Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(7): 16876–16884.
DOI: 10.3934/math.2024819
Received: 28 March 2024
Revised: 29 April 2024
Accepted: 06 May 2024
Published: 15 May 2024

Research article

Oscillation behavior of second-order self-adjoint q-difference equations

Ağacık Zafer and Zeynep Nilhan Gürkan*

College of Engineering and Technology, American University of the Middle East, Egaila 54200, Kuwait

* **Correspondence:** Email: zeynep.gurkan@aum.edu.kw; Tel: +965222514002081.

Abstract: In this study, we investigate the oscillation behavior of second-order self-adjoint q-difference equations, focusing on the renowned Leighton oscillation theorem. Through an example, we demonstrate that the q-version of Leighton's classical oscillation theorem does not hold and requires refinement. To address this, we introduce an oscillation-preserving transformation and establish alternative theorems to the ones existing in the literature. The strength of our work lies in the absence of any sign condition on the potential function. We also provide illustrative examples to support our findings and mention directions for future research.

Keywords: second-order equation; q-difference; quantum calculus; oscillation
Mathematics Subject Classification: 39A13, 39A21, 34C10

1. Introduction

The theory of q-difference equations and related q-calculus has a long history, dating back to the work of Jackson in 1910 [1]. Recently, studies have gained renewed interest due to their connections with applications in several research areas such as quantum mechanics [2], computational biology [3], and financial mathematics [4]. A comprehensive exploration of fundamental aspects of q-calculus can be found in [5, 6]. Concerning the asymptotic properties and the oscillation of solutions of q-difference equations, we may refer to [7–11] and the references cited therein. On the other hand, q-difference equations may be viewed as a special case of dynamic equations on time scales. Recent studies on the oscillation problem of such equations and their connections to the q-difference equations can be found in [12, 13]. To this end, we should note that the strength of the present work lies in the absence of any sign condition on the function $p(t)$ in Eq (1.1), and its applicability to the canonical and noncanonical cases.

We study the oscillation of a second-order linear q-difference equation

$$D_q(a(t)D_qx(t)) + p(t)x(qt) = 0, \quad t \geq t_0 \tag{1.1}$$

by exploring a connection with the pioneering work of Leighton [14]. Here, $q > 1$ is a real number, t and t_0 belong to the set $q^{\mathbb{N}} = \{q^k : \ k \in \mathbb{N}\}$ of discrete points, and the q-derivative, also known as the Jackson derivative [5], of a function x at a point $t \in q^{\mathbb{N}}$ is defined by

$$D_q x(t) = \frac{x(qt) - x(t)}{(q-1)t}. \tag{1.2}$$

Obviously, if x is differentiable at t, then $D_q x(t) = x'(t)$ as $q \to 1^+$. Therefore, Eq (1.1) can be considered the q analog of the second-order ordinary differential equation

$$(a(t)x'(t))' + p(t)x(t) = 0, \quad t \geq t_0. \tag{1.3}$$

As usual, a nontrivial solution of (1.1) or (1.3) is called oscillatory if it is neither eventually positive nor eventually negative in the neighborhood of infinity, and the equation is said to be oscillatory if all its solutions are oscillatory.

Leighton's famous oscillation theorem is the following:

Theorem 1.1 ([14]). *Let a and p be continuous functions on $[t_0, \infty)$ with $a(t) > 0$. If*

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} = \int_{t_0}^{\infty} p(t)\,\mathrm{d}t = \infty, \tag{1.4}$$

then Eq (1.3) is oscillatory.

Before we state the q-version of Theorem 1.1, we need to recall some background material from q-calculus. The q analog of a number α, denoted by $[\alpha]_q$, is defined as

$$[\alpha]_q = \frac{q^\alpha - 1}{q-1}, \quad \alpha \neq 0.$$

Using this and the definition of the q-derivative formula (1.1), the so-called power rule becomes:

$$D_q t^r = [r]_q\, t^{r-1}, \quad r \neq 0.$$

On the other hand, the q-definite integral of a function $f : q^{\mathbb{N}} \to \mathbb{R}$ is defined by

$$\int_a^b f(t)\,\mathrm{d}_q t = (q-1) \sum_{t=a}^{b/q} t\, f(t), \quad a, b \in q^{\mathbb{N}}\ (a < b). \tag{1.5}$$

Improper integrals are defined as usual by the limits of corresponding proper integrals.

