AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(7): 16848–16862.
DOI: 10.3934/math.2024817
Received: 19 March 2024
Revised: 24 April 2024
Accepted: 06 May 2024
Published: 14 May 2024

Research article

On linear transformation of generalized affine fractal interpolation function

Najmeddine Attia[1,2,*] **and Rim Amami**[3]

[1] Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia

[2] Analysis, Probability and Fractals Laboratory LR18ES17, Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir, 5000-Monastir, Tunisia

[3] Department of Basic Sciences, Deanship of Preparatory Year and Supporting Studies, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam 34212, Saudi Arabia

* **Correspondence:** Email: nattia@kfu.edu.sa; Tel: +966556411269.

Abstract: In this work, we investigate a class of generalized affine fractal interpolation functions (FIF) with variable parameters, where ordinate scaling is substituted by a real-valued control function. Let $\mathcal{S}$ be an iterated function system (IFS) with the attractor G_Δ, where Δ is a given data set. We consider an affine transformation $\omega(\Delta)$ of Δ, and we define the IFS $\hat{\mathcal{S}}$ with the attractor $G_{\omega(\Delta)}$. We give a sufficient condition so that $G_{\omega(\Delta)} = \omega(G_\Delta)$. In addition, we compare the definite integrals of the corresponding FIF and study the additivity property. Some examples will be given, highlighting the effectiveness of our results.

Keywords: iterated function system; generalized affine fractal interpolation function; linear transformation
Mathematics Subject Classification: 28A80, 47H10, 65D05

1. Introduction and main results

The fractal interpolation function (FIF) interpolates some experimental data using a non-smooth curve. Since most time series often exhibit sudden fluctuations or changes, it is natural to use the FIF when studying these data. In fact, the concept of FIF is essentially the key to construct fractals and it was first introduced via iterative function systems (IFS) on compact subsets of $\mathbb{R}$ [1]. Since then, this theory has become common practice in several fields of applied sciences [2–6]. Furthermore, various important properties of FIF have been demonstrated, such as stability [7, 8] and smoothness [9–13].

Let $N \geq 2$, $J = \{1, \ldots, N\}$, $(\mathbb{X}, d)$ be a complete metric space and $\{w_i : \mathbb{X} \to \mathbb{X}\}_{i \in J}$ be a finite set of continuous mappings. Now, consider the IFS

$$\{\mathbb{X}, w_i, \quad i \in J\},$$

and the Hutchinson operator $W : \mathsf{H}(\mathbb{X}) \longrightarrow \mathsf{H}(\mathbb{X})$ by

$$W(B) = \bigcup_{n=1}^{N} w_n(B)\,, \qquad \forall\, B \in \mathsf{H}(\mathbb{X}), \tag{1.1}$$

where $w_n(B) = \{w_n(x)\,,\ x \in B\}$ and $\mathsf{H}(\mathbb{X})$ denotes the set of all compact subsets of $\mathbb{X}$. For $k \in \mathbb{N}^*$, let W^k denote the k-fold auto composition of W. Any compact set $G \in \mathsf{H}(\mathbb{X})$ such that $W(G) = G$ is called an attractor for the IFS, and the IFS admits always at least one attractor [1]. Moreover, if each w_n is a contraction, i.e., if there exists $c \in [0, 1)$ such that $d(w(x), w(y)) \le c\, d(x, y)$, for all $x, y \in \mathbb{X}$, then $\mathcal{S}$ is called hyperbolic. In this case, the Hutchinson operator W is a contraction mapping, that is,

$$d_H(W(A), W(B)) \le c\, d_H(A, B), \qquad \forall A, B \in \mathsf{H}(\mathbb{X}),$$

and hence, by Banach's fixed point theorem, admits a unique attractor G, which is the limit set of the IFS, i.e., $G = \lim_{k\to\infty} W^k(B)$, for an arbitrary $B \in \mathsf{H}(\mathbb{X})$ [1] (see, for instance, some extensions of the Hutchinson framework [14–18]). The construction of the attractor is based on Banach's fixed point theorem or some of its generalizations [11, 19–23].

Let $x_0 < x_1 < \cdots < x_N$, $y_i \in [\mathsf{a}, \mathsf{b}]$, with $-\infty < \mathsf{a} < \mathsf{b} < \infty$. We define $J_0 := \{0, \dots, N\}$, $I = [x_0, x_N]$ and $\Delta = \Big\{(x_n, y_n) \in I \times \mathbb{R}\ ;\ n \in J_0\Big\}$ as a given data set. We also define, for $n \in J$, the set $I_n = [x_{n-1}, x_n]$, a contractive homeomorphism $\mathsf{L}_n : I \longrightarrow I_n$, and a continuous mapping $F_n : \mathsf{K} := I \times [\mathsf{a}, \mathsf{b}] \longrightarrow \mathbb{R}$. Assume that

$$\mathsf{L}_n(x_0) = x_{n-1}, \quad \mathsf{L}_n(x_N) = x_n \ \text{ and } \ |\mathsf{L}_n(x) - \mathsf{L}_n(x')| \le l|x - x'|, \tag{1.2}$$

$$F_n(x_0, y_0) = y_{n-1}, \qquad F_n(x_N, y_N) = y_n, \tag{1.3}$$

and

$$|F_n(x, y) - F_n(x, y')| \le |\mathsf{s}_n||y - y'|, \tag{1.4}$$

for all $x, x' \in I$, $y, y' \in [\mathsf{a}, \mathsf{b}]$, and for some $l \in [0, 1)$ and $\mathsf{s}_n \in (-1, 1)$. For $n \in J$, we define the mapping

$$W_n(x, y) = \big(\mathsf{L}_n(x), F_n(x, y)\big),$$

for all $(x, y) \in \mathsf{K}$. Then, the IFS $\{K, W_n : n \in J\}$ has a unique attractor G_Δ which is the graph of the continuous function $f : I \longrightarrow \mathbb{R}$, called FIF, such that $f(x_n) = y_n$ for all $n \in J$ [1]. For $n \in J$, let $\alpha_n, \psi_n, \hat{\psi}_n : I \longrightarrow \mathbb{R}$ be continuous functions. Here, we investigate the generalized affine FIF defined by

$$\mathcal{S} = \begin{cases} \mathsf{L}_n(x) = a_n x + e_n, \\ F_n(x, y) = \alpha_n(x) y + \psi_n(x), \end{cases} \tag{1.5}$$

where $n \in J$, a_n and e_n are determined by (1.2), and the conditions (1.3) and (1.4) hold. This system is extensively studied (see, for instance, [7, 12, 24–27]), especially when the functions $\{\alpha_n\}_n$ are constant [8–10, 24, 28–30]. We consider a small perturbation on the data set Δ. For this, we define the following affine transformation:

$$\omega(x, y) = (px, sy),$$

where p and s are positive real numbers. Let $\hat{\Delta} = \omega(\Delta) := \{(px_n, sy_n),\ n \in J_0\}$ and consider the generalized affine FIFs $\hat{f}$, interpolating $\hat{\Delta}$ and defined by the following IFS:

$$\hat{\mathcal{S}} = \begin{cases} \hat{\mathsf{L}}_n(x) = \hat{a}_n x + \hat{e}_n, \\ \hat{F}_n(x,y) = \alpha_n(x)y + \hat{\psi}_n(x). \end{cases} \tag{1.6}$$

