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Research article

Double fuzzy α-ð-continuous multifunctions

M. N. Abu_Shugair[1,2], A. A. Abdallah[1], S. E. Abbas[2] and Ismail Ibedou[3,*]

[1] Department of Mathematics, College of Science, Jazan University, P.O. Box 114, Jazan 45142, Kingdom of Saudi Arabia

[2] Mathematics Department, Faculty of Science, Sohag University, Sohag 82524, Egypt

[3] Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt

* **Correspondence:** Email: ismail.ibedou@gmail.com, ismail.abdelaziz@fsc.bu.edu.eg; Tel: +201104420786.

Abstract: Various types of double fuzzy continuity of fuzzy multifunctions are introduced in this paper. These types are double fuzzy upper and lower α-ð-continuous, almost α-ð-continuous, weakly α-ð-continuous and almost weakly α-ð-continuous multifunctions. Double fuzzy ideals are playing the main role in defining these types of continuous multifunctions. All implications associated with these types are ensured; also, many examples are introduced to illustrate these implications, and to explain the advantages of these new types of continuity for some previous definitions.

Keywords: multifunctions; double fuzzy ideal topological space; double fuzzy α-ð-continuous multifunction
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1. Introduction

The fuzzy concept has penetrated nearly all branches of mathematics since the concept was defined by Zadeh [23]. Fuzzy sets have applications in several fields, such as information systems [9] and control [8]. The theory of fuzzy topological spaces was first defined and developed by Chang [9], and since then, various notions of general topology have been generalized to the fuzzy topological spaces defined by Chang. Sostak [18] and Kubiak [14] developed fuzzy topology as an extension of the fuzzy topology introduced by Chang [9]. It has been developed in several ways. Sostak [19] published a survey article on the areas of development of fuzzy topological spaces. In a previous study [8], the author introduced the idea of intuitionistic fuzzy sets; subsequently, Coker [10] introduced the concept of intuitionistic fuzzy topological spaces. Furthermore, as a generalization of fuzzy topological spaces, Mondal and Samanta [16] introduced the concept of the intuitionistic gradation of openness.

In 2005, Garcia and Rodabaugh [11] proposed the termination of the term intuitionistic. They proved that the term intuitionistic is unsuitable in mathematics and related applications and replaced it with the notation "double". Several topologists have studied various concepts in a double fuzzy (DF-) topological spaces [12, 15]. A fuzzy multifunction is a fuzzy set valued function [7, 21]. Fuzzy multifunctions arise in many applications, for instance, the budget multifunction occurs in artificial intelligence, economic theory and decision theory. The biggest difference between fuzzy multifunctions and fuzzy functions has to do with the definition of an inverse image. For a fuzzy multifunction there are two types of inverses. These two definitions of the inverse have led to two definitions of continuity (DFU) and (DFL) continuous multifunctions; for more details the reader is referred to [2, 3, 13, 20]. Fuzzy multifunctions are being used and applied in many fields, like economics, artificial intelligence, decision theory, uncertainty, etc.

The goals of this paper were as follows: (1) to introduce DF-local multifunctions related to DF-ideals and study its properties; (2) to submit new types of DF-continuity based on a DF-ideal and study the common properties of continuity; and (3) to discuss the implications between these new types of continuity. Many examples are introduced to ensure the non reversed implications. The use of a DF-ideal in defining these new types of continuity extend the usual corresponding definitions of fuzzy continuity; thus, the introduced types of DF-continuity are extensions of the corresponding usual ones. These types of DF-ideal continuous multifunctions are called *almost, weak* and *almost weak*.

The layout of the paper is divided into 6 sections. Section 1 is an introduction. Section 2 presents the main definition of DF-local functions that are joined to a DF-ideal. Section 3 investigates the notions of DFU and DFL almost α-ð-continuity and introduces many characteristic properties of these defined multifunctions. Section 4 investigates the notions of DFU and DFL weak α-ð-continuity and discusses its properties, as well as investigates the implications associated with the previous definitions of DFU and DFL almost α-ð-continuity. Section 5 investigates the notions of DFU and DFL almost weak α-ð-continuity, and discusses its properties, as well as investigates the implications associated with the previous definitions of DFU and DFL almost α-ð-continuity and DFU and DFL weak α-ð-continuity. In Section 6, we introduce a generalization of several types of DF-multifunctions by using arbitrary operators. Section 7 presents the conclusion.

Throughout this paper, let X be a universal set, I be the closed unit interval $[0,1]$, $I_0 = (0,1]$, and $I_1 = [0,1)$. I^X refers to the set of all fuzzy sets in X. $\underline{0}$ and $\underline{1}$ refer to the empty and the whole fuzzy sets, respectively on X. The complement $\underline{1} - \lambda$ of a fuzzy set $\lambda \in I^X$ is defined by $\underline{1} - \lambda(x) = 1 - \lambda(x)$. A fuzzy point x_t in X is a fuzzy set, so $x_t(z) = 0\ \forall z \neq x$ and $x_t(x) = t$.
$x_t \in \lambda$ if and only if $t \leq \lambda(x), x \in X$. In [5], the authors defined the fuzzy difference between two fuzzy sets as follows:

$$\mu \overline{\wedge} \lambda = \underline{0} \text{ if } \mu \leq \lambda, \text{ and } \mu \overline{\wedge} \lambda = \mu \wedge \lambda^c \text{ otherwise.}$$

Recall that a DF-ideal $(ð, ð^\circ)$ on X [1] as $ð, ð^\circ : I^X \to I$ satisfies the following conditions:
(1) $ð(\lambda) + ð^\circ(\lambda) \leq 1$.
(2) $\lambda_1 \leq \lambda_2$ implies that $ð(\lambda_1) \geq ð(\lambda_2)$ and $ð^\circ(\lambda_1) \leq ð^\circ(\lambda_2)$.
(3) $ð(\lambda_1 \vee \lambda_2) \geq ð(\lambda_1) \wedge ð(\lambda_2)$ and $ð^\circ(\lambda_1 \vee \lambda_2) \leq ð^\circ(\lambda_1) \vee ð^\circ(\lambda_2)$.
The special DF-ideals $\left(ð^0, ð^{\circ 0}\right)$, $\left(ð^1, ð^{\circ 1}\right)$ are defined by:
$ð^0(\underline{0}) = ð^{\circ 1}(\underline{1}) = 1$, $ð^{\circ 0}(\underline{0}) = ð^1(\underline{1}) = 0$; otherwise, we have that $ð^0(\nu) = ð^{\circ 1}(\nu) = 0$ and $ð^1(\nu) = ð^{\circ 0}(\nu) = 1$. Let $\left(ð_1, ð_1^\circ\right)$ and $\left(ð_2, ð_2^\circ\right)$ be DF-ideals on X. Then, $\left(ð_1, ð_1^\circ\right) \leq \left(ð_2, ð_2^\circ\right)$ iff $ð_2(\nu) \leq ð_1(\nu)$ and $ð_2^\circ(\nu) \geq ð_1^\circ(\nu)$ for each $\nu \in I^X$.

Let (X, τ, τ°) be a DF-topological space ; then, the closure $C_{\tau,\tau^\circ} : I^X \times I_0 \times I_1 \to I^X$ and the interior $I_{\tau,\tau^\circ} : I^X \times I_0 \times I_1 \to I^X$ are respectively denoted by $C_{\tau,\tau^\circ}(\lambda, p, q)$ and $I_{\tau,\tau^\circ}(\lambda, p, q)$ for any fuzzy set $\lambda \in I^X$. Let $(X, (\tau, \tau^\circ), (ð, ð^\circ))$ be a DF-ideal topological space, $\lambda \in I^X$, $p \in I_0$ and $q \in I_1$. Then, the (p, q)-fuzzy local function $\Psi(\lambda, p, q)$ [6] is defined by

$$\Psi(\lambda, p, q) = \bigwedge\{\mu \in I^X : ð(\lambda \overline{\wedge} \mu) \geq p,\ ð^\circ(\lambda \overline{\wedge} \mu) \leq q,\ \tau(\mu^c) \geq p, \tau^\circ(\mu^c) \leq q\}.$$

Moreover, we define operators cl°, $int^\circ : I^X \times I_0 \times I_1 \to I^X$ as follows:
$cl^\circ(\lambda, p, q) = \lambda \vee \Psi(\lambda, p, q)$, $int^\circ(\lambda, p, q) = \lambda \wedge (\Psi(\lambda^c, p, q))^c$.
Also, λ is called (p, q)-fuzzy α-ð-open iff $\lambda \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\lambda, p, q), p, q), p, q)$.

A map $\Phi : (X, \tau) \multimap (Y, \sigma)$ is called a fuzzy multifunction [4] iff $\Phi(x) \in I^Y$ for each $x \in X$. The degree of membership of y in $\Phi(x)$ is denoted by $\Phi(x)(y) = G_\Phi(x, y)$ for any $(x, y) \in X \times Y$. Φ is called crisp iff $G_\Phi(x, y) = \underline{1}$ for each $x \in X$ and $y \in Y$. Φ is called normalized iff for each $x \in X$, there exists $y_0 \in Y$ such that $G_\Phi(x, y_0) = \underline{1}$. The image $\Phi(\lambda)$, the lower inverse $\Phi^l(\lambda)$ and the upper inverse $\Phi^u(\lambda)$ of $\lambda \in I^X$ are defined respectively as follows:
$\Phi(\lambda)(y) = \bigvee_{x\in X}(G_\Phi(x, y) \wedge \lambda(x))$, $\Phi^l(\lambda)(x) = \bigvee_{y\in Y}(G_\Phi(x, y) \wedge \lambda(y))$, $\Phi^u(\lambda)(x) = \bigwedge_{y\in Y}((G_\Phi)^c(x, y) \vee \lambda(y))$.

Moreover, Φ is called compact valued [2] iff $\Phi(x_t)$ is (p, q)-fuzzy compact for each $x_t \in dom(\Phi)$.

Let $\Phi : X \multimap Y$ and $\Xi : Y \multimap Z$ be two fuzzy multifunctions. Then, the composition $\Xi \circ \Phi : X \multimap Z$ is defined by $((\Xi \circ \Phi)(x))(z) = \bigvee_{y\in Y}(G_\Phi(x, y) \wedge G_\Xi(y, z))$.

All related definitions and properties of image, upper, lower and compositions of fuzzy multifunctions can be found in [4].

Definition 1.1. [17] Let $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ)$ be a DF-multifunction, $p \in I_0$ and $q \in I_1$; then, Φ is as follows:

(1) DFU semi-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists $\lambda \in I^X$, $\tau(\lambda) \geq p$, $\tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \wedge dom(\Phi) \leq \Phi^u(\mu)$.

(2) DFL semi-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^l(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists $\lambda \in I^X$, $\tau(\lambda) \geq p$, $\tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^l(\mu)$.

(3) DFU (DFL) semi-continuous iff it is DFU (DFL) semi-continuous at every fuzzy point $x_t \in dom(\Phi)$.

Definition 1.2. [6] Let $\Phi : (X, \tau, \tau^\circ, ð, ð^\circ) \multimap (Y, \sigma, \sigma^\circ)$ be a DF-multifunction, $p \in I_0$ and $q \in I_1$. Then, Φ is as follows:

(1) DFU ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists $\lambda \in I^X, \tau(\lambda) \geq p$, $\tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \wedge dom(\Phi) \leq \Psi(\Phi^u(\mu), p, q)$.

(2) DFL ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^l(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists $\lambda \in I^X, \tau(\lambda) \geq p, \tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \leq \Psi(\Phi^l(\mu), p, q)$.

(3) DFU ð-continuous (resp. DFL ð-continuous) iff it is DFU ð-continuous (resp. DFL ð-continuous) at every fuzzy point $x_t \in dom(\Phi)$.

2. DF-α-ð-continuous multifunctions

This section focuses on the definitions of the DF-α-ð-continuous multifunctions related to a DF-ideal.

