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*Research article*

# Research on the ellipsoidal boundary of reachable sets of neutral systems with bounded disturbances and discrete time delays

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**Abstract:** This research focuses on the challenge of defining the ellipsoidal boundaries of the reachable set (RS) for neutral-type dynamical systems with time delays. A novel analytical approach is proposed, leveraging the development of new Lyapunov functions and matrix inequality techniques. These methods provide powerful tools for determining the ellipsoidal boundaries of the system's RS. A comparative analysis, supported by numerical examples, demonstrates that the approach outlined in this study can accurately identify smaller yet effective RS boundaries compared to existing literature. This precise boundary determination offers significant theoretical support for state estimation and control design in dynamical systems, thereby enhancing their effectiveness and reliability in real-world applications.

**Keywords:** time delays; reachable set; neutral-type systems; bounded disturbances; Lyapunov function; ellipsoid boundary
**Mathematics Subject Classification:** 34K40, 37B25

## 1. Introduction

Time delays are prevalent in both natural phenomena and various systems encountered in everyday life, characterized by a temporal lag in system responses [1–5]. They manifest in diverse contexts, spanning from biological processes to communication networks and industrial control systems. Specifically, neutral type delay systems [6], incorporating neutral terms, present heightened complexity and pose greater challenges compared to conventional delay systems. Exploring the dynamic characteristics of neutral type delay systems is not only of considerable theoretical significance but also of substantial practical value in various applications. Consequently, it has become a central focus of current academic research [7]. Understanding these systems' behaviors contributes to advancing fundamental knowledge in dynamical systems theory while also facilitating the development of robust control strategies for real-world applications. By delving deeply into their dynamics, we can enhance our comprehension and predictive capabilities regarding the behavior of these systems. In turn, this furnishes a scientific foundation and technical backing for addressing real-world challenges effectively. Such insights empower us to develop tailored solutions and robust strategies that account for the complexities inherent in neutral type delay systems, thus facilitating their practical implementation across various domains.

Since the 1970s, stability analysis in continuous dynamic systems has garnered considerable attention [8–12]. In the last three decades, stability investigations concerning delay differential systems have continued to be a focal point within the global control theory community. Among these, neutral type delay systems stand out for providing a nuanced and precise understanding of the fundamental principles underlying dynamic changes in phenomena. Their distinctive attributes offer significant advantages in addressing complex real-world challenges, such as those encountered in turbine jet engine systems and lateral cutting applications [13, 14]. Indeed, the incorporation of neutral terms complicates the analysis of dynamical properties in these systems, presenting greater challenges compared to ordinary delay differential systems. As a result, progress in research on neutral type differential systems has been relatively slow. In [15], Shen et al. proposed a method based on a fuzzy model to study a nonlinear Markov jump singular perturbation system, adopting deep learning optimization and a new online parallel learning algorithm. This method is very interesting, and the effectiveness of the proposed method was shown. As interesting as research work [15], in [16], a novel hybrid reinforcement $Q$-learning control method was developed for adaptive fuzzy $H_\infty$ control of discrete-time nonlinear Markov jump systems, along with an innovative online parallel $Q$-learning algorithm that enhances learning efficiency by eliminating initial stability conditions and achieving faster convergence compared to traditional methods. The intricate interplay between delayed and non-delayed components necessitates the development of specialized analytical techniques and methodologies to unravel their behaviors accurately. Since M. A. Cruz and J. K. Hale initially introduced the concept of neutral functional differential equations and their associated theories, substantial theoretical advancements have been made in investigating their stability [17–19]. These pioneering contributions laid the groundwork for subsequent research endeavors, driving forward our understanding of the stability properties inherent in such systems. The majority of these findings have been derived using techniques such as Lyapunov's direct method [20], state space analysis, and the characteristic equation method. Through the application of these methodologies, researchers have been able to uncover fundamental stability properties and develop strategies for effectively analyzing and controlling these complex systems.

The term "reachable set" (RS) encompasses the set of states that a system can potentially attain within its state space [21–24]. This which concept is fundamental in understanding the dynamic evolution and potential trajectories of a given system under different conditions and inputs. In dynamic systems, the system's state is typically represented by a set of variables, and the values of these variables form the state space [25]. Research on RS is crucial for understanding the system's dynamic behavior, stability analysis, control design, and performance evaluation. Researching the RS of system states is of paramount significance [26–28]. The RS denotes the collection of states that a system in a state space may potentially attain, playing a pivotal role in comprehending the system's dynamic behavior and stability. By scrutinizing the RS, we can assess the stability of the system and ascertain the existence and stability characteristics of equilibrium points. Moreover, the RS serves as a fundamental framework for devising effective control strategies. By comprehending the accessibility of various system states, we can identify suitable control inputs to steer the system towards desired states or steer clear of undesired ones. Furthermore, investigations into RS contribute to assessing system performance, enabling comparisons between the system's RS and the anticipated workspace to determine if the system meets the operational requirements in real-world scenarios. In summary, studying the RS of system states provides crucial reference points for system design, control, and performance evaluation [22, 29–32].

In [33], the focus is on addressing the challenge of determining an ellipsoidal RS that encompasses the states of linear time-delay systems under the influence of bounded peak disturbances. Feng and Zheng investigated the RS estimation problem for discrete-time delayed Takagi-Sugeno fuzzy systems with bounded input disturbances and nonzero initial conditions. By utilizing an approach based on reciprocally convex combinations to estimate the forward difference of terms that are triple-summable, a condition for RS estimation was formulated. Numerical examples validated the effectiveness of the proposed results [34]. [35] investigated the boundedness issue concerning reachable sets for linear discrete-time systems that are affected by state delays and bounded disturbances. A novel approach involved minimizing the projection distance of ellipsoids along each axis with different exponential convergence rates, rather than simply minimizing their radii, to obtain smaller boundaries. As a result, the intersection points of these ellipsoids produce a more compact boundary for the RS. Numerical demonstrations confirmed the efficacy of the proposed methodology. In [36], Feng delved into issues concerning RS estimation and synthesis for time-delay systems. The non-uniform delay-partitioning method and the triple integral technique were introduced to propose an improved RS estimation condition, which is formulated as a linear matrix inequality. This enhanced condition offers advancements in accurately estimating and synthesizing the RS for time-delay systems. Using this criterion, a sufficient condition for the existence of a state-feedback controller is established, ensuring that the reachable set of the closed-loop system is bounded by a predefined ellipsoid. Two numerical examples were provided to demonstrate the efficacy of the results. Jian and Duan explored the finite-time synchronization issue of fuzzy inertial neural networks with time-varying coefficients and proportional delays by employing suitable variable transformations. They proposed criteria based on algebraic inequalities to attain finite-time synchronization. The effectiveness of the proposed method was validated through simulations of two numerical examples, with estimation of the settling time included [37]. [38] introduced a novel approach that integrates spatiotemporal trajectory planning and control using a combination of RS and optimization techniques. The method encompasses a risk assessment model that accounts for uncertainty in predictive position distribution, along with a strategy for constructing spatiotemporal corridors that incorporate risk fields. Additionally, the authors devised a trajectory optimization strategy employing the Iterative Linear Quadratic Regulator (ILQR) and considering coupled lateral-longitudinal dynamics. This approach facilitates rolling iterative optimization within the defined spatiotemporal corridors.