It follows that

$$\int_a^b t^{r-1}\,\mathrm{d}_\mathrm{q} t = \left.\frac{t^r}{[r]_q}\right|_{t=a}^{b}, \quad r \neq 0, \tag{1.6}$$

and that the improper integral

$$\int_a^\infty \frac{1}{t^\alpha}\,\mathrm{d}_\mathrm{q} t \tag{1.7}$$

is convergent if and only if $\alpha > 1$.

The quotient and product rules of q-derivatives read as

$$D_q\left(\frac{f}{g}\right) = \frac{gD_qf - fD_qg}{g^\sigma g} \tag{1.8}$$

and

$$D_q(fg) = gD_qf + f^\sigma D_qg, \tag{1.9}$$

where $f^\sigma = f \circ \sigma$, $\sigma = q\,t$.

In view of the above notations, we now state the q version of the famous Leighton oscillation theorem.

Theorem 1.2 ([15]). *Let $a, p : q^{\mathbb{N}} \to \mathbb{R}$ with $a(t) > 0$. If*

$$\int_{t_0}^{\infty} \frac{d_qt}{a(t)} = \int_{t_0}^{\infty} p(t)\, d_qt = \infty, \tag{1.10}$$

then the q-difference equation (1.1) *is oscillatory.*

Our focus in this work will be on condition (1.10), which cannot be satisfied for most equations. As an example, let us consider the simple second-order q-Euler equation

$$D_q^2x(t) + \frac{c}{q(q-1)t^2}x(qt) = 0, \quad t \geq t_0. \tag{1.11}$$

Using (1.6), one can easily see that

$$\int_{t_0}^{\infty} p(t)d_qt = \frac{c}{q(q-1)} \int_{t_0}^{\infty} \frac{1}{t^2}\, d_qt = \frac{c}{(q-1)t_0} < \infty, \tag{1.12}$$

which means Theorem 1.2 fails to apply. On the other hand, it is known (see [15]) that the q-difference equation (1.11) is oscillatory if $c > 1$. Motivated by the above example, our aim in this work is to find an alternative theorem or an improvement that can be applied when condition (1.10) fails. Indeed, we will allow

$$\int_{t_0}^{\infty} \frac{d_qt}{a(t)} < \infty \tag{1.13}$$

and

$$\int_{t_0}^{\infty} p(t)\, d_qt < \infty. \tag{1.14}$$

We should note that there is already a large body of research papers in the literature in which the authors extend, generalize, and improve Theorem 1.1, as well as provide alternative theorems when condition (1.4) does not hold; see [16–22] and the references therein. Our methodology in this study is fundamentally inspired by the approach utilized in [16] to explore the oscillation and boundedness of solutions of (1.3), where the author used an equivalence transformation $x = uz$. Note that if the arbitrary function u is positive within a designated interval of interest, then the zeros of the functions x and z coincide under this transformation.

2. Main results

Our main theorem is as follows:

Theorem 2.1. *Let* $a, p : q^{\mathbb{N}} \to \mathbb{R}$ *with* $a(t) > 0$. *If there exists a function* $u : q^{\mathbb{N}} \to (0, \infty)$ *such that*

$$\int_{t_0}^{\infty} \frac{d_q t}{a(t)u(t)u(qt)} = \infty \tag{2.1}$$

and

$$\int_{t_0}^{\infty} [p(t)u^2(qt) + D_q(a(t)D_q u(t))u(qt)]\, d_q t = \infty, \tag{2.2}$$

then the q-difference equation (1.1) *is oscillatory.*

Proof. Let x be a solution of (1.1). Define $z = x/u$. Clearly, x is oscillatory if and only if z is oscillatory. Applying the quotient rule (1.8), we write

$$D_q z(t) = \frac{u(t)D_q x(t) - x(t)\, D_q u(t)}{u(t)u(qt)},$$

and so,

$$a(t)u(t)u(qt)D_q z(t) = u(t)a(t)D_q x(t) - x(t)\, a(t)D_q u(t). \tag{2.3}$$

Using the product rule (1.9), we see that

$$\begin{aligned} D_q[a(t)u(t)u(qt)D_q z(t)] &= u(qt)D_q[a(t)D_q x(t)] - x(qt)[D_q(a(t)D_q u(t))] \\ &= -[u(qt)p(t) + D_q(a(t)D_q u(t))]x(qt) \\ &= -[u^2(qt)p(t) + u(qt)D_q(a(t)D_q u(t))]z(qt). \end{aligned}$$