We will assume that

$$\hat{\mathsf{L}}_n(px_0) = px_{n-1}, \quad \hat{\mathsf{L}}_n(px_N) = px_n \ \text{ and } \ |\hat{\mathsf{L}}_n(x) - \hat{\mathsf{L}}_n(x')| \le \hat{l}|x - x'|,$$
$$\hat{F}_n(px_0, sy_0) = sy_{n-1}, \qquad \hat{F}_n(px_N, sy_N) = sy_n,$$

and

$$|\hat{F}_n(x,y) - \hat{F}_n(x,y')| \le |\hat{\mathsf{s}}_n||y - y'|,$$

for all $x, x' \in I$, $y, y' \in [\mathsf{a}, \mathsf{b}]$, and for some $\hat{l} \in [0,1)$ and $\hat{\mathsf{s}}_n \in (-1,1)$. Our first main result gives a sufficient condition to have $\omega(G_\Delta) = G_{\omega(\Delta)}$, where $\omega(A) = \{(px, sy);\ (x,y) \in A\}$, for any compact set A in $\mathbb{R}^2$.

Theorem 1.1. *Let G_Δ and $G_{\omega(\Delta)}$ be the attractors of the IFSs $\mathcal{S}$ and $\hat{\mathcal{S}}$, respectively. Assume, for all $(x,y) \in \mathsf{K}$ and $n \in J$, that*

$$s[\alpha_n(x) - \alpha_n(px)]y + s\psi_n(x) - \hat{\psi}_n(px) = 0,$$

then $\omega(G_\Delta) = G_{\omega(\Delta)}$.

Remark 1.1. *Assume that α_n is a constant function and ψ_n is affine for each $n \in J$. Then, in the case of equally spaced interpolation points, one can get a smooth or non-smooth fractal function depending on the choice of α_n. More precisely, we have the box dimension of Δ which is given by [31]*

$$D_{G_\Delta} := 1 + \frac{\log\left(\sum_{n=1}^{N} |\alpha_n|\right)}{\log(N)}. \tag{1.7}$$

Therefore, if $\hat{\psi}_n$ is affine for each $n \in J$, then $D_{G_\Delta} = D_{\omega(G_\Delta)}$. Note that the condition of Theorem 1.1 may be satisfied, for example, if $p = 1$ or α_n is a constant function for each $n \in J$. This makes it possible, in particular, to obtain FIFs with different box dimensions.

In Section 2, we will prove Theorem 1.1 and consider some examples. Let $C(I,\mathbb{R}) = \{f : I \longrightarrow \mathbb{R}$, continuous $\}$ and assume that $C(I,K)$ is endowed with the uniform norm. We define the bounded and nonidentity linear operator $b : C(I,\mathbb{R}) \longrightarrow C(I,\mathbb{R})$ such that

$$b(g)(x_0) = g(x_0) \quad \text{and} \quad b(g)(x_N) = g(x_N), \tag{1.8}$$

for all $g \in C(I,\mathbb{R})$. Now, let $h \in C(I,\mathbb{R})$ be the piecewise linear interpolation function through the set points Δ and consider the IFS $\mathcal{S}$ with

$$\psi_n(x) = h \circ \mathsf{L}_n(x) - \alpha_n(x) b(h)(x), \qquad n \in J. \tag{1.9}$$

Let Γ_h be the graph of the function h, and define the function $\hat{h}$ such that $\omega(\Gamma_h) = \Gamma_{\hat{h}}$. It is clear that $\hat{h}$ is the piecewise linear interpolation function through the set points $\omega(\Delta)$. Similarly, we define the generalized affine FIF $\hat{f}$, interpolating $\hat{\Delta}$, and defined by the IFS $\hat{\mathcal{S}}$ with

$$\hat{\psi}_n(x) = \hat{h} \circ \hat{\mathsf{L}}_n(x) - \alpha_n(x)\hat{b}(\hat{h})(x), \qquad n \in J, \tag{1.10}$$

such that $\hat{b}(sh)(x) = sb(h)(x)$.

Corollary 1.1. *Let G_Δ and $G_{\omega(\Delta)}$ be the attractors of the IFSs $\mathcal{S}$ and $\hat{\mathcal{S}}$, respectively, such that ψ_n and $\hat{\psi}_n$ are defined by* (1.9) *and* (1.10). *Assume that $p = 1$ or, for $n \in J$, α_n is a constant function, then $\omega(G_\Delta) = G_{\omega(\Delta)}$.*

Section 4 is devoted to studying the additivity property. Let $m \geq 2$ and Δ_k, for $0 \leq k \leq m$, be the data set such that $\Delta_0 = \Delta$ and $\Delta_k = \left\{(x_n, y_n^k) \in I \times \mathbb{R}\ ;\ n \in J_0\right\}$, for $k \neq 0$. We define the sequence of IFSs $(\mathcal{S}_k)$ such that $\mathcal{S}_0 = \mathcal{S}$ and

$$\mathcal{S}_k = \begin{cases} \mathsf{L}_n(x) = a_n x + e_n, \\ F_{k,n}(x, y) = \alpha_n(x) y + \psi_{k,n}(x), \end{cases} \tag{1.11}$$

$n \in J, 1 \leq k \leq m$. Now, we define the data $\Delta_S = \left\{(x_n, \sum_{k=1}^m y_n^k) \in I \times \mathbb{R}\ ;\ n \in J_0\right\}$ and let G_{Δ_S} be the attractor of the IFS

$$\begin{cases} \mathsf{L}_n(x) = a_n x + e_n, \\ S_n(x, y) = \alpha_n(x) y + \sum_{k=1}^m \psi_{k,n}(x), \end{cases} \tag{1.12}$$

where $n \in J$.