Definition 2.1. Let $\Phi : (X, \tau, \tau^\circ, ð, ð^\circ) \multimap (Y, \sigma, \sigma^\circ)$ be a DF-multifunction, $p \in I_0$ and $q \in I_1$. Then, Φ is as follows:

(1) DFU α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(\Phi) \leq \Phi^u(\mu)$.

(2) DFL α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^l(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^l(\mu)$.

(3) DFU α-ð-continuous (DFL α-ð-continuous) iff it is DFU α-ð-continuous (DFL α-ð-continuous) at every fuzzy point $x_t \in dom(\Phi)$.

If we take $cl^\circ = C_{\tau,\tau^\circ}$, then we have the definition of DF-α-continuity.

Remark 2.1. If Φ is a normalized multifunction, then Φ is DFU α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^u(\mu)$.

Remark 2.2. (1) DFU (resp. DFL) semi-continuous $\Rightarrow$ DFU (resp. DFL) α-ð-continuous.

(2) DFU (resp. DFL) α-ð-continuous $\Rightarrow$ DFU (resp. DFL) α-continuous.

(3) DFU (resp. DFL) α- $ð^0$-continuous $\Leftrightarrow$ DFU (resp. DFL) α- continuous.

The converses are not true.

Example 2.1. *Let* $X = \{x_1, x_2, x_3\}$ *and* $Y = \{y_1, y_2, y_3\}$. *For* $\mu_1, \mu_2 \in I^X$ *and* $\mu_3 \in I^Y$ *defined as* $\mu_1 = \{0.9, 0.5, 0.5\}$, $\mu_2 = \{0.9, 0.9, 0.5\}$ *and* $\mu_3 = \{0.9, 0.9, 0.5\}$, *define DF-topologies* τ, τ°: $I^X \to I$, σ, σ°: $I^Y \to I$ *as follows:*

$\tau(\underline{0}) = \tau(\underline{1}) = 1,\ \tau(\mu_1) = \frac{1}{3}$ and $\tau(\lambda) = 0$ otherwise,

$\tau^\circ(\underline{0}) = \tau^\circ(\underline{1}) = 0,\ \tau^\circ(\mu_1) = \frac{2}{3}$ and $\tau^\circ(\lambda) = 1$ otherwise,

$\sigma(\underline{0}) = \sigma(\underline{1}) = 1,\ \sigma(\mu_3) = \frac{1}{3}$ and $\sigma(\lambda) = 0$ otherwise,

$\sigma^\circ(\underline{0}) = \sigma^\circ(\underline{1}) = 0,\ \sigma^\circ(\mu_3) = \frac{2}{3}$ and $\sigma^\circ(\lambda) = 1$ otherwise.

Then,

(1) $\Phi : X \multimap Y$ *is a fuzzy multifunction defined by* $G_\Phi(x_1, y_1) = 1$, $G_\Phi(x_1, y_2) = 1$, $G_\Phi(x_1, y_3) = 0$, $G_\Phi(x_2, y_1) = 0$, $G_\Phi(x_2, y_2) = 1$, $G_\Phi(x_2, y_3) = 0$, $G_\Phi(x_3, y_1) = 0$, $G_\Phi(x_3, y_2) = 0.3$, $G_\Phi(x_3, y_3) = 1$.
For a DF-ideal $ð, ð^\circ$: $I^X \to I$ *defined as follows:*

$ð(\underline{0}) = 1,\ ð(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.5}$ and $ð(\nu) = 0$ otherwise;

$ð^\circ(\underline{0}) = 0,\ ð^\circ(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.5}$ and $ð^\circ(\nu) = 1$ otherwise,

Φ is DFU and DFL α-ð-continuous multifunction but is neither a DFU nor DFL semi-continuous multifunction because

$$\Phi^u(\mu_3) = \mu_2 \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\mu_2, \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}) = \underline{1},$$

$$\Phi^l(\mu_3) = \mu_2 \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\mu_2, \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}) = \underline{1},$$

and

$$\Phi^u(\mu_3) = \mu_2 \nleq I_{\tau,\tau^\circ}(\Phi^u(\mu_3), \frac{1}{3}, \frac{2}{3}) = \mu_1,$$

$$\Phi^l(\mu_3) = \mu_2 \nleq I_{\tau,\tau^\circ}(\Phi^l(\mu_3), \frac{1}{3}, \frac{2}{3}) = \mu_1.$$

(2) $\Phi : X \multimap Y$ is a fuzzy multifunction defined by $G_\Phi(x_1,y_1)=1$, $G_\Phi(x_1,y_2)=1$, $G_\Phi(x_1,y_3)=0$, $G_\Phi(x_2,y_1)=1$, $G_\Phi(x_2,y_2)=0.4$, $G_\Phi(x_2,y_3)=0$, $G_\Phi(x_3,y_1)=0$, $G_\Phi(x_3,y_2)=0.5$, $G_\Phi(x_3,y_3)=1$.
For a DF-ideal ð, ð°: $I^X \to I$ defined as follows:

ð($\underline{0}$) = 1, ð(ν) = $\frac{2}{3}$ if $\underline{0} < \nu \leq \underline{0.9}$ and ð(ν) = 0 otherwise;

ð°($\underline{0}$) = 0, ð°(ν) = $\frac{1}{4}$ if $\underline{0} < \nu \leq \underline{0.9}$ and ð°(ν) = 1 otherwise.

Φ is a DFU and DFL α-continuous multifunction but is neither a DFU nor DFL α-ð-continuous multifunction because

$$\Phi^u(\mu_3)=\mu_2 \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(\mu_3),\frac{1}{3},\frac{2}{3}),\frac{1}{3},\frac{2}{3}),\frac{1}{3},\frac{2}{3})=\underline{1},$$

$$\Phi^l(\mu_3)=\mu_2 \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^l(\mu_3),\frac{1}{3},\frac{2}{3}),\frac{1}{3},\frac{2}{3}),\frac{1}{3},\frac{2}{3})=\underline{1},$$

and

$$\Phi^u(\mu_3)=\mu_2 \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(\mu_3),\frac{1}{3},\frac{2}{3}),\frac{1}{3},\frac{2}{3}),\frac{1}{3},\frac{2}{3})=\mu_1,$$

$$\Phi^l(\mu_3)=\mu_2 \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(\mu_3),\frac{1}{3},\frac{2}{3}),\frac{1}{3},\frac{2}{3}),\frac{1}{3},\frac{2}{3})=\mu_1.$$

Theorem 2.1. *For a DF-multifunction $\Phi : (X,\tau,\tau^\circ,ð,ð^\circ) \multimap (Y,\sigma,\sigma^\circ)$, $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$, the following statements are equivalent:*

(1) Φ is DFL α-ð-continuous.

(2) $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(\mu),p,q),p,q),p,q)$ if $\sigma(\mu)\geq p$ and $\sigma^\circ(\mu)\leq q$.

(3) $C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(\Phi^u(\mu),p,q),p,q),p,q) \leq \Phi^u(\mu)$ if $\sigma(\mu^c)\geq p$ and $\sigma^\circ(\mu^c)\leq q$.

(4) $\Phi^l(I_{\sigma,\sigma^\circ}(\mu,p,q)) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(\mu,p,q)),p,q),p,q),p,q)$.

Proof. (1) $\Longrightarrow$ (2) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu)\geq p$, $\sigma^\circ(\mu)\leq q$ and $x_t \in \Phi^l(\mu)$. Then, there exists a (p,q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^l(\mu,p,q)$.

Thus, $x_t \in \lambda \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(\mu),p,q),p,q),p,q)$; hence
$\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(\mu),p,q),p,q),p,q)$.

(2) $\Longrightarrow$ (3) Let $\mu \in I^Y$ with $\sigma(\mu^c)\geq p$ and $\sigma^\circ(\mu^c)\leq q$. Then, by (2),

$$\begin{aligned}[\Phi^u(\mu)]^c &= \Phi^l(\mu^c) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(\mu^c),p,q),p,q),p,q)\\ &= [C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(\Phi^u(\mu),p,q),p,q),p,q)]^c.\end{aligned}$$

Thus, $C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(\Phi^u(\mu),p,q),p,q),p,q) \leq \Phi^u(\mu)$.

(3) $\Longrightarrow$ (4) Since $C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^\circ}(\mu,p,q)),p,q),p,q),p,q) \leq \Phi^u(C_{\sigma,\sigma^\circ}(\mu,p,q))$ for each $\mu \in I^Y$, it follows that

$$\Phi^l(I_{\sigma,\sigma^\circ}(\mu,p,q)) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(\mu,p,q)),p,q),p,q),p,q).$$

(4) $\Longrightarrow$ (1) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu)\geq p$, $\sigma^\circ(\mu)\leq q$ and $x_t \in \Phi^l(\mu)$. Then, by (4) and $\mu = I_{\sigma,\sigma^\circ}(\mu,p,q)$, we have

$$x_t \in \Phi^l(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(\mu),p,q),p,q),p,q).$$

Thus, Φ is DFL α-ð-continuous. □

The proof of the following theorem is similar to that of Theorem 2.1.

Theorem 2.2. *For a normalized DF-multifunction* $\Phi : (X, \tau, \tau^\circ, ð, ð^\circ) \multimap (Y, \sigma, \sigma^\circ)$, $\mu \in I^Y$, $p \in I_0$ *and* $q \in I_1$, *the following statements are equivalent:*

(1) Φ is DFU α-ð-continuous.
(2) $\Phi^u(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(\mu), p, q), p, q), p, q)$ if $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$.
(3) $C_{\tau,\tau^\circ}\left(int^\circ(C_{\tau,\tau^\circ}(\Phi^l(\mu), p, q), p, q), p, q\right) \leq \Phi^l(\mu)$ if $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$.
(4) $\Phi^u(I_{\sigma,\sigma^\circ}(\mu, p, q)) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(I_{\sigma,\sigma^\circ}(\mu, p, q)), p, q), p, q), p, q)$.

Corollary 2.1. (1) *Let* $\Phi : X \multimap Y$ *and* $H : Y \multimap Z$ *be two DF-multifunctions. Then,* $H \circ \Phi$ *is DFL* α-ð-*continuous if* Φ *is DFL* α-ð-*continuous and H is DFL semi-continuous.*

(2) *Let* $\Phi : X \multimap Y$ *and* $H : Y \multimap Z$ *be two normalized DF-multifunctions. Then,* $H \circ \Phi$ *is DFU* α-ð-*continuous if* Φ *is DFU* α-ð-*continuous and H is DFU semi-continuous.*

3. DF-almost α-ð-continuous multifunctions

This section investigates the definitions of DFU and DFL almost α-ð-continuity and introduces many characteristic properties of the defined multifunctions.

Definition 3.1. Let $\Phi : (X, \tau, \tau^\circ, ð_1, ð_1^\circ) \multimap (Y, \sigma, \sigma^\circ, ð_2, ð_2^\circ)$ be a DF-multifunction, $p \in I_0$ and $q \in I_1$. Then, Φ is as follows:

(1) DFU almost α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(\Phi) \leq \Phi^u(I_{\sigma,\sigma^\circ}(cl^\circ(\mu, p, q), p, q))$.

(2) DFL almost α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^l(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\mu, p, q), p, q))$.

(3) DFU almost α-ð-continuous (DFL almost α-ð-continuous) iff it is DFU almost α-ð-continuous (DFL almost α-ð-continuous) at every fuzzy point $x_t \in dom(\Phi)$.

If we take $cl^\circ = C_{\tau,\tau^\circ}$, then we have the definition of DF-almost α- continuous.

Remark 3.1. If Φ is a normalized multifunction, then Φ is DFU almost α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^u(I_{\sigma,\sigma^\circ}(cl^\circ(\mu, p, q), p, q))$.