This leads to the second-order q-difference equation

$$D_q(a_1(t)D_q z(t)) + p_1(t)z(qt) = 0, \quad t \geq t_0, \tag{2.4}$$

where

$$a_1(t) = a(t)\, u(t)u(qt); \tag{2.5}$$

$$p_1(t) = p(t)\, u^2(qt) + D_q(a(t)D_q u(t))u(qt). \tag{2.6}$$

Using the conditions (2.1) and (2.2), it follows from Theorem 1.2 that Eq (2.4) is oscillatory. Thus the q-difference equation (1.1) is also oscillatory. □

Remark 2.1. *If* $u \equiv 1$, *then Theorem 2.1 reduces to Theorem 1.2.*

Let us re-consider (1.11). Put $u(t) = t^{1/2}$. Then, from (2.5) and (2.6),

$$a_1(t) = q^{1/2}t;$$
$$p_1(t) = \left[\frac{c}{q-1} + \frac{2q^{1/2} - (q+1)}{(q-1)^2}\right]\frac{1}{t}.$$

Moreover, if $c > 1$, then $p_1(t)$ becomes positive for all $q > 1$. In view of (1.7), we compute

$$\int_{t_0}^{\infty} \frac{\mathrm{d_q}t}{a_1(t)} = \frac{1}{q^{1/2}} \int_{t_0}^{\infty} \frac{\mathrm{d_q}t}{t} = \infty$$

and

$$\int_{t_0}^{\infty} p_1(t)\,\mathrm{d_q}t = \left(\frac{c}{q-1} - \frac{q+1-2q^{1/2}}{(q-1)^2}\right) \int_{t_0}^{\infty} \frac{\mathrm{d_q}t}{t} = \infty. \tag{2.7}$$

Thus, the conditions (2.1) and (2.2) of Theorem 2.1 are satisfied. Therefore, we may also conclude that (1.11) is oscillatory if $c > 1$. Indeed, we only need in (2.7) that

$$c > \frac{q+1-2q^{1/2}}{q-1}$$

for each fixed $q > 1$. This is an improvement over condition $c > 1$. For example, if $q = 4$, then $c > 1/3$ is sufficient for the oscillation. In Figure 1, we display the graph of an exemplary solution $x(t)$ of (1.11) with initial conditions $x(0) = 1$ and $x(1) = 2$ to illustrate its oscillation behavior on different intervals, where $q = 4$ and $c = 2/3 < 1$.

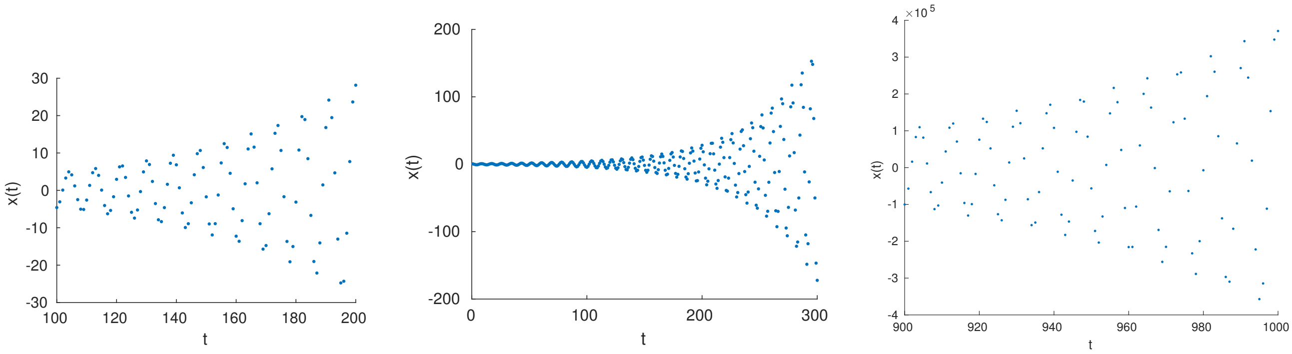

Figure 1. The graph of a solution $x(t)$ for $q = 4$ and $c = 2/3$.

The next result is a Leighton-type theorem under condition (1.13). In this case, we may define

$$\phi(t) = \int_t^{\infty} \frac{\mathrm{d_q}s}{a(s)}, \quad t \geq t_0. \tag{2.8}$$