Building on the insights from the literature mentioned above, this study concentrates on addressing the elliptical boundary problem of RS for time-delay neutral systems. By introducing novel Lyapunov functions and employing matrix inequality techniques, we propose several methodologies for determining the elliptical boundaries of RS. The primary contributions of this study are outlined as follows

(1) First of all, the investigation of neutral time-delay systems reveals intricate dynamical behaviors,

rendering stability analysis and control design notably intricate. Delving into neutral time-delay systems holds paramount importance in tackling delay-associated challenges in practical engineering scenarios. Such research endeavors contribute significantly to bolstering system stability, robustness, and overall performance.

(2) Additionally, the elliptical boundary problem of RS for time-delay neutral systems was investigated. Through the construction of innovative Lyapunov functionals and the application of matrix inequality techniques, diverse approaches have been developed to determine the smallest feasible elliptical boundaries of RS. This research significantly advances both comprehension and computational methodologies for analyzing reachable sets in time-delay neutral systems. Such advancements are pivotal for applications in control, optimization, and ensuring system safety.

(3) Ultimately, compared with other studies (see [33] and [39]), the effectiveness of the proposed method is further validated through numerical simulations conducted using MATLAB/Simulink. These simulations demonstrate the capability of the method to identify a smaller and more efficient boundary for the RS.

The remainder of this paper is structured as follows: Section 2 provides the model of neutral-type time-delay systems, introduces key lemmas, and delineates the control objectives briefly. Section 3 introduces a RS method utilizing ellipsoidal boundaries, applies the Lyapunov direct method, and verifies stability. In Section 4, numerical simulation results, from MATLAB, are presented to validate that the proposed approach produces smaller RS boundaries through comparative analysis. Finally, Section 5 offers concluding remarks and outlines avenues for future research.

## 2. System description and main lemmas

### 2.1. System model

The neutral delay system studied in this section can be described as

$$\begin{cases} \dot{x}(t) - C\dot{x}(t-\tau) = Ax(t) + Bx(t-h(t)) + Dw(t), \\ x(t_0+\theta) = \varphi(\theta), \quad \forall\theta \in [-\tau^*, 0], \end{cases} \tag{2.1}$$

where, $x(t) \in \mathfrak{R}^n$ denotes the system's state vector, while $w(t) \in \mathfrak{R}^l$ represents the disturbance. The function $\varphi(\cdot)$ is a differentiable function defining initial values for the system. The parameter $\tau > 0$ signifies a time-varying neutral type delay, and $h(t)$ denotes a time-varying discrete delay. Additionally, both the neutral type delay and the disturbances are subject to specific conditions

$$0 \leqslant h(t) \leqslant h \leqslant +\infty, \tag{2.2}$$

$$\dot{h}(t) \leqslant h_d \leqslant 1, \tag{2.3}$$

$$w^{\mathrm{T}}(t)w(t) \leqslant 1, \tag{2.4}$$

where, $h$ and $h_D$ are constants, and $\tau^* = \max(\tau, h)$ denotes the maximum delay in the system, incorporating both the neutral type delay $\tau$ and the discrete delay $h$. The matrices $A$, $B$, $C$, and $D$ are known real matrices with dimensions $A, B, C \in \mathfrak{R}^{n\times n}$ and $D \in \mathfrak{R}^{n\times l}$, defining the linear dynamics and interactions within the system and with external disturbances. In addition, $w_m$ is the maximum value of $w$ and satisfies $0 < w_m \leq 1$.

The RS of the system is denoted as

$$\Re_x = \{x(t) \in \mathbb{R}^n \mid x(t), w(t) \text{ satisfy: } (2-1), (2-2)\}. \tag{2.5}$$

The RS $\Re_x$ is represented by the ellipsoidal boundary defined as $\Im(P_1, 1)$:

$$\Im(P_1, 1) = \left\{x(t) \in \mathbb{R}^n \mid x(t) \text{ satisfies } (2-1), (2-2), \text{ and } x^{\mathrm{T}}(t)P_1x(t) \leq 1\right\}. \tag{2.6}$$

**Remark 1.** *This section delves into the dynamics of a specific class of delay systems known as neutral type systems. These systems are characterized by a differential equation capturing the current rate of change of the state vector $x(t)$, as well as its rate of change at a previous time, incorporating both a neutral type delay $\tau$ and a discrete delay $h(t)$. The state vector $x(t) \in \mathbb{R}^n$ evolves under the influence of a disturbance vector $w(t) \in \mathbb{R}^l$. Moreover, the system initializes from an initial state defined by a differentiable function $\varphi(\cdot)$ over the interval $[-\tau, 0]$, where $\tau = \max(\tau, h)$ denotes the maximum delay experienced by the system.*

**Remark 2.** *The delays within the system, including both the neutral and discrete components, are bounded, thereby ensuring that the system's dynamics remain well-defined within these constraints. This boundedness of delays is crucial for maintaining the stability and predictability of the system's behavior under varying conditions. The matrices A, B, C, and D are known real matrices that characterize the interactions within the system and with external disturbances. To analytically describe the reachability set, an ellipsoidal boundary $\Im(P_1, 1)$ is defined, encompassing all states $x(t)$ that not only satisfy the system's differential equations but also adhere to a quadratic constraint defined by the matrix $P_1$.*