Theorem 1.2. *Let f^s be the FIF defined through the IFS* (1.12). *Then for all $n \in J_0$, $f^s(x_n) = \sum_{k=1}^m y_n^k$. Moreover, we have*

$$G_{\Delta_S} = \sum_{k=1}^m G_{\Delta_k} := \left\{(x, \sum_{k=1}^m y_k),\ (x, y_k) \in G_{\Delta_k}\right\}.$$

2. FIF under affine transformation

2.1. Proof of Theorem 1.1

For $(x, y) \in \mathsf{K}$ and $n \in J$, we define first $\hat{W}_n(x, y) = \left(\mathsf{L}_n(x), \hat{F}_n(x, y)\right)$, and the Hutchinson operators $\mathcal{W}$ and $\hat{\mathcal{W}}$ by

$$\mathcal{W}(A) = \bigcup_{n=1}^N W_n(A) \quad \text{and} \quad \hat{\mathcal{W}}(A) = \bigcup_{n=1}^N \hat{W}_n(A)$$

for all $A \in \mathsf{H}(\mathsf{K})$. Therefore, $G_\Delta = \mathcal{W}(G_\Delta)$ and $G_\Delta = \lim_{k\to\infty} \mathcal{W}^k(A)$ for any $A \in \mathsf{H}(\mathsf{K})$. Now, let (A_k) be the sequence on $\mathsf{H}(\mathsf{K})$ such that A_0 be the polygonal interpolation of Δ and $A_k = \mathcal{W}(A_{k-1})$, for all $k \geq 1$. It follows that

$$G_\Delta = \lim_{k\to\infty} A_k \quad \text{and} \quad \omega(G_\Delta) = \lim_{k\to\infty} \omega(A_k).$$

In addition, $\omega \circ W_n$ maps A_k into the n-th piece of $\omega(A_{k+1})$. We also define the sequence (A'_k) on $\mathsf{H}(\mathsf{K})$ such that $A'_0 = \omega(A_0)$ and, for $k \geq 1$, $A'_k = \hat{\mathcal{W}}(A'_{k-1})$. Therefore, by definition of $G_{\omega(\Delta)}$, we have $G_{\omega(\Delta)} = \lim_{k\to\infty} A'_k$. Then,

$$A'_1 = \bigcup_{n=1}^N \hat{W}_n(A'_0) = \bigcup_{n=1}^N \hat{W}_n \circ \omega(A_0) = \bigcup_{n=1}^N \omega \circ W_n(A_0) = \omega(A_1),$$

and hence $A'_k = \omega(A_k)$, for all $k \geq 1$. As a consequence, we obtain

$$G_{\omega(\Delta)} = \lim_{k\to\infty} A'_k = \lim_{k\to\infty} \omega(A_k) = \omega(G_\Delta).$$

Now, $\hat{W}_n$ maps $\omega(A_k)$ into $\omega(A_{k+1})$ if, and only if $\omega \circ W_n = \hat{W}_n \circ \omega$, for all $n \in J$. For all $n \in J$, we have

$$\begin{aligned} \omega \circ W_n(x,y) &= \omega\Big(a_n x + e_n, \alpha_n(x) y + \psi_n(x)\Big) \\ &= \Big(pa_n x + pe_n, s\alpha_n(x)y + s\psi_n(x)\Big). \end{aligned}$$

Similarly, we have

$$\hat{W}_n \circ \omega(x,y) = \hat{W}_n(px, sy) = \Big(\hat{a}_n px + \hat{e}_n, s\alpha_n(px)y + \hat{\psi}_n(px)\Big).$$

It follows that, for $n \in J$, $\omega \circ W_n - \hat{W}_n \circ \omega = 0$ if and only if

$$\begin{cases} pa_n x + pe_n = \hat{a}_n px + \hat{e}_n, \\ s\alpha_n(x)y + s\psi_n(x) = s\alpha_n(px)y + \hat{\psi}_n(px). \end{cases} \tag{2.1}$$

In addition, using (1.2), we get, for all $n \in J$,

$$a_n = \frac{\Delta x_{n-1}}{x_N - x_0} = \hat{a}_n = \frac{p\Delta x_{n-1}}{p(x_N - x_0)},$$

$$e_n = x_n - a_n x_N \qquad \text{and} \qquad \hat{e}_n = px_n - \hat{a}_n p x_N,$$

where $\Delta x_n = x_n - x_{n-1}$. Thus, under our hypothesis, we have $\omega(G_\Delta) = G_{\omega(\Delta)}$.

2.2. Application: linear case

In this paragraph, we will consider the linear case, that is, when the functions ψ_n and $\hat{\psi}_n$ are defined by

$$\psi_n(x) = c_n x + d_n \qquad \text{and} \qquad \hat{\psi}_n(x) = \hat{c}_n x + \hat{d}_n. \tag{2.2}$$

From Theorem 1.1, we may deduce the following corollary:

Corollary 2.1. *Let G and $\hat{G}$ be the attractors of the IFSs $\mathcal{S}$ and, $\hat{\mathcal{S}}$ respectively, such that ψ_n and $\hat{\psi}_n$ are defined by* (2.2)*. Assume that $p = 1$ or, for $n \in J$, α_n is a constant function, then $\omega(G) = \hat{G}$.*

Proof. Assume that $p = 1$ or, for $n \in J$, α_n is a constant function, then the condition of Theorem 1.1 is reduced to $s\psi_n(x) - \hat{\psi}_n(px) = 0$. Using Eq (1.3), we get

$$c_n = \frac{\Delta y_{n-1} + \alpha_n(x_0)y_0 - \alpha_n(x_N)y_N}{x_N - x_0}, \quad d_n = y_n - c_n x_N - \alpha_n(x_N)y_N,$$

and

$$\hat{c}_n = \frac{s\Delta y_{n-1} + s\alpha_n(x_0)y_0 - s\alpha_n(x_N)y_N}{x_N - x_0}, \quad \hat{d}_n = sy_n - \hat{c}_n x_N - s\alpha_n(x_N)y_N.$$

It follows, for all $n \in J$, that $\hat{c}_n = sc_n$ and $\hat{d}_n = sd_n$, which implies that $\hat{\psi}_n(x) = s\psi_n(x)$ as required. □

Example 2.1. *(α_n are not constant functions) Let* $\Delta = \{(0,0), (0.25, 0.5), (0.5, 0.2), (0.75, 0.4), (1, 0.5)\}$ *and G be the attractor of the following IFS:*

$$\mathcal{S} = \begin{cases} \mathsf{L}_n(x) = 0.25x + x_{n-1}, \\ F_n(x,y) = 0.5\sin(x)y + c_n x + d_n, \end{cases}$$

