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Research article

Normalized multi-bump solutions of nonlinear Kirchhoff equations

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Abstract: We are concerned with the existence and concentration of multi-bump solutions for the nonlinear Kirchhoff equation

$$-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla v|^2 \,\mathrm{d}x\right)\Delta v + \lambda v = K(x)\,|v|^{2\sigma}\,v, \quad x \in \mathbb{R}^3$$

with an L^2-constraint in the L^2-subcritical case $\sigma \in \left(0, \frac{2}{3}\right)$ and the L^2-supercritical case $\sigma \in \left(\frac{2}{3}, 2\right)$. Here $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier, ε is a small positive parameter and $K > 0$ possesses several local maximum points. By employing the variational gluing method and the penalization technique, we prove the existence of multi-bump solutions that are concentrated at local maximum points of K for the problem above.

Keywords: Kirchhoff equation; normalized solutions; concentration behavior
Mathematics Subject Classification: 35A15, 35B25, 35B40

1. Introduction and main results

1.1. Background and motivation

In this paper, we mainly focus our interest on the existence and concentration of normalized solutions of the following nonlinear elliptic problem involving a Kirchhoff term:

$$\begin{cases} -\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla v|^2 \,\mathrm{d}x\right)\Delta v - K(x)\,|v|^{2\sigma}\,v = -\lambda v & \text{in } \mathbb{R}^3, \\ |v|_2^2 = \int_{\mathbb{R}^3} v^2 \mathrm{d}x = m_0 \varepsilon^\alpha, \quad v(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where a, b, α are positive real numbers and $\sigma \in (0, 2)$, λ is unkown and appears as a Lagrange multiplier. Equation (1.1) is related to the stationary solutions of

$$u_{tt} - \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u = g(x,\, t). \tag{1.2}$$

Equation (1.2) was first proposed by Kirchhoff in [13] and regarded as an extension of the classical D'Alembert's wave equation, which describes free vibrations of elastic strings. Kirchhoff-type problems also appear in other fields like biological systems. To better understand the physical background, we refer the readers to [1, 2, 4, 14]. From a mathematical point of view, problem (1.1) is not a pointwise identity because of the appearance of the term $(\int_{\mathbb{R}^3} |\nabla u|^2)\Delta u$. Due to such a characteristic, Kirchhoff- type equations constitute nonlocal problems. Compared with the semilinear states (i.e., setting $b = 0$ in the above two equations), the nonlocal term creates some additional mathematical difficulties which make the study of such problems particularly interesting.

In the literature about the following related unconstrained Kirchhoff problems, there have been a lot of results on the existence and concentration of solutions for small values of ε.

$$-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\right)\Delta u + V(x)u = f(u), x \in \mathbb{R}^3. \tag{1.3}$$

In physics, such solutions are called the semiclassical states for small values of ε. In [10], the existence, multiplicity and concentration behavior of positive solutions to the Kirchhoff problem (1.3) have been studied by He and Zou, where $V(x)$ is a continuous function and f is a subcritical nonlinear term. For the critical case, Wang et al., in [28] obtained some multiplicity and concentration results of positive solutions for the Kirchhoff problem (1.3). And He et al., in [11] obtained the concentration of solutions in the critical case. Recently, multi-peak solutions were established by Luo et al., in [18] for the following problem:

$$-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\right)\Delta u + V(x)u = |u|^{p-2}\, u, x \in \mathbb{R}^3. \tag{1.4}$$

In [15] Li et al., revisited the singular perturbation problem (1.4), where $V(x)$ satisfies some suitable assumptions. They established the uniqueness and nondegeneracy of positive solutions to the following limiting Kirchhoff problem:

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\right)\Delta u + u = |u|^{p-2}\, u, x \in \mathbb{R}^3.$$

By the Lyapunov-Schmidt reduction method and a local Pohozaev identity, single-peak solutions were obtained for (1.4). In the past decades, other related results have also been widely studied, such as the existence of ground states, positive solutions, multiple solutions and sign-changing solutions to (1.4). We refer the reader to [7, 9, 10, 16, 29] and the references therein.

In recent years, the problems on normalized solutions have attracted much attention from many researchers. In [25, 26], Stuart considered the problem given by

$$\begin{cases} -\Delta u + \lambda u = f(u), \quad x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = c \end{cases} \tag{1.5}$$

in the mass-subcritical case and obtained the existence of normalized solutions by seeking a global minimizer of the energy functional. In [12], Jeanjean considered the mass supercritical case and studied the existence of normalized solutions to problem (1.5) by using the mountain pass lemma. For the Sobolev critical case, Soave in [24] considered normalized ground state solutions of problem (1.5) with $f(u) = \mu|u|^{q-2}u + |u|^{2^*-2}u$, where $2^* = 2N/(N-2), N \geq 3$ is the Sobolev critical exponent. For $f(u) = g(u) + |u|^{2^*-2}u$ with a mass critical or supercritical state but Sobolev subcritical nonlinearity g, we refer the reader to [19]. Now, we would like to mention some related results on Kirchhoff problems. The authors of [29, 30] considered the problem in the mass subcritical and mass critical cases:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla v|^2 \, dx\right)\Delta v = \lambda v + f(v) & \text{in } \mathbb{R}^N, \\ |v|_2^2 = \int_{\mathbb{R}^N} v^2 dx = c^2, \end{cases} \tag{1.6}$$

with $a, b > 0$ and $p \in (2, 2^*)$. The existence and non-existence of normalized solutions are obtained. In [20], the Kirchhoff problem (1.6) was investigated for $f(u) = \mu|u|^{q-2}u + |u|^{2^*-2}u$ and $N = 3$. With the aid of a subcritical approximation approach, the existence of normalized ground states can be obtained for $\mu > 0$ large enough. Moreover, the asymptotic behavior of ground state solutions is also considered as $c \to \infty$. As for further results on Sobolev critical Kirchhoff equations and high energy normalized solutions, we refer the reader to [21, 22, 32].

In what follows, we turn our attention to normalized multi-bump solutions of the Kirchhoff problem (1.1). For the related results on Schrödinger equations, we refer the reader to the references [27, 31]. In [31], the following nonlinear Schrödinger equation was studied by Zhang and Zhang:

$$\begin{cases} -\hbar^2\Delta v - K(x)\,|v|^{2\sigma}\, v = -\lambda v & \text{in } \mathbb{R}^N, \\ |v|_2^2 = \int_{\mathbb{R}^N} v^2 dx = m_0\hbar^\alpha, \quad v(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.7}$$

For the case that the parameter $\hbar$ goes to 0, the authors of [31] constructed normalized multi-bump solutions around the local maximum points of K by employing the variational gluing methods of Séré [23] and Zelati and Rabinowitz [5, 6], as well as the penalization technique [31]. Soon afterward, Tang et al., in [27] considered normalized solutions to the nonlinear Schrödinger problem

$$-\Delta u + \lambda a(x)u + \mu u = |u|^{2\sigma}\, u, \quad x \in \mathbb{R}^N \tag{1.8}$$

with an L^2-constraint. By taking the limit as $\lambda \to +\infty$, they derive the existence of normalized multi-bump solutions with each bump concentrated around the local minimum set of $a(x)$.

1.2. *Main result of this paper*

Motivated by [27, 31], the present paper is devoted to the existence and concentration behavior of the multi-bump solutions for the Kirchhoff problem (1.1). In contrast to the nonlinear Schrödinger problems, the Kirchhoff term brings us some additional difficulties. We intend to obtain the existence of multi-bump solutions for (1.1).

Before stating our main result, we give the following assumptions:

(A) $\alpha \in (3, \frac{2}{\sigma})$ if $\sigma \in (0, \frac{2}{3})$ and $\alpha \in (\frac{2}{\sigma}, 3)$ if $\sigma \in (\frac{2}{3}, 2)$.

(K) $K \in (\mathbb{R}^3, (0, +\infty)) \cap L^\infty(\mathbb{R}^3)$ and there are $\ell \geq 2$ mutually disjoint bounded domains $\Omega_i \subset \mathbb{R}^3$, $i = 1, 2, \cdots, \ell$ such that

$$k_i := \max_{x\in\Omega_i} K(x) > \max_{x\in\partial\Omega_i} K(x).$$

Denote $\mathcal{K}_i = \{x \in \Omega_i | K(x) = k_i\}$, which is nonempty and compact and set

$$\beta := \frac{2 - \alpha\sigma}{2 - 3\sigma}.$$

Now, we state our main result as follows.