Theorem 3.1. *For a DF-multifunction* $\Phi : (X, \tau, \tau^\circ, ð_1, ð_1^\circ) \multimap (Y, \sigma, \sigma^\circ, ð_2, ð_2^\circ)$, $\mu \in I^Y$, $p \in I_0$ *and* $q \in I_1$, *the following statements are equivalent:*

(1) Φ is DFL almost α-ð-continuous.
(2) $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\mu, p, q), p, q)), p, q), p, q), p, q)$ if $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$.
(3) $C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^\circ}(int^\circ(\mu, p, q), p, q)), p, q), p, q), p, q) \leq \Phi^u(\mu)$ if $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$.
(4) $\Phi^l(I_{\sigma,\sigma^\circ}(\mu, p, q)) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(I_{\sigma,\sigma^\circ}(\mu, p, q), p, q), p, q)), p, q), p, q), p, q)$.

Proof. (1) $\Longrightarrow$ (2) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\mu, p, q), p, q))$.

Thus, $x_t \in \lambda \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\mu,p,q),p,q)),p,q),p,q),p,q)$, and hence $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\mu,p,q),p,q)),p,q),p,q),p,q)$.

(2) $\Longrightarrow$ (3) Let $\mu \in I^Y$ with $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$. Then, by (2) ,

$$\begin{aligned}[\Phi^u(\mu)]^c &= \Phi^l(\mu^c) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\mu^c,p,q),p,q)),p,q),p,q),p,q) \\ &= [C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^\circ}(int^\circ(\mu,p,q),p,q)),p,q),p,q),p,q)]^c.\end{aligned}$$

Thus, $C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^\circ}(int^\circ(\mu,p,q),p,q)),p,q),p,q),p,q) \leq \Phi^u(\mu)$.

(3) $\Longrightarrow$ (4) Since

$$C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^\circ}(int^\circ(C_{\sigma,\sigma^\circ}(\mu,p,q),p,q),p,q)),p,q),p,q),p,q) \leq \Phi^u(C_{\sigma,\sigma^\circ}(\mu,p,q))$$

for each $\mu \in I^Y$, then

$$\Phi^l(I_{\sigma,\sigma^\circ}(\mu,p,q)) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(I_{\sigma,\sigma^\circ}(\mu,p,q),p,q),p,q)),p,q),p,q),p,q).$$

(4) $\Longrightarrow$ (1) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, by (4) and $\mu = I_{\sigma,\sigma^\circ}(\mu,p,q)$, we have that
$x_t \in \Phi^l(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\mu,p,q),p,q)),p,q),p,q),p,q)$.
Thus, Φ is DFL almost α-ð-continuous. □

The proof of the following theorem is similar to that of Theorem 3.1.

Theorem 3.2. *For a normalized DF-multifunction* $\Phi : (X,\tau,\tau^\circ,ð_1,ð_1^\circ) \multimap (Y,\sigma,\sigma^\circ,ð_2,ð_2^\circ)$, $\mu \in I^Y$, $p \in I_0$ *and* $q \in I_1$, *the following statements are equivalent:*

(1) Φ is DFU almost α-ð-continuous.

(2) $\Phi^u(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(I_{\sigma,\sigma^\circ}(cl^\circ(\mu,p,q),p,q)),p,q),p,q),p,q)$ if $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$.

(3) $C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(\Phi^l(C_{\sigma,\sigma^\circ}(int^\circ(\mu,p,q),p,q)),p,q),p,q),p,q) \leq \Phi^l(\mu)$ if $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$.

(4) $\Phi^u(I_{\sigma,\sigma^\circ}(\mu,p,q)) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(I_{\sigma,\sigma^\circ}(cl^\circ(I_{\sigma,\sigma^\circ}(\mu,p,q),p,q),p,q)),p,q),p,q),p,q)$.

Remark 3.2. (1) DFU (resp. DFL) α-ð-continuous $\Rightarrow$ DFU (resp. DFL) almost α-ð-continuous.

(2) DFU (resp. DFL) almost α-ð-continuous $\Rightarrow$ DFU (resp. DFL) almost α-continuous.

(3) DFU (resp. DFL) almost α- $ð^0$-continuous $\Longleftrightarrow$ DFU (resp. DFL) almost α-continuous.

(4) DFU (resp. DFL) semi-continuous $\Rightarrow$ DFU (resp. DFL) almost α-ð-continuous.

The coming examples show that these implications are not reversed.

Example 3.1. *Let* $X = \{x_1,x_2,x_3\}$, $Y = \{y_1,y_2,y_3\}$ *and* $\Phi : X \multimap Y$ *be a DF-multifunction defined by* $G_\Phi(x_1,y_1) = 1$, $G_\Phi(x_1,y_2) = 0.1$, $G_\Phi(x_1,y_3) = 0.3$, $G_\Phi(x_2,y_1) = 0.5$, $G_\Phi(x_2,y_2) = 1$, $G_\Phi(x_2,y_3) = 0.1$, $G_\Phi(x_3,y_1) = 0$, $G_\Phi(x_3,y_2) = 0$, $G_\Phi(x_3,y_3) = 1$. *Define DF-topologies* τ,τ°: $I^X \to I$, σ,σ°: $I^Y \to I$, *and DF-ideals* $ð_1,ð_1^\circ$: $I^X \to I$, $ð_2,ð_2^\circ$: $I^Y \to I$ *as follows:*

$\tau(\underline{0}) = \tau(\underline{1}) = 1$, $\tau(\underline{0.4}) = \frac{1}{2}$, $\tau(\underline{0.9}) = \frac{1}{4}$ and $\tau(\lambda) = 0$ otherwise;
$\tau^\circ(\underline{0}) = \tau^\circ(\underline{1}) = 0$, $\tau^\circ(\underline{0.4}) = \frac{1}{2}$, $\tau^\circ(\underline{0.9}) = \frac{3}{4}$ and $\tau^\circ(\lambda) = 1$ otherwise;
$\sigma(\underline{0}) = \sigma(\underline{1}) = 1$, $\sigma(\underline{0.7}) = \frac{1}{4}$ and $\sigma(\lambda) = 0$ otherwise;
$\sigma^\circ(\underline{0}) = \sigma^\circ(\underline{1}) = 0$, $\sigma^\circ(\underline{0.7}) = \frac{3}{4}$ and $\sigma^\circ(\lambda) = 1$ otherwise;
$ð_1(\underline{0}) = 1$, $ð_1(\nu) = \frac{1}{2}$ if $\underline{0} < \nu \leq \underline{0.3}$ and $ð_1(\nu) = 0$ otherwise;

$\eth^\circ_1(\underline{0}) = 0,\ \eth^\circ_1(\nu) = \frac{1}{2}$ if $\underline{0} < \nu \leq \underline{0.3}$ and $\eth^\circ_1(\nu) = 1$ otherwise;
$\eth_2(\underline{0}) = 1,\ \eth_2(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.3}$ and $\eth_2(\nu) = 0$ otherwise;
$\eth^\circ_2(\underline{0}) = 0,\ \eth^\circ_2(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.3}$ and $\eth^\circ_2(\nu) = 1$ otherwise.

Then, $\Phi : (X, \tau, \tau^\circ, \eth_1, \eth^\circ_1) \multimap (Y, \sigma, \sigma^\circ, \eth_2, \eth^\circ_2)$ is DFU and DFL almost α-ð-continuous but is neither DFU nor DFL α-ð-continuous because

$$\underline{0.7} = \Phi^u(\underline{0.7}) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(I_{\sigma,\sigma^\circ}(cl^\circ(\underline{0.7}, \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{1},$$

$$\underline{0.7} = \Phi^l(\underline{0.7}) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\underline{0.7}, \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{1},$$

and

$$\underline{0.7} = \Phi^u(\underline{0.7}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(\underline{0.7}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{0.4},$$

$$\underline{0.7} = \Phi^l(\underline{0.7}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(\underline{0.7}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{0.4}.$$

Example 3.2. *Let* $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ *and* $\Phi : X \multimap Y$ *be a DF-multifunction defined by* $G_\Phi(x_1, y_1) = 0.4$, $G_\Phi(x_1, y_2) = 0$, $G_\Phi(x_1, y_3) = 1$, $G_\Phi(x_2, y_1) = 0.2$, $G_\Phi(x_2, y_2) = 1$, $G_\Phi(x_2, y_3) = 0.3$, $G_\Phi(x_3, y_1) = 1$, $G_\Phi(x_3, y_2) = 0.3$, $G_\Phi(x_3, y_3) = 0.5$. *Define DF-topologies* $\tau, \tau^\circ : I^X \to I$, $\sigma, \sigma^\circ : I^Y \to I$, *and DF-ideals* $\eth_1, \eth^\circ_1 : I^X \to I$, $\eth_2, \eth^\circ_2 : I^Y \to I$ *as follows:*

$\tau(\underline{0}) = \tau(\underline{1}) = 1,\ \tau(\underline{0.3}) = \frac{1}{2}$ and $\tau(\lambda) = 0$ otherwise;
$\tau^\circ(\underline{0}) = \tau^\circ(\underline{1}) = 0,\ \tau^\circ(\underline{0.3}) = \frac{1}{2}$ and $\tau^\circ(\lambda) = 1$ otherwise;
$\sigma(\underline{0}) = \sigma(\underline{1}) = 1,\ \sigma(\underline{0.7}) = \frac{1}{3}$ and $\sigma(\lambda) = 0$ otherwise;
$\sigma^\circ(\underline{0}) = \sigma^\circ(\underline{1}) = 0,\ \sigma^\circ(\underline{0.7}) = \frac{2}{3}$ and $\sigma^\circ(\lambda) = 1$ otherwise;
$\eth_1(\underline{0}) = 1,\ \eth_1(\nu) = \frac{1}{2}$ if $\underline{0} < \nu \leq \underline{0.3}$ and $\eth_1(\nu) = 0$ otherwise;
$\eth^\circ_1(\underline{0}) = 0,\ \eth^\circ_1(\nu) = \frac{1}{2}$ if $\underline{0} < \nu \leq \underline{0.3}$ and $\eth^\circ_1(\nu) = 1$ otherwise;
$\eth_2(\underline{0}) = 1,\ \eth_2(\nu) = \frac{1}{3}$ if $\underline{0} < \nu \leq \underline{0.8}$ and $\eth_2(\nu) = 0$ otherwise;
$\eth^\circ_2(\underline{0}) = 0,\ \eth^\circ_2(\nu) = \frac{1}{2}$ if $\underline{0} < \nu \leq \underline{0.8}$ and $\eth^\circ_2(\nu) = 1$ otherwise.

