4) = \frac{\pi^4}{30}.$$

In the next section, we introduce the algebraic structure for MZVs, as originally proposed by Hoffman [10]. Furthermore, we present a combinatorial generalization of the iterated integral associated with a 2-poset, represented by a Hasse diagram. These integrals are referred to as Yamamoto's integrals [19, 20]. In Section 3, for given positive integers n and $a_1, a_2, \ldots, a_n$, we determine the value of

$$\sum_{m_1,\ldots,m_n\geq 1} \frac{1}{m_1^{a_1} m_2^{a_2} \cdots m_n^{a_n} \binom{m_1+\cdots+m_n}{m_n}}$$

in terms of MZVs. Then, in Section 4, we express the value of

$$\sum_{m_1,\ldots,m_n\geq 1} \frac{H_{m_1-1}^{(a_1)} H_{m_2-1}^{(a_2)} \cdots H_{m_n-1}^{(a_n)}}{m_1^{b_1} m_2^{b_2} \cdots m_n^{b_n} \binom{m_1+\cdots+m_n}{m_n}}$$

using MZVs, where n and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are positive integers. Finally, we present the conclusion of our main results in Section 5, along with additional concrete examples. For example,

$$\sum_{m,n\geq 1} \frac{H_n^{(2)}}{mn\binom{m+n}{m}} = \zeta(2,2)+\zeta(4), \qquad \sum_{m,n\geq 1} \frac{H_n^{(2)}}{m^2n^2\binom{m+n}{m}} = \zeta(2,2,2)+\zeta(4,2),$$

$$\sum_{m,n\geq 1} \frac{H_n^{(3)}}{m^3n^3\binom{m+n}{m}} = \zeta(3,3,1,2)+\zeta(6,1,2), \qquad \sum_{m,n,k\geq 1} \frac{H_k}{mnk\binom{m+n+k}{k}} = 2\zeta(4)+2\zeta(1,3).$$

2. Algebraic settings and integrals associated with 2-posets

We review the algebraic setup of MZVs introduced by Hoffman [10]. Let $\mathbb{Q}\langle x, y\rangle$ be the $\mathbb{Q}$-algebra of polynomials in two noncommutative variables which is graded by the degree (where each of the variables x and y is assumed to be of degree 1); we identify the algebra $\mathbb{Q}\langle x, y\rangle$ with the graded $\mathbb{Q}$-vector space $\mathfrak{H}$ spanned by the monomials in the variables x and y.

We also introduce the graded $\mathbb{Q}$-vector spaces $\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}$ by $\mathfrak{H}^1 = \mathbb{Q}\mathbf{1} \bigoplus x\mathfrak{H}$, $\mathfrak{H}^0 = \mathbb{Q}\mathbf{1} \bigoplus x\mathfrak{H}y$, where $\mathbf{1}$ denotes the unit (the empty word of weight 0 and length 0) of the algebra $\mathfrak{H}$. A word starts with x and ends with y, and we refer to such words as "admissible." In other words, the subalgebra $\mathfrak{H}^0$ is generated by admissibile words. Let $Z : \mathfrak{H}^0 \to \mathbb{R}$ be the $\mathbb{Q}$-linear map that assigns to each word $u_1u_2\cdots u_k$ in $\mathfrak{H}^0$, where $u_i \in \{x, y\}$, the multiple integral

$$\int_{0<t_1<\cdots<t_k<1} w_{u_1}(t_1)w_{u_2}(t_2)\cdots w_{u_k}(t_k). \tag{2.1}$$

Here, $w_x(t) = dt/(1-t)$, $w_y(t) = dt/t$. As the word $u_1u_2\cdots u_k$ is in $\mathfrak{H}^0$, we always have $w_{u_1}(t) = dt/(1-t)$ and $w_{u_k}(t) = dt/t$, so the integral converges. The space $\mathfrak{H}^1$ can be regarded as the subalgebra of $\mathbb{Q}\langle x, y\rangle$ generated by the words $z_s = xy^{s-1}$ $(s = 1, 2, 3, \ldots)$.

Let us define the bilinear product ⧢ (the *shuffle product*) on $\mathfrak{H}$ by the rules

$$1 ⧢ w = w ⧢ 1 = w, \tag{2.2}$$

for any word w, and

$$w_1x_1 ⧢ w_2x_2 = (w_1 ⧢ w_2x_2)x_1 + (w_1x_1 ⧢ w_2)x_2, \tag{2.3}$$

for any words w_1, w_2, any letters $x_i = x$ or y $(i = 1, 2)$, and then extend the above rules to the whole algebra $\mathfrak{H}$ and the whole subalgebra $\mathfrak{H}^1$ by linearity. It is known that each of the above products is commutative and associative. We denote the algebras $(\mathfrak{H}^1, +, ⧢)$ by $\mathfrak{H}^1_{⧢}$. By the standard shuffle product identity of iterated integrals, the evaluation map Z is again an algebra homomorphism for the multiplication ⧢ [11]:

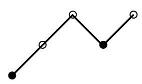
$$Z(w_1 ⧢ w_2) = Z(w_1)Z(w_2). \tag{2.4}$$

Yamamoto [19] introduced a combinatorial generalization of the iterated integral, the integral associated with a 2-poset. We review the definitions and basic properties of 2-labeled posets (we will call them 2-posets for short in this paper) and the associated integrals first introduced by Yamamoto [19].

Definition 2.1. [12, Definition 3.1] A *2-poset* is a pair (X, δ_X), where $X = (X, \leq)$ is a finite partially ordered set (poset for short) and δ_X is a map from X to $\{0, 1\}$. We often omit δ_X and simply say "a 2-poset X." The δ_X is called the *label map* of X.

