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Research article

Functional differential equations of the neutral type: Oscillatory features of solutions

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Abstract: This article delves into the behavior of solutions to a general class of functional differential equations that contain a neutral delay argument. This category encompasses the half-linear case and the multiple-delay case of neutral equations. The motivation to study this type of equation lies not only in the exciting analytical issues it presents but also in its numerous vital applications in physics and biology. We improved some of the inequalities that play a crucial role in developing the oscillation test. Then, we used an improved technique to derive several criteria that ensure the oscillation of the solutions of the studied equation. Additionally, we established a criterion that did not require imposing monotonic constraints on the delay functions and took into account their effect. We have supported the novelty and effectiveness of the results by analyzing and comparing them with previous results in the literature.

Keywords: differential equations; oscillation theory; neutral delay argument; Philos-type criteria
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1. Introduction

The qualitative theory of differential equations deals with the properties of solutions to these equations. The focus of this theory is to examine traits such as stability, oscillation, bifurcation, periodicity, synchronization, symmetry, and more. Researchers established this theory to gather adequate information on nonlinear models that arise while modeling physical, biological, and other processes, as seen in [1–3]. Oscillation theory is a branch of qualitative theory that establishes criteria for both oscillatory and non-oscillatory solutions to differential equations. It is a crucial mathematical tool used in many cutting-edge disciplines and technologies. In recent decades, there has been a lot of activity in studying oscillation conditions for particular functional differential equations, as seen in [4–9].

1.1. NDDEs with several delays

Functional differential equations (FDEs) are equations that express the derivative of an unknown function at a particular point in time in terms of the function's values at earlier points in time. A specific type of FDE is a delay differential equation (DDE), which has numerous applications in physiology, electrical engineering, and biology. For instance, in a predator-prey model, the birth rate of predators may be influenced by past predator or prey numbers rather than just the present. Neutral delay differential equations (NDDEs) are a type of DDE that have the highest derivative in the solution both with and without delay. In addition to its theoretical significance, the qualitative analysis of these equations holds great practical significance. This is because neutral differential equations are involved in a number of phenomena, such as the study of vibrating masses attached to elastic bars, the solution of variational problems with time delays, and problems involving electric networks with lossless transmission lines (such as those found in high-speed computers, where these lines are used to connect switching circuits), see [10]. Recently, many researchers have been interested in studying the stability of some practical models of neutral differential equations, see [11–13].

In this study, we establish novel standards for evaluating the oscillatory characteristics of nonlinear second-order NDDEs with several delays. Namely, we consider the nonlinear NDDE:

$$\left(r(\nu)\,\psi(u(\nu))\,[\mathcal{Z}'(\nu)]^{\alpha}\right)' + \sum_{j=1}^{\kappa} q_j(\nu)\,F\left(u\left(\sigma_j(\nu)\right)\right) = 0, \tag{1.1}$$

where $\nu \geq \nu_0$ and $\mathcal{Z} := u + p \cdot (u \circ \tau)$. During the study, we made the following assumptions:

(B1) $r \in \mathbf{C}^1([\nu_0, \infty), \mathbb{R}^+)$, $\kappa \in \mathbb{Z}^+$, and $\alpha \in \mathbb{Q}^+$ is a quotient of odd numbers.

(B2) $p, q \in \mathbf{C}([\nu_0, \infty), [0, \infty))$, and $p(\nu) \leq p_0 < 1$.

(B3) τ, $\sigma_j \in \mathbf{C}([\nu_0, \infty), \mathbb{R})$, $\tau(\nu) \leq \nu$, $\sigma_j(\nu) \leq \nu$, $\lim_{\nu\to\infty} \tau(\nu) = \infty$, and $\lim_{\nu\to\infty} \sigma_j(\nu) = \infty$, for $j = 1, 2, ..., \kappa$.

(B4) $\psi \in \mathbf{C}^1(\mathbb{R}, (l, L])$, where l and L are positive constants, and $\delta = \sqrt[\alpha]{(L/l)}$.

(B5) $F \in \mathbf{C}(\mathbb{R}, \mathbb{R})$, and $F(u)/u^{\alpha} \geq k$ for $u \neq 0$, where k is a positive constant.

By a solution of NDDE (1.1), we mean a real-valued function $u \in \mathbf{C}^1([\nu_u, \infty), \mathbb{R})$, $\nu_u \geq \nu_0$, which satisfies (1.1) on $[\nu_u, \infty)$, and has the properties $r\psi(u)[\mathcal{Z}']^{\alpha} \in \mathbf{C}^1([\nu_u, \infty), \mathbb{R})$ and $\sup\{|u(\nu)| : \nu \geq \nu_*\} > 0$ for all $\nu_* \geq \nu_u$. A solution u of FDE (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Furthermore, we study the behavior of (1.1) in the canonical case, that is,

$$\int_{\nu_0}^{\infty} r^{-1/\alpha}(\mu)\,\mathrm{d}\mu = \infty. \tag{1.2}$$

The paper is structured into three main sections: the introduction, main results, and conclusion. In the introduction section, we introduce the equation under study along with the fundamental assumptions, and then we review the previous work in the relevant literature. The main results section is further divided into three subsections: improved properties of positive solutions, oscillation theorems, and examples and discussion.

1.2. The function class $\mathbb{H}$

In 1987, Philos [14] defined a new class of functions $\mathbb{H}$ to extend the results of Kamenev [15]. Assume that

$$\mathcal{D}_0 = \{(\nu,\mu) : \nu > \mu > \nu_0\} \text{ and } \mathcal{D} = \{(\nu,\mu) : \nu \geq \mu \geq \nu_0\}.$$

A function $\mathcal{P} \in \mathbf{C}(\mathcal{D},\mathbb{R})$ is said to belong to the function class $\mathbb{H}$, written by $\mathcal{P} \in \mathbb{H}$, if

(i) $\mathcal{P}(\nu,\nu) = 0$ for $\nu \geq \nu_0$, $\mathcal{P}(\nu,\mu) > 0$ on $\mathcal{D}_0$.
(ii) $\mathcal{P}(\nu,\mu)$ has a continuous and nonpositive partial derivative $\partial\mathcal{P}/\partial\mu$ on $\mathcal{D}_0$ such that the condition

$$\frac{\partial\mathcal{P}(\nu,\mu)}{\partial\mu} = -\rho(\nu,\mu)\left[\mathcal{P}(\nu,\mu)\right]^{\alpha/(\alpha+1)},$$

for all $(\nu,\mu) \in \mathcal{D}_0$, is satisfied for some $\rho \in \mathbf{C}(\mathcal{D},\mathbb{R})$.

Since then, Philos class $\mathbb{H}$ has been used to study the oscillations of differential equations of different orders and types, see for example, [16–18].

1.3. Literature review

In the first part of the related work, we will discuss the findings related to the oscillation of NDDE (1.1) or a similar case. In the second part, we will present some work that helped to enhance the relationships and inequalities used in the study of oscillation.