$n \in J = \{1, \ldots, 4\}$, *where* c_n *and* d_n *are determined by the condition* (1.3). *Then* G *is the graph of the function* f *interpolating* Δ. *First, we consider the cases* $p = 1$ *and* $s = 2$, *so that* $\hat{\Delta} = \{(0,0), (0.25, 1), (0.5, 0.4), (0.75, 0.8), (1, 1)\}$. *Now, let the FIF* $\hat{f}$ *interpolates* $\hat{\Delta}$ *and define using the following IFS:*

$$\hat{\mathcal{S}} = \begin{cases} \hat{\mathsf{L}}_n(x) = \hat{a}_n x + \hat{e}_n, \\ \hat{F}_n(x,y) = 0.5\sin(x)y + \hat{c}_n x + \hat{d}_n, \end{cases}$$

$n \in J$, *where,* $\hat{a}_n$ *and* $\hat{e}_n$ *are determined by* (1.2)*, and the condition* (1.3) *holds. In Figure 1, we plot the FIFs of the systems* $\mathcal{S}$ *and* $\hat{\mathcal{S}}$ *constructed by using the Chaos Game algorithm [31]. As we can see, we have* $\omega(G) = \hat{G}$. *In Figure 2, we plot the FIFs* f *and* $\hat{f}$ *when* $p = s = 2$. *It is clear that these graphs differ from each other* ($\omega(G) \neq \hat{G}$)*, which is expected since the condition of Corollary 2.1 is not satisfied* ($p \neq 1$).

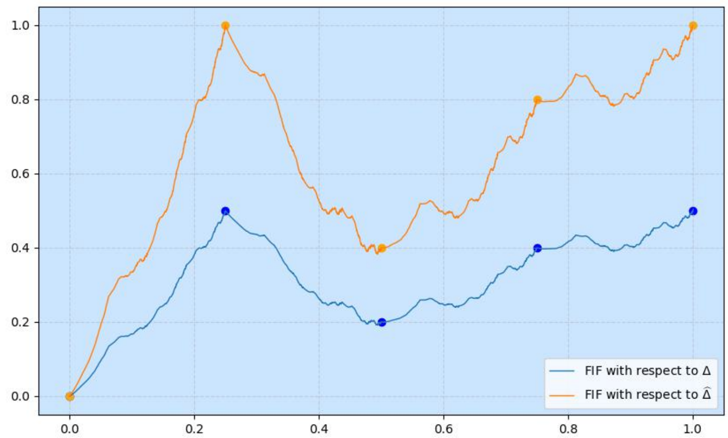

Figure 1. FIF with $p = 1$, $s = 2$.

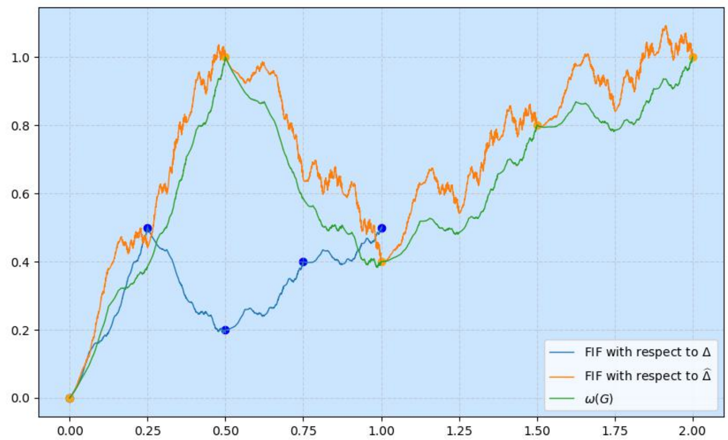

Figure 2. FIF with $p = s = 2$.

Example 2.2. *(α_n are constant functions) In this example, we consider the case when α_n are constant functions. Let* $\Delta = \{(0,0),(0.25,0.5),(0.5,0.2),(0.75,0.4),(1,0.5)\}$ *and G be the attractor of the following IFS:*

$$\mathcal{S} = \begin{cases} \mathsf{L}_n(x) = 0.25x + x_{n-1}, \\ F_n(x,y) = \alpha_n y + c_n x + d_n, \end{cases}$$

$n \in J = \{1,\dots,4\}$, where c_n and d_n are determined by the condition (1.3). *Now, let the FIF $\hat{f}$ interpolate $\hat{\Delta}$ and define using the following IFS:*

$$\hat{\mathcal{S}} = \begin{cases} \hat{\mathsf{L}}_n(x) = \hat{a}_n x + \hat{e}_n, \\ \hat{F}_n(x,y) = \alpha_n y + \hat{c}_n x + \hat{d}_n, \end{cases}$$

$n \in J$, where $\hat{a}_n$ and $\hat{e}_n$ are determined by (1.2), *and the condition* (1.3) *holds. Let $p = s = 2$, then we have $\omega(G) = \hat{G}$ (see Figure 3 for $\alpha_n = 0.5$ and Figure 4 for $\alpha_1 = \alpha_2 = 0.2$, $\alpha_3 = \alpha_4 = 0.6$).*

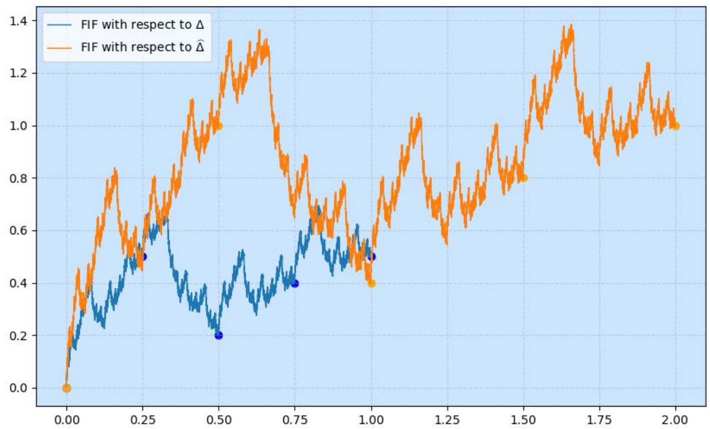

Figure 3. FIF with $p = s = 2$ and $\alpha_n = 0.5$.