Theorem 1.1. *Assume that* (A) *and* (K). *There is* $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0, \varepsilon_0)$, *it follows that* (1.1) *admits a solution* $(\lambda_\varepsilon, v_\varepsilon) \in \mathbb{R} \times H^1(\mathbb{R}^3)$ *with the following properties:*

(a) v_ε *admits exactly* ℓ *local maximum points* $P_{i,\varepsilon}$, $i = 1, 2, \cdots, \ell$ *that satisfy*

$$\lim_{\varepsilon \to 0} \mathrm{dist}(P_{i,\varepsilon}, \mathcal{K}_i) = 0.$$

(b) $\mu = \varepsilon^{\frac{2\sigma(3-\alpha)}{2-3\sigma}} \lambda_\varepsilon \to \mu_0$ *and* $\|\varepsilon^{\frac{3-\alpha}{2-3\sigma}} v_\varepsilon(\varepsilon^\beta \cdot) - \sum_{i=1}^{\ell} u_i(\cdot - \varepsilon^{-\beta} P_{i,\varepsilon})\|_{H^1} \to 0$ *as* $\varepsilon \to 0$, *where*

$$\mu_0 = m_0^{\frac{2\sigma}{2-3\sigma}} a^{-\frac{3\sigma}{2-3\sigma}} (\sum_{i=1}^{\ell} \theta_i^{-\frac{1}{\sigma}} |U|_2^2)^{-\frac{2\sigma}{2-3\sigma}},$$

$$u_i = \theta_i^{-\frac{1}{2\sigma}} \mu^{\frac{1}{2\sigma}} U(\sqrt{\frac{\mu}{a}} \cdot), \ i = 1, 2, \cdots, \ell,$$

and $U \in H^1(\mathbb{R}^3)$ *is a positive solution to*

$$\begin{cases} -\Delta U + U = |U|^{2\sigma} U & \text{in } \mathbb{R}^3, \\ U(0) = \max_{x \in \mathbb{R}^3} U(x), \quad \lim_{x \to \infty} U(x) = 0. \end{cases} \tag{1.9}$$

(c) *There are constants* $C, c > 0$ *that are independent of* ε *such that*

$$|v_\varepsilon| \le C \varepsilon^{-\frac{3-\alpha}{2-3\sigma}} \exp\left\{-c\varepsilon^{-\beta} \mathrm{dist}(x, \cup_{i=1}^{\ell} \mathcal{K}_i)\right\}.$$

1.3. *The strategy for the proof*

The proof of Theorem 1.1 is similar to that in [31]. By virtue of the change of variables techinque, we have

$$u(\cdot) = \varepsilon^{\frac{3-\alpha}{2-3\sigma}} v(\varepsilon^\beta \cdot).$$

Equation(1.1) is transformed into the following problem:

$$\begin{cases} -(a + \varepsilon^{\frac{(3-\alpha)(\sigma-2)}{2-3\sigma}} b |\nabla u|_2^2) \Delta u - K(\varepsilon^\beta x) |u|^{2\sigma} u = -\lambda \varepsilon^{\frac{2\sigma(3-\alpha)}{2-3\sigma}} u & \text{in } \mathbb{R}^3, \\ |u|_2^2 = m_0, \quad u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

Let

$$\hbar := \varepsilon^\beta, \ \ \mu = \varepsilon^{\frac{2\sigma(3-\alpha)}{2-3\sigma}} \lambda, \ \ d = \frac{(3-\alpha)(\sigma-2)}{2-\alpha\sigma}.$$

Then, under the assumption (A) and given $\beta > 0$ and $d > 0$, we have the following:

$$\begin{cases} -(a + \hbar^d b |\nabla u|_2^2) \Delta u - K(\hbar x) |u|^{2\sigma} u = -\mu u & \text{in } \mathbb{R}^3, \\ |u|_2^2 = m_0, \quad u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.10}$$

Define the energy functional

$$E_\hbar(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{\hbar^d b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \frac{1}{2\sigma+2}\int_{\mathbb{R}^3} K(\hbar x)\,|u|^{2\sigma+2}\,.$$

Then, a solution $(\mu_\hbar,\, u_\hbar)$ of (1.10) can be obtained as a critical point of $E_\hbar$ that is restrained on

$$\mathcal{M} := \left\{u \in H^1(\mathbb{R}^3)|\,|u|_2^2 = m_0\right\}.$$

By adopting similar deformation arguments in [5, 6, 23, 31], we show that the Lagrange multiplier $\mu_\hbar$ satisfies

$$\mu_\hbar = \mu_0 + o_\hbar(1), \quad u_\hbar = \sum_{i=1}^{\ell} u_i(\cdot - q_{i,\hbar}) + o_\hbar(1) \quad \text{in } H^1(\mathbb{R}^3),$$

where $q_{i,\hbar}$ satisfies the condition that $\mathrm{dist}(\hbar q_{i,\hbar}, \mathcal{K}_i) \to 0$ as $\hbar \to 0$, $i = 1,\, 2,\, \cdots,\, \ell$.

This paper is organized as follows: In Section 2, we study the existence and variational structure of solutions to the limit equation of Eq (1.1). In Section 3, we introduce the penalized function which satisfies the Palais-Smale condition. In Section 4, we prove the existence of a critical point of the penalized function in the subcritical and supercritical cases. In Section 5, we show that the critical point is a solution to the original problem through the application of a decay estimate.

Notation : In this paper, we make use of the following notations:

- $|u|_p := (\int_{\mathbb{R}^3} |u|^p)^{\frac{1}{p}}$, where $u \in L^p(\mathbb{R}^3)$, $p \in [1,\, \infty)$;
- $\|u\| := (\int_{\mathbb{R}^3} |\nabla u|^2 + |u|^2)^{\frac{1}{2}}$, where $u \in H^1(\mathbb{R}^3)$;
- $b^\pm = \max\{0,\, \pm b\}$ for $b \in \mathbb{R}$;
- $B(x,\, \rho)$ denotes an open ball centered at $x \in \mathbb{R}^3$ with radius $\rho > 0$;
- For a domain $D \subset \mathbb{R}^3$, we denote $\frac{1}{\hbar}D := \left\{x \in \mathbb{R}^3 | \hbar x \in D\right\}$;
- Unless stated otherwise, δ and C are general constants.

2. The limit system

Let $m_0,\, \theta_1,\, \theta_2,\, \cdots,\, \theta_\ell$ be a series of positive numbers. We consider the following system:

$$\begin{cases} -a\Delta v_i - \theta_i\, |v_i|^{2\sigma}\, v_i = -\mu v_i & \text{in } \mathbb{R}^3, \\ \sum_{i=1}^{\ell} |v_i|_2^2 = m_0, & \\ v_i(x) > 0, \quad \lim_{|x|\to\infty} v_i(x) = 0, & i = 1,\, 2,\, \cdots,\, \ell. \end{cases} \tag{2.1}$$

Next, we refer the reader to [31] to show Lemmas 2.1–2.3 as follows.

Lemma 2.1. *For $\sigma \in \left(0,\, \frac{2}{3}\right) \cup \left(\frac{2}{3},\, 2\right)$, system* (2.1) *has a unique solution $(\mu,\, v_1,\, v_2,\, \cdots,\, v_\ell) \in \mathbb{R} \times H^1(\mathbb{R}^3)^\ell$ up to translations of each v_i, $i = 1,\, 2,\, \cdots,\, \ell$, where*

$$\mu = m_0^{\frac{2\sigma}{2-3\sigma}} a^{-\frac{3\sigma}{2-3\sigma}} \Big(\sum_{i=1}^{\ell} \theta_i^{-\frac{1}{\sigma}}\, |U|_2^2\Big)^{-\frac{2\sigma}{2-3\sigma}}, \quad v_i(x) = \theta_i^{-\frac{1}{2\sigma}} \mu^{\frac{1}{2\sigma}} U\Big(\sqrt{\frac{\mu}{a}}x\Big), \tag{2.2}$$

and $U \in H^1(\mathbb{R}^3)$ is the unique positive radial solution to (1.9).

By using (2.2), we can obtain the mass distribution for each v_i, $i = 1, 2, \cdots, \ell$ in the limit system (2.1), as follows:

$$|v_i|_2^2 = \frac{m_0 \theta_i^{-\frac{1}{\sigma}}}{\sum_{i=1}^{\ell} \theta_i^{-\frac{1}{\sigma}}}$$

and for each $i = 1, 2, \cdots, \ell$, v_i is the ground state of

$$I_{\theta_i}(u) = \frac{a}{2} |\nabla u|_2^2 - \frac{\theta_i}{2\sigma + 2} |u|_{2\sigma+2}^{2\sigma+2}$$

on

$$\mathcal{M}_i := \left\{ u \in H^1(\mathbb{R}^3) | \, |u|_2^2 = |v_i|_2^2 \right\}.$$

Lemma 2.2. $\sum_{i=1}^{\ell} I_{\theta_i}(v_i)$ *is continuous and strictly decreasing with respect to* m_0 *and* θ_i, $i = 1, 2, \cdots, \ell$, *where* v_i *is determined as in Lemma 2.1.*

We next characterize the energy level of $\sum_{i=1}^{\ell} I_{\theta_i}(v_i)$. Let

$$s = (s_1, s_2, \cdots, s_\ell) \in (0, +\infty)^\ell$$

and for each $s_i > 0$, the minimizing problem

$$b_{s_i} = \inf \left\{ I_{\theta_i}(v) | \, |v|_2^2 = s_i^2, |\nabla v|_2^2 = \frac{3\theta_i \sigma}{(2\sigma + 2)a} |v|_{2\sigma+2}^{2\sigma+2} \right\}$$

is achieved for each $i = 1, 2, \cdots, \ell$ given some radial function w_{s_i}. In particular, $v_i = w_{s_i^0}$ for $s_i^0 = |v_i|_2$. Moreover, if $\sigma \in \left(0, \frac{2}{3}\right)$, then

$$b_{s_i} = \inf \left\{ I_{\theta_i}(v) | v \in H^1(\mathbb{R}^3), |v|_2^2 = s_i^2 \right\}$$

and if $\sigma \in \left(\frac{2}{3}, 2\right)$, then

$$b_{s_i} = \inf \left\{ \sup_{t>0} I_{\theta_i}(t^{\frac{3}{2}} v(t\cdot)) | v \in H^1(\mathbb{R}^3), |v|_2^2 = s_i^2 \right\}.$$

Set

$$S_+^{\ell-1} := \left\{ s = (s_1, s_2, \cdots, s_\ell) \in (0, \sqrt{m_0})^\ell | \sum_{i=1}^{\ell} s_i^2 = m_0, \, i = 1, 2, \cdots, \ell \right\},$$

and define $E(s) := \sum_{i=1}^{\ell} I_{\theta_i}(w_{s_i})$ for $s \in S_+^{\ell-1}$.