In 2000, Manojlovic´ [19] presented an oscillation condition for the nonlinear ordinary differential equation

$$\left(r(\nu)\psi(u(\nu))\left|u'(\nu)\right|^{\alpha-1}u'(\nu)\right)' + q(\nu)F(u(\nu)) = 0, \tag{1.3}$$

using Philos class $\mathbb{H}$, where F satisfies the condition

$$\frac{F'(u)}{\left(\psi(u)\left|F(u)\right|^{\alpha-1}\right)^{1/\alpha}} \geq M > 0. \tag{1.4}$$

Džurina and Lacková [20] and Şahiner [21] studied the oscillatory behavior of NDDE

$$\left(r(\nu)\psi(u(\nu))\mathcal{Z}'(\nu)\right)' + q(\nu)F(u(\sigma(\nu))) = 0. \tag{1.5}$$

In [20], Džurina and Lacková used the condition

$$\int_{\nu_0}^{\infty} q(\mu)F(\pm NR(\sigma(\mu)))\,\mathrm{d}\mu = \infty, \tag{1.6}$$

for all $N > 0$, where

$$R(\nu) := \int_{\nu_0}^{\nu}\frac{1}{r(\mu)}\mathrm{d}\mu,$$

and presented the following criterion for oscillation:

Theorem 1.1. *([20], Corollary 2.4) Suppose that $F(u) = u$ and $\sigma'(\nu) \geq 0$. If*

$$\int_{\nu_0}^{\infty}\left[q(\mu)R(\sigma(\mu)) - \frac{L\sigma'(\mu)}{4(1-p_0)R(\sigma(\mu))r(\sigma(\mu))}\right]\mathrm{d}\mu = \infty,$$

then every solution of (1.5) is oscillatory.

On the other hand, Şahiner [21] used Philos class $\mathbb{H}$ and three different forms of restrictions on F, namely (B5) with $\alpha = 1$, (1.4) with $\alpha = 1$, and $F'(u) \geq A > 0$. Next, we present one of Şahiner's results in which he used condition (B5).

Theorem 1.2. *([21], Theorem 2.3) Suppose that $\sigma'(\nu) \geq 0$ and there are $\xi \in \mathbf{C}([\nu_0, \infty), \mathbb{R}^+)$ and $\mathcal{P} \in \mathbb{H}$ such that*

$$\limsup_{\nu\to\infty} \frac{1}{\mathcal{P}(\nu, \nu_0)} \int_{\nu_0}^{\nu} \left[\mathcal{P}(\nu, \mu) \xi(\mu) Q(\mu) - \frac{L}{4k} \frac{r(\sigma(\mu)) \xi(\mu)}{\sigma'(\mu)} G^2(\nu, \mu) \right] \mathrm{d}\mu = \infty,$$

where $Q(\nu) := q(\nu)[1 - p(\sigma(\nu))]$, and

$$G(\nu, \mu) = \rho(\nu, \mu) - \frac{\xi'(\mu)}{\xi(\mu)} \sqrt{\mathcal{P}(\nu, \mu)}. \tag{1.7}$$

Then, every solution of (1.5) is oscillatory.

Using integral average conditions of Philos-type, Xu and Weng [22] investigated the oscillatory properties of NDDE with distributed deviating argument

$$(r(\nu) \psi(u(\nu)) \mathcal{Z}'(\nu))' + \int_a^b q(\nu, \mu) F(u(\sigma(\nu, \mu))) \mathrm{d}\delta(\mu) = 0,$$

where $\tau(\nu) = \nu - \tau_0$, $\tau_0 > 0$. For the more general equation

$$\left(r(\nu) \psi(u(\nu)) |\mathcal{Z}'(\nu)|^{\alpha-1} \mathcal{Z}'(\nu) \right)' + q(\nu) F(u(\sigma(\nu))) = 0, \tag{1.8}$$

Ye and Xu [23] established oscillation criteria under the condition (B5), and considered the canonical and noncanonical cases. The following theorem presents the oscillation criteria for the canonical case.

Theorem 1.3. *([23], Theorem 2.1) Suppose that $\sigma'(\nu) \geq 0$ and there is $\xi \in \mathbf{C}([\nu_0, \infty), \mathbb{R}^+)$ such that*

$$\int_{\nu_0}^{\infty} \left[\xi(\mu) q(\mu) [1 - p(\sigma(\mu))]^{\alpha} - \frac{L}{k(\alpha+1)^{\alpha+1}} \frac{r(\sigma(\mu)) [\xi'(\mu)]^{\alpha+1}}{[\sigma'(\mu)]^{\alpha} \xi^{\alpha}(\mu)} \right] \mathrm{d}\mu = \infty. \tag{1.9}$$

Then, every solution of (1.8) is oscillatory.

Li et al. [24] obtained oscillation criteria for NDDE

$$(r(\nu) [(u(\nu) - p(\nu) u(\tau(\nu)))']^{\alpha})' + q(\nu) u(\sigma(\nu)) = 0, \tag{1.10}$$

where $0 \leq p(\nu) \leq p_0 < 1$. Arul and Shobha [25] enhanced the findings, achieving improved oscillation results for the solutions of Eq (1.10).

In [26], numerous findings on oscillation were derived for the second order differential equation

$$(r(\nu) [(u(\nu) - p(\nu) u(\tau(\nu)))']^{\gamma})' + q(\nu) u^{\beta}(\sigma(\nu)) = 0, \tag{1.11}$$

where γ and β are ratios of positive odd integers, $\gamma \geq \beta$. As a more generalized equation than (1.11), Grace et al. [27] established some results for oscillation of solutions of NDDE

$$\left(r(\nu) (y'(\nu))^{\beta} \right)' + q(\nu) u^{\gamma}(\sigma(\nu)) = 0,$$

where

$$y(t) := u(v) + p_1(v)u^{\alpha_1}(\tau(v)) - p_2(t)u^{\alpha_2}(\tau(v)).$$

Baculikova et al. [28] studied the oscillatory behavior of solutions of NDDE

$$(a(v)((x(v) - p(v)x^{\alpha}(\tau(v)))'))' + q(v)x^{\beta}(\sigma(v)) = 0,$$

where $0 < \alpha \leq 1$.

In the past ten years, there has been a significant amount of research aimed at developing methods and relationships to study the oscillations of functional differential equations. Grace et al. [29] and Moaaz et al. [30–32] have studied the oscillation of NDDE

$$(r(v)\,[\mathcal{Z}'(v)]^{\alpha})' + q(v)u^{\beta}(\sigma(v)) = 0, \tag{1.12}$$

and enhanced the well-known results published in the literature.

By improving the relationships between u and $\mathcal{Z}$, Hassan et al. [33] and Moaaz et al. [34, 35] improved the oscillation results for several classes of functional differential equations.

Lemma 1.1. *([35], Lemma 1) Suppose that u is a solution of (1.1) and $u(v) > 0$ for $v \geq v_1$. Then, eventually,*

$$u(v) > \sum_{\ell=0}^{m}\left(\prod_{i=0}^{2\ell} p(\tau_i(v))\right)\left[\frac{\mathcal{Z}(\tau_{2\ell}(v))}{p(\tau_{2\ell}(v))} - \mathcal{Z}(\tau_{2\ell+1}(v))\right],$$

for any integer $m \geq 0$, where

$$\tau_0(v) := v,\ \tau_{\ell}(v) = (\tau \circ \tau_{\ell-1})(v),\ \textit{for}\ \ell = 1, 2, \ldots .$$

Oscillation theory is an important area of study that offers fascinating analytical problems. Recently, researchers have been working on developing oscillation criteria for various types of functional differential equations. This research can be found in sources such as [13, 36–39].