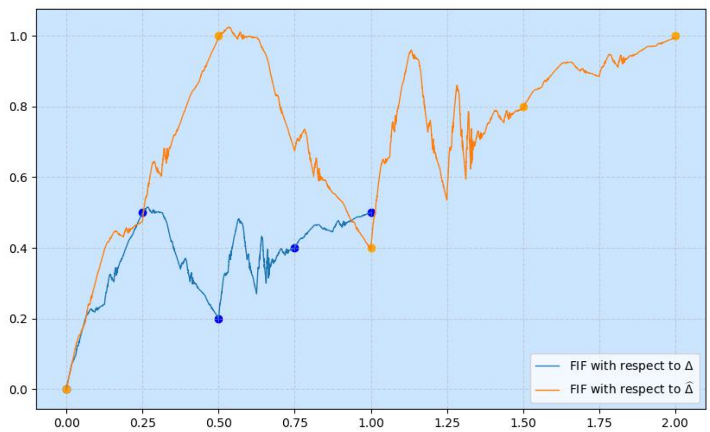

Figure 4. FIF with $p = s = 2$, $\alpha_1 = \alpha_2 = 0.2$, $\alpha_3 = \alpha_4 = 0.6$.

3. Proof of Corollary 1.1

Since α_n are constant functions or $p = 1$, then

$$\begin{aligned} & s[\alpha_n(x) - \alpha_n(px)]y + s\psi_n(x) - \hat{\psi}_n(px) \\ = \;& sh \circ \mathsf{L}_n(x) - s\alpha_n(x)b(h)(x) - \hat{h} \circ \hat{\mathsf{L}}_n(px) + \alpha_n(x)\hat{b}(\hat{h})(px) \\ = \;& \alpha_n(x)[\hat{b}(\hat{h})(px) - sb(h)(x)] + sh \circ \mathsf{L}_n(x) - \hat{h} \circ \hat{\mathsf{L}}_n(px). \end{aligned}$$

On the interval $I_n = [x_n, x_{n+1}]$, we have

$$h(x) = \frac{\Delta y_n}{\Delta x_n}x + \frac{y_n x_{n+1} - x_n y_{n+1}}{\Delta x_n}$$

and

$$\hat{h}(x) \;=\; \frac{s}{p}\frac{\Delta y_n}{\Delta x_n}x + s\frac{y_n x_{n+1} - x_n y_{n+1}}{\Delta x_n}.$$

Note that for all $x \in I_n$, we have $\hat{h}(px) - sh(x) = 0$, and, in particular

$$\begin{aligned} & s[\alpha_n(x) - \alpha_n(px)]y + s\psi_n(x) - \hat{\psi}_n(px) \\ = \;& \alpha_n(x)[\hat{b}(sh)(x) - sb(h)(x)] + sh(a_n x + e_n) - \hat{h}(\hat{a}_n px + \hat{e}_n) \\ = \;& \alpha_n(x)[\hat{b}(sh)(x) - sb(h)(x)] + sh(a_n x + e_n) - \hat{h}(a_n px + pe_n) \\ = \;& 0. \end{aligned}$$

Example 3.1. *In this example, we consider* $\Delta = \{(0,0), (0.25, .5), (0.5, .2), (0.75, .4), (1, 1)\}$ *and let* h *be the linear interpolation of the data* Δ*. Let* $b : C(I, \mathbb{R}) \longrightarrow C(I, \mathbb{R})$ *be defined by* $b(g) = g^2$ *for all* $g \in C(I, \mathbb{R})$*. Since* $h(0) = 0$ *and* $h(1) = 1$*, then*

$$b(h)(0) = h(0) = 0 \qquad \textit{and} \qquad b(h)(1) = h(1) = 1.$$

Now, let G *be the attractor of the following IFS:*

$$\mathcal{S} = \begin{cases} \mathsf{L}_n(x) = 0.25x + x_{n-1}, \\ F_n(x, y) = 0.5y + h \circ \mathsf{L}_n(x) - 0.5b(h)(x), \end{cases}$$

$n \in J = \{1, \ldots, 4\}$*, where* c_n *and* d_n *are determined by the condition* (1.3)*. Then* G *is the graph of the function* f *interpolating* Δ*. We consider the case* $p = s = 2$*, so that* $\hat{\Delta} = \{(0,0), (0.5, 1), (1, .4), (1.5, .8), (2, 2)\}$*. Let* $\hat{h}$ *be the linear interpolation of the data* $\hat{\Delta}$ *and let* $\hat{b} : C(I, \mathbb{R}) \longrightarrow C(I, \mathbb{R})$ *be defined by* $\hat{b}(g) = g^2/s$ *for all* $g \in C(I, \mathbb{R})$*. Since* $\hat{h}(0) = 0$ *and* $\hat{h}(2) = 2$*, we get*

$$\hat{b}(\hat{h})(0) = h(0) = 0 \qquad \textit{and} \qquad \hat{b}(\hat{h})(2) = h(2) = 2.$$

We consider the FIF $\hat{f}$ *interpolating* $\hat{\Delta}$ *defined using the following IFS:*

$$\hat{\mathcal{S}} = \begin{cases} \hat{\mathsf{L}}_n(x) = \hat{a}_n x + \hat{e}_n, \\ \hat{F}_n(x, y) = 0.5y + \hat{h} \circ \hat{\mathsf{L}}_n(x) - 0.5\hat{b}(\hat{h})(x), \end{cases} \tag{3.1}$$

$n \in J$, where $\hat{a}_n$ and $\hat{e}_n$ are determined by (1.2)*, and the condition* (1.3) *holds. In Figure 5, we plot the FIFs constructed through $\mathcal{S}$ and $\hat{\mathcal{S}}$. Moreover, since $\hat{b}(sh)(x) = sb(h)(x)$, the condition* (1.10) *is satisfied, and $\omega(G) = \hat{G}$ by Corollary 1.1.*

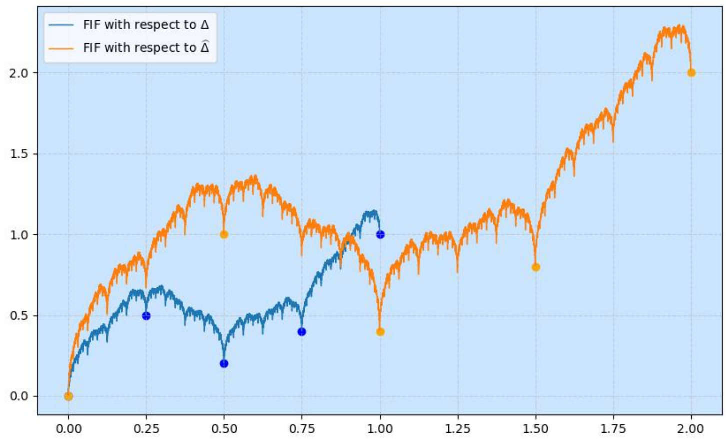

Figure 5. FIF with $p = s = 2$ and $\hat{b}(g) = g^2/s$.