Then, $\Phi : (X, \tau, \tau^\circ, \eth_1, \eth^\circ_1) \multimap (Y, \sigma, \sigma^\circ, \eth_2, \eth^\circ_2)$ is DFU and DFL almost α-continuous but is neither DFU nor DFL almost α-ð-continuous because

$$\underline{0.7} = \Phi^u(\underline{0.7}) \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(I_{\sigma,\sigma^\circ}(C_{\sigma,\sigma^\circ}(\underline{0.7}, \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3})), \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}) = \underline{1},$$

$$\underline{0.7} = \Phi^l(\underline{0.7}) \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(C_{\sigma,\sigma^\circ}(\underline{0.7}, \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3})), \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}) = \underline{1},$$

and

$$\underline{0.7} = \Phi^u(\underline{0.7}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(I_{\sigma,\sigma^\circ}(cl^\circ(\underline{0.7}, \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3})), \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}) = \underline{0.3},$$

$$\underline{0.7} = \Phi^l(\underline{0.7}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\underline{0.7}, \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3})), \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}) = \underline{0.3}.$$

Example 3.3. *Let* $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ *and* $\Phi : X \multimap Y$ *be a DF-multifunction defined by* $G_\Phi(x_1, y_1) = 0.5$, $G_\Phi(x_1, y_2) = 1$, $G_\Phi(x_1, y_3) = 0$, $G_\Phi(x_2, y_1) = 1$, $G_\Phi(x_2, y_2) = 0.3$, $G_\Phi(x_2, y_3) = 0.2$, $G_\Phi(x_3, y_1) = 0$, $G_\Phi(x_3, y_2) = 0.4$, $G_\Phi(x_3, y_3) = 1$. *Define* $\mu_1, \mu_2 \in I^X$ *and* $\mu_3 \in I^Y$ *as follows:*
$\mu_1 = \{0.8, 0.5, 0.5\}$, $\mu_2 = \{0.8, 0.8, 0.5\}$ *and* $\mu_3 = \{0.8, 0.8, 0.5\}$. *Define DF-topologies* $\tau, \tau^\circ : I^X \to I$, $\sigma, \sigma^\circ : I^Y \to I$, *and DF-ideals* $\eth_1, \eth^\circ_1 : I^X \to I$, $\eth_2, \eth^\circ_2 : I^Y \to I$ *as follows:*

$\tau(\underline{0}) = \tau(\underline{1}) = 1,\ \tau(\mu_1) = \frac{1}{4}$ and $\tau(\lambda) = 0$ otherwise;
$\tau^\circ(\underline{0}) = \tau^\circ(\underline{1}) = 0,\ \tau^\circ(\mu_1) = \frac{3}{4}$ and $\tau^\circ(\lambda) = 1$ otherwise;
$\sigma(\underline{0}) = \sigma(\underline{1}) = 1,\ \sigma(\mu_3) = \frac{1}{3}$ and $\sigma(\lambda) = 0$ otherwise;
$\sigma^\circ(\underline{0}) = \sigma^\circ(\underline{1}) = 0,\ \sigma^\circ(\mu_3) = \frac{2}{3}$ and $\sigma^\circ(\lambda) = 1$ otherwise;
$ð_1(\underline{0}) = 1,\ ð_1(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.5}$ and $ð_1(\nu) = 0$ otherwise;
$ð_1^\circ(\underline{0}) = 0,\ ð_1^\circ(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.5}$ and $ð_1^\circ(\nu) = 1$ otherwise,
$ð_2(\underline{0}) = 1,\ ð_2(\nu) = \frac{1}{4}$ if $\underline{0} < \nu < \underline{0.5}$ and $ð_2(\nu) = 0$ otherwise;
$ð_2^\circ(\underline{0}) = 0,\ ð_2^\circ(\nu) = \frac{3}{4}$ if $\underline{0} < \nu < \underline{0.5}$ and $ð_2^\circ(\nu) = 1$ otherwise.

Then, $\Phi : (X, \tau, \tau^\circ, ð_1, ð_1^\circ) \multimap (Y, \sigma, \sigma^\circ, ð_2, ð_2^\circ)$ is DFU and DFL almost α-ð-continuous but is neither DFU nor DFL semi-continuous because

$$\mu_2 = \Phi^u(\mu_3) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(I_{\sigma,\sigma^\circ}(cl^\circ(\mu_3, \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{1},$$

$$\mu_2 = \Phi^l(\mu_3) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\mu_3, \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{1},$$

and

$$\mu_2 = \Phi^u(\mu_3) \nleq I_{\tau,\tau^\circ}(\Phi^u(\mu_3), \frac{1}{4}, \frac{3}{4}) = \mu_1,$$

$$\mu_2 = \Phi^l(\mu_3) \nleq I_{\tau,\tau^\circ}(\Phi^l(\mu_3), \frac{1}{4}, \frac{3}{4}) = \mu_1.$$

Corollary 3.1. (1) *Let $\Phi : X \multimap Y$ and $H : Y \multimap Z$ be two DF-multifunctions. Then, $H \circ \Phi$ is DFL almost α-ð-continuous if Φ is DFL α-ð-continuous and H is DFL almost ð-continuous.*

(2) *Let $\Phi : X \multimap Y$ and $H : Y \multimap Z$ be two normalized DF-multifunctions. Then, $H \circ \Phi$ is DFU almost α-ð-continuous if Φ is DFU α-ð-continuous and H is DFU almost ð-continuous.*

4. DF-weakly α-ð-continuous multifunctions

This section presents the notions of DFU and DFL weakly α-ð-continuity and discusses its relations with the previous definitions of DFU and DFL almost α-ð-continuity.

Definition 4.1. Let $\Phi : (X, \tau, \tau^\circ, ð_1, ð_1^\circ) \multimap (Y, \sigma, \sigma^\circ, ð_2, ð_2^\circ)$ be a DF-multifunction, $p \in I_0$ and $q \in I_1$. Then, Φ is as follows:

(1) DFU weakly α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(\Phi) \leq \Phi^u(cl^\circ(\mu, p, q))$.

(2) DFL weakly α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^l(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^l(cl^\circ(\mu, p, q))$.

(3) DFU weakly α-ð-continuous (DFL weakly α-ð-continuous) iff it is DFU weakly α-ð-continuous (DFL weakly α-ð-continuous) at every fuzzy point $x_t \in dom(\Phi)$.

If we take $cl^\circ = C_{\tau,\tau^\circ}$, we have the definition of DF-weakly α-continuous.

Remark 4.1. If Φ is a normalized multifunction, then Φ is DFU weakly α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^u(cl^\circ(\mu, p, q))$.

Theorem 4.1. *For a DF-multifunction* $\Phi : (X, \tau, \tau^{\circ}, ð_1, ð_1^{\circ}) \multimap (Y, \sigma, \sigma^{\circ}, ð_2, ð_2^{\circ})$, $\mu \in I^Y$, $p \in I_0$ *and* $q \in I_1$, *the following statements are equivalent:*

(1) Φ is DFL weakly α-ð-continuous.
(2) $\Phi^l(\mu) \leq I_{\tau,\tau^{\circ}}(cl^{\circ}(I_{\tau,\tau^{\circ}}(\Phi^l(cl^{\circ}(\mu, p, q)), p, q), p, q), p, q)$ if $\sigma(\mu) \geq p$ and $\sigma^{\circ}(\mu) \leq q$.
(3) $C_{\tau,\tau^{\circ}}(int^{\circ}(C_{\tau,\tau^{\circ}}(\Phi^u(int^{\circ}(\mu, p, q)), p, q), p, q), p, q) \leq \Phi^u(\mu)$ if $\sigma(\mu^c) \geq p$ and $\sigma^{\circ}(\mu^c) \leq q$.
(4) $\Phi^l(I_{\sigma,\sigma^{\circ}}(\mu, p, q)) \leq I_{\tau,\tau^{\circ}}(cl^{\circ}(I_{\tau,\tau^{\circ}}(\Phi^l(cl^{\circ}(I_{\sigma,\sigma^{\circ}}(\mu, p, q), p, q)), p, q), p, q), p, q)$.

Proof. (1) $\Longrightarrow$ (2) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^{\circ}(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^l(cl^{\circ}(\mu, p, q))$.

Thus, $x_t \in \lambda \leq I_{\tau,\tau^{\circ}}(cl^{\circ}(I_{\tau,\tau^{\circ}}(\Phi^l(cl^{\circ}(\mu, p, q)), p, q), p, q), p, q)$, and hence $\Phi^l(\mu) \leq I_{\tau,\tau^{\circ}}(cl^{\circ}(I_{\tau,\tau^{\circ}}(\Phi^l(cl^{\circ}(\mu, p, q)), p, q), p, q), p, q)$.

(2) $\Longrightarrow$ (3) Let $\mu \in I^Y$ with $\sigma(\mu^c) \geq p$ and $\sigma^{\circ}(\mu^c) \leq q$. Then, by (2),

$$\begin{aligned}[\Phi^u(\mu)]^c &= \Phi^l(\mu^c) \leq I_{\tau,\tau^{\circ}}(cl^{\circ}(I_{\tau,\tau^{\circ}}(\Phi^l(cl^{\circ}(\mu^c, p, q)), p, q), p, q), p, q) \\ &= [C_{\tau,\tau^{\circ}}(int^{\circ}(C_{\tau,\tau^{\circ}}(\Phi^u(int^{\circ}(\mu, p, q)), p, q), p, q), p, q)]^c.\end{aligned}$$

Thus, $C_{\tau,\tau^{\circ}}(int^{\circ}(C_{\tau,\tau^{\circ}}(\Phi^u(int^{\circ}(\mu, p, q)), p, q), p, q), p, q) \leq \Phi^u(\mu)$.

(3) $\Longrightarrow$ (4) Since

$$C_{\tau,\tau^{\circ}}(int^{\circ}(C_{\tau,\tau^{\circ}}(\Phi^u(int^{\circ}(C_{\sigma,\sigma^{\circ}}(\mu, p, q), p, q)), p, q), p, q), p, q) \leq \Phi^u(C_{\sigma,\sigma^{\circ}}(\mu, p, q))$$

for each $\mu \in I^Y$. Then,

$$\Phi^l(I_{\sigma,\sigma^{\circ}}(\mu, p, q)) \leq I_{\tau,\tau^{\circ}}(cl^{\circ}(I_{\tau,\tau^{\circ}}(\Phi^l(cl^{\circ}(I_{\sigma,\sigma^{\circ}}(\mu, p, q), p, q)), p, q), p, q), p, q).$$

(4) $\Longrightarrow$ (1) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^{\circ}(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, by (4) and $\mu = I_{\sigma,\sigma^{\circ}}(\mu, p, q)$, we have that $x_t \in \Phi^l(\mu) \leq I_{\tau,\tau^{\circ}}(cl^{\circ}(I_{\tau,\tau^{\circ}}(\Phi^l(cl^{\circ}(\mu, p, q)), p, q), p, q), p, q)$. Thus, Φ is DFL weakly α-ð-continuous. □

The following theorem is proved similarly as in the case of Theorem 4.1.

Theorem 4.2. *For a normalized DF-multifunction* $\Phi : (X, \tau, \tau^{\circ}, ð_1, ð_1^{\circ}) \multimap (Y, \sigma, \sigma^{\circ}, ð_2, ð_2^{\circ})$, $\mu \in I^Y$, $p \in I_0$ *and* $q \in I_1$, *the following statements are equivalent:*

(1) Φ is DFU weakly α-ð-continuous.
(2) $\Phi^u(\mu) \leq I_{\tau,\tau^{\circ}}(cl^{\circ}(I_{\tau,\tau^{\circ}}(\Phi^u(cl^{\circ}(\mu, p, q)), p, q), p, q), p, q)$ if $\sigma(\mu) \geq p$ and $\sigma^{\circ}(\mu) \leq q$.
(3) $C_{\tau,\tau^{\circ}}(int^{\circ}(C_{\tau,\tau^{\circ}}(\Phi^l(int^{\circ}(\mu, p, q)), p, q), p, q), p, q) \leq \Phi^l(\mu)$ if $\sigma(\mu^c) \geq p$ and $\sigma^{\circ}(\mu^c) \leq q$.
(4) $\Phi^u(I_{\sigma,\sigma^{\circ}}(\mu, p, q)) \leq I_{\tau,\tau^{\circ}}(cl^{\circ}(I_{\tau,\tau^{\circ}}(\Phi^u(cl^{\circ}(I_{\sigma,\sigma^{\circ}}(\mu, p, q), p, q)), p, q), p, q), p, q)$.

Remark 4.2. (1) DFU (resp. DFL) almost α-ð-continuous ⇒ DFU (resp. DFL) weakly α-ð-continuous.
(2) DFU (resp. DFL) weakly α-ð-continuous ⇒ DFU (resp. DFL) weakly α-continuous.
(3) DFU (resp. DFL) weakly α- $ð^0$-continuous $\Longleftrightarrow$ DFU (resp. DFL) weakly α-continuous.

In general, the converse is not true as we will see in the following examples.