### *2.2. Main useful lemmas*

**Definition 1.** *Nonlinear systems exhibit diverse equilibrium point configurations, which can range from none to one or multiple. Unlike their linear counterparts, nonlinear systems often display a richer array of equilibrium behaviors, frequently featuring multiple equilibrium points or lacking any at all. Linear systems, on the other hand, are characterized by a more straightforward equilibrium point analysis.*

$$\dot{x} = Ax,$$

*and when A is non-singular, the system has a unique equilibrium point at $x = 0$.*

**Definition 2.** *[40] If there exist constants $\alpha > 0$ and $\gamma \geqslant 1$ such that for any $x(t)$ the inequality*

$$\|x(t)\| \leqslant \gamma \sup_{-\gamma^* < s < 0} \sqrt{\|\varphi(s)\|^2 + \|\phi(s)\|^2} \mathrm{e}^{-\alpha t}, \tag{2.7}$$

*hold, then the system Eq (2.1) is said to be exponentially stable, with a decay rate of $\alpha$.*

**Lemma 1.** *[41] For the provided symmetric positive definite matrices $\Sigma_1$ and $\Sigma_2$, along with any matrix $\Sigma_3$, the condition $\Sigma_1 + \Sigma_3^{\mathrm{T}}\Sigma_2^{-1}\Sigma_3 < 0$ is met if and only if both of the following matrix inequalities hold:*

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^{\mathrm{T}} \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0, \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^{\mathrm{T}} & \Sigma_1 \end{bmatrix} < 0. \tag{2.8}$$

**Lemma 2.** *[42] For any matrix $\Phi \in \mathbb{R}^{n\times n}$ and a constant $\gamma > 0$, if the function $w : [0,\gamma] \to \mathbb{R}^n$ is integrable, then the following equation holds:*

$$\left(\int_0^{\gamma} w(s)\mathrm{d}s\right)^{\mathrm{T}} \Phi\left(\int_0^{\gamma} w(s)\mathrm{d}s\right) \leqslant \gamma \int_0^{\gamma} w^{\mathrm{T}}(s)\Phi w(s)\mathrm{d}s. \tag{2.9}$$

**Lemma 3.** *[43] Let $V(x(0)) = 0$ and $w^{\mathrm{T}}(t)w(t) \leqslant w_m^2$. If*

$$\dot{V}(t, x_t) + \alpha V(t, x_t) - \beta w^{\mathrm{T}}(t)w(t) \leqslant 0, \tag{2.10}$$

*then, for all $t \geqslant t_0$, it holds that*

$$V(t, x_t) \leqslant \frac{\beta}{\alpha} w_m^2. \tag{2.11}$$

**Lemma 4.** *[44] Given $\alpha \in \mathbb{R}^{n_a}$, $\beta \in \mathbb{R}^{n_b}$, and $N \in \mathbb{R}^{n_a\times n_b}$, if the block matrix $\begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix} \geqslant 0$, then for any matrices $X \in \mathbb{R}^{n_a\times n_a}$, $Y \in \mathbb{R}^{n_a\times n_b}$, and $Z \in \mathbb{R}^{n_b\times n_b}$, the following inequality holds:*

$$-2\alpha^{\mathrm{T}} N\beta \leqslant \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X & Y - N \\ Y^{\mathrm{T}} - N^{\mathrm{T}} & Z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{2.12}$$

**Lemma 5.** *[40] Given matrices $Q = Q^{\mathrm{T}}$, $H$, $E$, and $R = R^{\mathrm{T}}$ of appropriate dimensions, it holds that for all $F^{\mathrm{T}}F \leqslant R$*

$$Q + HFE + E^{\mathrm{T}}F^{\mathrm{T}}H^{\mathrm{T}} < 0. \tag{2.13}$$

*If and only if $\exists \varepsilon > 0$,*

$$Q + \varepsilon HH^{\mathrm{T}} + \eta^{-1}E^{\mathrm{T}}RE < 0. \tag{2.14}$$

### *2.3. Control objective*

This study addresses the intricate task of delineating the ellipsoidal boundaries of RS for neutral-type systems, particularly in the presence of bounded and nonlinear disturbances. It investigates methodologies aimed at precisely defining the RS boundaries, ensuring comprehensive coverage under specified conditions. Leveraging the direct Lyapunov method, the concept of free-weighting matrices, and techniques involving matrix inequalities, a suitable Lyapunov function is selected to tackle this challenge. A novel approach for determining the ellipsoidal boundaries of the system's RS is proposed. Through numerical examples, this paper showcases that the proposed method excels in accurately identifying smaller and more effective RS boundaries compared to existing research.

## 3. Ellipsoidal boundary of RS

In this section, we will address the RS boundary problem for the aforementioned neutral-type system using the direct Lyapunov method and various techniques involving matrix inequalities. We will present the following theorem:

**Theorem 1.** *There exist matrices $Y$, $Q_{12}$, $P_2$, and $P_3$ are postulated to exist, alongside matrices $X$, $Z$, $Q_{11}$, $Q_{22}$, $P_1 > 0$, $R > 0$, $S_1 \geqslant 0$, $S_2 > 0$, $G > 0$ and $M > 0$ and the real number $\alpha > 0$, satisfying the following linear matrix inequality:*

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & P_2^{\mathrm{T}}C - Y & 0 & \alpha Y & P_2^{\mathrm{T}}D \\ * & \Phi_{22} & P_3^{\mathrm{T}}B & P_3^{\mathrm{T}}C & 0 & Y + Z & P_3^{\mathrm{T}}D \\ * & * & \Phi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & -\mathrm{e}^{-\alpha\tau}R & 0 & -Z & 0 \\ * & * & * & * & -\mathrm{e}^{-\alpha h}S_2 & 0 & 0 \\ * & * & * & * & * & \alpha Z - \mathrm{e}^{-\alpha\tau}G & 0 \\ * & * & * & * & * & * & -\frac{\alpha}{w^2}I \end{bmatrix} \leqslant 0, \tag{3.1}$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{bmatrix} \geqslant 0, \tag{3.2}$$

$$\begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z + \frac{1}{\tau}\mathrm{e}^{-\alpha\tau}R \end{bmatrix} \geqslant 0, \tag{3.3}$$