We investigated the oscillatory behavior of NDDE (1.1) in this study. We extended the results in [35] and showed a new relationship between u and $\mathcal{Z}$. We established many criteria that ensure the oscillation of every solution to the studied equation. The motivations and novelty of the results are summarized as follows:

- Expanding previous results in the literature to include the half-linear case as well as the case of multiple delays
- Removing some monotonic constraints on delay functions
- Provide criteria that take into account the effects of delay functions
- Providing more efficient and accurate oscillation parameters than previous related results

2. Main results

For convenience, we define the following: U^+ represents the class of all eventually positive solutions to NDDE (1.1), $\widetilde{\alpha} := \alpha^{\alpha}/(\alpha+1)^{\alpha+1}$, $\sigma_{\min}(v) := \min\left\{\sigma_j(v) : j = 1, 2, ..., \kappa\right\}$, and

$$\eta_c(v) := \int_c^v r^{-1/\alpha}(\mu)\,\mathrm{d}\mu.$$

To prove the main results, we will need the following lemma:

Lemma 2.1. *([40], Lemma 2.3) Assume that $G(\theta) = c_1\theta - c_2\theta^{1+1/\alpha}$, where $c_1, c_2 > 0$. Then, G has the maximum value at $\theta_{\max} := (\alpha c_1/((\alpha+1)c_2))^{\alpha}$ and $G(\theta) \leq G(\theta_{\max}) = \widetilde{\alpha} c_1^{\alpha+1} c_2^{-\alpha}$ for $\theta \in \mathbb{R}$.*

2.1. Improved properties of positive solutions

In this section, we deduce some new and improved relationships for positive solutions to the studied equation.

Lemma 2.2. *Suppose that $u \in U^+$. Then,*

P1 $\mathcal{Z}$ and $\mathcal{Z}'$ are positive, and $(r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^{\alpha})' < 0$, eventually.

P2 $\mathcal{Z}(\nu) \geq \frac{1}{\delta} r^{1/\alpha}(\nu)\mathcal{Z}'(\nu)\eta_{\nu_1}(\nu)$ and $\left(\mathcal{Z}/\eta_{\nu_1}^{\delta}\right)' \leq 0$.

Proof. Let $u \in U^+$. Then, there is a $\nu_1 \geq \nu_0$ such that $u(\tau(\nu)) > 0$ and $u\left(\sigma_j(\nu)\right) > 0$ for $\nu \geq \nu_1$ and $j = 1, 2, ..., \kappa$. From the definition of $\mathcal{Z}$, we get that $\mathcal{Z}(\nu) > 0$ for $\nu \geq \nu_1$. It follows from (1.1), (B2), and (B5) that

$$(r(\nu)\,\psi(u(\nu))\,[\mathcal{Z}'(\nu)]^{\alpha})' = -\sum_{j=1}^{\kappa} q_j(\nu) F\left(u\left(\sigma_j(\nu)\right)\right) \leq 0.$$

Thus, $(r \cdot (\psi \circ u)[\mathcal{Z}']^{\alpha})' \leq 0$ and $\mathcal{Z}'$ is of a constant sign.

Suppose that $\mathcal{Z}'(\nu) < 0$ for $\nu \geq \nu_2$. So,

$$r(\nu)\psi(u(\nu))[\mathcal{Z}'(\nu)]^{\alpha} \leq -c_0 < 0.$$

Since $\psi(u) \leq L$, we find

$$\mathcal{Z}'(\nu) \leq -\sqrt[\alpha]{\frac{c_0}{L}}\frac{1}{r^{1/\alpha}(\nu)}.$$

Integrating this inequality leads to

$$\mathcal{Z}(\nu) \leq \mathcal{Z}(\nu_2) - \sqrt[\alpha]{\frac{c_0}{L}}\eta_{\nu_2}(\nu).$$

Using (1.2), we obtain $\lim_{\nu\to\infty}\mathcal{Z}(\nu) = -\infty$, which contradicts the fact that $\mathcal{Z}(\nu) > 0$.

Now, we have

$$\mathcal{Z}'(\mu) \geq \frac{[r(\mu)\psi(u(\mu))]^{1/\alpha}\mathcal{Z}'(\mu)}{\sqrt[\alpha]{L}r^{1/\alpha}(\mu)}, \text{ for } \mu \geq \nu_1.$$

Integrating this inequality leads to

$$\mathcal{Z}(\nu) \geq \mathcal{Z}(\nu_1) + \int_{\nu_1}^{\nu}\frac{[r(\mu)\psi(u(\mu))]^{1/\alpha}\mathcal{Z}'(\mu)}{\sqrt[\alpha]{L}r^{1/\alpha}(\mu)}\mathrm{d}\mu.$$

Using the properties in (P1), we arrive at

$$\begin{aligned}\mathcal{Z}(\nu) &\geq \frac{1}{\sqrt[\alpha]{L}}[r(\nu)\psi(u(\nu))]^{1/\alpha}\mathcal{Z}'(\nu)\int_{\nu_1}^{\nu}\frac{1}{r^{1/\alpha}(\mu)}\mathrm{d}\mu \\ &\geq \frac{1}{\delta}r^{1/\alpha}(\nu)\mathcal{Z}'(\nu)\eta_{\nu_1}(\nu).\end{aligned}$$

Then,

$$\begin{aligned}\left(\frac{\mathcal{Z}}{\eta_{v_1}^{\delta}}\right)' &= \frac{\eta_{v_1}^{\delta}\mathcal{Z}' - \delta\eta_{v_1}^{\delta-1}r^{-1/\alpha}\mathcal{Z}}{\eta_{v_1}^{2\delta}} \\ &= \frac{1}{\eta_{v_1}^{\delta+1}}\left[\eta_{v_1}\mathcal{Z}' - \delta r^{-1/\alpha}\mathcal{Z}\right] \\ &\leq 0.\end{aligned}$$

This completes the proof. □

Lemma 2.3. *Suppose that $u \in U^+$. Then,*

$$u(v) > \mathcal{Z}(v)\sum_{\ell=0}^{m}\left(\prod_{i=0}^{2\ell-1}p(\tau_i(v))\right)\frac{\eta_{v_1}^{\delta}(\tau_{2\ell}(v))}{\eta_{v_1}^{\delta}(v)}\left[1-p(\tau_{2\ell}(v))\right], \tag{2.1}$$

for $v \geq v_1$ and any integer $m \geq 0$.