Example 4.1. *Let* $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ *and* $\Phi : X \multimap Y$ *be a DF-multifunction defined by* $G_{\Phi}(x_1, y_1) = 0$, $G_{\Phi}(x_1, y_2) = 0.3$, $G_{\Phi}(x_1, y_3) = 1$, $G_{\Phi}(x_2, y_1) = 0.5$, $G_{\Phi}(x_2, y_2) = 1$, $G_{\Phi}(x_2, y_3) = 0$, $G_{\Phi}(x_3, y_1) = 1$, $G_{\Phi}(x_3, y_2) = 0.2$, $G_{\Phi}(x_3, y_3) = 0.4$. *Define DF-topologies* $\tau, \tau^{\circ}: I^X \to I$, $\sigma, \sigma^{\circ}: I^Y \to I$, *and DF-ideals* $ð_1, ð_1^{\circ}: I^X \to I$, $ð_2, ð_2^{\circ}: I^Y \to I$ *as follows:*

$\tau(\underline{0}) = \tau(\underline{1}) = 1,\ \tau(\underline{0.6}) = \frac{1}{2}$ and $\tau(\lambda) = 0$ otherwise;
$\tau^\circ(\underline{0}) = \tau^\circ(\underline{1}) = 0,\ \tau^\circ(\underline{0.6}) = \frac{1}{2}$ and $\tau^\circ(\lambda) = 1$ otherwise;
$\sigma(\underline{0}) = \sigma(\underline{1}) = 1,\ \sigma(\underline{0.3}) = \sigma(\underline{0.4}) = \frac{1}{4}$ and $\sigma(\lambda) = 0$ otherwise;
$\sigma^\circ(\underline{0}) = \sigma^\circ(\underline{1}) = 0,\ \sigma^\circ(\underline{0.3}) = \sigma^\circ(\underline{0.4}) = \frac{3}{4}$ and $\sigma^\circ(\lambda) = 1$ otherwise;
$\eth_1(\underline{0}) = 1,\ \eth_1(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.3}$ and $\eth_1(\nu) = 0$ otherwise;
$\eth_1^\circ(\underline{0}) = 0,\ \eth_1^\circ(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.3}$ and $\eth_1^\circ(\nu) = 1$ otherwise;
$\eth_2(\underline{0}) = 1,\ \eth_2(\nu) = \frac{1}{3}$ if $\underline{0} < \nu < \underline{0.3}$ and $\eth_2(\nu) = 0$ otherwise;
$\eth_2^\circ(\underline{0}) = 0,\ \eth_2^\circ(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.3}$ and $\eth_2^\circ(\nu) = 1$ otherwise.

Then, $\Phi : (X, \tau, \tau^\circ, \eth_1, \eth_1^\circ) \multimap (Y, \sigma, \sigma^\circ, \eth_2, \eth_2^\circ)$ is DFU and DFL weakly α-ð-continuous but is neither DFU nor DFL almost α-ð-continuous because

$$\underline{0.3} = \Phi^u(\underline{0.3}) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(cl^\circ(\underline{0.3}, \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{1},$$

$$\underline{0.3} = \Phi^l(\underline{0.3}) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\underline{0.3}, \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{1},$$

and

$$\underline{0.4} = \Phi^u(\underline{0.4}) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(cl^\circ(\underline{0.4}, \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{1},$$

$$\underline{0.4} = \Phi^l(\underline{0.4}) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\underline{0.4}, \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{1},$$

but

$$\underline{0.3} = \Phi^u(\underline{0.3}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(I_{\sigma,\sigma^\circ}(cl^\circ(\underline{0.3}, \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{0},$$

$$\underline{0.3} = \Phi^l(\underline{0.3}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\underline{0.3}, \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{0},$$

and

$$\underline{0.4} = \Phi^u(\underline{0.4}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(I_{\sigma,\sigma^\circ}(cl^\circ(\underline{0.4}, \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{0},$$

$$\underline{0.4} = \Phi^l(\underline{0.4}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(I_{\sigma,\sigma^\circ}(cl^\circ(\underline{0.4}, \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{0}.$$

Example 4.2. *Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $\Phi : X \multimap Y$ be a DF-multifunction defined by $G_\Phi(x_1, y_1) = 0.1$, $G_\Phi(x_1, y_2) = 1$, $G_\Phi(x_1, y_3) = 0.3$, $G_\Phi(x_2, y_1) = 0$, $G_\Phi(x_2, y_2) = 0.4$, $G_\Phi(x_2, y_3) = 1$, $G_\Phi(x_3, y_1) = 1$, $G_\Phi(x_3, y_2) = 0$, $G_\Phi(x_3, y_3) = 0.2$. Define DF-topologies τ, τ°: $I^X \to I$, σ, σ°: $I^Y \to I$, and DF-ideals $\eth_1, \eth_1^\circ$: $I^X \to I$, $\eth_2, \eth_2^\circ$: $I^Y \to I$ as follows:*

$\tau(\underline{0}) = \tau(\underline{1}) = 1,\ \tau(\underline{0.7}) = \frac{1}{2}$ and $\tau(\lambda) = 0$ otherwise;
$\tau^\circ(\underline{0}) = \tau^\circ(\underline{1}) = 0,\ \tau^\circ(\underline{0.7}) = \frac{1}{2}$ and $\tau^\circ(\lambda) = 1$ otherwise;
$\sigma(\underline{0}) = \sigma(\underline{1}) = 1,\ \sigma(\underline{0.6}) = \frac{1}{4}$ and $\sigma(\lambda) = 0$ otherwise;
$\sigma^\circ(\underline{0}) = \sigma^\circ(\underline{1}) = 0,\ \sigma^\circ(\underline{0.6}) = \frac{3}{4}$ and $\sigma^\circ(\lambda) = 1$ otherwise;
$\eth_1(\underline{0}) = 1,\ \eth_1(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.3}$ and $\eth_1(\nu) = 0$ otherwise;
$\eth_1^\circ(\underline{0}) = 0,\ \eth_1^\circ(\nu) = \frac{1}{2}$ if $\underline{0} < \nu < \underline{0.3}$ and $\eth_1^\circ(\nu) = 1$ otherwise;
$\eth_2(\underline{0}) = 1,\ \eth_2(\nu) = \frac{1}{4}$ if $\underline{0} < \nu \leq \underline{0.6}$ and $\eth_2(\nu) = 0$ otherwise;
$\eth_2^\circ(\underline{0}) = 0,\ \eth_2^\circ(\nu) = \frac{3}{4}$ if $\underline{0} < \nu \leq \underline{0.6}$ and $\eth_2^\circ(\nu) = 1$ otherwise.

Then, $\Phi : (X, \tau, \tau^\circ, ð_1, ð_1^\circ) \multimap (Y, \sigma, \sigma^\circ, ð_2, ð_2^\circ)$ is DFU and DFL almost α-continuous but is neither DFU nor DFL almost α-ð-continuous because

$$\underline{0.6} = \Phi^u(\underline{0.6}) \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^\circ}(\underline{0.6}, \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{1},$$

$$\underline{0.6} = \Phi^l(\underline{0.6}) \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^l(C_{\sigma,\sigma^\circ}(\underline{0.6}, \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{1},$$

but

$$\underline{0.6} = \Phi^u(\underline{0.6}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(cl^\circ(\underline{0.6}, \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{0},$$

$$\underline{0.6} = \Phi^l(\underline{0.6}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\underline{0.6}, \frac{1}{4}, \frac{3}{4})), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}), \frac{1}{4}, \frac{3}{4}) = \underline{0}.$$

5. DF-almost weakly α-ð-continuous multifunctions

This section presents the notions of DFU and DFL almost weakly α-ð-continuity and discusses its relations with the previous definitions of DFU and DFL α-ð-continuity (almost and weakly).

Definition 5.1. Let $\Phi : (X, \tau, \tau^\circ, ð_1, ð_1^\circ) \multimap (Y, \sigma, \sigma^\circ, ð_2, ð_2^\circ)$ be a DF-multifunction, $p \in I_0$ and $q \in I_1$. Then, Φ is as follows:

(1) DFU almost weakly α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(\Phi) \leq C_{\tau,\tau^\circ}(\Phi^u(cl^\circ(\mu, p, q)), p, q)$.

(2) DFL almost weakly α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^l(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq C_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\mu, p, q)), p, q)$.

(3) DFU almost weakly α-ð-continuous (DFL almost weakly α-ð-continuous) iff it is DFU almost weakly α-ð-continuous (DFL almost weakly α-ð-continuous) at every fuzzy point $x_t \in dom(\Phi)$.

If we take $cl^\circ = C_{\tau,\tau^\circ}$, then we have the definition of DF-almost weakly α-continuous.

Remark 5.1. If Φ is a normalized multifunction, then Φ is DFU almost weakly α-ð-continuous at a fuzzy point $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists a (p, q)-fuzzy α-ð-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq C_{\tau,\tau^\circ}(\Phi^u(cl^\circ(\mu, p, q)), p, q)$.

Theorem 5.1. *For a DF-multifunction $\Phi : (X, \tau, \tau^\circ, ð_1, ð_1^\circ) \multimap (Y, \sigma, \sigma^\circ, ð_2, ð_2^\circ)$, $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$, the following statements are equivalent:*

(1) Φ is DFL almost weakly α-ð-continuous.

(2) $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\mu, p, q)), p, q), p, q), p, q), p, q)$ if $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$.

(3) $C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(int^\circ(\mu, p, q)), p, q), p, q), p, q), p, q) \leq \Phi^u(\mu)$ if $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$.

(4) $\Phi^l(I_{\sigma,\sigma^\circ}(\mu, p, q)) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^\circ(I_{\sigma,\sigma^\circ}(\mu, p, q), p, q)), p, q), p, q), p, q), p, q)$.

Proof. (1) $\Longrightarrow$ (2) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, there exists $\lambda \in I^X$ with $\lambda \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\lambda, p, q), p, q), p, q)$ and $x_t \in \lambda$ such that $\lambda \leq C_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\mu, p, q)), p, q)$.

Thus, $x_t \in \lambda \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\mu,p,q)),p,q),p,q),p,q),p,q)$, and hence $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\mu,p,q)),p,q),p,q),p,q),p,q)$.

(2) $\Longrightarrow$ (3) Let $\mu \in I^Y$ with $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$. Then, by (2) ,

$$\begin{aligned}[\Phi^u(\mu)]^c &= \Phi^l(\mu^c) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\mu^c,p,q)),p,q),p,q),p,q),p,q)\\ &= [C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(int^\circ(\mu,p,q)),p,q),p,q),p,q),p,q)]^c .\end{aligned}$$

Thus, $C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(int^\circ(\mu,p,q)),p,q),p,q),p,q),p,q) \leq \Phi^u(\mu)$.

(3) $\Longrightarrow$ (4) Since $C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(int^\circ(C_{\sigma,\sigma^\circ}(\mu,p,q),p,q)),p,q),p,q),p,q),p,q) \leq \Phi^u(C_{\sigma,\sigma^\circ}(\mu,p,q))$ for each $\mu \in I^Y$, it follows that

$$\Phi^l(I_{\sigma,\sigma^\circ}(\mu,p,q)) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^\circ(I_{\sigma,\sigma^\circ}(\mu,p,q),p,q)),p,q),p,q),p,q),p,q).$$

(4) $\Longrightarrow$ (1) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, by (4) and $\mu = I_{\sigma,\sigma^\circ}(\mu,p,q)$, we have that $x_t \in \Phi^l(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\mu,p,q)),p,q),p,q),p,q),p,q)$.

Thus, Φ is DFL almost weakly α-ð-continuous. □

The following theorem is proved similarly as in the case of Theorem 5.1.