Remark 3.1. *Take, in Example 3.1, the linear operator $\hat{b} : C(I, \mathbb{R}) \longrightarrow C(I, \mathbb{R})$ defined by $\hat{b}(g) = \sqrt{2|g|}$ for all $g \in C(I, \mathbb{R})$. Clearly, we have $\hat{b}(\hat{h})(0) = \hat{h}(0) = 0$ and $\hat{b}(\hat{h})(2) = \hat{h}(2) = 2$.*

We consider the FIF $\hat{f}$ interpolating $\hat{\Delta}$ defined using the following IFS:

$$\hat{\mathcal{S}} = \begin{cases} \hat{\mathsf{L}}_n(x) = \hat{a}_n x + \hat{e}_n, \\ \hat{F}_n(x, y) = 0.5y + \hat{h} \circ \hat{\mathsf{L}}_n(x) - 0.5\hat{b}(\hat{h})(x), \end{cases}$$

$n \in J$, where $\hat{a}_n$ and $\hat{e}_n$ are determined by (1.2)*, and the condition* (1.3) *holds. In Figure 6, we plot the FIFs constructed through $\mathcal{S}$ and $\hat{\mathcal{S}}$. Here, we have $\omega(G) \neq \hat{G}$ which is expected since the condition of Corollary 1.1 is not satisfied : $\hat{b}(sh)(x) \neq sb(h)(x)$.*

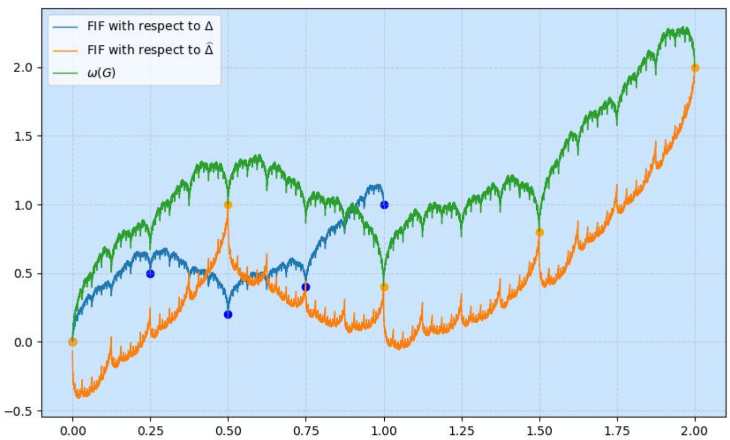

Figure 6. FIF with $p = s = 2$ and $\hat{b}(g) = \sqrt{2|g|}$.

Consider the notation of Corollary 1.1 and assume, for $n \in J$, that α_n is a constant function. Let $\mathcal{J}$ be the set of continuous functions $g : I \longrightarrow \mathbb{R}$, such that $g(x_0) = x_0$ and $g(x_N) = x_N$. It is well known that $(\mathcal{J}, \rho)$ is a complete metric space, where ρ is a metric defined as $\rho(g, h) = \|g - h\|_\infty$ for all $g, h \in \mathcal{J}$. We define on $\mathcal{J}$ the Read-Bajraktarevic operator T by

$$\mathsf{T}(g(x)) = F_n\Big(\mathsf{L}_n^{-1}(x), g(\mathsf{L}_n^{-1}(x))\Big),$$

for all $x \in I_n, n \in J$. We can prove easily that T is a contraction mapping; that is,

$$\|\mathsf{T}(f) - \mathsf{T}(g)\| \quad \leq \quad \alpha \|f - g\|_\infty,$$

where $\alpha := \max_n |\alpha_n|$. Thus, T possesses a unique fixed-point f on $\mathcal{J}$. Moreover, f is the unique function satisfying, for $x \in I_n, n \in J$,

$$f(x) = F_n\Big(\mathsf{L}_n^{-1}(x), f(\mathsf{L}_n^{-1}(x))\Big). \tag{3.2}$$

In the next section, we will compare the definite integrals $\int_I f(x)dx$ and $\int_{\omega(I)} \hat{f}(x)dx$, where $\omega(I) = \{px, x \in I\}$.

Proposition 3.1. *Assume that* $\delta = \sum_{n=1}^N a_n\alpha_n$*, then*

$$(1 - \delta) \int_{\omega(I)} \hat{f}(x)dx - ps(1 - \delta) \int_I f(x)dx = \int_I p\, s\, \delta\, b(h)(x) - p\, \delta\, \hat{b}(sh)(x)dx.$$

In particular, if $\delta = 0$*, then* $\int_{\omega(I)} \hat{f}(x)dx = ps \int_I f(x)dx$.

Proof. Using (3.2), we obtain $f(x) = h(x) + \alpha_n[f - b(h)](\mathsf{L}_n^{-1}(x))$, and then

$$\int_{x_0}^{x_N} f(x)dx = \int_{x_0}^{x_N} h(x)dx + \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} \alpha_n(x)[f - b(h)](\mathsf{L}_n^{-1}(x))dx.$$

Letting $z = \mathsf{L}_n^{-1}(x)$, we get

$$\int_{x_0}^{x_N} f(x)dx = \int_{x_0}^{x_N} h(x)dx + \delta \int_{x_0}^{x_N} [f - b(h)](z)dz.$$

It follows that

$$(1 - \delta) \int_{x_0}^{x_N} f(x)dx = \int_{x_0}^{x_N} h(x)dx - \delta \int_{x_0}^{x_N} b(h)(z)dz.$$

Similarly, we have

$$\int_{px_0}^{px_N} \hat{f}(x)dx = \int_{px_0}^{px_N} \hat{h}(x)dx + \delta \int_{px_0}^{px_N} [\hat{f} - \hat{b}(\hat{h})](z)dz,$$

and then

$$(1 - \delta) \int_{px_0}^{px_N} \hat{f}(x)dx = \int_{px_0}^{px_N} \hat{h}(x)dx - \delta \int_{px_0}^{px_N} \hat{b}(\hat{h})(z)dz.$$

Note that, for all $x \in I_n$, we have $\hat{h}(px) - sh(x) = 0$. It follows that

$$
\begin{aligned}
(1-\delta)\int_{px_0}^{px_N}\hat{f}(x)dx &= ps\int_{x_0}^{x_N}h(z)dz-\delta\int_{px_0}^{px_N}\hat{b}(\hat{h})(z)dz\\
&= ps(1-\delta)\int_{x_0}^{x_N}f(x)dx+ps\delta\int_{x_0}^{x_N}b(h)(z)dz-\delta\int_{px_0}^{px_N}\hat{b}(\hat{h})(z)dz\\
&= ps(1-\delta)\int_{x_0}^{x_N}f(x)dx+ps\delta\int_{x_0}^{x_N}b(h)(z)dz-p\delta\int_{x_0}^{x_N}\hat{b}(\hat{h})(px)dx\\
&= ps(1-\delta)\int_{x_0}^{x_N}f(x)dx+ps\delta\int_{x_0}^{x_N}b(h)(x)dx-p\delta\int_{x_0}^{x_N}\hat{b}(sh)(x)dx,
\end{aligned}
$$

as required. □

Remark 3.2. *(1) Clearly, if $p = 1$, then the result can be deduced immediately from Corollary 1.1 and we have*

$$\int_I \hat{f}(x)dx = s\int_I f(x)dx.$$

(2) Consider the case when $\sum_{n=1}^{N}\alpha_n = 0$. Then, for any uniform partition, that is Δx_n is constant for all n, we get $\delta = 0$, and then $\int_{\omega(I)}\hat{f}(x)dx = ps\int_I f(x)dx$.