Lemma 2.3. *Denote* $s^0 = (s_1^0, s_2^0, \cdots, s_\ell^0) = (|v_1|_2, |v_2|_2, \cdots, |v_\ell|_2)$. *For each* $s \in S_+^{\ell-1} \setminus \left\{ s^0 \right\}$, *the following statements hold:*

(a) *If* $\sigma \in (0, \frac{2}{3})$, *then* $\sum_{i=1}^{\ell} I_{\theta_i}(v_i) = E(s^0) > E(s)$;

(b) *If* $\sigma \in (\frac{2}{3}, 2)$, *then* $\sum_{i=1}^{\ell} I_{\theta_i}(v_i) = E(s^0) < E(s)$.

3. Existence of constrained localized Palais-Smale sequences

In this section, we adopt the penalization argument and the deformation approach in [31] to obtain a constrained localized Palais-Smale sequence. Denote (μ_0, u_i) as the solution of the limit system (2.1) with $m_0 = 1$ and $\theta_i = k_i$, $i = 1, 2, \cdots, \ell$, where $(k_i)_{i=1}^{\ell}$ denotes positive numbers given by (K). Next, we set $b_0 := \sum_{i=1}^{\ell} I_i(u_i)$, where

$$I_i(u) := I_{k_i}(u) = \frac{a}{2}|\nabla u|_2^2 - \frac{k_i}{2\sigma+2}|u|_{2\sigma+2}^{2\sigma+2}.$$

Then, we will find a positive solution $(\mu_\hbar, u_\hbar)$ to the following system:

$$\begin{cases} -(a+\hbar^d b|\nabla u|_2^2)\Delta u - K(\hbar x)|u|^{2\sigma}u = -\mu u & \text{in } \mathbb{R}^3, \\ |u|_2^2 = 1, \quad u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{3.1}$$

satisfying

$$\mu_\hbar = \mu_0 + o_\hbar(1), \quad u_\hbar(x) = \sum_{i=1}^{\ell} u_i(x - q_{i,\hbar}) + o_\hbar(1) \quad \text{in } H^1(\mathbb{R}^3)$$

with $\hbar q_{i,\hbar} \to q_i \in \mathcal{K}_i$.

Set $\mathcal{M} := \left\{u \in H^1(\mathbb{R}^3) | \, |u|_2 = 1\right\}$ and for $i = 1, 2, \cdots, \ell$ and $\tau > 0$, define

$$(\mathcal{K}_i)^\tau := \left\{x \in \mathbb{R}^3 | \mathrm{dist}(x, \mathcal{K}_i) \le \tau\right\} \subset \Omega_i.$$

Define the following equation for each $\rho \in (0, \frac{1}{10}\min_{1\le i\le \ell} \|u_i\|_{L^2(B_1(0))})$:

$$Z(\rho) = \left\{u = \sum_{i=1}^{\ell} u_i(x - q_{i,\hbar}) + v \in \mathcal{M} | \hbar q_{i,\hbar} \in (\mathcal{K}_i)^\tau, \|v\| \le \rho\right\}.$$

For $u \in H^1(\mathbb{R}^3)$, consider the penalized energy functional $I_\hbar : H^1(\mathbb{R}^3) \to \mathbb{R}$ is given by

$$I_\hbar(u) := E_\hbar(u) + G_\hbar(u),$$

where

$$G_\hbar(u) = \left(\hbar^{-1}\int_{\mathbb{R}^3} \chi_\hbar(x)(|\nabla u|^2 + u^2)\mathrm{d}x - 1\right)_+^2,$$

and

$$\chi_\hbar = \begin{cases} 0 & x \notin \mathbb{R}^3 \setminus \cup_{i=1}^{\ell} \frac{1}{\hbar}\Omega_i, \\ 1 & x \in \mathbb{R}^3 \setminus \cup_{i=1}^{\ell} \frac{1}{\hbar}\Omega_i. \end{cases}$$

We also denote

$$J(u) = \frac{1}{2}|u|_2^2 \quad \text{for } u \in H^1(\mathbb{R}^3).$$

Note that if $u_\hbar \in \mathcal{M}$ with $\|u_\hbar\|^2_{H^1(\mathbb{R}^3 \setminus \cup_{i=1}^{\ell} \frac{1}{\hbar}\Omega_i)} < \hbar$ is a critical point of $I_\hbar|_{\mathcal{M}}$, then it solves (3.1) for some $\mu_\hbar$. Denote the tangent space of $\mathcal{M}$ at $u \in \mathcal{M}$ by

$$T_u\mathcal{M} = \left\{v \in H^1(\mathbb{R}^3) | \int_{\mathbb{R}^3} uv = 0\right\}.$$

Lemma 3.1. *For any $L \in \mathbb{R}$, there exists $\hbar_L > 0$ such that for any fixed $\hbar \in (0, \hbar_L)$, if a sequence $\{u_{n,\hbar}\} \subset Z(\rho)$ such that*

$$I_\hbar(u_{n,\hbar}) \le L, \quad \left\| I_\hbar|'_{\mathcal{M}}(u_{n,\hbar}) \right\|_{T^*_{u_{n,\hbar}}\mathcal{M}} \to 0, \tag{3.2}$$

as $n \to \infty$, then $u_{n,\hbar}$ has a strong convergent subsequence in $H^1(\mathbb{R}^3)$.

Proof. Set $u_{n,\hbar} = \sum_{i=1}^{\ell} u_i(x - z_{n,i,\hbar}) + v_{n,\hbar}$ with $\hbar z_{n,i,\hbar} \in (\mathcal{K}_i)^\tau$ and $\left\|v_{n,\hbar}\right\| \le \rho$. It follows from $u_{n,\hbar} \in Z(\rho)$ that $\left\|u_{n,\hbar}\right\| \le \rho + \sum_{i=1}^{\ell} \|u_i\|$, which is bounded. Then, by

$$I_\hbar(u_{n,\hbar}) + \frac{1}{2\sigma+2}\int_{\mathbb{R}^3} K(\hbar x)\left|u_{n,\hbar}\right|^{2\sigma+2} = \frac{a}{2}\left|\nabla u_{n,\hbar}\right|_2^2 + \frac{\hbar^d b}{4}\left|\nabla u_{n,\hbar}\right|_2^4 + G_\hbar(u_{n,\hbar}),$$

we have that $G_\hbar(u_{n,\hbar}) \le I_\hbar(u_{n,\hbar}) + \frac{1}{2\sigma+2}\int_{\mathbb{R}^3} K(\hbar x)\left|u_{n,\hbar}\right|^{2\sigma+2} \le C_L$ for some $C_L > 0$ that is independent of $\hbar$ and n. From the assumption (3.2), for some $\mu_{n,\hbar} \in \mathbb{R}$, we deduce that

$$I'_\hbar(u_{n,\hbar}) + \mu_{n,\hbar} J'(u_{n,\hbar}) \to 0 \quad \text{in } H^{-1}, \quad \text{as } n \to \infty. \tag{3.3}$$

We have

$$\begin{aligned}
\left|\mu_{n,\hbar}\right| &= I'_\hbar(u_{n,\hbar})u_{n,\hbar} + o(1) \\
&\le a\int_{\mathbb{R}^3}\left|\nabla u_{n,\hbar}\right|^2 + \hbar^d b\left(\int_{\mathbb{R}^3}\left|\nabla u_{n,\hbar}\right|^2\right)^2 - \int_{\mathbb{R}^3} K(\hbar x)\left|u_{n,\hbar}\right|^{2\sigma+2} + G'_\hbar(u_{n,\hbar})u_{n,\hbar} \\
&\le C(\left\|u_{n,\hbar}\right\|^2 + \left\|u_{n,\hbar}\right\|^4 + \left\|u_{n,\hbar}\right\|^{2\sigma+2} + G_\hbar(u_{n,\hbar}) + G_\hbar(u_{n,\hbar})^{\frac{1}{2}}) \\
&\le C_L^*,
\end{aligned}$$

where $C_L^* > 0$ is independent of $\hbar$ and n. Then up to a subsequence, $\mu_{n,\hbar} \to \mu_\hbar$ in $\mathbb{R}$ and $u_{n,\hbar} \rightharpoonup u_\hbar = \sum_{i=1}^{\ell} u_i(x - z_{i,\hbar}) + v_\hbar$ in $H^1(\mathbb{R}^3)$ with $z_{n,i,\hbar} \to z_{i,\hbar} \in \frac{1}{\hbar}(\mathcal{K}_i)^\tau$ and $v_{n,\hbar} \rightharpoonup v_\hbar$.