A 2-poset (X, δ_X) is called *admissible* if $\delta_X(x) = 0$ for all maximal elements $x \in X$ and $\delta_X(x) = 1$ for all minimal elements $x \in X$.

A 2-poset is depicted as a Hasse diagram in which an element x with $\delta(x) = 0$ (resp. $\delta(x) = 1$) is represented by ◦ (resp. •). For example, the diagram



represents the 2-poset $X = \{x_1, x_2, x_3, x_4, x_5\}$ with order $x_1 < x_2 < x_3 > x_4 < x_5$ and label $(\delta_X(x_1), \ldots, \delta_X(x_5)) = (1, 0, 0, 1, 0)$.

Definition 2.2. [12, Definition 3.2] For an admissible 2-poset X, we define the associated integral

$$I(X) = \int_{\Delta_X} \prod_{x \in X} \omega_{\delta_X(x)}(t_x), \tag{2.5}$$

where

$$\Delta_X = \{(t_x)_x \in [0, 1]^X \mid t_x < t_y \text{ if } x < y\} \quad \text{and} \quad \omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1-t}.$$

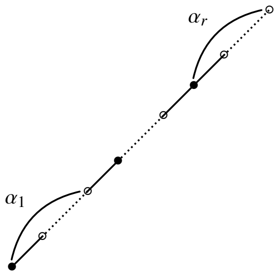
Note that the admissibility of a 2-poset corresponds to the convergence of the associated integral. We also recall an algebraic setup for 2-posets (cf. Remark at the end of §2 of [19]). Let $\mathfrak{P}$ be the $\mathbb{Q}$-algebra generated by the isomorphism classes of 2-posets, whose multiplication is given by the disjoint union of 2-posets. Then, the integral (2.5) defines a $\mathbb{Q}$-algebra homomorphism $I\colon \mathfrak{P}^0 \to \mathbb{R}$ from the subalgebra $\mathfrak{P}^0$ of $\mathfrak{P}$ generated by the classes of admissible 2-posets. We refer to this type of integral as Yamamoto's integral.

It is known that [20] there is a $\mathbb{Q}$-linear map

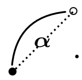
$$W : \mathfrak{P} \to \mathfrak{H}, \tag{2.6}$$

which transforms a 2-poset into a finite sum of words in x and y. This transformation is characterized by the following two conditions: The first condition states that for a totally ordered $X = x_1 < x_2 < \cdots < x_k$, $W(X) = z_{\delta(x_1)} z_{\delta(x_2)} \cdots z_{\delta(x_k)}$, and the second condition asserts that if a and b are noncomparable in X, then $W(X)$ can be expressed as $W(X^b_a) + W(X^a_b)$, where X^b_a represents the 2-poset obtained from X by adjoining a new relation $a < b$. This W sends $\mathfrak{P}^0$ onto $\mathfrak{H}^0$ and satisfies $I = Z \circ W : \mathfrak{P}^0 \to \mathbb{R}$.

Here, we present three known examples in [19, 20]: For an index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ (admissible or not), we write the 'totally ordered' diagram:

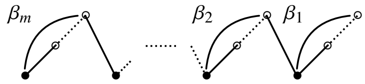


as

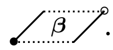


.

For an index $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$, we write the following diagram:

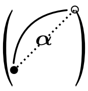
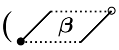


as

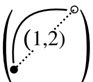
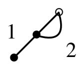
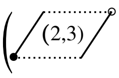
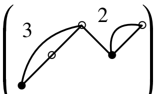


.

Then, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are admissible, we have [20, Propositions 2.4 and 2.7]

$\zeta(\boldsymbol{\alpha}) = I\Big($  $\Big)$, and $\zeta^\star(\boldsymbol{\beta}) = I\,($  $)$. (2.7)

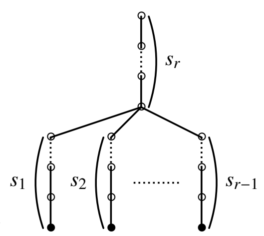
For example,

$\zeta(1,2) = I\Big($  $\Big) = I\Big($  $\Big)$, $\zeta^\star(2,3) = I\Big($  $\Big) = I\Big($  $\Big)$.

The last example is the MZV of Mordell-Tornheim type which is defined by the series

$$\zeta_{MT}(s_1, \ldots, s_{r-1}; s_r) = \sum_{m_1, \ldots, m_r \geq 1} \frac{1}{m_1^{s_1} \cdots m_{r-1}^{s_{r-1}} (m_1 + \cdots + m_{r-1})^{s_r}}.$$

They have the following integral form [20, Proposition 2.8]

$\zeta_{MT}(s_1, \ldots, s_{r-1}; s_r) = I\Bigg($ 

$\Bigg)$.

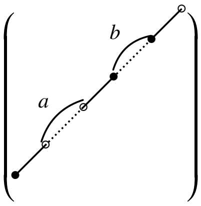
3. Series with reciprocal of binomial coefficients

We first address the basic form of double series, employing primarily the approach of evaluating Yamamoto's integral using both integral and series modes separately.

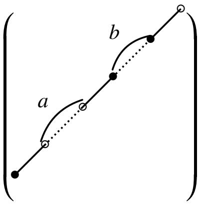
Theorem 3.1. *For integers $a \geq 0$ and $b \geq 0$, we have*

$$\sum_{m,n \geq 1} \frac{1}{m^{a+1} n^{b+1} \binom{m+n}{m}} = \zeta(a+1, \{1\}^{b-1}, 2). \tag{3.1}$$

Proof. Using Eq (2.7), we know that

I  $= \zeta(a+1, \{1\}^{b-1}, 2).$ (3.2)

On the other hand, we evaluate this Yamamoto's integral as follows.