Theorem 5.2. *For a normalized DF-multifunction* $\Phi : (X,\tau,\tau^\circ,ð_1,ð_1^\circ) \multimap (Y,\sigma,\sigma^\circ,ð_2,ð_2^\circ)$, $\mu \in I^Y$, $p \in I_0$ *and* $q \in I_1$, *the following statements are equivalent:*

(1) Φ is DFU almost weakly α-ð-continuous.
(2) $\Phi^u(\mu) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^u(cl^\circ(\mu,p,q)),p,q),p,q),p,q),p,q)$ if $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$.
(3) $C_{\tau,\tau^\circ}(int^\circ(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^l(int^\circ(\mu,p,q)),p,q),p,q),p,q),p,q) \leq \Phi^l(\mu)$ if $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$.
(4) $\Phi^u(I_{\sigma,\sigma^\circ}(\mu,p,q)) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^u(cl^\circ(I_{\sigma,\sigma^\circ}(\mu,p,q),p,q)),p,q),p,q),p,q),p,q)$.

Remark 5.2. DFU (resp. DFL) weakly α-ð-continuous $\Rightarrow$ DFU (resp. DFL) almost weakly α-ð-continuous.

In general, the converse is not true as we will show in the following example.

Example 5.1. *Let* $X = \{x_1,x_2,x_3\}$, $Y = \{y_1,y_2,y_3\}$ *and* $\Phi : X \multimap Y$ *be a DF-multifunction defined by* $G_\Phi(x_1,y_1) = 0.2$, $G_\Phi(x_1,y_2) = 0.3$, $G_\Phi(x_1,y_3) = 1$, $G_\Phi(x_2,y_1) = 1$, $G_\Phi(x_2,y_2) = 0.4$, $G_\Phi(x_2,y_3) = 0$, $G_\Phi(x_3,y_1) = 0.3$, $G_\Phi(x_3,y_2) = 1$, $G_\Phi(x_3,y_3) = 0.2$. *Define DF-topologies* $\tau,\tau^\circ: I^X \to I$, $\sigma,\sigma^\circ: I^Y \to I$, *and DF-ideals* $ð_1,ð_1^\circ: I^X \to I$, $ð_2,ð_2^\circ: I^Y \to I$ *as follows:*

$\tau(\underline{0}) = \tau(\underline{1}) = 1$, $\tau(\underline{0.8}) = \frac{1}{3}$ and $\tau(\lambda) = 0$ otherwise;
$\tau^\circ(\underline{0}) = \tau^\circ(\underline{1}) = 0$, $\tau^\circ(\underline{0.8}) = \frac{2}{3}$ and $\tau^\circ(\lambda) = 1$ otherwise;
$\sigma(\underline{0}) = \sigma(\underline{1}) = 1$, $\sigma(\underline{0.6}) = \frac{2}{3}$ and $\sigma(\lambda) = 0$ otherwise;
$\sigma^\circ(\underline{0}) = \sigma^\circ(\underline{1}) = 0$, $\sigma^\circ(\underline{0.6}) = \frac{1}{3}$ and $\sigma^\circ(\lambda) = 1$ otherwise;
$ð_1(\underline{0}) = 1$, $ð_1(\nu) = \frac{1}{4}$ if $\underline{0} < \nu < \underline{0.4}$ and $ð_1(\nu) = 0$ otherwise;
$ð_1^\circ(\underline{0}) = 0$, $ð_1^\circ(\nu) = \frac{3}{4}$ if $\underline{0} < \nu < \underline{0.4}$ and $ð_1^\circ(\nu) = 1$ otherwise;
$ð_2(\underline{0}) = 1$, $ð_2(\nu) = \frac{1}{4}$ if $\underline{0} < \nu < \underline{0.7}$ and $ð_2(\nu) = 0$ otherwise;
$ð_2^\circ(\underline{0}) = 0$, $ð_2^\circ(\nu) = \frac{3}{4}$ if $\underline{0} < \nu < \underline{0.7}$ and $ð_2^\circ(\nu) = 1$ otherwise.

Then, $\Phi : (X,\tau,\tau^\circ,\eth_1,\eth_1^\circ) \multimap (Y,\sigma,\sigma^\circ,\eth_2,\eth_2^\circ)$ is DFU and DFL almost weakly α-ð-continuous but is not DFU or DFL weakly α-ð-continuous because

$$\underline{0.6} = \Phi^u(\underline{0.6}) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^u(cl^\circ(\underline{0.6},\frac{1}{4},\frac{3}{4})),\frac{1}{4},\frac{3}{4}),\frac{1}{4},\frac{3}{4}),\frac{1}{4},\frac{3}{4}),\frac{1}{4},\frac{3}{4}) = \underline{1},$$

$$\underline{0.6} = \Phi^l(\underline{0.6}) \leq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\underline{0.6},\frac{1}{4},\frac{3}{4})),\frac{1}{4},\frac{3}{4}),\frac{1}{4},\frac{3}{4}),\frac{1}{4},\frac{3}{4}),\frac{1}{4},\frac{3}{4}) = \underline{1},$$

but

$$\underline{0.6} = \Phi^u(\underline{0.6}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^u(cl^\circ(\underline{0.6},\frac{1}{4},\frac{3}{4})),\frac{1}{4},\frac{3}{4}),\frac{1}{4},\frac{3}{4}),\frac{1}{4},\frac{3}{4}) = \underline{0},$$

$$\underline{0.6} = \Phi^l(\underline{0.6}) \nleq I_{\tau,\tau^\circ}(cl^\circ(I_{\tau,\tau^\circ}(\Phi^l(cl^\circ(\underline{0.6},\frac{1}{4},\frac{3}{4})),\frac{1}{4},\frac{3}{4}),\frac{1}{4},\frac{3}{4}),\frac{1}{4},\frac{3}{4}) = \underline{0}.$$

6. *DF* $(\star,\bullet,\triangle,\triangledown,(\eth,\eth^\circ))$-continuous multifunctions

Definition 6.1. (1) Let $\Phi : (X,\tau,\tau^\circ,\eth,\eth^\circ) \multimap (Y,\sigma,\sigma^\circ)$ be a DF-multifunction. Then, Φ is DFL $(\star,\bullet,\triangle,\triangledown,(\eth,\eth^\circ))$-continuous iff for every $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$,

$$\eth[\star(\Phi^l\,(\triangledown(\mu,p,q)),p,q) \;\;\overline{\wedge}\;\; \bullet(\Phi^l\,(\triangle(\mu,p,q)),p,q)] \geq \sigma(\mu),$$
$$\eth^\circ[\star(\Phi^l\,(\triangledown(\mu,p,q)),p,q) \;\;\overline{\wedge}\;\; \bullet(\Phi^l\,(\triangle(\mu,p,q)),p,q)] \leq \sigma^\circ(\mu).$$

(2) Let $\Phi : (X,\tau,\tau^\circ,\eth_1,\eth_1^\circ) \multimap (Y,\sigma,\sigma^\circ,\eth_2,\eth_2^\circ)$ be a normalized DF-multifunction. Then, Φ is DFU $(\star,\bullet,\triangle,\triangledown,(\eth,\eth^\circ))$ -continuous iff for every $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$,

$$\eth[\star(\Phi^u\,(\triangledown(\mu,p,q)),p,q) \;\;\overline{\wedge}\;\; \bullet(\Phi^u\,(\triangle(\mu,p,q)),p,q)] \geq \sigma(\mu),$$
$$\eth^\circ[\star(\Phi^u\,(\triangledown(\mu,p,q)),p,q) \;\;\overline{\wedge}\;\; \bullet(\Phi^u\,(\triangle(\mu,p,q)),p,q)] \leq \sigma^\circ(\mu).$$

We can see that the above definition generalizes the concept of DFL (resp. DFU) semi-continuous, when we choose $\star$ = identity operator, $\bullet$ = interior operator, $\triangle$ = identity operator, $\triangledown$ = identity operator and $(\eth,\eth^\circ)=\left(\eth^0,\eth^{\circ 0}\right)$. Because, if we supposed that there exist $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$ such that $\tau\left(\Phi^l(\mu)\right) < p \leq \sigma(\mu)$ and $\tau^\circ\left(\Phi^l(\mu)\right) > q \geq \sigma^\circ(\mu)$, and since

$$\eth^0[(\Phi^l(\mu) \,\overline{\wedge}\, I_{\tau,\tau^\circ}(\Phi^l(\mu),p,q)] \geq \sigma(\mu),\ \eth^{\circ 0}[(\Phi^l(\mu) \,\overline{\wedge}\, I_{\tau,\tau^\circ}(\Phi^l(\mu),p,q)] \leq \sigma^\circ(\mu)$$

for every $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$, it follows that $\Phi^l(\mu) \,\overline{\wedge}\, I_{\tau,\tau^\circ}(\Phi^l(\mu),p,q) = \underline{0}$; hence $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(\Phi^l(\mu),p,q)$, and it follows that $\tau\left(\Phi^l(\mu)\right) \geq p$ and $\tau^\circ\left(\Phi^l(\mu)\right) \leq q$, which is a contradiction. Then, $\tau\left(\Phi^l(\mu)\right) \geq \sigma(\mu)$ and $\tau^\circ\left(\Phi^l(\mu)\right) \leq \sigma^\circ(\mu)$. So, Φ is DFL semi-continuous. The other case is similarly proved.

Remark 6.1. (1) DFL (resp. normalized DFU) almost continuous multifunction $\Leftrightarrow$ DFL (resp. normalized DFU) $(id_X, I_{\tau,\tau^\circ}, I_{\sigma,\sigma^\circ}(C_{\sigma,\sigma^\circ}), id_Y, \left(\eth^0,\eth^{\circ 0}\right))$-continuous multifunction.

(2) DFL (resp. normalized DFU) weakly continuous multifunction $\Leftrightarrow$ DFL (resp. normalized DFU) $(id_X, I_{\tau,\tau^\circ}, C_{\sigma,\sigma^\circ}, id_Y, \left(\eth^0,\eth^{\circ 0}\right))$-continuous multifunction.

(3) DFL (resp. normalized DFU) almost weakly continuous multifunction $\Leftrightarrow$ DFL (resp. normalized DFU) $(id_X, I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}), C_{\sigma,\sigma^\circ}, id_Y, \left(\eth^0,\eth^{\circ 0}\right))$-continuous multifunction.

(4) DFL (resp. normalized DFU) ð-continuous multifunction ⇔ DFL (resp. normalized DFU) (id_X, $I_{\tau,\tau^\circ}(\Psi_{\tau,\tau^\circ})$, I_{σ,σ°, id_Y, $\left(ð^0, ð^{\circ 0}\right)$)-continuous multifunction.

(5) DFL (resp. normalized DFU) almost ð-continuous multifunction ⇔ DFL (resp. normalized DFU) (id_X, I_{τ,τ°, $I_{\sigma,\sigma^\circ}\left(cl^*_{\sigma,\sigma^\circ}\right)$, id_Y, $\left(ð^0, ð^{\circ 0}\right)$)-continuous multifunction.

(6) DFL (resp. normalized DFU) weakly ð-continuous multifunction ⇔ DFL (resp. normalized DFU) (id_X, I_{τ,τ°, $cl^*_{\sigma,\sigma^\circ}$, id_Y , $\left(ð^0, ð^{\circ 0}\right)$)-continuous multifunction.

(7) DFL (resp. normalized DFU) almost weakly ð-continuous multifunction ⇔ DFL (resp. normalized DFU) (id_X, $I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ})$, $cl^*_{\sigma,\sigma^\circ}$, id_Y, $\left(ð^0, ð^{\circ 0}\right)$)-continuous multifunction.

(8) DFL (resp. normalized DFU) α-continuous multifunction ⇔ DFL (resp. normalized DFU) (id_X, $I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}))$, I_{σ,σ°, id_Y, $\left(ð^0, ð^{\circ 0}\right)$)-continuous multifunction.

(9) DFL (resp. normalized DFU) α-ð-continuous multifunction ⇔ DFL (resp. normalized DFU) (id_X, $I_{\tau,\tau^\circ}(cl^*_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}))$, I_{σ,σ°, id_Y, $\left(ð^0, ð^{\circ 0}\right)$)-continuous multifunction.