Next, for any $\varphi \in H^1(\mathbb{R}^3)$, note that $\lim_{n\to\infty} I'_\hbar(u_{n,\hbar})\varphi + \mu_{n,\hbar} J'(u_{n,\hbar})\varphi = 0$, $(\mu_\hbar, u_\hbar)$ satisfies

$$\begin{aligned}
&a\int_{\mathbb{R}^3} \nabla u_\hbar \nabla\varphi + \hbar^d b\int_{\mathbb{R}^3}|\nabla u_\hbar|^2\int_{\mathbb{R}^3}\nabla u_\hbar\nabla\varphi - \int_{\mathbb{R}^3} K(\hbar x)|u_\hbar|^{2\sigma}u_\hbar\varphi \\
&+ \int_{\mathbb{R}^3}\mu_\hbar u_\hbar\varphi + Q_\hbar\int_{\mathbb{R}^3}\chi_\hbar(\nabla u_\hbar\nabla\varphi + u_\hbar\varphi) = 0,
\end{aligned} \tag{3.4}$$

where $Q_\hbar = 4\hbar^{-1}\lim_{n\to\infty} G_\hbar(u_{n,\hbar})^{\frac{1}{2}} \ge 0$. Then, we claim that $\hbar_L$ and μ_L are two positive constants such that $\mu_\hbar > \mu_L$ for each $\hbar \in (0, \hbar_L)$. Otherwise, we assume that $\mu_\hbar \to \mu \le 0$ as $\hbar \to 0$ up to a subsequence. Because $u_\hbar$ is bounded in $H^1(\mathbb{R}^3)$, we can assume that $u_\hbar(\cdot + z_{1,\hbar}) \rightharpoonup u$. Note that

$$\liminf_{\hbar\to 0}\left\|u_\hbar(\cdot + z_{i,\hbar})\right\|_{L^2(B_1(0))} \ge \|u_i\|_{L^2(B_1(0))} - \rho > 0.$$

We can obtain that $u \ne 0$ if $\rho > 0$ is small. Then set $\varphi = \psi(x - z_{1,\hbar})$ in (3.4) for each $\psi \in C_0^\infty(\mathbb{R}^3)$ and take the limit $\hbar \to 0$, that is

$$\begin{aligned}
&\lim_{\hbar\to 0}\Big[a\int_{\mathbb{R}^3}\nabla u_\hbar\nabla\psi(x - z_{1,\hbar}) + \hbar^d b\int_{\mathbb{R}^3}|\nabla u_\hbar|^2\int_{\mathbb{R}^3}\nabla u_\hbar\nabla\psi(x - z_{1,\hbar}) \\
&- \int_{\mathbb{R}^3} K(\hbar x)|u_\hbar|^{2\sigma}u_\hbar\psi(x - z_{1,\hbar}) + \int_{\mathbb{R}^3}\mu_\hbar u_\hbar\psi(x - z_{1,\hbar}) \\
&+ Q_\hbar\int_{\mathbb{R}^3}\chi_\hbar(\nabla u_\hbar\nabla\psi(x - z_{1,\hbar}) + u_\hbar\psi(x - z_{1,\hbar}))\Big] = 0.
\end{aligned}$$

Using the boundedness of $u_\hbar$ and $d > 0$, we have

$$\hbar^d b \int_{\mathbb{R}^3} |\nabla u_\hbar|^2 \int_{\mathbb{R}^3} \nabla u_\hbar \nabla \psi(x - z_{1,\hbar}) = o(1).$$

We see that u is a nontrivial solution to $-a\Delta u + \mu u = k_0 |u|^{2\sigma} u$ in $H^1(\mathbb{R}^3)$ for some $k_0 > 0$, which is impossible by Lemma 2.1.

Setting $\varphi = u_{n,\hbar} - u_\hbar$ in (3.4), we have

$$\begin{aligned} & a \int_{\mathbb{R}^3} \nabla u_\hbar \nabla (u_{n,\hbar} - u_\hbar) + \hbar^d b \int_{\mathbb{R}^3} |\nabla u_\hbar|^2 \int_{\mathbb{R}^3} \nabla u_\hbar \nabla (u_{n,\hbar} - u_\hbar) \\ & - \int_{\mathbb{R}^3} K(\hbar x) |u_\hbar|^{2\sigma} u_\hbar (u_{n,\hbar} - u_\hbar) + \int_{\mathbb{R}^3} \mu_\hbar u_\hbar (u_{n,\hbar} - u_\hbar) \\ & + Q_\hbar \int_{\mathbb{R}^3} \chi_\hbar (\nabla u_\hbar \nabla (u_{n,\hbar} - u_\hbar) + u_\hbar (u_{n,\hbar} - u_\hbar)) = 0. \end{aligned} \tag{3.5}$$

Then it follows from (3.3) that

$$\langle I'_\hbar(u_{n,\hbar}) + \mu_{n,\hbar} J'(u_{n,\hbar}),\, u_{n,\hbar} - u_\hbar \rangle = o(1) \left\| u_{n,\hbar} - u_\hbar \right\|.$$

That is,

$$\begin{aligned} & a \int_{\mathbb{R}^3} \nabla u_{n,\hbar} \nabla (u_{n,\hbar} - u_\hbar) + \hbar^d b \int_{\mathbb{R}^3} \left|\nabla u_{n,\hbar}\right|^2 \int_{\mathbb{R}^3} \nabla u_{n,\hbar} \nabla (u_{n,\hbar} - u_\hbar) \\ & - \int_{\mathbb{R}^3} K(\hbar x) \left|u_{n,\hbar}\right|^{2\sigma} u_{n,\hbar} (u_{n,\hbar} - u_\hbar) + \int_{\mathbb{R}^3} \mu_{n,\hbar} u_{n,\hbar} (u_{n,\hbar} - u_\hbar) \\ & + Q_{n,\hbar} \int_{\mathbb{R}^3} \chi_\hbar (\nabla u_{n,\hbar} \nabla (u_{n,\hbar} - u_\hbar) + u_{n,\hbar} (u_{n,\hbar} - u_\hbar)) \\ & = o(1) \left\| u_{n,\hbar} - u_\hbar \right\|. \end{aligned} \tag{3.6}$$

We can show that for n large enough,

$$\begin{aligned} & \int_{\mathbb{R}^3} \left|\nabla u_{n,,\hbar}\right|^2 \int_{\mathbb{R}^3} \nabla u_{n,\hbar} \nabla (u_{n,\hbar} - u_\hbar) - \int_{\mathbb{R}^3} |\nabla u_\hbar|^2 \int_{\mathbb{R}^3} \nabla u_\hbar \nabla (u_{n,\hbar} - u_\hbar) \\ = & \int_{\mathbb{R}^3} \left|\nabla u_{n,\hbar}\right|^2 \int_{\mathbb{R}^3} \nabla u_{n,\hbar} \nabla (u_{n,\hbar} - u_\hbar) - \int_{\mathbb{R}^3} \left|\nabla u_{n,\hbar}\right|^2 \int_{\mathbb{R}^3} \nabla u_\hbar \nabla (u_{n,\hbar} - u_\hbar) \\ & + \int_{\mathbb{R}^3} \left|\nabla u_{n,\hbar}\right|^2 \int_{\mathbb{R}^3} \nabla u_\hbar \nabla (u_{n,\hbar} - u_\hbar) - \int_{\mathbb{R}^3} |\nabla u_\hbar|^2 \int_{\mathbb{R}^3} \nabla u_\hbar \nabla (u_{n,\hbar} - u_\hbar) \\ = & \int_{\mathbb{R}^3} \left|\nabla u_{n,\hbar}\right|^2 \int_{\mathbb{R}^3} \left|\nabla u_{n,\hbar} - \nabla u_\hbar\right|^2 \\ & + (\int_{\mathbb{R}^3} \left|\nabla u_{n,\hbar}\right|^2 - \int_{\mathbb{R}^3} |\nabla u_\hbar|^2) \int_{\mathbb{R}^3} \nabla u_\hbar \nabla (u_{n,\hbar} - u_\hbar) \\ \geq & o_n(1), \end{aligned} \tag{3.7}$$

where using the fact that $u_{n,\hbar} \rightharpoonup u_\hbar$ in $H^1(\mathbb{R}^3)$, it follows $\int_{\mathbb{R}^3} \nabla u_\hbar \nabla (u_{n,\hbar} - u_\hbar) \to 0$. Thus from (3.5)–(3.7), we have

$$a \int_{\mathbb{R}^3} \left|\nabla (u_{n,\hbar} - u_\hbar)\right|^2 + \mu_\hbar \int_{\mathbb{R}^3} |u_{n,\hbar} - u_\hbar|^2$$

$$- \int_{\mathbb{R}^3} K(\hbar x) \left|u_{n,\hbar} - u_\hbar\right|^{2\sigma+2} + Q_h \int_{\mathbb{R}^3} \chi_h \left[\left|\nabla(u_{n,\hbar} - u_\hbar)\right|^2 + |u_{n,\hbar} - u_h|^2\right]$$
$$+ \hbar^d b \int_{\mathbb{R}^3} \left|\nabla u_{n,\hbar}\right|^2 \int_{\mathbb{R}^3} \left|\nabla u_{n,\hbar} - \nabla u_\hbar\right|^2 = o(1).$$