*where*

$$\Phi_{11} = \alpha P_1 + \alpha X + P_2^{\mathrm{T}}A + A^{\mathrm{T}}P_2 + M - \mathrm{e}^{-\alpha h}S_1, \tag{3.4}$$

$$\Phi_{12} = P_1 - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 + X + Y, \tag{3.5}$$

$$\Phi_{13} = P_2^{\mathrm{T}}B + Q_{12} + \mathrm{e}^{-\alpha h}S_1, \tag{3.6}$$

$$\Phi_{22} = R + \tau^2 G + h^2(S_1 + S_2) + h\mathrm{e}^{-\alpha h}Q_{11} - P_3 - P_3^{\mathrm{T}}, \tag{3.7}$$

$$\Phi_{33} = hQ_{22} - Q_{12} - Q_{12}^{\mathrm{T}} - \min\left((1 - h_d)\mathrm{e}^{-\alpha h}, 1 - h_d\right)M - \mathrm{e}^{-\alpha h}S_1. \tag{3.8}$$

*Therefore, the ellipsoid $\mathfrak{J}(P_1, 1)$ constitutes the boundary of the RS for the neutral type system Eq (2.1).*

**Remark 3.** *This statement underscores the significance of employing linear matrix inequalities (LMIs) and specific matrix conditions to establish the boundaries of the Reachability Set (RS) in control theory. Defining these boundaries is pivotal for comprehending the system's capabilities and limitations, especially in the context of neutral-type systems where factors like time delays can introduce complexity to the dynamics. By characterizing the ellipsoid $\mathfrak{J}(P_1, 1)$, we obtain valuable insights into the spectrum of states achievable by the system under specified conditions. This understanding is indispensable for devising control strategies, ensuring stability, and optimizing system performance.*

*Proof.* In order to derive a smaller boundary for the RS of the system described by Eq (2.1), we select the following Lyapunov functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t), \tag{3.9}$$

where

$$V_1(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & \dot{x}^{\mathrm{T}}(t) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \tag{3.10}$$

$$V_2(t) = \int_{t-h(t)}^{t} \mathrm{e}^{\alpha(s-t)}x^{\mathrm{T}}(s)Mx(s)\mathrm{d}s + \int_{t-\tau}^{t} \mathrm{e}^{\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s, \tag{3.11}$$

$$V_3(t) = h\int_{-h}^{0} \mathrm{d}\theta \int_{t+\theta}^{t} \mathrm{e}^{\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)\left(S_1 + S_2\right)\dot{x}(s)\mathrm{d}s + \tau\int_{-\tau}^{0} \mathrm{d}\theta \int_{t+\theta}^{t} \mathrm{e}^{\alpha(s-t)}x^{\mathrm{T}}(s)Gx(s)\mathrm{d}s, \tag{3.12}$$

$$\begin{aligned} V_4(t) = \int_0^t \mathrm{d}\theta \int_{\theta-h(\theta)}^{\theta} \mathrm{e}^{\alpha(\theta-t)}\left[\dot{x}^{\mathrm{T}}(s) \quad x^{\mathrm{T}}(\theta-h(\theta))\right]\left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{array}\right]\left[\begin{array}{c} \dot{x}(s) \\ x(\theta-h(\theta)) \end{array}\right]\mathrm{d}s \\ + \mathrm{e}^{\alpha h}\int_0^t \mathrm{d}\theta \int_{\theta-h(\theta)}^{\theta} \mathrm{e}^{\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)Q_{11}\dot{x}(s)\mathrm{d}s, \end{aligned} \tag{3.13}$$

$$V_5(t) = \left[\begin{array}{cc} x^{\mathrm{T}}(t) & \left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}} \end{array}\right]\left[\begin{array}{cc} G_{11} & G_{12} \\ G_{12}^{\mathrm{T}} & G_{22} \end{array}\right]\left[\begin{array}{c} x(t) \\ \int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s \end{array}\right]. \tag{3.14}$$

The matrices $Q_{11}$, $Q_{22}$, $Q_{12}$, $P_1 > 0$, $P_2$, $P_3$, $R > 0$, $S_1 \geqslant 0$, $S_2 > 0$, $G > 0$, $M > 0$, $X$, $Y$, and $Z$ and the scalar $\varepsilon > 0$ are solutions to matrix inequalities (3.1) and (3.2). Initially, it is evident that for $t - h \leqslant s \leqslant t$, the following condition holds:

$$0 < \mathrm{e}^{-\alpha h} \leqslant \mathrm{e}^{\alpha(s-t)} \leqslant 1, \tag{3.15}$$

$$0 \leqslant h - t + s \leqslant h. \tag{3.16}$$

Therefore, it can be inferred that

$$V_3(t) \geqslant 0. \tag{3.17}$$

By applying a lemma, we establish the validity of the inequality:

$$V_2(t) \geqslant \mathrm{e}^{-\alpha\tau}\int_{t-\tau)}^{t} \dot{x}^{\mathrm{T}}(s)Rx(s)\mathrm{d}s \geqslant \frac{1}{\tau}\mathrm{e}^{-\alpha\tau}\left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}} R\left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right). \tag{3.18}$$

Moreover, it leads us to the conclusion that

$$V_4(t) \geqslant 0. \tag{3.19}$$

Furthermore, the following inequality holds:

$$\sum_{i=2}^{5} V_i(t) \geqslant \left[\begin{array}{cc} x^{\mathrm{T}}(t) & \left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}} \end{array}\right]\left[\begin{array}{cc} X & Y \\ Y^T & Z + \frac{1}{\tau}\mathrm{e}^{-\alpha\tau}R \end{array}\right]\left[\begin{array}{c} x(t) \\ \int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s \end{array}\right] \geqslant 0. \tag{3.20}$$

Therefore, we can deduce that

$$V(t) = \sum_{i=1}^{5} V_i \geqslant V_1\left(x(t) = x^{\mathrm{T}}(t)P_1x(t)\right). \tag{3.21}$$

Following the trajectory of the system as defined in Eq (2.1), we proceed to differentiate the Lyapunov functional $V(t)$, resulting in

$$V(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) + \dot{V}_5(t). \tag{3.22}$$