I  $= \int_{0<t_1<t_2<\cdots<t_{a+1}<s_1<\cdots<s_{b+1}<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{a+1}}{t_{a+1}}\frac{ds_1}{1-s_1}\cdots\frac{ds_b}{1-s_b}\frac{ds_{b+1}}{s_{b+1}}.$

We treat this integral as

$$\int_0^1 A(s_1)B(s_1)\frac{ds_1}{1-s_1},$$

where

$$A(s_1) = \int_{0<t_1<t_2<\cdots<t_{a+1}<s_1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{a+1}}{t_{a+1}} \quad \text{and} \quad B(s_1) = \int_{s_1<s_2<\cdots<s_{b+1}<1} \frac{ds_2}{1-s_2}\cdots\frac{ds_b}{1-s_b}\frac{ds_{b+1}}{s_{b+1}}.$$

We represent $A(s_1)$ and $B(s_1)$ as a power series:

$$A(s_1) = \sum_{m\geq 1}\frac{s_1^m}{m^{a+1}} \quad \text{and} \quad B(s_1) = \sum_{n\geq 1}\frac{(1-s_1)^n}{n^b}.$$

Therefore,

$$\int_0^1 A(s_1)B(s_1)\frac{ds_1}{1-s_1} = \sum_{n,m\geq 1}\frac{1}{m^{a+1}n^b}\int_0^1 s_1^m(1-s_1)^{n-1}\,ds_1.$$

Since the integral in the righthand side is the beta function, we obtain

$$\sum_{m,n\geq 1}\frac{\Gamma(m+1)\Gamma(n)}{m^{a+1}n^b\Gamma(m+n+1)} = \sum_{m,n\geq 1}\frac{1}{m^{a+1}n^{b+1}\binom{m+n}{m}}.$$

Combing Eq (3.2), we conclude our result. □

Next, we generalize the result stated in the theorem above to a more general form.

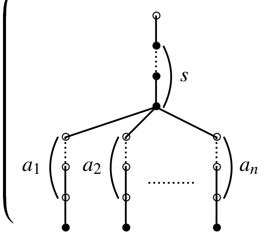
Theorem 3.2. *Given $n+1$ nonnegative integers $a_1, a_2, \ldots, a_n$ and s with $\sum_{i=1}^n a_i = w$, we have*

$$\begin{aligned}&\sum_{m_1,\ldots,m_{n+1}\geq 1}\frac{1}{m_1^{a_1+1}m_2^{a_2+1}\cdots m_n^{a_n+1}m_{n+1}^{s+1}\binom{m_1+\cdots+m_{n+1}}{m_{n+1}}} \\ &= \sum_{\substack{d_1+\cdots+d_n=w\\ d_i>=0,\forall i}}\zeta(d_1+1,\ldots,d_n+1,\{1\}^{s-1},2)\sum_{\sigma\in S_n}\sigma_a\left\{\prod_{j=2}^n\binom{\sum_{k=j}^n d_k - \sum_{k=j+1}^n a_k}{a_j}\right\},\end{aligned} \tag{3.3}$$

where S_n is the symmetric group of n objects and $\sigma_{\boldsymbol{a}}$ is induced permutations of $\sigma \in S_n$ on the set $\{a_1, a_2, \ldots, a_n\}$.

Proof. Consider the following Yamamoto's integral

$$A = I\left(\quad \right).$$



We use the map W which is defined in Eq (2.6) to transform the n legs in the 2-poset as $(xy^{a_1}, xy^{a_2}, \ldots, xy^{a_n})$. Since (see [3, Eq (4.6)])

$$xy^{a_1} ⧢ xy^{a_2} ⧢ \cdots ⧢ xy^{a_n} = \sum_{\substack{d_1+\cdots+d_n=w \\ d_i>=0,\forall i}} \prod_{\ell=1}^{n} xy^{d_\ell} \sum_{\sigma\in S_n} \sigma_{\boldsymbol{a}} \left\{ \prod_{j=2}^{n} \binom{\sum_{k=j}^{n} d_k - \sum_{k=j+1}^{n} a_k}{a_j} \right\},$$

where S_n is the symmetric group of n objects and $\sigma_{\boldsymbol{a}}$ is induced permutations of $\sigma \in S_n$ on the set $\{a_1, a_2, \ldots, a_n\}$. Then, we shuffle them together as a totally ordered diagram, and by Eq (2.7) we have

$$A = \sum_{\substack{d_1+\cdots+d_n=w \\ d_i>=0,\forall i}} \zeta(d_1+1, \ldots, d_n+1, \{1\}^{s-1}, 2) \sum_{\sigma\in S_n} \sigma_a \left\{ \prod_{j=2}^{n} \binom{\sum_{k=j}^{n} d_k - \sum_{k=j+1}^{n} a_k}{a_j} \right\}. \tag{3.4}$$

On the other hand, we write the Yamamoto's integral as follows.

$$A = \int_0^1 B_1(t)B_2(t)\cdots B_n(t)\cdot C(t)\frac{dt}{1-t},$$

where

$$B_i(t) = \int_{0<t_1^{(i)}<t_2^{(i)}<\cdots<t_{a_i+1}^{(i)}<t} \frac{dt_1^{(i)}}{1-t_1^{(i)}} \frac{dt_2^{(i)}}{t_2^{(i)}} \cdots \frac{dt_{a_i+1}^{(i)}}{t_{a_i+1}^{(i)}} \quad \text{and} \quad C(t) = \int_{t<u_1<\cdots<u_s<1} \frac{du_1}{1-u_1} \cdots \frac{du_{s-1}}{1-u_{s-1}} \frac{du_s}{u_s}.$$