Noting also that $\int_{\mathbb{R}^3} K(\hbar x) \left|u_{n,\hbar} - u_\hbar\right|^{2\sigma+2} \leq C\|u_{n,\hbar} - u_\hbar\|^{2\sigma+2}$ and

$$\begin{aligned}\|u_{n,\hbar} - u_\hbar\| &= \|\sum_{i=1}^{\ell} u_i(\cdot - z_{n,i,\hbar}) + v_{n,\hbar} - \sum_{i=1}^{\ell} u_i(\cdot - z_{i,\hbar}) - v_\hbar\| \\ &\leq \sum_{i=1}^{\ell} \|u_i(\cdot - z_{n,i,\hbar}) - u_i(\cdot - z_{i,\hbar})\| + \|v_{n,\hbar}\| + \|v_\hbar\| \\ &\leq 2\rho + o_n(1),\end{aligned}$$

the following inequality holds:

$$\begin{aligned}C^*\|u_{n,\hbar} - u_\hbar\|^2 &\leq a \int_{\mathbb{R}^3} \left|\nabla(u_{n,\hbar} - u_\hbar)\right|^2 + \mu_\hbar \int_{\mathbb{R}^3} |u_{n,\hbar} - u_\hbar|^2 \\ &\leq C\|u_{n,\hbar} - u_\hbar\|^{2\sigma+2} + o(1),\end{aligned}$$

where C^* is a positive constant since $a > 0$ and $\mu_\hbar > 0$. Making ρ smaller if necessary given $C\|u_{n,\hbar} - u_\hbar\|^{2\sigma} < C^*/2$, it follows that $u_{n,\hbar} \to u_\hbar$ in $H^1(\mathbb{R}^3)$. This completes the proof of Lemma 3.1. □

Proposition 3.2. *For some $\rho > 0$ small and by letting $\{\hbar_n\} \subset \mathbb{R}$, $\{\mu_n\} \subset \mathbb{R}$ and $\{u_n\} \subset Z(\rho)$ satisfy that*

$$\hbar_n \to 0^+, \quad \limsup_{n\to\infty} I_{\hbar_n}(u_n) \leq b_0, \tag{3.8}$$

$$\left\|I'_{\hbar_n}(u_n) + \mu_n J'(u_n)\right\|_{H^{-1}} \to 0, \tag{3.9}$$

as $n \to \infty$. Then, $\mu_n \to \mu_0$ holds, $\lim_{n\to\infty} I_{\hbar_n}(u_n) = b_0$ and for some $z_{n,i} \in \mathbb{R}^3$, $i = 1, 2, \cdots, \ell$, we have

$$\left\|u_n - \sum_{i=1}^{\ell} u_i(\cdot - z_{n,i})\right\| \to 0 \text{ and } \operatorname{dist}(\hbar_n z_{n,i}, \mathcal{K}_i) \to 0.$$

Proof. The proof is similar to that in [31]. For the sake of completeness, we shall give the details.
Step 1. We claim that $\mu_n \to \tilde{\mu} > 0$.

As $\{u_n\} \subset Z(\rho)$, we can write that $u_n = \sum_{i=1}^{\ell} u_i(x - z_{n,i}) + v_n$ with $z_{n,i} \in \frac{1}{\hbar}(\mathcal{K}_i)^\tau$ and $\|v_n\| \leq \rho$. It follows from $u_n \in Z(\rho)$ and the boundedness of $I_{\hbar_n}(u_n)$ that $\|u_n\|$ and $G_{\hbar_n}(u_n)$ are bounded. Besides, by (3.9) and $J'(u_n)u_n = 1$, we know that μ_n is bounded. Then up to a subsequence, we can assume that $\mu_n \to \tilde{\mu}$ in $\mathbb{R}$ and $u_n(\cdot + z_{n,i}) \rightharpoonup w_i \in H^1(\mathbb{R}^3)$. For $\rho < \frac{1}{10} \min_{1\leq i\leq \ell} \|u_i\|_{L^2(B_1(0))}$, we have

$$\liminf_{n\to\infty} \left\|u_n(\cdot + z_{n,i})\right\|_{L^2(B_1(0))} \geq \|u_i\|_{L^2(B_1(0))} - \rho > 0.$$

Notice that for any $R > 0$, we can obtain that $\|u_i - w_i\|_{L^2(B_R(0))} \leq \rho$. Hence,

$$\|u_i\|_2 - \rho \leq \|w_i\|_2 \leq \|u_i\|_2 + \rho. \tag{3.10}$$

Then, if ρ is small enough, we know that $w_i \neq 0$. Next, testing (3.9) with $\varphi(x - z_{n,i})$ for each $\varphi \in C_0^\infty(\mathbb{R}^3)$, we deduce that

$$\hbar_n^d b \int_{R^3} \left|\nabla u_n(x + z_{n,i})\right|^2 \int_{R^3} \nabla u_n(x + z_{n,i}) \nabla\varphi = o(1).$$

Thus, w_i is a solution to $-a\Delta w_i + \tilde{\mu} w_i = \tilde{k}_i |w_i|^{2\sigma} w_i$ in $H^1(\mathbb{R}^3)$ with $\lim\limits_{n\to\infty} K(\hbar_n z_{n,i}) \to \tilde{k}_i \in [\underline{k}, \bar{k}]$, where $\underline{k} = \min_{x \in U_{i=1}^{\ell} \bar{\Omega}_i} K(x) > 0$ and $\bar{k} = \max_{1\le i\le \ell} k_i$. Then, combining the Pohozaev identity with

$$a\,|w_i|_2^2 + \tilde{\mu}\,|w_i|_2^2 = \tilde{k}_i\,|w_i|_{2\sigma+2}^{2\sigma+2},$$

it follows that there exists a positive contant $\tilde{\mu}$.

Step 2. $u_n - \sum_{i=1}^{\ell} w_i(\cdot - z_{n,i}) \to 0$ in $L^{2\sigma+2}(\mathbb{R}^3)$ and $\mathrm{dist}(\hbar_n z_{n,i}, \mathcal{K}_i) \to 0$.

We show that

$$\tilde{v}_n := u_n - \sum_{i=1}^{\ell} w_i(\cdot - z_{n,i}) \to\ 0 \text{ in } L^{2\sigma+2}(\mathbb{R}^3).$$

Otherwise, by Lions' lemma [17], there exists a sequence of points $\{z_n\} \subset \mathbb{R}^3$ such that

$$\limsup_{n\to\infty} \left\| u_n - \sum_{i=1}^{\ell} w_i(\cdot - z_{n,i}) \right\|_{L^2(B_1(z_n))}^2 > 0.$$

Noting that $\left|z_n - z_{n,i}\right| \to \infty$ $i = 1, 2, \cdots, \ell$, we have

$$\limsup_{n\to\infty} \int_{B_1(0)} |u_n(\cdot + z_n)|^2 > 0. \tag{3.11}$$

By (3.8), $G_{\hbar_n}(u_n) \le C$ holds for some $C > 0$ that is independent of $\hbar$. Then, we have that $\mathrm{dist}\left(\hbar_n z_n, \cup_{i=1}^{\ell} \Omega_i\right) \to 0$. Up to a subsequence, we assume that $\tilde{v}_n(x + z_n) \rightharpoonup v_0 \neq 0$ in $H^1(\mathbb{R}^3)$ and $K(\hbar_n z_n) \to k_0 \in [\underline{k}, \bar{k}]$, where $k_0 = k(y_0)$, $y_0 \in \cup_{i=1}^{\ell} \Omega_i$. Let $D := \left\{x \in \mathbb{R}^3 | x_3 \ge -M\right\}$. For some i_0, if $\lim\limits_{n\to\infty} \frac{\mathrm{dist}(\hbar_n z_n, \partial\Omega_{i_0})}{\hbar_n} = M < \infty$, we get that $\hbar_n z_n \to z_0$ as $n \to \infty$, where $z_0 \in \partial\Omega_{i_0}$. Next, without loss of generality we can assume that $v_0 \in H_0^1(D)$. Testing (3.9) with $\varphi(\cdot - z_n)$ for any $\varphi \in C_0^\infty(D)$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\Big[a \int_{\mathbb{R}^3} \nabla u_n \nabla\varphi(x - z_n) + \hbar_n^d b \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} \nabla u_n \nabla\varphi(x - z_n) \\
&- \int_{\mathbb{R}^3} K(\hbar_n x)|u_n|^{2\sigma} u_n \varphi(x - z_n) + \int_{\mathbb{R}^3} \mu_n u_n \varphi(x - z_n) \\
&+ Q_{\hbar_n} \int_{\mathbb{R}^3} \chi_{\hbar_n}(\nabla u_n \nabla\varphi(x - z_n) + u_n \varphi(x - z_n))\Big] = 0.
\end{aligned}$$