Through Eq (3.9), we have

$$\begin{aligned}
\dot{V}_1(t) =&\begin{bmatrix} x^{\mathrm{T}}(t) & \dot{x}^{\mathrm{T}}(t) \end{bmatrix}\begin{bmatrix} P_1 & P_2^{\mathrm{T}} \\ 0 & P_3^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} x(t) \\ 0 \end{bmatrix} \\
=&\begin{bmatrix} x^{\mathrm{T}}(t) & \dot{x}^{\mathrm{T}}(t) \end{bmatrix}\begin{bmatrix} P_1 & P_2^{\mathrm{T}} \\ 0 & P_3^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \begin{pmatrix} Ax(t) - \dot{x}(t) + Bx(t-h(t)) \\ +C\dot{x}(t-d(t)) + Dw(t) \end{pmatrix} \end{bmatrix} \\
=&x^{\mathrm{T}}(t)\left[P_2^{\mathrm{T}}A + A^{\mathrm{T}}P_2\right]x(t) + 2x^{\mathrm{T}}(t)\left[P_1 - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3\right]\dot{x}(t) \\
&+ 2x^{\mathrm{T}}(t)P_2^{\mathrm{T}}Bx(t-h(t)) + 2x^{\mathrm{T}}(t)P_2^{\mathrm{T}}C\dot{x}(t-\tau) \\
&+ 2x^{\mathrm{T}}(t)P_2^{\mathrm{T}}Dw(t) - \dot{x}^{\mathrm{T}}(t)\left[P_3 + P_3^{\mathrm{T}}\right]\dot{x}(t) \\
&+ 2\dot{x}^{\mathrm{T}}(t)P_3^{\mathrm{T}}Bx(t-h(t)) + +2\dot{x}^{\mathrm{T}}(t)P_3^{\mathrm{T}}C\dot{x}(t-\tau) + 2x^{\mathrm{T}}(t)P_3^{\mathrm{T}}Dw(t),
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\dot{V}_2(t) =&x^{\mathrm{T}}(t)Mx(t) + \dot{x}^{\mathrm{T}}(t)R\dot{x}(t) - (1-\dot{h}(t))\mathrm{e}^{-\alpha h(t)}x^{\mathrm{T}}(t-h(t))Mx(t-h(t)) \\
&- \mathrm{e}^{-\alpha\tau}\dot{x}^{\mathrm{T}}(t-\tau)R\dot{x}(t-\tau) - \alpha\int_{t-h(t)}^{t}\mathrm{e}^{2\alpha(s-t)}x^{\mathrm{T}}(s)Mx(s)\mathrm{d}s \\
&- \alpha\int_{t-\tau}^{t}\mathrm{e}^{2\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s \\
\leqslant&x^{\mathrm{T}}(t)Mx(t) + \dot{x}^{\mathrm{T}}(t)R\dot{x}(t) - \mathrm{e}^{-\alpha\tau}\dot{x}^{\mathrm{T}}(t-\tau)R\dot{x}(t-\tau) \\
&- \min\left((1-h_d)\,\mathrm{e}^{-\alpha h}, 1-h_d\right)x^{\mathrm{T}}(t-h(t))Mx(t-h(t)) - \alpha V_2,
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
\dot{V}_3(t) \leqslant&\begin{bmatrix} x^{\mathrm{T}}(t) & \dot{x}^{\mathrm{T}}(t) \end{bmatrix}\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix}\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&- (1-\tau_D)\cdot\begin{bmatrix} x^{\mathrm{T}}(t-\tau) & \dot{x}^{\mathrm{T}}(t-\tau) \end{bmatrix}\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix}\begin{bmatrix} x(t-\tau) \\ \dot{x}(t-\tau) \end{bmatrix} \\
\leqslant&x^{\mathrm{T}}(t)S_{11}x(t) + 2x^{\mathrm{T}}(t)S_{12}\dot{x}(t) + 2\dot{x}^{\mathrm{T}}(t)S_{22}\dot{x}(t) - (1-\tau_D) \\
&\cdot\left[x^{\mathrm{T}}(t-\tau)S_{11}x(t-\tau) + 2x^{\mathrm{T}}(t-\tau)S_{12}\dot{x}(t-\tau) + \dot{x}^{\mathrm{T}}(t-\tau)S_{22}\dot{x}(t-\tau)\right],
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\dot{V}_4(t) =&\int_{t-h(t)}^{t}\dot{x}^{\mathrm{T}}(s)Q_{11}\dot{x}(s)\mathrm{d}s + 2\left(\int_{t-h(t)}^{t}\dot{x}^{\mathrm{T}}(s)\mathrm{d}s\right)^{\mathrm{T}}Q_{12}x(t-h(t)) \\
&+ h(t)x^{\mathrm{T}}(t-h(t))Q_{22}x(t-h(t)) + h\mathrm{e}^{\alpha h}\dot{x}^{T}(s)Q_{11}\dot{x}(s) \\
&- \int_{t-h}^{t}\dot{x}^{\mathrm{T}}(s)Q_{11}\dot{x}(s)\mathrm{d}s - \alpha V_4 \\
\leqslant&\int_{t-h(t)}^{t}\dot{x}^{\mathrm{T}}(s)Q_{11}\dot{x}(s)\mathrm{d}s + 2[x(t) - x(t-h(t))]^{\mathrm{T}}Q_{12}x(t-h(t)) \\
&+ hx^{\mathrm{T}}(t-h(t))Q_{22}x(t-h(t)) + h\mathrm{e}^{\alpha h}\dot{x}^{\mathrm{T}}(s)Q_{11}\dot{x}(s) \\
&- \int_{t-h}^{t}\dot{x}^{\mathrm{T}}(s)Q_{11}\dot{x}(s)\mathrm{d}s - \alpha V_4 \\
=&2x^{\mathrm{T}}(t)Q_{12}x(t-h(t)) + h\mathrm{e}^{\alpha h}\dot{x}^{\mathrm{T}}(s)Q_{11}\dot{x}(s) \\
&+x^{\mathrm{T}}(t-h(t))\left[hQ_{22} - Q_{12} - Q_{12}^{\mathrm{T}}\right]x(t-h(t)) - \alpha V_4\Big],
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
\dot{V}_5(t) =& \begin{bmatrix} x^{\mathrm{T}}(t) & \left(\int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) - \dot{x}(t-\tau) \end{bmatrix} \\
\leqslant& 2\left[x^{\mathrm{T}}(t)[X+Y]\dot{x}(t) - x^{\mathrm{T}}(t)Y\dot{x}(t-\tau) + \dot{x}^{\mathrm{T}}(t)[Y+Z]\cdot\left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right)\right. \\
&\left. -\dot{x}^{\mathrm{T}}(t-\tau)Z\left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right)\right].
\end{aligned} \tag{3.27}$$