Theorem 2.2. *Let* (1.13) *hold. If*

$$\int_{t_0}^{\infty} p(t)\phi^2(qt)\,d_qt = \infty, \tag{2.9}$$

then the q-difference equation (1.1) *is oscillatory.*

Proof. It suffices to show that the conditions of Theorem 2.1 are satisfied for $u = \phi$. First, we calculate that

$$D_q\phi(t) = -\frac{1}{(q-1)t}\int_t^{qt} \frac{\mathrm{d_q}s}{a(s)},$$

and so, using the definition (1.5), we have $D_q\phi(t) = -1/a(t)$. Then, $D_q[a(t)D_q\phi(t)] = 0$, and hence,

$$p_1(t) = p(t)\,\phi^2(qt).$$

Using (2.9), we see that (2.2) is valid. We only need to show that (2.1) is satisfied as well. Indeed,

$$\int_{t_0}^{\infty} \frac{\mathrm{d_q}t}{a(t)u(t)u(qt)} = -\int_{t_0}^{\infty} \frac{D_q\,\phi(t)}{\phi(t)\phi(qt)}\mathrm{d_q}t = \int_{t_0}^{\infty} D_q\left(\frac{1}{\phi(t)}\right)\mathrm{d_q}t = \left.\frac{1}{\phi(t)}\right|_{t_0}^{\infty} = \infty.$$

□

Example 2.1. Consider the equation

$$D_q(t^r D_q x(t)) + \frac{1}{t^k}x(qt) = 0, \quad t \geq t_0, \tag{2.10}$$

where r and k are real numbers such that $r > 1$ and $2r + k \leq 3$.

Note that $a(t) = t^r$ and $p(t) = 1/t^k$. Thus, by (1.7), we see that

$$\int_{t_0}^{\infty} \frac{1}{a(t)}\ \mathrm{d_q}t = \int_{t_0}^{\infty} \frac{1}{t^r}\ \mathrm{d_q}t < \infty.$$

Also, we calculate from (2.8) that $\phi(t) = c_1 t^{r-1}$, where $c_1 = 1/[1-r]_q$. So, we also have

$$\int_{t_0}^{\infty} p(t)\phi^2(qt)\ \mathrm{d_q}t = c_1^2\int_{t_0}^{\infty} \frac{1}{t^{2r+k-2}}\ \mathrm{d_q}t = \infty.$$

Since the conditions of Theorem 2.2 are all satisfied, we may deduce that the q-difference equation (2.10) is oscillatory. Note that Theorem 1.2 does not apply to (2.10).

Example 2.2. Consider the equation

$$D_q(t^r D_q x(t)) + \frac{\lambda}{t^k}x(qt) = 0, \quad t \geq t_0, \tag{2.11}$$

where $r < 1$, $k > 1$, and $\lambda > 0$ are real numbers.

In view of (1.7), we see that

$$\int_{t_0}^{\infty} \frac{1}{a(t)}\ \mathrm{d_q}t = \int_{t_0}^{\infty} \frac{1}{t^r}\ \mathrm{d_q}t = \infty$$

and

$$\int_{t_0}^{\infty} p(t)\ \mathrm{d_q}t = \int_{t_0}^{\infty} \frac{\lambda}{t^k}\ \mathrm{d_q}t < \infty.$$

This means that Theorem 1.2 fails for (2.11). Setting $u(t) = t^m$ with $m \geq k - 1$, we obtain from (2.5) and (2.6) that

$$a_1(t) = q^m t^{r+2m};$$
$$p_1(t) = \lambda q^{2m} t^{2m-k} - \frac{q^{r+2m-1} - q^{r+3m-1} + q^{2m} - q^m}{(q-1)^2}\ t^{r+2m-2}.$$

It follows that the conditions of Theorem 2.1 are all satisfied if

$$r + 2m < 1, \quad 2m \geq k - 1 \tag{2.12}$$

or

$$r + k = 2, \quad 2m = k - 1, \quad \lambda > q^{-2m}\,[m]_q^2. \tag{2.13}$$

Let $\varepsilon, \delta > 0$ with $\delta > \varepsilon$, and put $k = 1 + \varepsilon$ and $r = 1 - \delta$. It is easily seen that (2.12) is satisfied with $m = (\varepsilon + \delta)/4$. Therefore, we deduce from Theorem 2.1 that the q-difference equation (2.11) is oscillatory. It should be noted that neither (2.12) nor (2.13) holds when $r > 1$ and $k > 1$.

3. Conclusions

In this study, we have explored Leighton's well-known oscillation theorem under the condition (1.4) of being unsatisfied. Through the application of an equivalence transformation, we have demonstrated the possibility of an alternative theorem. Furthermore, we have provided examples to illustrate the results under critical conditions (1.13) and (1.14). Example 2.2 also highlights the unresolved nature of the problem when both (1.13) and (1.14) are simultaneously satisfied. As a direction for future research, we encourage the reader to investigate the feasibility of obtaining an oscillation theorem similar to Theorem 2.2 under the condition (1.14). It appears that such research and related ones would further contribute to the advancement of the oscillation theory of q-difference equations.