Then by applying $\|u_n\|_{H^1(\mathbb{R}^3 \setminus \frac{1}{\hbar_n} \cup_{i=1}^{\ell} \Omega_i)} \le C\hbar_n$ and

$$\hbar_n^d b \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} \nabla u_n \nabla\varphi(x - z_n) = o(1),$$

we can obtain that v_0 is a solution of $-a\Delta u + \tilde{\mu} u = k_0 |u|^{2\sigma} u$ in $H_0^1(D)$, which is impossible since this equation does not have a nontrivial solution on the half space according to [8]. Thus $\lim_{n\to\infty} \text{dist}(\hbar_n z_n, \partial\Omega_{i_0}) = +\infty$ and $z_n \in \frac{1}{\hbar_n}\Omega_{i_0}$. Now we test (3.9) with $\varphi(\cdot - z_n)$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$ to get

$$-a\Delta v_0 + \tilde{\mu} v_0 = k_0 |v_0|^{2\sigma} v_0,$$

where $\tilde{\mu} > 0$, and $|v_0|_2^2 > C_1$ for some $C_1 > 0$ that is independent of ρ.

If we have chosen ρ small enough, then by the Brézis-Lieb lemma,

$$\begin{aligned} 1 = \lim_{n\to\infty} |u_n|_2^2 &= \lim_{n\to\infty} \left| u_n(\cdot + z_{n,1}) - v_0(\cdot + z_{n,1}) \right|_2^2 + |v_0|_2^2 + o(1) \\ &\geq \sum_{i=1}^{\ell} |w_i|_2^2 + |v_0|_2^2 \\ &\geq \sum_{i=1}^{\ell} |u_i|_2^2 - 2\rho \sum_{i=1}^{\ell} |u_i|_2^2 + \ell\rho^2 + C_1 \\ &> 1, \end{aligned}$$

which is a contradiction.

Step 3. $\left\| u_n - \sum_{i=1}^{\ell} w_i(\cdot - z_{n,i}) \right\| \to 0$ and $\lim_{n\to\infty} I_{\hbar_n}(u_n) = b_0$.

Testing (3.9) with $u_n - \sum_{i=1}^{\ell} w_i(\cdot - z_{n,i})$, given

$$\hbar_n^d b \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} \nabla u_n \nabla (u_n - \sum_{i=1}^{\ell} w_i(x - z_{n,i})) = o(1),$$

we can get that

$$a(|\nabla u_n|_2^2 - \sum_{i=1}^{\ell} |\nabla w_i|_2^2) + \tilde{\mu}(|u_n|_2^2 - \sum_{i=1}^{\ell} |w_i|_2^2) \leq o_n(1).$$

Next, we have

$$a|\nabla(u_n - \sum_{i=1}^{\ell} w_i(\cdot - z_{n,i}))|_2^2 + \tilde{\mu}|u_n - \sum_{i=1}^{\ell} w_i(\cdot - z_{n,i})|_2^2 = o_n(1),$$

i.e., $u_n - \sum_{i=1}^{\ell} w_i(\cdot - z_{n,i}) \to 0$ in $H^1(\mathbb{R}^3)$.

On the other hand, by Lemma 2.2, we obviously get that $\lim_{n\to\infty} I_{\hbar_n}(u_n) = b_0$. □

4. Existence of critical points

In this section, let ρ be fixed in Proposition 3.2. We present the result as follows.

Proposition 4.1. *There exists $\hbar_0 > 0$ such that for $\hbar \in (0, \hbar_0)$, $I_\hbar|_{\mathcal{M}}$ has a critical point $u_\hbar \in Z(\rho)$. Moreover,* $\lim_{\hbar\to 0} I(u_\hbar) = b_0$ *and the Lagrange multiplier $\mu_\hbar \in \mathbb{R}$ satisfies*

$$\lim_{\hbar\to 0} \mu_\hbar = \mu_0, \quad I'_\hbar(u_\hbar) + \mu_\hbar J'(u_\hbar) = 0. \tag{4.1}$$

Remark 4.2. *By Proposition 3.2, it is easy to verify that* (4.1) *holds if $u_\hbar$ is a critical point of $I_\hbar|_{\mathcal{M}}$ such that* $\limsup_{\hbar\to 0} I_{u_\hbar} \le b_0$.

The proof of Proposition 4.1 can be obtained as in [31] by considering the following contradiction: $\{\hbar_n\}$ with $\hbar_n \to 0$ such that for some sequence $b_{\hbar_n} \to b_0$, $I_\hbar$ admits no critical points in $\{u \in Z(\rho) | I_{\hbar_n}(u) \le b_{\hbar_n}\}$. For brevity, we denote $\hbar = \hbar_n$. Then from Lemma 3.1 and Proposition 3.2, there respectively exist $\kappa_0 > 0$ and $\nu > 0$ independent of $\hbar$ and $\nu_\hbar > 0$ such that

$$\begin{aligned} &\left\| I_\hbar|'_{\mathcal{M}}(u) \right\|_{T_u^*\mathcal{M}} \ge \nu_\hbar, \text{for } u \in Z(\rho) \cap [b_0 - 2\kappa_0 \le I_\hbar \le b_\hbar], \\ &\left\| I_\hbar|'_{\mathcal{M}}(u) \right\|_{T_u^*\mathcal{M}} \ge \nu, \text{for } u \in (Z(\rho) \setminus Z(\rho/4)) \cap [b_0 - 2\kappa_0 \le I_\hbar \le b_\hbar], \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} [b_1 \le I_\hbar] &= \left\{ u \in H^1(\mathbb{R}^3) | b_1 \le I_\hbar(u) \right\}, \\ [I_\hbar \le b_2] &= \left\{ u \in H^1(\mathbb{R}^3) | I_\hbar(u) \le b_2 \right\}, \\ [b_1 \le I_\hbar \le b_2] &= \left\{ u \in H^1(\mathbb{R}^3) | b_1 \le I_\hbar(u) \le b_2 \right\}, \end{aligned}$$

for any b_1, $b_2 \in \mathbb{R}$.

Thanks to (4.2), one can get the following deformation lemma.

Lemma 4.3. *Let $\nu_\hbar$ and ν be given as in* (4.2). *For any $\kappa \in \left(0, \min\left\{\kappa_0, \frac{\rho\nu}{16}\right\}\right)$, there exists $\hbar_\kappa > 0$ such that for $\hbar \in (0, \hbar_\kappa)$ there is a deformation $\eta : \mathcal{M} \to \mathcal{M}$ that satisfied the following conditions:*

(*a*) $\eta(u) = u$ *if* $u \in \mathcal{M} \setminus (Z(\rho) \cap [b_0 - 2\kappa \le I_\hbar])$.
(*b*) $I_\hbar(\eta(u)) \le I_\hbar(u)$ *if* $u \in \mathcal{M}$.
(*c*) $\eta(u) \in Z(\rho) \cap [I_\hbar \le b_0 - \kappa]$ *if* $u \in Z(\rho/4) \cap [I_\hbar \le b_\hbar]$.

To give the proof of Lemma 4.3, we borrow some ideas from [5,6,31] in the L^2-subcritical case and L^2-supercritical case.

4.1. L^2-subcritical case $\sigma \in \left(0, \frac{2}{3}\right)$

For every $\delta > 0$, we denote

$$S_\delta := \left\{ s \in S_+^{\ell-1} | \left| s - s^0 \right| \le \delta \right\},$$

where $s^0 = (|u_1|_2, \cdots, |u_\ell|_2)$. Fix $q_i \in \mathcal{K}_i$ and $q_{i,\hbar} = \frac{1}{\hbar} q_i$ for $i = 1, 2, \cdots, \ell$ and define the $(\ell - 1)$-dimensional initial path by

$$\xi_\hbar(s) = B_\hbar \sum_{i=1}^{\ell} w_{s_i}(\cdot - q_{i,\hbar}),$$

where $B_\hbar := \left| \sum_{i=1}^{\ell} w_{s_i}(\cdot - q_{i,\hbar}) \right|_2^{-1}$. Note that we can fix $\delta > 0$ small enough such that

$$\xi_\hbar(s) \in Z(\rho/4) \text{ for } s \in S_\delta$$

and

$$B_\hbar \to 1 \text{ as } \hbar \to 0 \text{ uniformly in } S_\delta.$$

Define

$$b_\hbar := \max_{s \in S_\delta} I_\hbar(\xi_\hbar(s)).$$

Lemma 4.4. $\lim_{\hbar\to 0} b_\hbar = b_0$ *and fix any* $\kappa \in (0, \min\left\{\kappa_0, \frac{\rho\nu}{16}\right\})$ *such that*

$$\sup_{s\in\partial S_\delta} I_\hbar(\xi_\hbar(s)) < b_0 - 2\kappa, \tag{4.3}$$

where $\partial S_\delta := \left\{s \in S_+^\ell \middle| \left|s - s^0\right| = \delta\right\}$.