Proof. Let $u \in U^+$. Then, there is a $v_1 \geq v_0$ such that $u(\tau(v)) > 0$ and $u\left(\sigma_j(v)\right) > 0$ for $v \geq v_1$ and $j = 1, 2, ..., \kappa$. From Lemma 2.2, we have the properties in (P1) and (P2) satisfied. It follows from these properties that

$$\mathcal{Z}\circ\tau_{2\ell} \geq \mathcal{Z}\circ\tau_{2\ell+1},$$

and

$$(\mathcal{Z}\circ\tau_{2\ell}) \geq \frac{\left(\eta_{v_1}^{\delta}\circ\tau_{2\ell}\right)}{\eta_{v_1}^{\delta}}\cdot\mathcal{Z},$$

for $\ell = 0, 1, ..., m$. Thus, from Lemma 1.1, we arrive at

$$\begin{aligned} u(v) &> \sum_{\ell=0}^{m}\left(\prod_{i=0}^{2\ell}p(\tau_i(v))\right)\left[\frac{\mathcal{Z}(\tau_{2\ell}(v))}{p(\tau_{2\ell}(v))} - \mathcal{Z}(\tau_{2\ell+1}(v))\right] \\ &> \sum_{\ell=0}^{m}\left(\prod_{i=0}^{2\ell}p(\tau_i(v))\right)\left[\frac{1}{p(\tau_{2\ell}(v))} - 1\right]\mathcal{Z}(\tau_{2\ell}(v)) \\ &> \mathcal{Z}(v)\sum_{\ell=0}^{m}\left(\prod_{i=0}^{2\ell-1}p(\tau_i(v))\right)\frac{\eta_{v_1}^{\delta}(\tau_{2\ell}(v))}{\eta_{v_1}^{\delta}(v)}\left[1-p(\tau_{2\ell}(v))\right].\end{aligned}$$

This completes the proof. □

2.2. *Oscillation theorems*

Theorem 2.1. *Suppose that $\sigma'_{\min}(v) \geq 0$, and there is a $\xi \in \mathbf{C}^1(\mathbb{I}, \mathbb{R}^+)$ such that*

$$\limsup_{v\to\infty}\frac{1}{\mathcal{P}(v,v_0)}\int_{v_0}^{v}\left[k\xi(\mu)\mathcal{P}(v,\mu)\Phi_m(\mu) - \frac{L\xi(\mu)r(\sigma_{\min}(\mu))}{\left[\sigma'_{\min}(\mu)\right]^{\alpha}}\frac{\left[\Psi(v,\mu)\right]^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right]\mathrm{d}\mu = \infty, \tag{2.2}$$

where

$$\Phi_m(\nu) := \sum_{j=1}^{\kappa} q_j(\nu) \left[\sum_{\ell=0}^{m} \left(\prod_{i=0}^{2\ell-1} p\left(\tau_i\left(\sigma_j(\nu)\right)\right) \right) \frac{\eta_{\nu_1}^{\delta}\left(\tau_{2\ell}\left(\sigma_j(\nu)\right)\right)}{\eta_{\nu_1}^{\delta}\left(\sigma_j(\nu)\right)} \left[1 - p\left(\tau_{2\ell}\left(\sigma_j(\nu)\right)\right)\right] \right]^{\alpha}$$

and

$$\Psi(\nu,\mu) := \rho(\nu,\mu) - \frac{\xi'(\mu)}{\xi(\mu)} \left[\mathcal{P}(\nu,\mu)\right]^{1/(\alpha+1)}.$$

Then, every solution of (1.1) is oscillatory.

Proof. Assume the contrary that Eq (1.1) has a nonoscillatory solution. Thus, there is a solution u of (1.1) such that $u \in U^+$. Moreover, there is a $\nu_1 \geq \nu_0$ such that $u(\tau(\nu)) > 0$ and $u\left(\sigma_j(\nu)\right) > 0$ for $\nu \geq \nu_1$ and $j = 1, 2, ..., \kappa$. Using (B5), Eq (1.1) becomes

$$\left(r(\nu)\, \psi(u(\nu))\, [\mathcal{Z}'(\nu)]^{\alpha}\right)' \leq -k \sum_{j=1}^{\kappa} q_j(\nu)\, u^{\alpha}\left(\sigma_j(\nu)\right). \tag{2.3}$$

From Lemma 2.3, we have that (1.12) holds. Combining (2.1) and (2.3), we obtain

$$\begin{aligned} & \left(r(\nu)\, \psi(u(\nu))\, [\mathcal{Z}'(\nu)]^{\alpha}\right)' \\ \leq\ & -k \sum_{j=1}^{\kappa} q_j(\nu) \left[\mathcal{Z}\left(\sigma_j(\nu)\right)\right]^{\alpha} \\ & \times \left[\sum_{\ell=0}^{m} \left(\prod_{i=0}^{2\ell-1} p\left(\tau_i\left(\sigma_j(\nu)\right)\right) \right) \frac{\eta_{\nu_1}^{\delta}\left(\tau_{2\ell}\left(\sigma_j(\nu)\right)\right)}{\eta_{\nu_1}^{\delta}\left(\sigma_j(\nu)\right)} \left[1 - p\left(\tau_{2\ell}\left(\sigma_j(\nu)\right)\right)\right] \right]^{\alpha}, \end{aligned}$$

and so

$$\left(r(\nu)\, \psi(u(\nu))\, [\mathcal{Z}'(\nu)]^{\alpha}\right)' \leq -k\Phi_m(\nu) \left[\mathcal{Z}(\sigma_{\min}(\nu))\right]^{\alpha}. \tag{2.4}$$

Now, we define the function

$$w := \xi \cdot \frac{r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^{\alpha}}{(\mathcal{Z}^{\alpha} \circ \sigma_{\min})} > 0. \tag{2.5}$$

Then,

$$w' = \frac{\xi'}{\xi} \cdot w + \xi \left[\frac{\left(r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^{\alpha}\right)'}{(\mathcal{Z}^{\alpha} \circ \sigma_{\min})} - \frac{r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^{\alpha}}{(\mathcal{Z}^{\alpha+1} \circ \sigma_{\min})} \cdot \alpha\left(\mathcal{Z}' \circ \sigma_{\min}\right) \cdot \sigma_{\min}' \right],$$

which with (2.4) gives

$$w' \leq \frac{\xi'}{\xi} \cdot w + \xi \left[-k\Phi_m - \frac{r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^{\alpha}}{[\mathcal{Z} \circ \sigma_{\min}]^{\alpha+1}} \cdot \alpha\left(\mathcal{Z}' \circ \sigma_{\min}\right) \cdot \sigma_{\min}' \right]. \tag{2.6}$$

Since $\left(r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^{\alpha}\right)' \leq 0$, we get that

$$\begin{aligned} (\mathcal{Z}' \circ \sigma_{\min}) &\geq \frac{\left(r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^{\alpha}\right)^{1/\alpha}}{(r \circ \sigma_{\min})^{1/\alpha} \cdot (\psi \circ u \circ \sigma_{\min})^{1/\alpha}} \\ &\geq \frac{\left(r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^{\alpha}\right)^{1/\alpha}}{L^{1/\alpha} (r \circ \sigma_{\min})^{1/\alpha}}. \end{aligned}$$

Substituting $(\mathcal{Z}' \circ \sigma_{\min})$ in (2.6), we conclude that

$$\begin{aligned} w' &\leq \frac{\xi'}{\xi} \cdot w + \xi \left[-k\Phi_m - \frac{\alpha}{L^{1/\alpha}} \left[\frac{r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^{\alpha}}{[\mathcal{Z} \circ \sigma_{\min}]^{\alpha}} \right]^{1+1/\alpha} \cdot \frac{\sigma'_{\min}}{(r \circ \sigma_{\min})^{1/\alpha}} \right] \\ &= \frac{\xi'}{\xi} \cdot w - k\xi \cdot \Phi_m - \frac{\alpha}{L^{1/\alpha}} \frac{\sigma'_{\min}}{\xi^{1/\alpha} \cdot (r \circ \sigma_{\min})^{1/\alpha}} w^{1+1/\alpha}. \end{aligned} \tag{2.7}$$