Then, we have

$$\begin{aligned}
&\dot{V}(t) + \alpha V(t) - \frac{\alpha}{w_m^2} w^{\mathrm{T}}(t)w(t) \\
\leqslant& x^{\mathrm{T}}(t)\left[\alpha P_1 + \alpha X + P_2^{\mathrm{T}}A + A^{\mathrm{T}}P_2 + M - \mathrm{e}^{-\alpha h}S_1\right]x(t) + 2x^{\mathrm{T}}(t)\left[P_1 - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 + X + Y\right]\dot{x}(t) \\
&+ 2x^{\mathrm{T}}(t)\left[P_2^{\mathrm{T}}B + Q_{12} + \mathrm{e}^{-\alpha h}S_1\right]x(t-h(t)) + 2x^{\mathrm{T}}(t)\left[P_2^{\mathrm{T}}C - Y\right]\dot{x}(t-\tau) \\
&+ 2\alpha x^{\mathrm{T}}(t)Y\cdot\left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right) + 2x^{\mathrm{T}}(t)P_2^{\mathrm{T}}Dw(t) \\
&+ \dot{x}^{\mathrm{T}}(t)\left[R + \tau^2 G + h^2(S_1 + S_2) + h\mathrm{e}^{-\alpha h}Q_{11} - P_3 - P_3^{\mathrm{T}}\right]\dot{x}(t) \\
&+ 2\dot{x}^{\mathrm{T}}(t)P_3^{\mathrm{T}}Bx(t-h(t)) + 2\dot{x}^{\mathrm{T}}(t)P_3^{\mathrm{T}}C\dot{x}(t-\tau) \\
&+ 2\dot{x}^{\mathrm{T}}(t)[Y+Z]\left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right) + 2\dot{x}^{\mathrm{T}}(t)P_3^{\mathrm{T}}Dw(t) \\
&+ 2x^{\mathrm{T}}(t-h(t))\left[hQ_{22} - Q_{12} - Q_{12}^{\mathrm{T}} - \min\left((1-h_d)\,\mathrm{e}^{-\alpha h}, 1-h_d\right)M - \mathrm{e}^{-\alpha h}S_1\right]x(t-h(t)) \\
&- \mathrm{e}^{-\alpha\tau}\dot{x}^{\mathrm{T}}(t-\tau)R\cdot\dot{x}(t-\tau) - 2\dot{x}^{\mathrm{T}}(t-\tau)Z\left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right) \\
&- \mathrm{e}^{-\alpha h}\left(\int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}} S_2\left(\int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s\right) \\
&- \left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}}[\alpha Z - \mathrm{e}^{-\alpha\tau}G]\left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right) - \frac{\alpha}{w_m^2}w^{\mathrm{T}}(t)w(t) \\
=& X^{\mathrm{T}}(t)\Phi X(t),
\end{aligned} \tag{3.28}$$

where

$$X(t) = \left[x^{\mathrm{T}}(t), \dot{x}^{\mathrm{T}}(t), x^{\mathrm{T}}(t-h(t)), \dot{x}^{\mathrm{T}}(t-\tau), \left(\int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}}, \left(\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}}, w^{\mathrm{T}}(t)\right]^{\mathrm{T}}.$$

The function $\Phi$ is defined as given in Theorem 1. Furthermore, through matrix inequalities (3.1) to (3.3) , we can derive the following inequality:

$$\dot{V}(t) + \alpha V(t) - \frac{\alpha}{w_m^2}w^{\mathrm{T}}(t)w(t) \leq 0. \tag{3.29}$$

According to Lemma 3, it follows that

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) \leq 1. \tag{3.30}$$

From inequality Eq (3.9), we deduce that $V_2(t) + V_3(t) + V_4(t) + V_5(t) > 0$, which implies that

$$V_1(t) = x^{\mathrm{T}}(t)P_1x(t) \leq 1. \tag{3.31}$$

Thus, the theorem is proven. □

The ellipsoid $\Im(P_1, 1)$ serves as the boundary of the RS for the neutral type system Eq (2.1). It is important to note, as mentioned in Theorem 1, that if solutions to the matrix inequalities exist, they are not necessarily unique. This is because when the value of $\log\det(P_1)^{1/2}$ is minimized, the volume of the ellipsoid $\Im(P_1, 1)$ is also minimized. Therefore, $\log\det(P_1)^{1/2}$ is commonly used to measure the volume of the ellipsoid $\Im(P_1, 1)$. However, finding the minimum value of $\log\det(P_1)^{1/2}$ can be quite challenging. Thus, this problem is often simplified to finding the maximum value of $\delta$ that satisfies $\delta I \leqslant P_1$, which can be formulated as

$$\begin{aligned} &\min \bar{\delta}\left(\bar{\delta} = \frac{1}{\delta}\right) \\ &\text{s.t.} \begin{cases} (a)\begin{bmatrix} \bar{\delta}I & I \\ I & P_1 \end{bmatrix} \geqslant 0, \\ (b) \quad P_1 > 0 \text{ and satisfies the matrix inequality (3.1).} \end{cases} \end{aligned} \tag{3.32}$$

**Remark 4.** *The derivation process primarily relied on Leibniz's integral rule for managing derivatives of integrals with varying limits, along with the chain rule for handling derivatives of composite functions. This method facilitated accurate computation of the time rate of change for integral expressions, even in scenarios involving time-sensitive integral bounds and integrands. Additionally, fundamental theorems of integration and differentiation, coupled with knowledge of derivatives of matrix multiplication, were leveraged. These mathematical tools are widely employed in the analysis of dynamic systems. By employing a comprehensive array of mathematical techniques, including these fundamental principles, the solutions of derivative problems concerning complex integral expressions are efficiently and precisely achieved. This underscores a profound grasp of essential mathematical concepts in dynamic systems and control theory.*

**Remark 5.** *When the scalar parameter $\alpha$ in Theorem 1's matrix inequality remains constant, (3.32) simplifies to a linear matrix inequality. Therefore, we can utilize the MATLAB function fminsearch.m to find a local optimal value of $\alpha$ such that the matrix inequality (3.32) has a feasible solution. This approach facilitates an efficient search for an optimal value of $\alpha$, guaranteeing the existence of a feasible solution for the matrix inequality under consideration.*