Proof. Since

$$\hbar^d b\left(\int_{\mathbb{R}^3} |\nabla\xi_\hbar|^2\right)^2 \to 0 \quad \text{as} \quad \hbar \to 0,$$

one can deduce that

$$I_\hbar(\xi_\hbar(s)) \to \sum_{i=1}^{\ell} I_i(w_{s_i}) \text{ as } \hbar \to 0 \text{ uniformly for } s \in S_\delta.$$

By Lemma 2.3(a), we have

$$\sup_{s\in\partial S_\delta} I_\hbar(\xi_\hbar(s)) < b_0 - 2\kappa.$$

□

Proof of Proposition 4.1 in the L^2-subcritical case. By Lemma 4.3 and (4.3), we have

$$\eta(\xi_\hbar(s)) = \xi_\hbar(s) \text{ for } s \in \partial S_\delta, \tag{4.4}$$

$$I_\hbar(\eta(\xi_\hbar(s))) \le b_0 - \kappa \textit{ and } \eta(\xi_\hbar(s)) \in Z(\rho) \text{ for } s \in S_\delta. \tag{4.5}$$

Define

$$\Psi_{i,\hbar} = \left(\int_{\frac{1}{\hbar}\Omega_i} |u|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\ell} \int_{\frac{1}{\hbar}\Omega_i} |u|^2\right)^{-\frac{1}{2}}, \text{ for } u \in \mathcal{M}.$$

Similar to the case in [31], there exists $s^1 \in S_\delta$ such that $\Psi_{i,\hbar}(\eta(\xi_\hbar(s^1))) = s_i^0 = |u_i|_2$. Denote

$$u_{0,\hbar} := \eta(\xi_\hbar(s^1)),\ u_{i,\hbar} := \gamma_{i,\hbar} u_{0,\hbar}, \tag{4.6}$$

where $\gamma_{i,\hbar} \in C_0^\infty(\frac{1}{\hbar}(\Omega_i'), [0, 1])$ is a cut-off function such that $\gamma_{i,\hbar} = 1$ on $\frac{1}{\hbar}\Omega_i$ and $\left|\nabla\gamma_{i,\hbar}\right| \le C\hbar$ for each $i = 1, 2, \cdots, \ell$ and some $C > 0$; Ω_i' is an open neighborhood of $\bar{\Omega}_i$. By (4.5), we have that $G_\hbar(u_{0,\hbar}) \le C$ for some $C > 0$ that is independent of $\hbar$, which implies that

$$\left\|u_{0,\hbar}\right\|_{H^1(\mathbb{R}^3\setminus\cup_{i=1}^{\ell}\frac{1}{\hbar}\Omega_i)} \le C\hbar. \tag{4.7}$$

Then

$$\left|u_{i,\hbar}\right|_2 = |u_i|_2 + o_\hbar(1) \tag{4.8}$$

and

$$I_i(u_i) \le I_i(u_{i,\hbar}) + o_\hbar(1). \tag{4.9}$$

Hence from (4.5)–(4.9), we have

$$b_0 - \kappa \ge I_\hbar(u_{0,\hbar}) \ge \sum_{i=1}^{\ell} I_i(u_{i,\hbar}) + o_\hbar(1) \ge \sum_{i=1}^{\ell} I_i(u_i) + o_\hbar(1) = b_0 + o_\hbar(1),$$

which is a contradiction. This completes the proof. □

4.2. L^2-supercritical case $\sigma \in \left(\frac{2}{3}, 2\right)$

Fix $q_i \in \mathcal{K}_i$ and denote $q_{i,\hbar} = \frac{1}{\hbar} q_i$; we set

$$\zeta_\hbar(s) = \bar{B}_\hbar \sum_{i=1}^{\ell} t_i^{3/2} u_i(t_i(\cdot - q_{i,\hbar})) \text{ for } t = (t_1,\, t_2,\, \cdots,\, t_\ell) \in (0,\, +\infty)^\ell\,,$$

where $\bar{B}_\hbar := \left|\sum_{i=1}^{\ell} t_i^{3/2} u_i(t_i(\cdot - q_{i,\hbar}))\right|_2^{-1}$.

Define

$$b_\hbar := \max_{t \in [1-\bar{\delta},\, 1+\bar{\delta}]^\ell} I_\hbar(\zeta_\hbar(t)).$$

Note that we can fix $\bar{\delta} > 0$ small enough such that

$$\zeta_\hbar(t) \in Z(\rho/4) \text{ for } t \in [1-\bar{\delta},\, 1+\bar{\delta}]^\ell,$$

and $\bar{B}_\hbar \to 1$ holds. Note also that

$$I_i(u_i) > I_i(t_i^{3/2} u_i(t_i \cdot)) \text{ for } t_i \in [1-\bar{\delta},\, 1+\bar{\delta}] \setminus \{1\}\,.$$

Since

$$\hbar^d b \left(\int_{\mathbb{R}^3} |\nabla \zeta_\hbar|^2 \right)^2 \to 0 \quad \text{as} \quad \hbar \to 0,$$

and

$$I_\hbar(\zeta_\hbar(t)) \to \sum_{i=1}^{\ell} I_i(t_i^{3/2} u_i(t_i \cdot)) \text{ as } \hbar \to 0 \text{ uniformly for } t \in [1-\bar{\delta},\, 1+\bar{\delta}]^\ell,$$

one can get the result as in [31].

Lemma 4.5. *$\lim_{\hbar\to 0} b_\hbar = b_0$ and fix any $\kappa \in (0,\, \min\left\{\kappa_0,\, \frac{\rho\nu}{16}\right\})$ such that*

$$\sup_{t \in \partial[1-\bar{\delta},\, 1+\bar{\delta}]^\ell} I_\hbar(\zeta_\hbar(t)) < b_0 - 2\kappa. \tag{4.10}$$

Proof of Proposition 4.1 in the L^2-supercritical case. By Lemma 4.3 and (4.10),

$$\eta(\zeta_\hbar(t)) = \zeta_\varepsilon(t) \text{ if } t \in \partial[1-\bar{\delta},\, 1+\bar{\delta}]^\ell, \tag{4.11}$$

$$I_\hbar(\eta(\zeta_\hbar(t))) \le b_0 - \kappa \text{ and } \eta(\zeta_\hbar(t)) \in Z(\rho) \text{ for } t \in [1-\bar{\delta},\, 1+\bar{\delta}]^\ell. \tag{4.12}$$

Define

$$\Phi_{i,\hbar} = \left(\int_{\frac{1}{\hbar}\Omega_i} |\nabla u|^2 \right)^{\frac{1}{2-3\sigma}} \left(\frac{3\sigma k_i}{(2+2\sigma)a} \int_{\frac{1}{\hbar}\Omega_i} |u|^{2\sigma+2} \right)^{-\frac{1}{2-3\sigma}}, \text{ for } u \in \mathcal{M}.$$

Similar to the case in [31], there exists $t^1 \in [1-\bar{\delta},\, 1+\bar{\delta}]^\ell$ such that

$$\Phi_{i,\hbar}(\eta(\zeta_\hbar(t^1))) = 1,\; i = 1,\, 2,\, \cdots,\, \ell. \tag{4.13}$$

We denote

$$\bar{u}_{0,\hbar} := \eta(\zeta_\hbar(t^1)), \quad \bar{u}_{i,\hbar} := \gamma_{i,\hbar}\bar{u}_{0,\hbar}\left(\sum_{i=1}^{\ell}\left|\gamma_{i,\hbar}\bar{u}_{0,\hbar}\right|_2^2\right)^{-\frac{1}{2}}.$$

Similar to (4.7) and (4.8), we have

$$\left\|\bar{u}_{0,\hbar}\right\|_{H^1(\mathbb{R}^3\setminus\cup_{i=1}^{\ell}\frac{1}{\hbar}\Omega_i)} = o_\hbar(1) \tag{4.14}$$

and

$$\sum_{i=1}^{\ell}\left|\gamma_{i,\hbar}\bar{u}_{0,\hbar}\right|_2^2 = 1 + o_\hbar(1). \tag{4.15}$$