By multiplying the inequality

$$w'(\mu) \leq \frac{\xi'(\mu)}{\xi(\mu)} w(\mu) - k\xi(\mu)\Phi_m(\mu) - \frac{\alpha}{L^{1/\alpha}} \frac{\sigma'_{\min}(\mu)}{\xi^{1/\alpha}(\mu)(r(\sigma_{\min}(\mu)))^{1/\alpha}} [w(\mu)]^{1+1/\alpha}$$

by $\mathcal{P}(\nu,\mu)$ and then integrating it from $\nu_2 \geq \nu_1$ to ν, we arrive at

$$\begin{aligned} \int_{\nu_2}^{\nu} \mathcal{P}(\nu,\mu) w'(\mu) \, \mathrm{d}\mu &\leq \int_{\nu_2}^{\nu} \frac{\xi'(\mu)}{\xi(\mu)} \mathcal{P}(\nu,\mu) w(\mu) \, \mathrm{d}\mu - k \int_{\nu_2}^{\nu} \xi(\mu) \mathcal{P}(\nu,\mu) \Phi_m(\mu) \, \mathrm{d}\mu \\ &\quad - \frac{\alpha}{L^{1/\alpha}} \int_{\nu_2}^{\nu} \frac{\sigma'_{\min}(\mu)}{\xi^{1/\alpha}(\mu)(r(\sigma_{\min}(\mu)))^{1/\alpha}} \mathcal{P}(\nu,\mu) [w(\mu)]^{1+1/\alpha} \, \mathrm{d}\mu, \end{aligned}$$

and so

$$\begin{aligned} k \int_{\nu_2}^{\nu} \xi(\mu) \mathcal{P}(\nu,\mu) \Phi_m(\mu) \, \mathrm{d}\mu &\leq \mathcal{P}(\nu,\nu_2) w(\nu_2) + \int_{\nu_2}^{\nu} [\mathcal{P}(\nu,\mu)]^{\alpha/(\alpha+1)} \Psi(\nu,\mu) w(\mu) \, \mathrm{d}\mu \\ &\quad - \frac{\alpha}{L^{1/\alpha}} \int_{\nu_2}^{\nu} \frac{\sigma'_{\min}(\mu)}{\xi^{1/\alpha}(\mu)(r(\sigma_{\min}(\mu)))^{1/\alpha}} \mathcal{P}(\nu,\mu) [w(\mu)]^{1+1/\alpha} \, \mathrm{d}\mu. \end{aligned}$$

Using Lemma 2.1 with $\theta = w$, $c_1 = [\mathcal{P}(\nu,\mu)]^{\alpha/(\alpha+1)} \Psi(\nu,\mu)$, and

$$c_2 = \frac{\alpha}{L^{1/\alpha}} \frac{\sigma'_{\min}(\mu)}{\xi^{1/\alpha}(\mu)(r(\sigma_{\min}(\mu)))^{1/\alpha}} \mathcal{P}(\nu,\mu),$$

we obtain

$$\begin{aligned} &\mathcal{P}(\nu,\nu_2) w(\nu_2) \\ &\geq k \int_{\nu_2}^{\nu} \xi(\mu) \mathcal{P}(\nu,\mu) \Phi_m(\mu) \, \mathrm{d}\mu - \int_{\nu_2}^{\nu} \left[\frac{L\widetilde{\alpha}}{\alpha^{\alpha}} \frac{\xi(\mu) r(\sigma_{\min}(\mu))}{\left[\sigma'_{\min}(\mu)\right]^{\alpha}} [\Psi(\nu,\mu)]^{\alpha+1} \right] \mathrm{d}\mu, \end{aligned}$$

or

$$\frac{1}{\mathcal{P}(\nu,\nu_2)} \int_{\nu_2}^{\nu} \left[k\xi(\mu) \mathcal{P}(\nu,\mu) \Phi_m(\mu) - \frac{L\xi(\mu) r(\sigma_{\min}(\mu))}{\left[\sigma'_{\min}(\mu)\right]^{\alpha}} \frac{[\Psi(\nu,\mu)]^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \right] \mathrm{d}\mu \leq w(\nu_2),$$

which contradicts assumption (2.2). This completes the proof. □

Theorem 2.2. *Suppose that $\sigma'_{\min}(\nu) \geq 0$, and there is a $\xi \in \mathbf{C}^1(\mathbb{I}, \mathbb{R}^+)$ such that*

$$\limsup_{\nu \to \infty} \int_{\nu_1}^{\nu} \left[k\xi(\mu) \Phi_m(\mu) - \frac{L}{(\alpha+1)^{\alpha+1}} \frac{[\xi'(\mu)]^{\alpha+1} (r(\sigma_{\min}(\mu)))}{\left[\xi(\mu) \sigma'_{\min}(\mu)\right]^{\alpha}} \right] \mathrm{d}\mu = \infty, \tag{2.8}$$

where Φ_m is defined as in Theorem 2.1. Then, every solution of (1.1) is oscillatory.

Proof. Assume the contrary that Eq (1.1) has a nonoscillatory solution. Thus, there is a solution u of (1.1) such that $u \in U^+$. Moreover, there is a $\nu_1 \geq \nu_0$ such that $u(\tau(\nu)) > 0$ and $u\big(\sigma_j(\nu)\big) > 0$ for $\nu \geq \nu_1$ and $j = 1, 2, ..., \kappa$.

Now, we define w as in (2.5). Proceeding as in the proof of Theorem 2.1, we arrive at (2.7). Using Lemma 2.1 with $\theta = w$, $c_1 = \xi'/\xi$, and

$$c_2 = \frac{\alpha}{L^{1/\alpha}} \frac{\sigma'_{\min}}{\xi^{1/\alpha} \cdot (r \circ \sigma_{\min})^{1/\alpha}},$$

we obtain

$$w' \leq -k\xi \cdot \Phi_m + \frac{\widetilde{\alpha} L}{\alpha^\alpha} \frac{[\xi']^{\alpha+1} \cdot (r \circ \sigma_{\min})}{\left[\xi \cdot \sigma'_{\min}\right]^\alpha}.$$

Integrating this inequality leads to

$$\int_{\nu_1}^{\nu} \left[k\xi(\mu) \Phi_m(\mu) - \frac{L}{(\alpha+1)^{\alpha+1}} \frac{[\xi'(\mu)]^{\alpha+1} \cdot (r(\sigma_{\min}(\mu)))}{\left[\xi(\mu) \sigma'_{\min}(\mu)\right]^\alpha} \right] \mathrm{d}\mu \leq w(\nu_1),$$

which contradicts assumption (2.8). This completes the proof. □

In the following theorem, we present a new criterion for the oscillation of (1.1), but unlike previous theorems, it does not need the constraint $\sigma'_{\min}(\nu) \geq 0$.