Author contributions

All authors contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors explicitly state that they did not utilize any Artificial Intelligence (AI) tools during the development of this article.

Acknowledgments

The authors are grateful to the editor and referees for their invaluable comments that helped improve the paper.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. F. H. Jackson, On q-definite integrals, *Quart. J. Pure Appl. Math.*, **41** (1910), 193–203.
2. R. Floreanini, L. Vinet, Quantum symmetries of q-difference equations, *J. Math. Phys.*, **36** (1995), 3134–3156. https://doi.org/10.1063/1.531017
3. M. Bohner, R. Chieochan, The Beverton-Holt q-difference equation, *J. Biol. Dyn.*, **7** (2013), 86–95. https://doi.org/10.1080/17513758.2013.804599
4. Q. A. Hamed, R. Al-Salih, W. Laith, The analogue of regional economic models in quantum calculus, *J. Phys.: Conf. Ser.*, **1530** (2020), 012075. https://doi.org/10.1088/1742-6596/1530/1/012075
5. G. Bangerezako, *An introduction to q-difference equations*, San Diego: Harcourt/Academic Press, 2008.
6. V. Kac, P. Cheung, *Quantum calculus*, Springer, 2002. https://doi.org/10.1007/978-1-4613-0071-7
7. M. Bohner, M. Ünal, Kneser's theorem in q-calculus, *J. Phys. A: Math. Gen.*, **38** (2005), 6729. https://doi.org/10.1088/0305-4470/38/30/008
8. S. Garoufalidis, J. S. Geronimo, Asymptotics of q-difference equations, In: T. Kohno, M. Morishita, *Primes and knots*, Contemporary Mathematics, **416** (2006), 83–114.
9. J. Baoguo, L. Erbe, A. Peterson, Oscillation of a family of q-difference equations, *Appl. Math. Lett.*, **22** (2009), 871–875. https://doi.org/10.1016/j.aml.2008.07.014
10. P. Rehak, On a certain asymptotic class of solutions to second-order linear q-difference equations, *J. Phys. A: Math. Theor.*, **45** (2012), 055202. https://doi.org/10.1088/1751-8113/45/5/055202
11. T. G. G. Soundarya, V. R. Sherine, Oscillation theory of q-difference equation, *J. Comput. Math.*, **5** (2021), 083–091. https://doi.org/10.26524/cm111
12. A. M. Hassan, H. Ramos, O. Moaaz, Second-order dynamic equations with noncanonical operator: oscillatory behavior, *Fractal Fract.*, **7** (2023), 134. https://doi.org/10.3390/fractalfract7020134
13. T. S. Hassan, R. A. El-Nabulsi, N. Iqbal, A. A. Menaem, New criteria for oscillation of advanced noncanonical nonlinear dynamic equations, *Mathematics*, **12** (2024), 824. https://doi.org/10.3390/math12060824
14. W. Leighton, On self-adjoint differential equations of second order, *J. Lond. Math. Soc.*, **s1-27** (1952), 37–47. https://doi.org/10.1112/jlms/s1-27.1.37

15. M. Bohner, A. Peterson, *Dynamic equations on time scales: an introduction with applications*, Birkhäuser, 2001. https://doi.org/10.1007/978-1-4612-0201-1

16. R. A. Moore, The behavior of solutions of a linear differential eqution of second order, *Pac. J. Math.*, **5** (1955), 125–145. https://doi.org/10.2140/PJM.1955.5.125

17. E. C. Tomastik, Oscillation of nonlinear second order differential equations, *SIAM J. Appl. Math.*, **5** (1967), 1275–1277.

18. N. P. Bhatia, An oscillation theorem, *Notices Amer. Math. Soc.*, **13** (1966), 243.

19. P. Hartman, On non-oscillatory linear differential equations of second order, *Amer. J. Math.*, **74** (1952), 389–400. https://doi.org/ 10.2307/2372004

20. I. V. Kamenev, An integral criterion for oscillation of linear differential equations of second order, *Math. Notes Acad. Sci. USSR*, **23** (1978), 136–138. https://doi.org/10.1007/BF01153154

21. W. J. Coles, Oscilllation criteria for nonlinear second order equations, *Ann. Mat. Pura Appl.*, **82** (1969), 123–133. https://doi.org/10.1007/BF02410793

22. E. Hille, Non-oscillation theorems, *Trans. Amer. Math. Soc.*, **64** (1948), 234–252. https://doi.org/ 10.1090/S0002-9947-1948-0027925-7

AIMS Press