From (4.13)–(4.15), we have

$$t_{i,\hbar} := \left(\left|\nabla\bar{u}_{i,\hbar}\right|_2^2\right)^{\frac{1}{2-3\sigma}}\left(\frac{3\sigma k_i}{(2+2\sigma)a}\left|\bar{u}_{i,\hbar}\right|_{2\sigma+2}^{2\sigma+2}\right)^{\frac{1}{3\sigma-2}} = \Phi_{i,\hbar}(\bar{u}_{0,\hbar}) + o_\hbar(1) = 1 + o_\hbar(1).$$

A direct calculation shows that

$$t^* := \left(\left|t_{1,\hbar}^{-\frac{3}{2}}\bar{u}_{1,\hbar}(t_{1,\hbar}^{-1}\cdot)\right|_2, \left|t_{2,\hbar}^{-\frac{3}{2}}\bar{u}_{2,\hbar}(t_{2,\hbar}^{-1}\cdot)\right|_2, \cdots, \left|t_{\ell,\hbar}^{-\frac{3}{2}}\bar{u}_{\ell,\hbar}(t_{\ell,\hbar}^{-1}\cdot)\right|_2\right) \in S_+^{\ell-1}$$

and

$$\left|\nabla\left(t_{i,\hbar}^{-\frac{3}{2}}\bar{u}_{i,\hbar}(t_{i,\hbar}^{-1}\cdot)\right)\right|_2^2 = \frac{3\sigma k_i}{(2+2\sigma)a}\left|t_{i,\hbar}^{-\frac{3}{2}}\bar{u}_{i,\hbar}(t_{i,\hbar}^{-1}\cdot)\right|_{2\sigma+2}^{2\sigma+2}.$$

Hence by the definition of b_{s_i}, we have

$$\sum_{i=1}^{\ell} I_i(u_i) = b_0 \le \sum_{i=1}^{\ell} I_i\left(t_{i,\hbar}^{-\frac{3}{2}}\bar{u}_{i,\hbar}(t_{i,\hbar}^{-1}\cdot)\right) = \sum_{i=1}^{\ell} I_i(\bar{u}_{i,\hbar}) + o_\hbar(1).$$

Similarly, one can get a contradiction. □

5. Completion of the proof

Let $u_\hbar$ be the critical point of the modified function $I_\hbar$ given in Proposition 4.1.

Completion of proof of Theorem 1.1.

Proof. We show that there exists $c > 0$ independent of $\hbar$ such that

$$\|u_\hbar\|^2_{H^1(\mathbb{R}^3\setminus\cup_{i=1}^{\ell}\frac{1}{\hbar}(\mathcal{K}_i)^\tau)} \le e^{-\frac{C}{\hbar}}. \tag{5.1}$$

We adopt some arguments from [3, 31]. Set $\left\lfloor 2\hbar^{-1}\tau\right\rfloor - 1 := n_\hbar$. For $n = 1, 2, \cdots, n_\hbar$, we take $\phi_n \in C^1(\mathbb{R}^3, [0, 1])$ such that

$$\begin{cases} \phi_n(x) = 0, & \text{if } x \in \mathbb{R}^3 \setminus E_n, \\ \phi_n(x) = 1, & \text{if } x \in E_{n+1}, \\ |\nabla\phi_n(x)| \le 2, & x \in \mathbb{R}^3, \end{cases}$$

where $E_n := \left\{x \in \mathbb{R}^3 | \mathrm{dist}(x, \cup_{i=1}^{\ell} \frac{1}{\hbar}(\mathcal{K}_i)^{\frac{\tau}{2}}) > n-1\right\}$. Then by Proposition 3.2,

$$\lim_{\hbar\to 0} \|u_\hbar\|_{H^1(E_1)} \le \lim_{\hbar\to 0} \sum_{i=1}^{\ell} \|u_i\|_{H^1(\mathbb{R}^3\setminus B_{\hbar\tau}(0))} = 0. \tag{5.2}$$

Note that for each $n = 1, 2, \cdots, n_\hbar$,

$$\mathrm{supp}\chi_\hbar = \mathbb{R}^3 \setminus \cup_{i=1}^{\ell} \frac{1}{\hbar}\Omega_i \subset \mathbb{R}^3 \setminus \cup_{i=1}^{\ell} \frac{1}{\hbar}(\mathcal{K}_i)^{\tau} \subset \phi_n^{-1}(1).$$

Since $\left\langle I_\hbar'(u_\hbar) + \mu_\hbar J'(u_\hbar), \phi_n u_\hbar \right\rangle = 0$, we have

$$\begin{aligned} & a\int_{\mathbb{R}^3} \nabla u_\hbar \nabla(\phi_n u_\hbar) + \hbar^d b \int_{\mathbb{R}^3} |\nabla u_\hbar|^2 \int_{\mathbb{R}^3} \nabla u_\hbar \nabla(\phi_n u_\hbar) \\ & - \int_{\mathbb{R}^3} K(\hbar x) |u_\hbar|^{2\sigma+2} \phi_n + \int_{\mathbb{R}^3} \mu_\hbar u_\hbar^2 \phi_n \\ =& -4\hbar^{-1} G_\hbar(u_\hbar)^{\frac{1}{2}} \int_{\mathbb{R}^3} \chi_\hbar (\nabla u_\hbar \nabla(\phi_n u_\hbar) + u_\hbar^2 \phi_n) \\ =& -4\hbar^{-1} G_\hbar(u_\hbar)^{\frac{1}{2}} \int_{\mathbb{R}^3\setminus\cup_{i=1}^{\ell}\frac{1}{\hbar}\Omega_i} (\nabla u_\hbar \nabla(\phi_n u_\hbar) + u_\hbar^2 \phi_n) \\ =& -4\hbar^{-1} G_\hbar(u_\hbar)^{\frac{1}{2}} \int_{\mathbb{R}^3\setminus\cup_{i=1}^{\ell}\frac{1}{\hbar}\Omega_i} (|\nabla u_\hbar|^2 + u_\hbar^2) \le 0. \end{aligned} \tag{5.3}$$

Therefore, by (5.3) and the Sobolev embedding,

$$\begin{aligned} & \min\left\{a, \frac{\mu_0}{2}\right\} \|u_\hbar\|^2_{H^1(E_{n+1})} \\ \le & \int_{\mathbb{R}^3} \phi_n (a|\nabla u_\hbar|^2 + \mu_\hbar u_\hbar^2) \\ \le & \int_{\mathbb{R}^3} K(\hbar x)|u_\hbar|^{2\sigma+2}\phi_n - a\int_{\mathbb{R}^3} u_\hbar \nabla u_\hbar \nabla\phi_n - \hbar^d b\int_{\mathbb{R}^3} |\nabla u_\hbar|^2 \int_{\mathbb{R}^3} \nabla u_\hbar \nabla(\phi_n u_\hbar) \\ \le & C\|u_\hbar\|^{2\sigma+2}_{H^1(E_n)} + a\|u_\hbar\|^2_{H^1(E_n)} - a\|u_\hbar\|^2_{H^1(E_{n+1})} - \hbar^d b\int_{\mathbb{R}^3} |\nabla u_\hbar|^2 \int_{\mathbb{R}^3} \nabla u_\hbar \nabla(\phi_n u_\hbar) \\ \le & (a + C\|u_\hbar\|^{2\sigma}_{H^1(E_1)} + o_\hbar(1))\|u_\hbar\|^2_{H^1(E_n)} - (a + o_\hbar(1))\|u_\hbar\|^2_{H^1(E_{n+1})}, \end{aligned}$$

where $-\hbar^d b \int_{\mathbb{R}^3} |\nabla u_\hbar|^2 \int_{\mathbb{R}^3} \nabla u_\hbar \nabla(\phi_n u_\hbar) \le o_\hbar(1)(2\|u_\hbar\|^2_{H^1(E_n)} - \|u_\hbar\|^2_{H^1(E_{n+1})})$ as $\hbar \to 0$. By (5.2), we have

$$\|u_\hbar\|^2_{H^1(E_{n+1})} \le \theta_\hbar^{-1} \|u_\hbar\|^2_{H^1(E_n)},$$

where

$$\theta_\hbar := \frac{a + \min\left\{a, \frac{\mu_0}{2}\right\} + o_\hbar(1)}{a + o_\hbar(1)} \to 1 + \min\left\{1, \frac{\mu_0}{2a}\right\} \text{ as } \hbar \to 0.$$

Nothing that $n_\hbar \ge \frac{\tau}{\hbar}$ for small values of $\hbar$, one can take some $\theta_0 > 1$ and obtain

$$\|u_\hbar\|^2_{H^1(\mathbb{R}^3\setminus\cup_{i=1}^{\ell}\frac{1}{\hbar}(\mathcal{K}_i)^{\tau})} \le \|u_\hbar\|^2_{H^1(E_{n_\hbar+1})} \le \theta_0^{-n_\hbar} \|u_\hbar\|^2_{H^1(E_1)} \le e^{-\frac{\tau \ln\theta_0}{\hbar}}.$$

It follows that for small values of $\hbar$, $G_\hbar(u_\hbar) = 0$. So $u_\hbar$ is a solution to the original problem (3.1) for small values of $\hbar$. □

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Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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