Theorem 2.3. *Suppose that there is a $\xi \in \mathbf{C}^1(\mathbb{I}, \mathbb{R}^+)$ such that*

$$\limsup_{\nu \to \infty} \int_{\nu_1}^{\nu} \left[k\xi(\mu) \Phi_m(\mu) \left[\frac{\eta_{\nu_1}(\sigma_{\min}(\mu))}{\eta_{\nu_1}(\mu)} \right]^{\delta\alpha} - \frac{L}{(\alpha+1)^{\alpha+1}} \frac{r(\mu)[\xi'(\mu)]^{\alpha+1}}{[\xi(\mu)]^\alpha} \right] \mathrm{d}\mu = \infty, \tag{2.9}$$

for $\nu_1 \geq \nu_0$, where Φ_m is defined as in Theorem 2.1. Then, every solution of (1.1) is oscillatory.

Proof. Assume the contrary that Eq (1.1) has a nonoscillatory solution. Thus, there is a solution u of (1.1) such that $u \in U^+$. Moreover, there is a $\nu_1 \geq \nu_0$ such that $u(\tau(\nu)) > 0$ and $u\big(\sigma_j(\nu)\big) > 0$ for $\nu \geq \nu_1$ and $j = 1, 2, ..., \kappa$.

Proceeding as in the proof of Theorem 2.1, we arrive at (2.4). Now, we define the function

$$w := \xi \cdot r \cdot (\psi \circ u) \cdot \left[\frac{\mathcal{Z}'}{\mathcal{Z}}\right]^\alpha > 0.$$

Then,

$$w' = \frac{\xi'}{\xi} \cdot w + \xi \left[\frac{(r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^\alpha)'}{\mathcal{Z}^\alpha} - \frac{r \cdot (\psi \circ u) \cdot [\mathcal{Z}']^\alpha}{\mathcal{Z}^{\alpha+1}} \cdot \alpha \mathcal{Z}' \right],$$

which with (2.4) gives

$$\begin{aligned} w' &\leq \frac{\xi'}{\xi} \cdot w + \xi \left[-k\Phi_m \cdot \left[\frac{(\mathcal{Z} \circ \sigma_{\min})}{\mathcal{Z}} \right]^\alpha - \alpha r \cdot (\psi \circ u) \cdot \left[\frac{\mathcal{Z}'}{\mathcal{Z}} \right]^{\alpha+1} \right] \\ &= \frac{\xi'}{\xi} \cdot w - k\xi \cdot \Phi_m \cdot \left[\frac{(\mathcal{Z} \circ \sigma_{\min})}{\mathcal{Z}} \right]^\alpha - \frac{\alpha}{r^{1/\alpha} \cdot \xi^{1/\alpha} \cdot (\psi \circ u)^{1/\alpha}} w^{1+1/\alpha} \end{aligned}$$

$$\leq \frac{\xi'}{\xi} \cdot w - k\xi \cdot \Phi_m \cdot \left[\frac{(\mathcal{Z} \circ \sigma_{\min})}{\mathcal{Z}} \right]^{\alpha} - \frac{\alpha L^{-1/\alpha}}{r^{1/\alpha} \cdot \xi^{1/\alpha}} w^{1+1/\alpha}. \tag{2.10}$$

It follows from Lemma 2.2 that $\left(\mathcal{Z}/\eta_{\nu_1}^{\delta}\right)' \leq 0$. Then,

$$\frac{(\mathcal{Z} \circ \sigma_{\min})}{\mathcal{Z}} \geq \frac{\left(\eta_{\nu_1}^{\delta} \circ \sigma_{\min}\right)}{\eta_{\nu_1}^{\delta}}.$$

So, (2.10) reduces to

$$w' \leq \frac{\xi'}{\xi} \cdot w - k\xi \cdot \Phi_m \cdot \left[\frac{\left(\eta_{\nu_1}^{\delta} \circ \sigma_{\min}\right)}{\eta_{\nu_1}^{\delta}} \right]^{\alpha} - \frac{\alpha L^{-1/\alpha}}{r^{1/\alpha} \cdot \xi^{1/\alpha}} w^{1+1/\alpha}.$$

Using Lemma 2.1 with $\theta = w$, $c_1 = \xi'/\xi$, and

$$c_2 = \frac{\alpha L^{-1/\alpha}}{r^{1/\alpha} \cdot \xi^{1/\alpha}},$$

we obtain

$$w' \leq -k\xi \cdot \Phi_m \cdot \left[\frac{\left(\eta_{\nu_1}^{\delta} \circ \sigma_{\min}\right)}{\eta_{\nu_1}^{\delta}} \right]^{\alpha} + \frac{L\widetilde{\alpha}}{\alpha^{\alpha}} \frac{r \cdot [\xi']^{\alpha+1}}{[\xi]^{\alpha}}.$$

Integrating this inequality leads to

$$\int_{\nu_1}^{\nu} \left[k\xi(\mu) \Phi_m(\mu) \left[\frac{\eta_{\nu_1}(\sigma_{\min}(\mu))}{\eta_{\nu_1}(\mu)} \right]^{\delta\alpha} - \frac{L}{(\alpha+1)^{\alpha+1}} \frac{r(\mu) [\xi'(\mu)]^{\alpha+1}}{[\xi(\mu)]^{\alpha}} \right] \mathrm{d}\mu \leq w(\nu_1),$$

which contradicts assumption (2.9). This completes the proof. □

By replacing $\mathcal{P}(\nu, \mu)$ with $(\nu - \mu)^n$, $n \in \mathbb{Z}^+$, in Theorem 2.1, we obtain an oscillation criterion for Eq (1.1) of the Kamenev type.

Corollary 2.1. *Suppose that $\sigma'_{\min}(\nu) \geq 0$, and there are $\xi \in \mathbf{C}^1(\mathbb{I}, \mathbb{R}^+)$ and $n \in \mathbb{Z}^+$ such that*

$$\limsup_{\nu\to\infty} \frac{1}{(\nu-\nu_0)^n} \int_{\nu_0}^{\nu} \left[k\xi(\mu)(\nu-\mu)^n \Phi_m(\mu) - \frac{L\xi(\mu) r(\sigma_{\min}(\mu))}{\left[\sigma'_{\min}(\mu)\right]^{\alpha}} \frac{[\Psi(\nu,\mu)]^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \right] \mathrm{d}\mu = \infty,$$

where Φ_m and Ψ are defined as in Theorem 2.1. Then, every solution of (1.1) is oscillatory.

In Theorem 2.2, by choosing $\xi(\mu) = \eta_{\nu_1}^{\alpha}(\sigma_{\min}(\mu))$, we get the following oscillation criterion for Eq (1.1).

Corollary 2.2. *Suppose that $\sigma'_{\min}(\nu) \geq 0$, and*

$$\limsup_{\nu\to\infty} \int_{\nu_1}^{\nu} \left[k\eta_{\nu_1}^{\alpha}(\sigma_{\min}(\mu)) \Phi_m(\mu) - \frac{L\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{\left[\sigma'_{\min}(\mu)\right]}{\eta_{\nu_1}(\sigma_{\min}(\mu)) \left[r(\sigma_{\min}(\mu))\right]^{1/\alpha}} \right] \mathrm{d}\mu = \infty, \tag{2.11}$$

where Φ_m is defined as in Theorem 2.1. Then, every solution of (1.1) is oscillatory.

For the linear special case of Eq (1.1), which is:

$$(r(\nu)\ \psi(u(\nu))\ \mathcal{Z}'(\nu))' + q(\nu)\,u(\sigma(\nu)) = 0, \tag{2.12}$$

we present the following oscillation criterion.