(10) DFL (resp. normalized DFU) almost α-ð-continuous multifunction ⇔ DFL (resp. normalized DFU) (id_X, $I_{\tau,\tau^\circ}(cl^*_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}))$, $I_{\sigma,\sigma^\circ}\left(cl^*_{\sigma,\sigma^\circ}\right)$, id_Y, $\left(ð^0, ð^{\circ 0}\right)$)-continuous multifunction.

(11) DFL (resp. normalized DFU) weakly α-ð-continuous multifunction ⇔ DFL (resp. normalized DFU) (id_X, $I_{\tau,\tau^\circ}(cl^*_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}))$, $cl^*_{\sigma,\sigma^\circ}$, id_Y, $\left(ð^0, ð^{\circ 0}\right)$)-continuous multifunction.

(12) DFL (resp. normalized DFU) almost weakly α-ð- continuous multifunction ⇔ DFL (resp. normalized DFU) (id_X, $I_{\tau,\tau^\circ}(cl^*_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ})))$, $cl^*_{\sigma,\sigma^\circ}$, id_Y, $\left(ð^0, ð^{\circ 0}\right)$)-continuous multifunction.

Definition 6.2. Let $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ)$ be a DF-multifunction (resp. normalized DF-multifunction). Then, Φ is DFL $\mathbb{k}$-continuous (resp. DFU $\mathbb{k}$-continuous) iff $\tau\left(\Phi^l(\mu)\right) \geq \sigma(\mu)$ and $\tau^\circ\left(\Phi^l(\mu)\right) \leq \sigma^\circ(\mu)$.
(resp. $\tau(\Phi^u(\mu)) \geq \sigma(\mu)$ and $\tau^\circ(\Phi^u(\mu)) \leq \sigma^\circ(\mu)$) for each $\mu \in I^Y$ that satisfies property $\mathbb{k}$.

Let $C_{\mathbb{k}} : I^Y \times I_0 \times I_1 \to I^Y$ be an operator on $(Y, \sigma, \sigma^\circ)$ defined as follows:

$$C_{\mathbb{k}}(\mu, p, q) = \begin{cases} \mu, & \text{if } \mu \text{ satisfies property } \mathbb{k} \text{ with } \sigma(\mu) \geq p,\ \sigma^\circ(\mu) \leq q,\ p \in I_0 \text{ and } s \in I_1 \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 6.1. (1) *Let $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ)$ be a DF-multifunction. Then, Φ is DFL $\mathbb{k}$ -continuous iff it is DFL (id_X, I_{τ,τ° , $C_{\mathbb{k}}$, id_Y, $(ð^0, ð^{0\circ})$)-continuous.*

(2) *Let $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ)$ be a normalized DF-multifunction. Then, Φ is DFU $\mathbb{k}$-continuous iff it is DFU (id_X, I_{τ,τ°, $C_{\mathbb{k}}$, id_Y, $(ð^0, ð^{0\circ})$)-continuous.*

Proof. (1) (⇒) Suppose that Φ is fuzzy lower $\mathbb{k}$-continuous and $\mu \in I^Y$.
Case 1. If μ satisfies property $\mathbb{k}$ with $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$, $C_{\mathbb{k}}(\mu, p, q) = \mu$, $\tau\left(\Phi^l(\mu)\right) \geq p$ and $\tau^\circ\left(\Phi^l(\mu)\right) \leq q$. Thus, we obtain that $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(\Phi^l(\mu), p, q) = I_{\tau,\tau^\circ}(\Phi^l(C_{\mathbb{k}}(\mu, p, q)), p, q)$. Then, $\Phi^l(\mu) \barwedge I_{\tau,\tau^\circ}(\Phi^l(C_{\mathbb{k}}(\mu, p, q)), p, q) = \underline{0}$. Hence,

$$ð^0[\Phi^l(\mu) \ \barwedge \ I_{\tau,\tau^\circ}(\Phi^l(C_{\mathbb{k}}(\mu, p, q)), p, q)] \geq \sigma(\mu),$$
$$ð^{\circ 0}[\Phi^l(\mu) \ \barwedge \ I_{\tau,\tau^\circ}(\Phi^l(C_{\mathbb{k}}(\mu, p, q)), p, q)] \leq \sigma^\circ(\mu).$$

Case 2. If μ does not satisfy property $\mathbb{k}$, $C_{\mathbb{k}}(\mu, p, q) = \underline{1}$. Thus, we obtain that
$\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(\Phi^l(\underline{1}), p, q) = I_{\tau,\tau^\circ}(\Phi^l(C_{\mathbb{k}}(\mu, p, q)), p, q)$. Then, $\Phi^l(\mu) \barwedge I_{\tau,\tau^\circ}(\Phi^l(C_{\mathbb{k}}(\mu, p, q)), p, q) = \underline{0}$. Hence,

$$ð^0[\Phi^l(\mu) \ \barwedge \ I_{\tau,\tau^\circ}(\Phi^l(C_{\mathbb{k}}(\mu, p, q)), p, q)] \geq \sigma(\mu),$$

$$\eth^{\circ 0}[\Phi^l(\mu) \quad \overline{\wedge} \quad I_{\tau,\tau^\circ}(\Phi^l(C_{\Bbbk}(\mu,p,q)),p,q)] \leq \sigma^\circ(\mu).$$

Then, Φ is DFL $(id_X, I_{\tau,\tau^\circ}, C_{\Bbbk}, id_Y, (\eth^0, \eth^{0*}))$-continuous.

($\Leftarrow$) Suppose that there exists $\mu \in I^Y$ such that $\tau\left(\Phi^l(\mu)\right) \ngeq \sigma(\mu)$ and $\tau^\circ\left(\Phi^l(\mu)\right) \nleq \sigma^\circ(\mu)$. There exists $p \in I_0$ and $q \in I_1$ such that $\tau\left(\Phi^l(\mu)\right) < p \leq \sigma(\mu)$ and $\tau^\circ\left(\Phi^l(\mu)\right) > q \geq \sigma^\circ(\mu)$. Since

$$\eth^0[\Phi^l(\mu) \quad \overline{\wedge} \quad I_{\tau,\tau^\circ}(\Phi^l(C_{\Bbbk}(\mu,p,q)),p,q)] \geq \sigma(\mu) \text{ and}$$
$$\eth^{\circ 0}[\Phi^l(\mu) \quad \overline{\wedge} \quad I_{\tau,\tau^\circ}(\Phi^l(C_{\Bbbk}(\mu,p,q)),p,q)] \leq \sigma^\circ(\mu),$$

it follows that $\Phi^l(\mu) \,\overline{\wedge}\, I_{\tau,\tau^\circ}(\Phi^l(C_{\Bbbk}(\mu,p,q)),p,q) = \underline{0}$ and $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(\Phi^l(C_{\Bbbk}(\mu,p,q)),p,q)$ for each $\mu \in I^Y$.

If μ satisfies property $\Bbbk$ with $\sigma(\mu) \geq r$, $\sigma^\circ(\mu) \leq s$, $C_{\Bbbk}(\mu,p,q) = \mu$, and hence $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(\Phi^l(\mu),p,q)$, then $\tau\left(\Phi^l(\mu)\right) \geq p$ and $\tau^\circ\left(\Phi^l(\mu)\right) \leq q$, which is a contradiction. Then, $\tau\left(\Phi^l(\mu)\right) \geq \sigma(\mu)$, $\tau^\circ\left(\Phi^l(\mu)\right) \leq \sigma^\circ(\mu)$, and hence Φ is DFL $\Bbbk$-continuous.

(2) It is similar to (1). □

Definition 6.3. Let $\star$ and $\bullet$ be fuzzy operators on (X,τ,τ°). Then, $\star \sqsubseteq \bullet$ iff $\star(\lambda,p,q) \leq \bullet(\lambda,p,q)$ $\forall \lambda \in I^X$ and $p \in I_0$, $q \in I_1$. Also, an operator $\star$ on X is called monotonic if $\lambda \leq \nu$; then, $\star(\lambda,p,q) \leq \star(\nu,p,q)$.

Theorem 6.2. (1) *Let* $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ)$ *be a DF-multifunction and* $(\eth,\eth^\circ)$ *be a DF ideal on X. Let* $\star$, $\bullet$ *be fuzzy operators on* (X,τ,τ°), $\bullet$ *be monotonic and* $\triangle$, $\triangle'$, $\triangledown$ *be fuzzy operators on* (Y,σ,σ°) *with* $\triangle \sqsubseteq \triangle'$. *If* Φ *is DFL* $(\star,\bullet,\triangle,\triangledown,(\eth,\eth^\circ))$*-continuous, then it is DFL* $(\star,\bullet,\triangle',\triangledown,(\eth,\eth^\circ))$*-continuous.*

(2) *Let* $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ)$ *be a normalized DF-multifunction and* $(\eth,\eth^\circ)$ *be a DF ideal on X. Let* $\star$, $\bullet$ *be fuzzy operators on* (X,τ,τ°) *and* $\triangle$, $\triangle'$, $\triangledown$ *are fuzzy operators on* (Y,σ,σ°) *with* $\triangle \sqsubseteq \triangle'$. *If* Φ *is DFU* $(\star,\bullet,\triangle,\triangledown,(\eth,\eth^\circ))$)*-continuous, then it is DFU* $(\star,\bullet,\triangle',\triangledown,(\eth,\eth^\circ))$)*-continuous.*

Proof. (1) If Φ is DFL $(\star,\bullet,\triangle,\triangledown,(\eth,\eth^\circ))$-continuous,

$$\eth[\star(\Phi^l(\triangledown(\mu,p,q)),p,q) \quad \overline{\wedge} \quad \bullet(\Phi^l(\triangle(\mu,p,q)),p,q)] \geq \sigma(\mu),$$
$$\eth^\circ[\star(\Phi^l(\triangledown(\mu,p,q)),p,q) \quad \overline{\wedge} \quad \bullet(\Phi^l(\triangle(\mu,p,q)),p,q)] \leq \sigma^\circ(\mu).$$

Since $\triangle \sqsubseteq \triangle'$, for every $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$,

$$\bullet(\Phi^l(\triangle(\mu,p,q)),p,q) \leq \bullet(\Phi^l(\triangle'(\mu,p,q)),p,q). \text{ Therefore,}$$

$$\begin{aligned}\star(\Phi^l(\triangledown(\mu,p,q)),p,q) \quad &\overline{\wedge} \quad \bullet(\Phi^l(\triangle'(\mu,p,q)),p,q)\\ &\leq \quad \star(\Phi^l(\triangledown(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^l(\triangle(\mu,p,q)),p,q).\end{aligned}$$

Thus,

$$\begin{aligned}\eth[\star(\Phi^l(\triangledown(\mu,p,q)),p,q) \quad &\overline{\wedge} \quad \bullet(\Phi^l\left(\triangle^{\bigstar}(\mu,p,q)\right),p,q)]\\ &\geq \quad \eth[\star(\Phi^l(\triangledown(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^l(\triangle(\mu,p,q)),p,q)] \geq \sigma(\mu),\end{aligned}$$

$$\eth^\circ[\star(\Phi^l(\triangledown(\mu,p,q)),p,q) \quad \overline{\wedge} \quad \bullet(\Phi^l\left(\triangle^{\bigstar}(\mu,p,q)\right),p,q)]$$

$$\leq \eth^\circ[\star(\Phi^l(\nabla(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^l(\triangle(\mu,p,q)),p,q)]$$
$$\leq \sigma^\circ(\mu).$$

Then, Φ is DFU $(\star, \bullet, \triangle', \nabla, (\eth, \eth^\circ))$-continuous.

(2) It is similar to (1). □

Definition 6.4. A fuzzy operator $\bullet$ on (X, τ, τ°) induces another operator $I_{\tau,\tau^\circ}(\bullet)$ defined as follows:

$I_{\tau,\tau^\circ}(\bullet)(\lambda, p, q) = I_{\tau,\tau^\circ}(\bullet(\lambda, p, q), p, q)$, $\forall \lambda \in I^X$. Observe that $I_{\tau,\tau^\circ}(\bullet) \sqsubseteq \bullet$.