We expand them into a power series:

$$B_i(t) = \sum_{m_i\ge1} \frac{t^{m_i}}{m_i^{a_i+1}} \quad \text{and} \quad C(t) = \sum_{m_{n+1}\ge1} \frac{(1-t)^{m_{n+1}}}{m_{n+1}^{s}}.$$

Therefore,

$$A = \sum_{m_1,\ldots,m_{n+1}\ge1} \frac{1}{m_1^{a_1+1}\cdots m_n^{a_n+1} m_{n+1}^{s}} \int_0^1 t^{m_1+\cdots+m_n}(1-t)^{m_{n+1}-1}\,dt.$$

Given that the integral on the righthand side is the beta function, we deduce that

$$A = \sum_{m_1,\ldots,m_{n+1}\ge1} \frac{1}{m_1^{a_1+1} m_2^{a_2+1}\cdots m_n^{a_n+1} m_{n+1}^{s+1}\binom{m_1+\cdots+m_{n+1}}{m_{n+1}}}. \tag{3.5}$$

Combining the two expressions for A given by Eqs (3.4) and (3.5), we obtain the desired formula. □

4. Series involving harmonic numbers

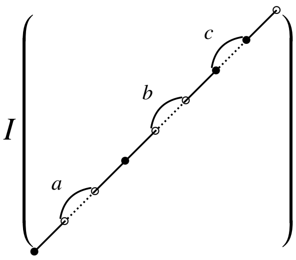
In this section, we begin by computing the double series form where the numerator involves harmonic numbers.

Theorem 4.1. *Given three nonnegative integers a, b, c, we have*

$$\sum_{m,n\geq 1} \frac{H_{n-1}^{(a+1)}}{n^{b+1}m^{c+1}\binom{m+n}{m}} = \zeta(a+1, b+1, \{1\}^{c-1}, 2). \tag{4.1}$$

Proof. Consider the following Yamamoto's integral:

$A = I$



.

Using Eq (2.7), we know that

$$A = \zeta(a+1, b+1, \{1\}^{c-1}, 2). \tag{4.2}$$

On the other hand, we evaluate this Yamamoto's integral as follows.

$$A = \int_\Delta \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{a+1}}{t_{a+1}}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\cdots\frac{ds_{b+1}}{s_{b+1}}\frac{dw_1}{1-w_1}\cdots\frac{dw_c}{1-w_c}\frac{dw_{c+1}}{w_{c+1}},$$

where Δ is a simplex in $[0,1]^{a+b+c+3}$ with

$$t_1 < \cdots < t_{a+1} < s_1 < \cdots < s_{b+1} < w_1 < \cdots < w_{c+1}.$$

We treat this integral as

$$\int_0^1 B(w_1)C(w_1)\frac{dw_1}{1-w_1},$$

where

$$\begin{aligned} B(w_1) &= \int_{0<t_1<\cdots<t_{a+1}<s_1<\cdots<s_{b+1}<w_1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{a+1}}{t_{a+1}}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\cdots\frac{ds_{b+1}}{s_{b+1}}, \\ C(w_1) &= \int_{w_1<w_2<\cdots<w_{c+1}<1} \frac{dw_2}{1-w_2}\cdots\frac{dw_c}{1-w_c}\frac{dw_{c+1}}{w_{c+1}}. \end{aligned}$$

We represent $B(w_1)$ and $C(w_1)$ as a power series:

$$B(w_1) = \sum_{1\leq k<n} \frac{w_1^n}{k^{a+1}n^{b+1}} \quad \text{and} \quad C(w_1) = \sum_{m\geq 1} \frac{(1-w_1)^m}{m^c}.$$

Therefore,

$$\int_0^1 B(w_1)C(w_1)\frac{dw_1}{1-w_1} = \sum_{n,m\geq 1} \frac{H_{n-1}^{(a+1)}}{n^{b+1}m^c}\int_0^1 w_1^n(1-w_1)^{m-1}\,dw_1.$$

Since the integral on the righthand side is the beta function, and by combining it with Eq (4.2), we arrive at our result. □

Next, we extend this to multiple series and sums where the numerator involves additional harmonic numbers. However, before proceeding further, we will introduce some basic concepts regarding the shuffle product identities that we will utilize.

The shuffle product can be described combinatorially. Utilizing the definition of the shuffle product $ш$ in $\mathfrak{H}$, we readily obtain

$$u_1u_2\cdots u_n \,ш\, u_{n+1}u_{n+2}\cdots u_{n+m} = \sum_{\sigma\in S_{n,m}} u_{\sigma(1)}u_{\sigma(2)}\cdots u_{\sigma(n+m)},$$

where $u_i \in \{x, y\}$ are letters, and

$$S_{n,m} = \left\{\sigma \in S_{n+m} \,\middle|\, \begin{array}{c} \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n), \\ \sigma^{-1}(n+1) < \sigma^{-1}(n+2) < \cdots < \sigma^{-1}(n+m) \end{array}\right\}.$$

Therefore,

$$xy^{a_1}xy^{b_1} \,ш\, xy^{a_2}xy^{b_2} \,ш\, \cdots \,ш\, xy^{a_n}xy^{b_n} = \sum_{\substack{\sum_{i=1}^{2n} d_i=\sum_{i=1}^{n}(a_i+b_i) \\ d_i>=0,\forall i}} C(d_1,\ldots,d_{2n}) \prod_{j=1}^{2n} xy^{d_j}, \tag{4.3}$$

where $C(d_1,\ldots,d_{2n})$ is a suitable constant depending on the variables $d_1,\ldots,d_{2n}$.