Corollary 2.3. *Suppose that there is a $\xi \in \mathbf{C}^1(\mathbb{I}, \mathbb{R}^+)$ such that*

$$\limsup_{\nu\to\infty} \int_{\nu_1}^{\nu} \left[\xi(\mu)\,\Phi_m(\mu) \left[\frac{\eta_{\nu_1}(\sigma_{\min}(\mu))}{\eta_{\nu_1}(\mu)} \right]^{\delta} - \frac{L}{4} \frac{r(\mu)\,[\xi'(\mu)]^2}{\xi(\mu)} \right] \mathrm{d}\mu = \infty,$$

for $\nu_1 \geq \nu_0$, where Φ_m is defined as in Theorem 2.1. Then, every solution of (2.12) is oscillatory.

2.3. Examples and discussion

In this section, we confirm with examples and remarks the novelty of the results and their efficiency in the oscillation test compared to previous relevant results.

Remark 2.1. *Let $\kappa = 1$ and $\alpha = 1$. Then, Eq (1.1) reduces to (1.5). It is simple to verify that*

$$\Phi_0(\nu) = q(\nu)\,[1 - p(\sigma(\nu))] = Q(\nu),$$

where $Q(\nu)$ is defined in Theorem 1.2. Moreover,

$$\begin{aligned} \Phi_m(\nu) &= Q(\nu) + q(\nu) \sum_{\ell=1}^{m} \left(\prod_{i=0}^{2\ell-1} p(\tau_i(\sigma(\nu))) \right) \frac{\eta_{\nu_1}^{\delta}(\tau_{2\ell}(\sigma(\nu)))}{\eta_{\nu_1}^{\delta}(\sigma(\nu))} [1 - p(\tau_{2\ell}(\sigma(\nu)))] \\ &\geq Q(\nu). \end{aligned} \tag{2.13}$$

Therefore, if $m = 0$, we note that Theorems 2.1 and 2.2 turns into Theorem 1.2 and 1.3, respectively. Also, Corollary 2.2 turn into Theorems 1.1.

In general, based on (2.13), Theorems 2.1 and 2.2 generalize and improve Theorems 1.1–1.3.

Remark 2.2. *Theorems 1.1–1.3, 2.1, and 2.2 require that constraint $\sigma'_{\min}(\nu) \geq 0$. However, Theorem 2.3 does not need this constraint.*

Example 2.1. *Consider the NDDE*

$$\left(\frac{1}{1 + \sin^2(u(\nu))} \left[(u(\nu) + p_0 u(\tau_0 \nu))' \right]^{\alpha} \right)' + \frac{1}{\nu^{\alpha+1}} \sum_{j=1}^{\kappa} a_j u^{\alpha}\left(\lambda_j \nu\right) = 0, \tag{2.14}$$

where $\alpha > 0$ is an odd integer, τ_0, $\lambda_j \in (0, 1]$ for $j = 1, 2, ..., \kappa$, $p_0 \in [0, 1)$, and $a_j \geq 0$. We see that $r(\nu) = 1$, $p(\nu) = p_0$, $\tau(\nu) = \tau_0 \nu$, $\sigma_j(\nu) = \lambda_j \nu$, $q_j(\nu) = a_j/\nu^{\alpha+1}$, $F(u) = u^{\alpha}$, and $\psi(u) = 1/\left(1 + \sin^2 u\right)$. It is simple to confirm that $k = 1$, $l = 1/2$, $L = 1$, $\delta = \sqrt[\alpha]{2}$, and $\tau_\ell(\nu) = \tau_0^{\ell} \nu$ for $\ell = 0, 1, ...$ Then,

$$\Phi_m(\nu) = \frac{1}{\nu^{\alpha+1}} [1 - p_0]^{\alpha} \sum_{j=1}^{\kappa} a_j \left[\sum_{\ell=0}^{m} p_0^{2\ell} \tau_0^{2\sqrt[\alpha]{2}\ell} \right]^{\alpha} = \frac{1}{\nu^{\alpha+1}} \widetilde{q},$$

where

$$\widetilde{q} := [1-p_0]^{\alpha} \sum_{j=1}^{\kappa} a_j \left[\sum_{\ell=0}^{m} p_0^{2\ell} \tau_0^{2\sqrt[\alpha]{2}\ell} \right]^{\alpha}.$$

By choosing $\xi(\mu) = \mu^{\alpha}$ and $\mathcal{P}(\nu,\mu) = (\nu-\mu)^{\alpha+1}$, condition (2.2) reduces to

$$\begin{aligned} & \limsup_{\nu\to\infty} \frac{1}{(\nu-\mu)^{\alpha+1}} \int_{\nu_0}^{\nu} \left[\widetilde{q} \frac{(\nu-\mu)^{\alpha+1}}{\mu} - \frac{\mu^{\alpha}\left[(\alpha+1)-\alpha\frac{(\nu-\mu)}{\mu}\right]^{\alpha+1}}{\lambda_0^{\alpha}(\alpha+1)^{\alpha+1}} \right] \mathrm{d}\mu \\ = & \left(\widetilde{q} - \frac{1}{\lambda_0^{\alpha}(\alpha+1)^{\alpha+1}} \right)(+\infty), \end{aligned}$$

where $\lambda_0 := \min_{1\le j\le\kappa} \lambda_j$, which is achieved if

$$\widetilde{q} > \frac{\alpha^{\alpha+1}}{\lambda_0^{\alpha}(\alpha+1)^{\alpha+1}}. \tag{2.15}$$

Moreover, condition (2.8) reduces to (2.15). On the other hand, condition (2.9) becomes

$$\limsup_{\nu\to\infty} \int_{\nu_1}^{\nu} \left[\widetilde{q}\lambda_0^{\alpha\sqrt[\alpha]{2}} - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \right] \frac{1}{\mu} \mathrm{d}\mu = \infty,$$

which is achieved if

$$\widetilde{q} > \frac{\alpha^{\alpha+1}}{\lambda_0^{\alpha\sqrt[\alpha]{2}}(\alpha+1)^{\alpha+1}}. \tag{2.16}$$

Therefore, using Theorems 2.1–2.3, every solution of (2.14) is oscillatory if one of the conditions (2.15) and (2.16) holds.

In order to determine which conditions are most efficient in the oscillation test, it is easy to note that

$$\frac{\alpha^{\alpha+1}}{\lambda_0^{\alpha}(\alpha+1)^{\alpha+1}} < \frac{\alpha^{\alpha+1}}{\lambda_0^{\alpha\sqrt[\alpha]{2}}(\alpha+1)^{\alpha+1}},$$

so criterion (2.15) provides the sharpest results in the oscillation test.

Remark 2.3. *To compare our results with previous results in the literature, we consider the special case*

$$\left(\frac{1}{1+\sin^2(u(\nu))} \left(u(\nu) + \frac{1}{2} u(0.9\,\nu) \right)' \right)' + \frac{a}{\nu^2} u(0.5\,\nu) = 0, \tag{2.17}$$

where $\lambda \in (0,1]$ and $a > 0$. Applying Theorems 1.1–1.3, we find that every solution of (2.17) is oscillatory if $a > 1$. However, condition (2.15) becomes $a > 0.83598$, which is better for oscillation. Moreover, Figure 1 shows one of the oscillatory solutions of Eq (2.14) when $p_0 = 0$, $\alpha = 1$, $k = 1$, $a_1 = 10$, and $\lambda_1 = 1$. We note that the solution has an infinite number of arbitrary zeros that converge to infinity. We also notice that the distance between the zeros of this oscillatory solution increases constantly. Also, oscillation waves rise with increasing t.