4. Additivity property

We will prove it with Theorem 1.2 for $m = 1$. First, note that the conditions (1.3) and (1.4) hold, and then the function f^s interpolates the data Δ_S. We define the Hutchinson operators $\mathcal{W}$ and $\mathcal{W}_1$ by

$$\mathcal{W}(A) = \bigcup_{n=1}^{N} W_n(A) \quad \text{and} \quad \mathcal{W}_1(A) = \bigcup_{n=1}^{N} W_{1,n}(A)$$

for all $A \in \mathsf{H}(\mathsf{K})$, where $W_{1,n}(x,y) = (\mathsf{L}_n(x), F_{1,n}(x,y))$. Let A_0 and A'_0 be the polygonal interpolation of Δ and Δ_1, respectively, and consider the sequences (A_k) and (A'_k) defined by $A_k = \mathcal{W}(A_{k-1})$ and $A'_k = \mathcal{W}_1(A'_{k-1})$, for all $k \geq 1$, respectively. It follows that

$$G_\Delta = \lim_{k\to\infty} A_k \quad \text{and} \quad G_{\Delta_1} = \lim_{k\to\infty} A'_k.$$

Now, we define $\mathcal{S}_n(x,y) = (\mathsf{L}_n(x), F_n(x,y) + F_{1,n}(x,y))$, for all $n \in J$, and the sequence (S_k) as $S_0 = A_0 + A'_0 = \{(x, y+y'), x \in I, y \in A_0, y' \in A'\}$ and for all $k \geq 1$, $S_k = \bigcup_{n=1}^{N}\mathcal{S}_n(S_{k-1})$. It follows that

$$
\begin{aligned}
S_1 &= \bigcup_{n=1}^{N}\mathcal{S}_n(S_0) = \bigcup_{n=1}^{N}\mathcal{S}_n(A_0 + A'_0)\\
&= \bigcup_{n=1}^{N}\Big\{(\mathsf{L}_n(x), F_n(x,y)), \quad (x,y)\in A_0\Big\} + \Big\{(\mathsf{L}_n(x), F_{1,n}(x,y)), \quad (x,y)\in A'_0\Big\}\\
&= \mathcal{W}(A_0) + \mathcal{W}'(A_0) = A_1 + A'_1.
\end{aligned}
$$

Similarly, for all $k \geq 1$, we have $S_k = A_k + A'_k = \mathcal{W}^k(A_0) + \mathcal{W}_1^k(A'_0)$. Hence,

$$G_{\Delta+\Delta_1} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \mathcal{W}^k(A_0) + \mathcal{W}_1^k(A'_0) = G_\Delta + G_{\Delta_1},$$

as required. Thus, the result of the theorem is acquired by induction.

Example 4.1. *Let $b : C(I,\mathbb{R}) \longrightarrow C(I,\mathbb{R})$ be the operator defined by $b(g) = g^2$ for all $g \in C(I,\mathbb{R})$. We consider the two data sets*

$$\begin{cases} \Delta = \{(0,0),(0.25,0.5),(0.5,0.2),(0.75,0.4),(1,1)\}, \\ \Delta_1 = \{(0,0),(0.25,0.75),(0.5,-0.2),(0.75,0.5),(1,1)\}, \end{cases}$$

and let h and h_1 be the linear interpolations of Δ and Δ_1, respectively. We define the IFSs

$$\mathcal{S} = \begin{cases} \mathsf{L}_n(x) = 0.25x + x_{n-1}, \\ F_n(x,y) = 0.5y + h \circ \mathsf{L}_n(x) - 0.5b(h)(x), \end{cases} \quad \mathcal{S}_1 = \begin{cases} \mathsf{L}_n(x) = 0.25x + x_{n-1}, \\ F_{1,n}(x,y) = 0.5y + h_1 \circ \mathsf{L}_n(x) - 0.5b(h_1)(x), \end{cases}$$

where $n \in J$. In Figure 7, we plot G_Δ, G_{Δ_1} and $G_{\Delta+\Delta_1}$.

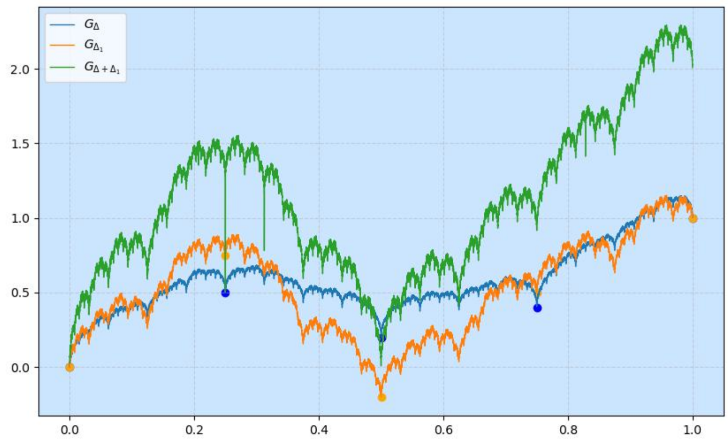

Figure 7. $G_{\Delta+\Delta_1} = G_\Delta + G_{\Delta_1}$.

5. Conclusions

In this work, a class of generalized affine FIFs with variable parameters $\alpha_n(x)$ is studied. Let $\mathcal{S}$ and $\hat{\mathcal{S}}$ be two IFSs with attractors, G_Δ and $G_{\hat{\Delta}}$ respectively, where Δ and $\hat{\Delta}$ are given data sets. Assume that $\hat{\Delta} = \omega(\Delta)$, where ω is some linear transformation, then we give a sufficient condition so that $G_{\omega(\Delta)} = \omega(G_\Delta)$. In addition, the definite integrals of the corresponding FIFs are considered, and the additivity property is studied. Some examples are given highlighting the effectiveness of our results. The findings presented in this paper suggest a compelling avenue for further investigation into the characteristics of these systems, particularly in exploring the connection between smoothly perturbing the systems' parameters and their corresponding FIFs.