Some explicit formulas can be found in [14].

Here, we express a formula of $xy^axy^b \,ш\, xy^cxy^d$ as follows (ref. [14, Eq (2.11)]).

$$\begin{aligned}
&xy^axy^b \,ш\, xy^cxy^d \\
&= \sum_{b_1+b_2+b_3=b} \binom{b_2+c}{b_2}\binom{b_3+d}{b_3} xy^a xy^{b_1} xy^{b_2+c} xy^{b_3+d} \\
&\quad + \sum_{d_1+d_2+d_3=d} \binom{d_2+a}{d_2}\binom{d_3+b}{d_3} xy^c xy^{d_1} xy^{d_2+a} xy^{d_3+b} \\
&\quad + \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b \\ c_1+c_2=c}} \binom{a_2+c_1}{a_2}\binom{c_2+b_1}{c_2}\binom{b_2+d}{b_2} xy^{a_1} xy^{a_2+c_1} xy^{c_2+b_1} xy^{b_2+d} \\
&\quad + \sum_{\substack{d_1+d_2=d \\ c_1+c_2=c \\ a_1+a_2=a}} \binom{c_2+a_1}{c_2}\binom{a_2+d_1}{a_2}\binom{d_2+b}{d_2} xy^{c_1} xy^{c_2+a_1} xy^{a_2+d_1} xy^{d_2+b} \\
&\quad + \sum_{\substack{a_1+a_2+a_3=a \\ d_1+d_2=d}} \binom{a_2+c}{a_2}\binom{a_3+d_1}{a_3}\binom{d_2+b}{d_2} xy^{a_1} xy^{a_2+c} xy^{a_3+d_1} xy^{d_2+b} \\
&\quad + \sum_{\substack{c_1+c_2+c_3=c \\ b_1+b_2=b}} \binom{c_2+a}{c_2}\binom{c_3+b_1}{c_3}\binom{b_2+d}{b_2} xy^{c_1} xy^{c_2+a} xy^{c_3+b_1} xy^{b_2+d}.
\end{aligned} \tag{4.4}$$

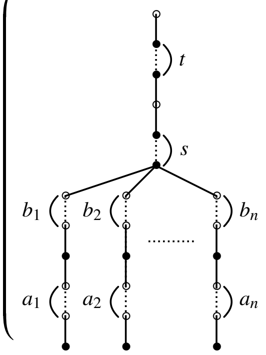
Theorem 4.2. *Given $2n+2$ nonnegative integers $a_1,\ldots,a_n,b_1,\ldots,b_n$, and s,t with $\sum_{i=1}^{n} a_i+b_i = w$, we have*

$$\sum_{m_1,\ldots,m_{n+1}\ge 1} \frac{H_{m_1-1}^{(a_1+1)}H_{m_2-1}^{(a_2+1)}\cdots H_{m_n-1}^{(a_n+1)}H_{m_{n+1}-1}^{(t+1)}}{m_1^{b_1+1}m_2^{b_2+1}\cdots m_n^{b_n+1}m_{n+1}^{s+1}\binom{m_1+\cdots+m_{n+1}}{m_{n+1}}} \tag{4.5}$$

$$= \sum_{\substack{d_1+\cdots+d_{2n}=w \\ d_i>=0,\forall i}} C(d_1,\ldots,d_{2n})\zeta(d_1+1,\ldots,d_{2n}+1,\{1\}^{s-1},2,\{1\}^{t-1},2),$$

where $C(d_1,\ldots,d_{2n})$ is defined in Eq (4.3).

Proof. Consider the following Yamamoto's integral:

$A = I$ ().

We use the map W which is defined in Eq (2.6) to transform the n legs in the 2-poset as $(xy^{a_1}xy^{b_1}, xy^{a_2}xy^{b_2},\ldots,xy^{a_n}xy^{b_n})$. Then, we shuffle them together (by Eq (4.3)) as a totally ordered diagram, and by Eq (2.7) we have

$$A = \sum_{\substack{d_1+\cdots+d_{2n}=w \\ d_i>=0,\forall i}} C(d_1,\ldots,d_{2n})\zeta(d_1+1,\ldots,d_{2n}+1,\{1\}^{s-1},2,\{1\}^{t-1},2). \tag{4.6}$$

On the other hand, we write the Yamamoto's integral as follows.

$$A = \int_0^1 B_1(t)B_2(t)\cdots B_n(t)\cdot C(t)\frac{dt}{1-t},$$

where

$$\begin{aligned} B_i(t) &= \int_{0<t_1<\cdots<t_{a_i+1}<s_1<\cdots<s_{b_i+1}<t} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{a_i+1}}{t_{a_i+1}}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\cdots\frac{ds_{b_i+1}}{s_{b_i+1}}, \\ C(t) &= \int_{t<w_2<\cdots<w_{s+1}<u_1<\cdots<u_{t+1}<1} \frac{dw_2}{1-w_2}\cdots\frac{dw_s}{1-w_s}\frac{dw_{s+1}}{w_{s+1}}\frac{du_1}{1-u_1}\cdots\frac{du_t}{1-u_t}\frac{du_{t+1}}{u_{t+1}}. \end{aligned}$$

We represent $B(w_1)$ and $C(w_1)$ as a power series:

$$B_i(t) = \sum_{1\le k_i<m_i}\frac{t^{m_i}}{k_i^{a_i+1}m_i^{b_i+1}} \quad \text{and} \quad C(t) = \sum_{1\le \ell<m_{n+1}}\frac{(1-t)^{m_{n+1}}}{\ell^{t+1}m_{n+1}^{s}}.$$