**Remark 6.** *In our study on system reachable sets, we prioritize system control and stability. Therefore, our research and design objective is to minimize the system RS. While there are various methods to investigate RS for the same system, based on the same LMI technique, we devise different Lyapunov functionals. We then compare these methods with others to identify an approach that yields a smaller RS for the system.*

## 4. Numerical simulation verification

Consider the RS boundary problem for the following time-delay system:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9+\rho \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1+0.5\rho \end{bmatrix} x(t-h(t)) + \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} w(t), \tag{4.1}$$

where,

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix},$$

$$w^{\mathrm{T}}(t)w(t) < w_m^2 = 1, |\rho| \leqslant 0.2.$$

The incorporation of a fixed constant time delay, represented by $h(t)$, introduces a memory effect into the system, potentially leading to reduced stability, oscillatory behavior, or even instability. Conversely, time-varying delays represented by $h(t)$ add complexity to the system dynamics, as the delay varies dynamically over time. Consequently, this study establishes two simulation scenarios to validate the efficacy of the proposed theory and to explore the impact on the RS. Hence, we select two distinct scenarios: one with a constant delay of $h(t) = 0.70$ and another with a time-varying delay of $h(t) = 0.2 + 0.2 * \sin(t)$.

Figure 1 depicts the time response plot of the system state vector when the time delay parameter is constant at $h = 0.70$. Figure 2 illustrates the trajectory plot of the system states under the constant time delay of $h = 0.70$. Meanwhile, Figure 3 contrasts the system state trajectories under the time delay parameter $h = 0.70$ with the ellipsoidal RS boundaries obtained from Theorem 1 proposed in this paper, the Kim method (see [33]), and the Zuo method (see [39]). Observing Figure 3, it is apparent that the approach outlined in Theorem 1 results in more compact ellipsoidal boundaries for the reachable sets (RS) of system states compared to two other methodologies from the literature. However, when the time delay follows the function $h(t) = 0.2 + 0.2 * \sin(t)$, the time response plot of the system state vector is depicted in Figure 4. Similar to Figure 2, Figure 5 showcases the trajectory plot of the system states with a constant time delay of $h(t) = 0.2 + 0.2 * \sin(t)$. Furthermore, Figure 6 provides a comparative analysis of system state trajectories under the time delay parameter $h(t) = 0.2 + 0.2 * \sin(t)$ with ellipsoidal RS boundaries derived from Theorem 1 proposed in this paper, the Kim method, and the Zuo method. Exploring these results sheds light on the efficacy and robustness of the proposed methodology across varying conditions, suggesting avenues for further refinement and application in real-world scenarios.