Author contributions

Najmeddine Attia and Rim Amami: Writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research, Vice President for Graduate Studies and Scientific Research at King Faisal University, Saudi Arabia, for financial support under the annual funding track [GRANT A270].

Conflict of interest

The authors declare no conflits of interest.

References

1. M. F. Barnsley, Fractal functions and interpolation, *Constr. Approx.*, **2** (1986), 303–329. https://doi.org/10.1007/BF01893434
2. G. E. Hardy, T. D. Rogers, A generalization of a fixed point theorem of Reich, *Can. Math. Bull.*, **16** (1973), 201–206. https://doi.org/10.4153/CMB-1973-036-0
3. M. A. Navascuès, M. V. Sebastian, Fitting curves by fractal interpolation: an application to the quantification of cognitive brain processes, In: *Thinking in patterns*, 2004, 143–154. https://doi.org/10.1142/9789812702746_0011
4. P. R. Massopust, *Fractal functions, fractal surfaces, and wavelets*, Academic Press, 1995.
5. A. Petrusel, I. A. Rus, M. A. Serban, Fixed points, fixed sets and iterated multifunction systems for nonself multivalued operators, *Set-Valued Var. Anal.*, **23** (2015), 223–237. https://doi.org/10.1007/s11228-014-0291-6
6. X. Y. Wang, F. P. Li, A class of nonlinear iterated function system attractors, *Nonlinear Anal.*, **70** (2009), 830–838. https://doi.org/10.1016/j.na.2008.01.013
7. N. Attia, N. Saidi, R. Amami, R. Amami, On the stability of fractal interpolation functions with variable parameters, *AIMS Math.,* **9** (2024), 2908–2924. https://doi.org/10.3934/math.2024143
8. A. K. B. Chand, G. P. Kapoor, Stability of affine coalescence hidden variable fractal interpolation functions, *Nonlinear Anal.*, **68** (2008), 3757–3770. https://doi.org/10.1016/j.na.2007.04.017
9. M. A. Navascuès, M. V. Sebastian, Smooth fractal interpolation, *J. Inequal. Appl.*, **2006** (2006), 1–20. https://doi.org/10.1155/JIA/2006/78734

10. A. K. B. Chand, G. P. Kapoor, Generalized cubic spline fractal interpolation functions, *SIAM J. Numer. Anal.*, **44** (2006), 655–676. https://doi.org/10.1137/040611070

11. J. Kim, H. Kim, H. Mun, Nonlinear fractal interpolation curves with function vertical scaling factors, *Indian J. Pure Appl. Math.*, **5**1 (2020), 483–499. https://doi.org/10.1007/s13226-020-0412-x

12. N. Attia, T. Moulahi, R. Amami, N. Saidi, Note on fractal interpolation function with variable parameters, *AIMS Math.,* **9** (2024), 2584–2601. https://doi.org/10.3934/math.2024127

13. N. Vijender, Bernstein fractal trigonometric approximation, *Acta Appl. Math.*, **159** (2019), 11–27. https://doi.org/10.1007/s10440-018-0182-1

14. E. Mihailescu, M. Urbański, Random countable iterated function systems with overlaps and applications, *Adv. Math.*, **298** (2016), 726–758. https://doi.org/10.1016/j.aim.2016.05.002

15. J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.*, **30** (1981), 713–747. https://doi.org/10.1512/iumj.1981.30.30055

16. K. Leśniak, Infinite iterated function systems: a multivalued approach, *Bull. Pol. Acad. Sci. Math.*, **52** (2004), 1–8.

17. A. Mihail, R. Miculescu, The shift space for an infinite iterated function system, *Math. Rep.*, **11** (2009), 21–32.

18. A. Mihail, R. Miculescu, Generalized IFSs on non-compact spaces, *Fixed Point Theory Appl.*, **2010** (2010), 1–15. https://doi.org/10.1155/2010/584215

19. S. Ri, A new nonlinear fractal interpolation function, *Fractals*, **25** (2017), 1750063. https://doi.org/10.1142/S0218348X17500633

20. M. A. Navascués, C. Pacurar, V. Drakopoulos, Scale-free fractal interpolation, *Fractal Fract.*, **6** (2022), 1–15. https://doi.org/10.3390/fractalfract6100602

21. S. Ri, New types of fractal interpolation surfaces, *Chaos Solitons Fract.*, **119** (2019), 291–297. https://doi.org/10.1016/j.chaos.2019.01.010

22. N. Attia, H. Jebali, On the construction of recurrent fractal interpolation functions using Geraghty contractions, *Electron. Res. Arch.*, **3**1 (2023), 6866–6880. https://doi.org/10.3934/era.2023347

23. N. Attia, M. Balegh, R. Amami, R. Amami, On the fractal interpolation functions associated with Matkowski contractions, *Electron. Res. Arch.*, **31** (2023), 4652–4668. https://doi.org/10.3934/era.2023238

24. P. Viswanathan, A. K. B. Chand, M. A. Navascuès, Fractal perturbation preserving fundamental shapes: bounds on the scale factors, *J. Math. Anal. Appl.*, **419** (2014), 804–817. https://doi.org/10.1016/j.jmaa.2014.05.019

25. M. Barnsley, A. N. Harrington, The calculus of fractal interpolation functions, *J. Approx. Theory*, **57** (1989), 14–34. https://doi.org/10.1016/0021-9045(89)90080-4

26. M. Navascués, Fractal polynomial interpolation, *Z. Anal. Anwend.*, **24** (2005), 401–418.

27. H. Y. Wang, J. S. Yu, Fractal interpolation functions with variable parameters and their analytical properties, *J. Approx. Theory*, **1**75 (2013), 1–18.

28. L. M. Kocić, Notes on fractal interpolation, *Novi Sad J. Math.*, **30** (2000), 59–68.

29. C. Gang, The smoothness and dimension of fractal interpolation functions, *Appl. Math.*, **11** (1996), 409–418. https://doi.org/10.1007/BF02662880

30. M. N. Akhtar, M. G. P. Prasad, M. A. Navascués, Box dimension of α-fractal function with variable scaling factors in subintervals, *Chaos Solitons Fract.*, **103** (2017), 440–449. https://doi.org/10.1016/j.chaos.2017.07.002

31. M. F. Barnsley, *Fractals everywhere*, Academic Press, 1988.