Therefore,

$$A = \sum_{m_1,\ldots,m_{n+1}\ge 1}\frac{H_{m_1-1}^{(a_1+1)}H_{m_2-1}^{(a_2+1)}\cdots H_{m_n-1}^{(a_n+1)}H_{m_{n+1}-1}^{(t+1)}}{m_1^{b_1+1}m_2^{b_2+1}\cdots m_n^{b_n+1}m_{n+1}^{s}}\int_0^1 t^{m_1+\cdots+m_n}(1-t)^{m_{n+1}-1}\,dt.$$

Given that the integral on the righthand side is the beta function, we deduce that

$$A = \sum_{m_1,\ldots,m_{n+1}\ge 1}\frac{H_{m_1-1}^{(a_1+1)}H_{m_2-1}^{(a_2+1)}\cdots H_{m_n-1}^{(a_n+1)}H_{m_{n+1}-1}^{(t+1)}}{m_1^{b_1+1}m_2^{b_2+1}\cdots m_n^{b_n+1}m_{n+1}^{s+1}\binom{m_1+\cdots+m_{n+1}}{m_{n+1}}}. \tag{4.7}$$

Combining the two expressions for A given by Eqs (4.6) and (4.7), we obtain the desired formula. □

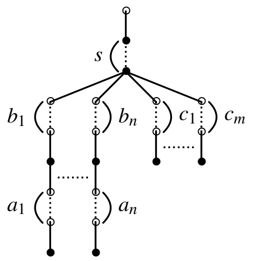
5. Main results and conclusions

Now, we treat the main result with the following fact [14]:

$$\left(ш_{i=1}^{n} xy^{a_i}xy^{b_i}\right) ш \left(ш_{k=1}^{m} xy^{c_k}\right) = \sum_{\substack{\sum_{i=1}^{2n+m} d_i=\sum_{i=1}^{n}(a_i+b_i)+\sum_{k=1}^{m} c_k \\ d_i>=0,\forall i}} E(d_1,\ldots,d_{2n+m}) \prod_{j=1}^{2n+m} xy^{d_j}, \tag{5.1}$$

where $E(d_1,\ldots,d_{2n+m})$ is a suitable constant depending on the variables $d_1,\ldots,d_{2n+m}$.

Given $2n+m+1$ nonnegative integers $a_1,\ldots,a_n,b_1,\ldots,b_n,c_1,\ldots,c_m,s$, with $\sum_{i=1}^{n}(a_i+b_i)+\sum_{k=1}^{m} c_k = w$, we have

I ()

$$= \sum_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_{m+1}\geq 1} \frac{H_{k_1-1}^{(a_1+1)}H_{k_2-1}^{(a_2+1)}\cdots H_{k_n-1}^{(a_n+1)}}{k_1^{b_1+1}\cdots k_n^{b_n+1}\ell_1^{c_1+1}\cdots \ell_m^{c_m+1}\ell_{m+1}^{s+1}\binom{k_1+\cdots+k_n+\ell_1+\cdots+\ell_{m+1}}{\ell_{m+1}}}$$

$$= \sum_{\substack{\sum_{i=1}^{2n+m} d_i=w \\ d_i>=0,\forall i}} E(d_1,\ldots,d_{2n+m})\zeta(d_1+1,\ldots,d_{2n+m}+1,\{1\}^{s-1},2).$$

Since $H_{k_i}^{(a_i+1)} = H_{k_i-1}^{(a_i+1)} + \frac{1}{k_i^{a_i+1}}$, in combination with Theorem 3.1, we conclude our main result.

Theorem 5.1. *For any nonnegative integers $n,m,a_1,\ldots,a_n,b_1,\ldots,b_n,c_1,\ldots,c_m,s$, the following infinite series*

$$\sum_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_{m+1}\geq 1} \frac{H_{k_1}^{(a_1+1)}H_{k_2}^{(a_2+1)}\cdots H_{k_n}^{(a_n+1)}}{k_1^{b_1+1}\cdots k_n^{b_n+1}\ell_1^{c_1+1}\cdots \ell_m^{c_m+1}\ell_{m+1}^{s+1}\binom{k_1+\cdots+k_n+\ell_1+\cdots+\ell_{m+1}}{\ell_{m+1}}} \tag{5.2}$$

can be expressed as a finite sum of MZVs.

In the following, we will provide concrete examples for clarification. Substitute $b = 0$ in Theorem 3.1, and we have

$$\sum_{m,n\geq 1} \frac{1}{m^{a+1}n\binom{m+n}{m}} = \zeta(a+2).$$

This identity is appeared in [7, Proposition 5].

Since $H_n^{(a+1)} = H_{n-1}^{(a+1)} + \frac{1}{n^{a+1}}$, utilizing Theorems 3.1 and 4.1, we deduce that

$$\sum_{m,n\geq 1} \frac{H_n^{(a+1)}}{m^{c+1}n^{b+1}\binom{m+n}{m}} = \zeta(a+1,b+1,\{1\}^{c-1},2) + \zeta(a+b+2,\{1\}^{c-1},2). \tag{5.3}$$

Letting $a = b = c = 0$ in Eq (5.3), we have (see [4, 7])

$$\sum_{m,n\geq 1} \frac{H_n}{mn\binom{m+n}{m}} = \zeta(1,2) + \zeta(3) = 2\zeta(3).$$

Let $a = 1$, $b = c = 0$, and we have

$$\sum_{m,n\geq 1} \frac{H_n^{(2)}}{mn\binom{m+n}{m}} = \zeta(2,2) + \zeta(4). \tag{5.4}$$