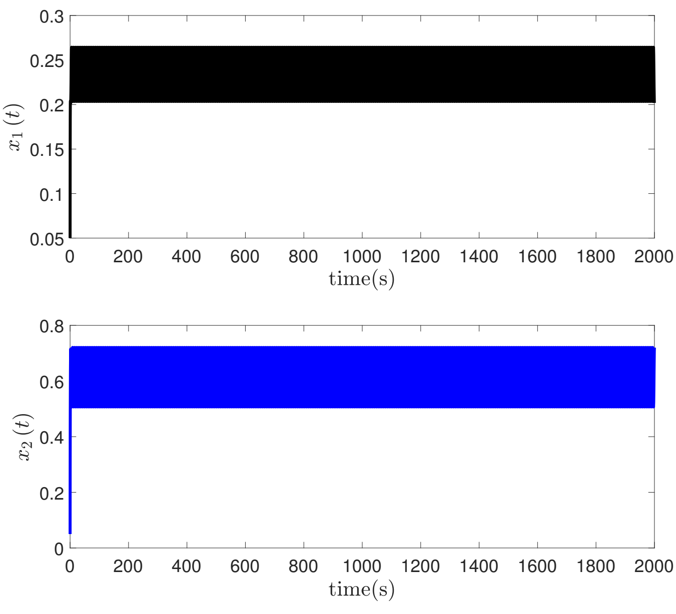


**Figure 1.** The response diagram of system state vector $x(t)$ when $h = 0.7$.

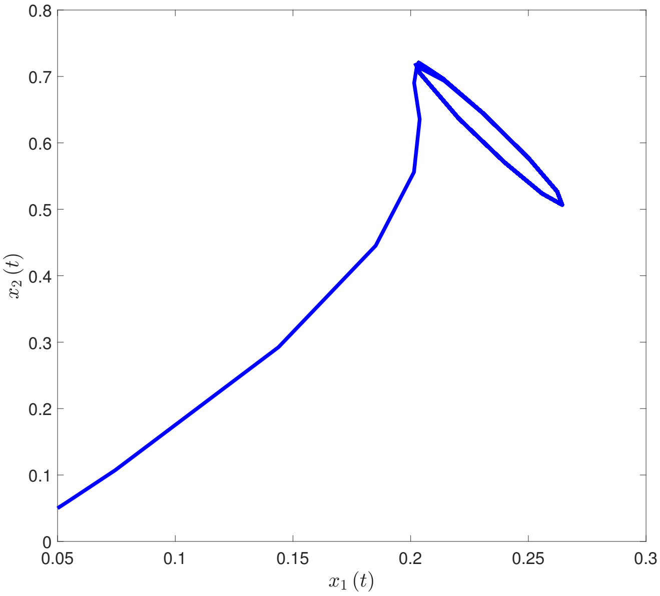


**Figure 2.** The diagram of system state trajectories under the time delay $h = 0.70$.

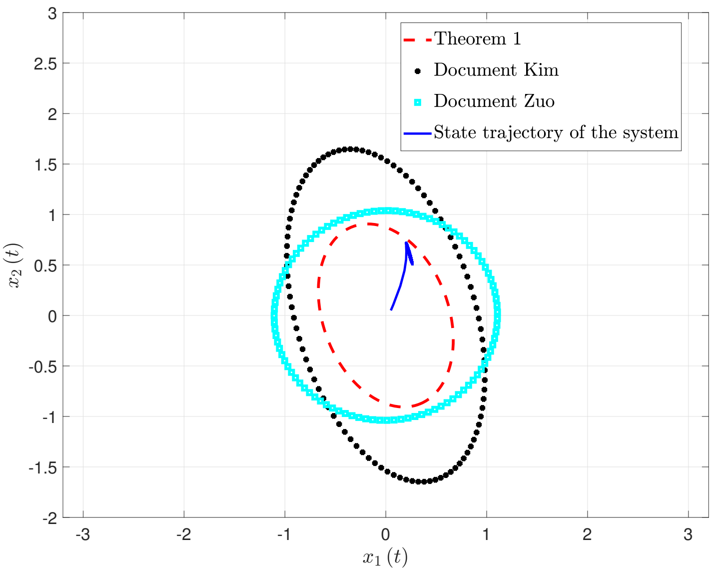


**Figure 3.** The RS system state trajectory diagram of different methods with time delay $h = 0.70$.

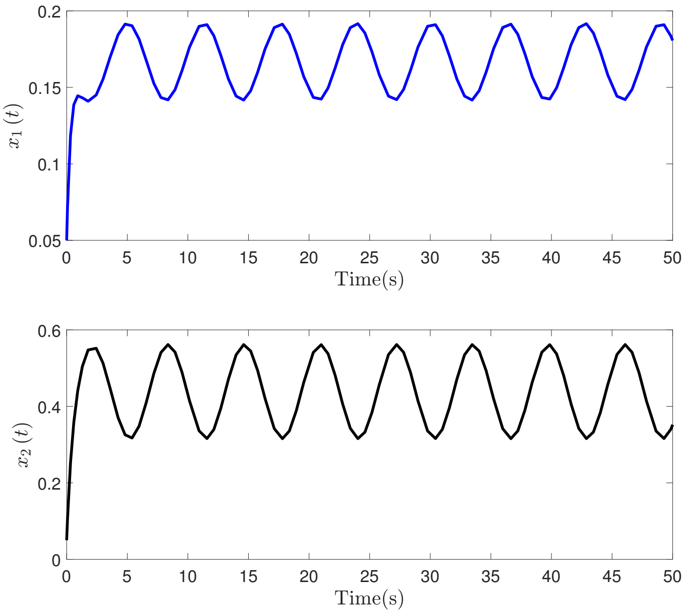


**Figure 4.** The response diagram of system state vector $x(t)$ when $h = 0.2 + 0.2 * \sin(\mathrm{t})$.

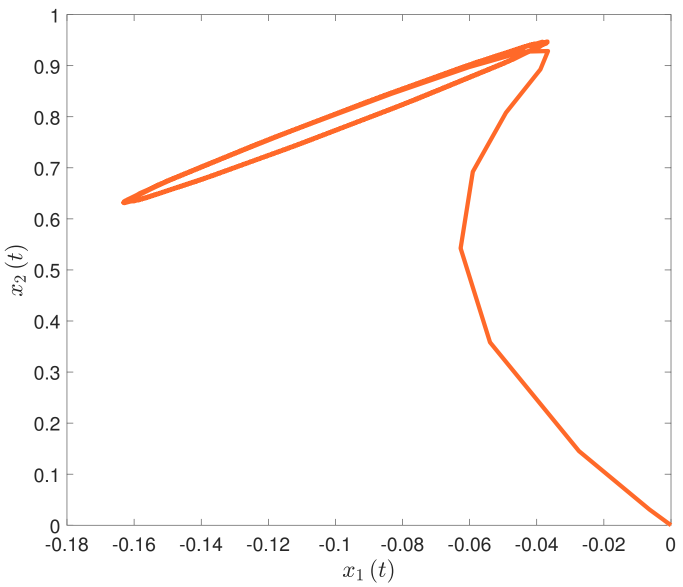


**Figure 5.** The diagram of system state trajectories under the time delay $h = 0.2 + 0.2 * \sin(\mathrm{t})$.

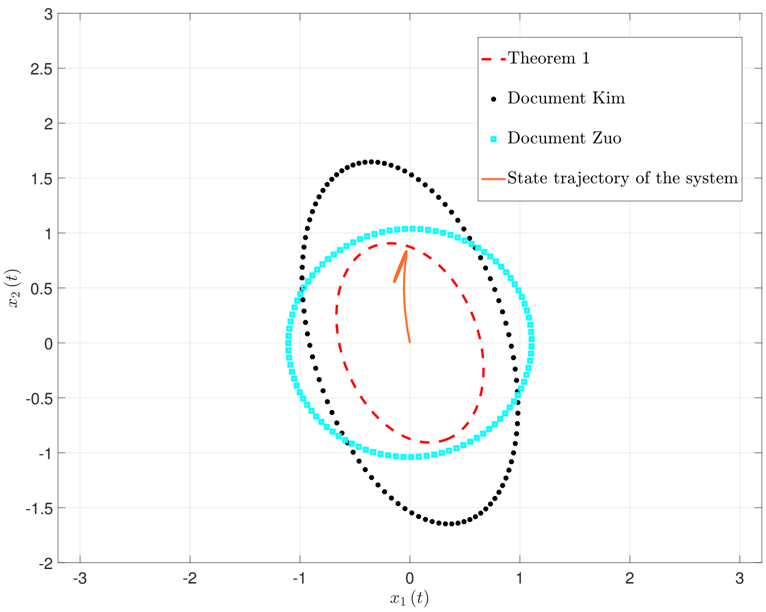


**Figure 6.** The RS system state trajectory diagram of different methods with time delay $h = 0.2 + 0.2 * \sin(t)$.

## 5. Conclusions

The study delves into the ellipsoidal boundary issue of reachable sets (RS) for neutral-type systems under bounded disturbances. Through the development of a novel Lyapunov functional and the application of matrix inequality techniques, the paper introduces several approaches for delineating the ellipsoidal boundary of reachable sets. Moreover, numerical demonstrations showcase that the proposed methodologies yield more compact and efficient RS boundaries compared to existing literature. Future endeavors might encompass further investigation into RS boundary challenges across different system paradigms, along with applying these techniques to enhance the design of safety measures in control systems.

## Authors contributions

Beibei Su: Software, Writing-original draft and Writing-review & editing; Liang Zhao: Conceptualization, Methodology, Supervision and Validation; Liang Du: Data curation; Qun Gu: Software and Writing-review & editing. All authors equally contributed to this manuscript and approved the final version.

## Use of AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflict of interest.

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