Theorem 6.3. *Let* $\star$, $\bullet$ *be fuzzy operators on* (X, τ, τ°), $\triangle, \nabla$ *be fuzzy operators on* $(Y, \sigma, \sigma^\circ)$, *and* $(\eth, \eth^\circ)$ *be a proper DF ideal on X. If* $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ)$ *is a DFU (resp. DFL)* $(\star, \bullet, \triangle, \nabla, (\eth, \eth^\circ))$*-continuous multifunction and*

$$\bullet[\Phi^u(\mu), p, q] \leq \bullet[\Phi^u(I_{\sigma,\sigma^\circ}(\mu, p, q)), p, q] \text{ (resp. } \bullet[\Phi^l(\mu), p, q] \leq \bullet[\Phi^l(I_{\sigma,\sigma^\circ}(\mu, p, q)), p, q])$$

for each $\mu \in I^Y$, $p \in I_0$ *and* $q \in I_1$. *Then,* Φ *is DFU (resp. lower)* $(\star, \bullet, I_{\sigma,\sigma^\circ}(\triangle), \nabla, (\eth, \eth^\circ))$*-continuous.*

Proof. If $\bullet(\Phi^u(\triangle(\mu, p, q)), p, q) \leq \bullet(\Phi^u(I_{\sigma,\sigma^\circ}(\triangle(\mu, p, q), p, q)), p, q)$, then

$$\star(\Phi^u(\nabla(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^u(I_{\sigma,\sigma^\circ}(\triangle(\mu,p,q),p,q)),p,q)$$
$$\leq \star(\Phi^u(\nabla(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^u(\triangle(\mu,p,q)),p,q).$$

Since Φ is DFU $(\star, \bullet, \triangle, \nabla, (\eth, \eth^\circ))$-continuous,

$$\eth[\star(\Phi^u(\nabla(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^u(\triangle(\mu,p,q)),p,q)] \geq \sigma(\mu) \text{ and}$$
$$\eth^\circ[\star(\Phi^u(\nabla(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^u(\triangle(\mu,p,q)),p,q)] \leq \sigma^\circ(\mu).$$

Hence,

$$\eth[\star(\Phi^u(\nabla(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^u(I_{\sigma,\sigma^\circ}(\triangle(\mu,p,q),p,q)),p,q)]$$
$$\geq \eth[\star(\Phi^u(\nabla(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^u(\triangle(\mu,p,q)),p,q)] \geq \sigma(\mu),$$

and

$$\eth^\circ[\star(\Phi^u(\nabla(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^u(I_{\sigma,\sigma^\circ}(\triangle(\mu,p,q),p,q)),p,q)]$$
$$\leq \eth^\circ[\star(\Phi^u(\nabla(\mu,p,q)),p,q) \,\overline{\wedge}\, \bullet(\Phi^u(\triangle(\mu,p,q)),p,q)]$$
$$\leq \sigma^\circ(\mu). \text{ Then,}$$

Φ is DFU $(\star, \bullet, I_{\sigma,\sigma^\circ}(\triangle), \nabla, (\eth, \eth^\circ))$ -continuous. The other case is similarly proved. □

Definition 6.5. Let (X, τ, τ°) be a DF-topological space, $\lambda \in I^X$, $p \in I_0$, and $q \in I_1$. Then, λ is called (p,q)-fuzzy $\triangle$-compact iff for every family $\{\mu_i \in I^X : \tau(\mu_i) \geq p, \tau^\circ(\mu_i) \leq q;\ i \in J\}$ such that $\lambda \leq \bigvee_{i \in J} \mu_i$, there exists a finite subset J_0 of J such that $\lambda \leq \bigvee_{i \in J_0} \triangle(\mu_i, p, q)$.

Theorem 6.4. *Let* $\Phi : X \multimap Y$ *be a crisp DFU* $(\star, I_{\tau,\tau^\circ}, \triangle, \nabla, (\eth^0, \eth^{\circ 0}))$*-continuous multifunction encompssing compact valued between DF-topological spaces* (X, τ, τ°), $(Y, \sigma, \sigma^\circ)$ *with* $\lambda \leq \star(\lambda, p, q)$ *and*
$\mu \leq \nabla(\mu, p, q)$ $\forall \mu \in I^X$. *Then,* $\Phi(\lambda)$ *is* (p,q)*-fuzzy* $\triangle$*-compact if* λ *is* (p,q)*-fuzzy compact.*

Proof. Let $\{\mu_i \in I^Y : \sigma(\mu_i) \geq p, \sigma^\circ(\mu_i) \leq q; i \in J\}$ and $\Phi(\lambda) \leq \bigvee_{i\in J} \mu_i$. Since $\lambda = \bigvee_{x_t\in\lambda} x_t$, then $\Phi(\lambda) = \Phi(\bigvee_{x_t\in\lambda} x_t) = \bigvee_{x_t\in\lambda} \Phi(x_t) \leq \bigvee_{i\in J} \mu_i$, that is, for each $x_t \in \lambda$, $\Phi(x_t) \leq \bigvee_{i\in J} \mu_i$. Since Φ is compact valued, there exists a finite subset J_{x_t} of J such that $\Phi(x_t) \leq \bigvee_{n\in J_{x_t}} \mu_n = \mu_{x_t}$; thus, we get that $x_t \leq \Phi^u(\Phi(x_t)) \leq \bigvee_{x_t\in\lambda} \Phi^u(\mu_{x_t})$ and $\lambda = \bigvee_{x_t\in\lambda} x_t \leq \bigvee_{x_t\in\lambda} \Phi^u(\mu_{x_t})$. Since Φ is a DFU $(\star, I_{\tau,\tau^\circ}, \triangle, \nabla, (\eth^0, \eth^{\circ 0}))$-continuous multifunction, $\Phi^u(\mu) \leq \star(\Phi^u(\nabla(\mu, p, q)), p, q) \leq I_{\tau,\tau^\circ}(\Phi^u(\triangle(\mu, p, q)), p, q) \leq \Phi^u(\triangle(\mu, p, q))$. Thus, $\lambda \leq \bigvee_{x_t\in\lambda} I_{\tau,\tau^\circ}(\Phi^u(\triangle(\mu_{x_t}, p, q)), p, q)$. Since λ is a (p, q)-fuzzy compact set, then there exists a finite index set M of J_{x_t} such that

$$\lambda \leq \bigvee_{m\in M} I_{\tau,\tau^\circ}(\Phi^u(\triangle(\mu_{x_{(t_m)}}, p, q)), p, q) \leq \bigvee_{m\in M} \Phi^u(\triangle(\mu_{x_{(t_m)}}, p, q)).$$

It follows that

$$\phi(\lambda) \leq \Phi(\bigvee_{m\in M} \Phi^u(\triangle(\mu_{x_{(t_m)}}, p, q))) = \bigvee_{m\in M} \Phi(\Phi^u(\triangle(\mu_{x_{(t_m)}}, p, q))) \leq \bigvee_{m\in M} \triangle(\mu_{x_{(t_m)}}, p, q).$$

Hence, $\Phi(\lambda)$ is a (p, q)-fuzzy $\triangle$-compact set. □

7. Conclusions

This article investigated DF-multifunctions based on a DF-ideal and analyzed their usual properties. Also, we have submitted new types of DF-continuity based on a DF-ideal, studied the common properties of continuity and discussed the implications associated with these new types of continuity. Some examples have been presented to explain that these implications may be not reversed. The use of DF-ideals in defining these new types of continuity extended the usual corresponding definitions of fuzzy continuity; thus, the introduced types of DF-continuity are extensions of the corresponding usual ones.

The conclusion regarding the resulting variety of continuous DF-multifunctions based on a DF-ideal and the associated implications is that these types of continuity constitute an extension and is meaningful more compared with the corresponding previous types of fuzzy continuity.

Author contributions

Abdallah and Abu_Shugair: Methodology and Funding; Abbas and Abu_Shugair: Validation and Formal analysis; Ibedou and Abbas: Investigation; Abu_Shugair and Ibedou: Writing-original draft; Ibedou and Abbas: Reviewing the final form; Ibedou: Visualization. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of the paper.

References

1. S. E. Abbas, A. H. Zakaria, Lattice valued double fuzzy ideal structure, *J. Fuzzy Math.*, **20** (2012), 899–910.
2. S. E. Abbas, M. A. Hebeshi, I. M. Taha, On fuzzy upper and lower semi-continuous multifunctions, *J. Fuzzy Math.*, **22** (2014), 951–962.
3. S. E. Abbas, M. A. Hebeshi, I. M. Taha, On upper and lower almost weakly continuous fuzzy multifunctions, *Iran. J. Fuzzy Syst.*, **12** (2015), 101–114.
4. S. E. Abbas, M. A. Hebeshi, I. M. Taha, On upper and lower contra-continuous fuzzy multifunctions, *Punjab Uni. J. Math.*, **47** (2015), 105–117.
5. S. E. Abbas, I. Ibedou, Fuzzy topological concepts via ideals and grills, *Ann. Fuzzy Math. Inform.*, **15** (2018), 137–148.
6. M. N. A. Shugair, A. A. Abdallah1, S. E. Abbas, E. El-Sanowsy, I. Ibedou, Double fuzzy ideal multifunctions, *Mathematics*, **2024** (2024), 1128.
7. J. Albrycht, M. Maltoka, On fuzzy multivalued functions, *Fuzzy Set. Syst.*, **12** (1984), 61–69.
8. K. T. Atanassov, New operations defined over the intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **61** (1994), 137–142.
9. C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, **24** (1968), 182–190.
10. D. Coker, An introduction to intuitionistic fuzzy topological spaces, *Fuzzy Set. Syst.*, **88** (1997), 81–89.
11. J. G. Garcia, S. E. Rodabaugh, Order-theoretic, topological, categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued intuitionistic sets, intuitionistic fuzzy sets and topologies, *Fuzzy Set. Syst.*, **156** (2005), 445–484.
12. A. Ghareeb, Normality of double fuzzy topological spaces, *Appl. Math. Lett.*, **24** (2011), 533–540.
13. M. A. Hebeshi, I. M. Taha, On upper and lower α-continuous fuzzy multifunctions, *J. Intell. Fuzzy Syst.*, **25** (2015), 2537–2546.
14. T. Kubiak, *On fuzzy topologies*, Ph.D. dissertation, Adam Mickiewicz University, Poznan, Poland, 1985.
15. F. M. Mohammed, M. S. M. Noorani, A. Ghareeb, Somewhat slightly generalized double fuzzy semi continuous functions, *Int. J. Math. Math. Sci.*, **2014** (2014), 756376.

16. T. K. Mondal, S. K. Samanta, On intuitionistic gradation of openness, *Fuzzy Set. Syst.*, **131** (2002), 323–336.

17. N. Rajesh, M. Vanishree, B. Brundha, Double fuzzy contra-continuous multifunctions, *Malaya J. Mat.*, **7** (2019), 862–866.

18. A. P. Sostak, *On a fuzzy topological structure*, In: Proceedings of the 13th Winter School on Abstract Analysis, Palermo, Italy: Circolo Matematico di Palermo, 1985, 89–103.

19. A. P. Sostak, Basic structures of fuzzy topology, *J. Math. Sci.*, **78** (1996), 662–701.

20. I. M. Taha, On fuzzy upper and lower α-ℓ-continuity and their decomposition, *J. Math. Comput. Sci.*, **11** (2021), 427–441.

21. E. Tsiporkova, B. De Baets, E. Kerre, A fuzzy inclusion based approach to upper inverse images under fuzzy multivalued mappings, *Fuzzy Set. Syst.*, **85** (1997), 93–108.

22. E. Tsiporkova, B. De Baets, E. Kerre, Continuity of fuzzy multivalued mappings, *Fuzzy Set. Syst.*, **94** (1998), 335–348.

23. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353.