Let $a = b = c = 1$, and we have

$$\sum_{m,n\geq 1} \frac{H_n^{(2)}}{m^2n^2\binom{m+n}{m}} = \zeta(2,2,2) + \zeta(4,2). \tag{5.5}$$

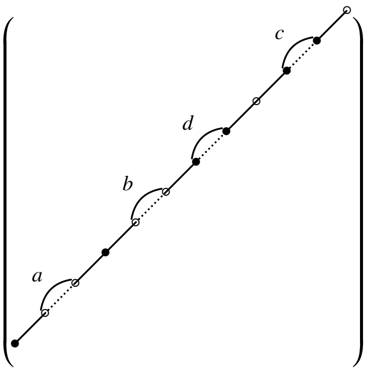
Let $a = b = c = 2$, and we have

$$\sum_{m,n\geq 1} \frac{H_n^{(3)}}{m^3n^3\binom{m+n}{m}} = \zeta(3,3,1,2) + \zeta(6,1,2). \tag{5.6}$$

In general, we set $a = b = c \geq 1$, then

$$\sum_{m,n\geq 1} \frac{H_n^{(a+1)}}{m^{a+1}n^{a+1}\binom{m+n}{m}} = \zeta(a+1, a+1, \{1\}^{a-1}, 2) + \zeta(2a+2, \{1\}^{a-1}, 2). \tag{5.7}$$

Following the similar method, we have

I 

$$= \sum_{m,n\geq 1} \frac{H_{n-1}^{(a+1)}H_{m-1}^{(c+1)}}{n^{b+1}m^{d+1}\binom{m+n}{m}} = \zeta(a+1, b+1, \{1\}^{d-1}, 2, \{1\}^{c-1}, 2), \tag{5.8}$$

where a, b, c, d are nonnegative integers. Similarly, we use Eq (5.4), Theorems 3.1 and 4.1 to get

$$\begin{aligned}\sum_{m,n\geq 1} \frac{H_n^{(a+1)}H_m^{(c+1)}}{n^{b+1}m^{d+1}\binom{m+n}{m}} &= \zeta(a+1, b+1, \{1\}^{d-1}, 2, \{1\}^{c-1}, 2) + \zeta(a+b+2, \{1\}^{d-1}, 2, \{1\}^{c-1}, 2) \quad (5.9)\\ &\quad +\zeta(a+1, b+1, \{1\}^{c+d}, 2) + \zeta(a+b+2, \{1\}^{c+d}, 2).\end{aligned}$$

In particular, we get

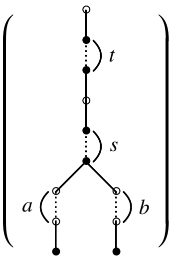
$$\sum_{m,n\geq 1} \frac{H_nH_m}{nm\binom{m+n}{m}} = \zeta(1,3) + \zeta(4) + \zeta(1,1,2) + \zeta(2,2) = 3\zeta(4) = \frac{\pi^4}{30}. \tag{5.10}$$

If we consider the triple infinite series as examples, then we first apply Theorem 3.1 to get

$$\sum_{m,n,k\geq 1} \frac{1}{m^{a+1}n^{b+1}k^{s+1}\binom{m+n+k}{k}} = \sum_{a_1+a_2=a} \binom{a_2+b}{a_2} \zeta(a_1+1, a_2+b+1, \{1\}^{s-1}, 2) \tag{5.11}$$

$$+ \sum_{b_1+b_2=b} \binom{b_2+a}{b_2} \zeta(b_1+1, b_2+a+1, \{1\}^{s-1}, 2).$$

Then we evaluate

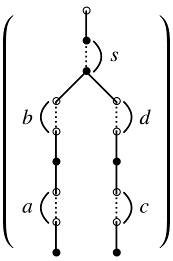
I 

$$= \sum_{m,n,k\geq 1} \frac{H_{k-1}^{(t+1)}}{m^{a+1} n^{b+1} k^{s+1} \binom{m+n+k}{k}} \tag{5.12}$$

$$= \sum_{a_1+a_2=a} \binom{a_2+b}{a_2} \zeta(a_1+1, a_2+b+1, \{1\}^{s-1}, 2, \{1\}^{t-1}, 2)$$

$$+ \sum_{b_1+b_2=b} \binom{b_2+a}{b_2} \zeta(b_1+1, b_2+a+1, \{1\}^{s-1}, 2, \{1\}^{t-1}, 2).$$

Second, we consider the following Yamamoto's integral

I 

$$= \sum_{m,n,k\geq 1} \frac{H_{m-1}^{(a+1)} H_{n-1}^{(c+1)}}{m^{b+1} n^{d+1} k^{s+1} \binom{m+n+k}{k}} \tag{5.13}$$

$$= \sum_{b_1+b_2+b_3=b} \binom{b_2+c}{b_2}\binom{b_3+d}{b_3} \zeta(a+1, b_1+1, b_2+c+1, b_3+d+1, \{1\}^{s-1}, 2)$$

$$+ \sum_{d_1+d_2+d_3=d} \binom{d_2+a}{d_2}\binom{d_3+b}{d_3} \zeta(c+1, d_1+1, d_2+a+1, d_3+b+1, \{1\}^{s-1}, 2)$$

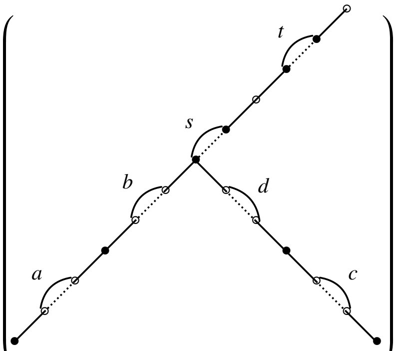
$$+ \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b \\ c_1+c_2=c}} \binom{a_2+c_1}{a_2}\binom{c_2+b_1}{c_2}\binom{b_2+d}{b_2} \zeta(a_1+1, a_2+c_1+1, c_2+b_1+1, b_2+d+1, \{1\}^{s-1}, 2)$$

$$+ \sum_{\substack{d_1+d_2=d \\ c_1+c_2=c \\ a_1+a_2=a}} \binom{c_2+a_1}{c_2}\binom{a_2+d_1}{a_2}\binom{d_2+b}{d_2} \zeta(c_1+1, c_2+a_1+1, a_2+d_1+1, d_2+b+1, \{1\}^{s-1}, 2)$$

$$+ \sum_{\substack{a_1+a_2+a_3=a \\ d_1+d_2=d}} \binom{a_2+c}{a_2}\binom{a_3+d_1}{a_3}\binom{d_2+b}{d_2} \zeta(a_1+1, a_2+c+1, a_3+d_1+1, d_2+b+1, \{1\}^{s-1}, 2)$$

$$+ \sum_{\substack{c_1+c_2+c_3=c \\ b_1+b_2=b}} \binom{c_2+a}{c_2}\binom{c_3+b_1}{c_3}\binom{b_2+d}{b_2} \zeta(c_1+1, c_2+a+1, c_3+b_1+1, b_2+d+1, \{1\}^{s-1}, 2).$$

Third, we evaluate

$I\Big($ 

$\Big)$

$$= \sum_{m,n,k\geq 1} \frac{H_{m-1}^{(a+1)} H_{n-1}^{(c+1)} H_{k-1}^{(t+1)}}{m^{b+1} n^{d+1} k^{s+1} \binom{m+n+k}{k}} \tag{5.14}$$

$$\begin{aligned}
&= \sum_{b_1+b_2+b_3=b} \binom{b_2+c}{b_2}\binom{b_3+d}{b_3} \zeta(a+1, b_1+1, b_2+c+1, b_3+d+1, \{1\}^{s-1}, 2, \{1\}^{t-1}, 2) \\
&\quad + \sum_{d_1+d_2+d_3=d} \binom{d_2+a}{d_2}\binom{d_3+b}{d_3} \zeta(c+1, d_1+1, d_2+a+1, d_3+b+1, \{1\}^{s-1}, 2, \{1\}^{t-1}, 2) \\
&\quad + \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b \\ c_1+c_2=c}} \binom{a_2+c_1}{a_2}\binom{c_2+b_1}{c_2}\binom{b_2+d}{b_2} \zeta(a_1+1, a_2+c_1+1, c_2+b_1+1, b_2+d+1, \{1\}^{s-1}, 2, \{1\}^{t-1}, 2) \\
&\quad + \sum_{\substack{d_1+d_2=d \\ c_1+c_2=c \\ a_1+a_2=a}} \binom{c_2+a_1}{c_2}\binom{a_2+d_1}{a_2}\binom{d_2+b}{d_2} \zeta(c_1+1, c_2+a_1+1, a_2+d_1+1, d_2+b+1, \{1\}^{s-1}, 2, \{1\}^{t-1}, 2) \\
&\quad + \sum_{\substack{a_1+a_2+a_3=a \\ d_1+d_2=d}} \binom{a_2+c}{a_2}\binom{a_3+d_1}{a_3}\binom{d_2+b}{d_2} \zeta(a_1+1, a_2+c+1, a_3+d_1+1, d_2+b+1, \{1\}^{s-1}, 2, \{1\}^{t-1}, 2) \\
&\quad + \sum_{\substack{c_1+c_2+c_3=c \\ b_1+b_2=b}} \binom{c_2+a}{c_2}\binom{c_3+b_1}{c_3}\binom{b_2+d}{b_2} \zeta(c_1+1, c_2+a+1, c_3+b_1+1, b_2+d+1, \{1\}^{s-1}, 2, \{1\}^{t-1}, 2).
\end{aligned}$$

Therefore, we have the formulas for

$$\sum_{m,n,k\geq 1} \frac{H_k^{(t+1)}}{m^{a+1} n^{b+1} k^{s+1} \binom{m+n+k}{k}}, \sum_{m,n,k\geq 1} \frac{H_m^{(a+1)} H_n^{(c+1)}}{m^{b+1} n^{d+1} k^{s+1} \binom{m+n+k}{k}}, \text{ and } \sum_{m,n,k\geq 1} \frac{H_m^{(a+1)} H_n^{(c+1)} H_k^{(t+1)}}{m^{b+1} n^{d+1} k^{s+1} \binom{m+n+k}{k}}.$$

As a final example, we list

$$\sum_{m,n,k\geq 1} \frac{H_k}{mnk\binom{m+n+k}{k}} = 2\zeta(4) + 2\zeta(1,3) = \frac{5}{2}\zeta(4) = \frac{\pi^4}{36}. \tag{5.15}$$

In this paper, we present a method for assessing the form described in Eq (5.2). Specifically, we derive this assessment by evaluating a particular Yamamoto's integral linked to a 2-poset Hasse diagram in two distinct manners: one employs the shuffle relations with its corresponding Lyndon words, while the other utilizes the corresponding summation expansions. Although we do not provide an explicit closed formula for the multiple series in Eq (5.2), if all the parameters are provided, our method enables us to derive the corresponding explicit expression as a linear combination of MZVs.

Author contributions

Kwang-Wu Chen: Conceptualization, Formal Analysis, Methodology, and Writing-original draft; Fu-Yao Yang: Conceptualization, Methodology, Validation, and Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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