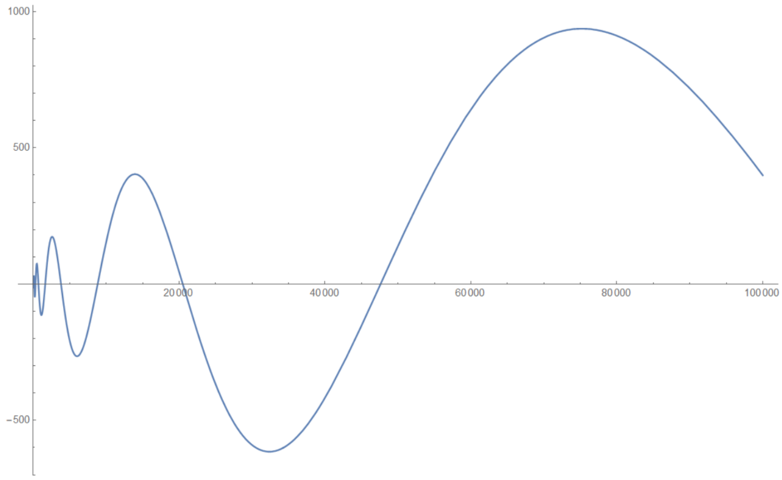

Figure 1. One of the oscillatory solutions to Eq (2.14).

Example 2.2. *Consider the delay equation*

$$\left(\frac{1+\mathbf{e}^{-u^2(\nu)}}{\mathbf{e}^{\nu}}\frac{\mathrm{d}}{\mathrm{d}\nu}\left[u(\nu)+p_0u(\nu-\tau_0)\right]'\right)'+\sum_{j=1}^{\kappa}\mathbf{e}^{a_j-\nu}u\left(\nu-\lambda_j\right)=0, \tag{2.18}$$

where τ_0, p_0, *and* λ_j *are positive, for* $j=1,2,...,\kappa$. *We see that* $\alpha=1$, $r(\nu)=\mathbf{e}^{-\nu}$, $p(\nu)=p_0$, $\tau(\nu)=\nu-\tau_0$, $\sigma_j(\nu)=\nu-\lambda_j$, $q(\nu)=\mathbf{e}^{a_j-\nu}$, $F(u)=u$, *and* $\psi(u)=1+\mathbf{e}^{-u^2}$. *It is simple to confirm that* $l=1$, $L=2$, $\delta=2$, *and*

$$\Phi_m(\nu)=[1-p_0]\,\mathbf{e}^{-\nu}\sum_{j=1}^{\kappa}\mathbf{e}^{a_j}\sum_{\ell=0}^{m}p_0^{2\ell}\mathbf{e}^{-4\ell\tau_0}=\widehat{q}\mathbf{e}^{-\nu},$$

where

$$\widehat{q}:=[1-p_0]\sum_{j=1}^{\kappa}\mathbf{e}^{a_j}\sum_{\ell=0}^{m}p_0^{2\ell}\mathbf{e}^{-4\ell\tau_0}.$$

By choosing $\xi(\mu)=\mathbf{e}^{\mu}$ *and* $\mathcal{P}(\nu,\nu_0)=(\nu-\mu)^2$, *condition (2.2) becomes*

$$\limsup_{\nu\to\infty}\frac{1}{(\nu-\nu_0)^2}\int_{\nu_0}^{\nu}\left[\widehat{q}(\nu-\mu)^2-\frac{1}{2\mathbf{e}^{\lambda_0}}\left[2-(\nu-\mu)\right]^2\right]\mathrm{d}\mu=\infty,$$

where $\lambda_0:=\max_{1\le j\le\kappa}\lambda_j$, *which is achieved if*

$$\widehat{q}>\frac{\mathbf{e}^{\lambda_0}}{2}. \tag{2.19}$$

Using Theorem 2.1, every solution of (2.18) is oscillatory if (2.19) holds.

Remark 2.4. *In 2018, Grace et al. [29] presented improved criteria for the oscillation of NDDE*

$$\left(r(\nu)\left[\mathcal{Z}'(\nu)\right]^{\alpha}\right)'+q(\nu)\,u^{\alpha}(\sigma(\nu))=0.$$

Applying the results in [29], we find that every solution of

$$\left[u(\nu)+p_0u(\tau_0\nu)\right]''+\frac{a}{\nu^2}u(\lambda\nu)=0 \tag{2.20}$$

is oscillatory if

$$a\lambda\left[1-p_0\right]\left[1+a\lambda\left(1-p_0\right)\right]\ln\frac{1}{\lambda}>\frac{1}{\mathbf{e}}, \tag{2.21}$$

(Theorem 3, [29]), or

$$a\lambda^{c}\left[1-p_0\right]>\frac{1}{4},$$

(Theorem 6, [29]), where $c=1/\left(1+a\lambda\left(1-p_0\right)\right)$. *For (2.20), condition (1.9) and (2.8) reduce to*

$$a>\frac{1}{4\lambda\left[1-p_0\right]}, \tag{2.22}$$

and

$$a\left[1-p_0\right]\sum_{\ell=0}^{m}p_0^{2\ell}\tau_0^{2\ell}>\frac{1}{4\lambda}, \tag{2.23}$$

respectively. Figure 2 compares the lower bounds of a-values with respect to $\lambda\in(0,1]$ *for conditions (2.21)–(2.23), when* $p_0=0.5$ *and* $\tau_0=0.9$. *We note that condition (2.21) provides the best results for oscillation in* $(0,0.214722)$, *while condition (2.23) is the best in* $(0,0.214722]$.

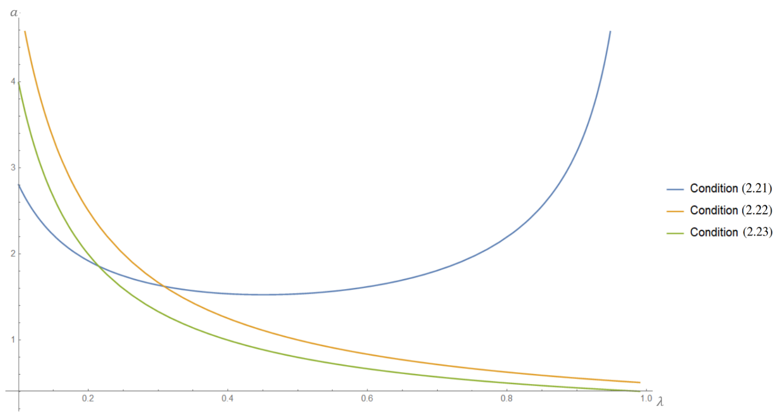

Figure 2. Oscillation criteria of Eq (2.20).

Remark 2.5. *Consider the special case of (2.20) in which*

$$\left[u(v)+\frac{9}{10}u\left(\frac{9}{10}v\right)\right]''+\frac{a}{v^2}u\left(\frac{1}{2}v\right)=0. \tag{2.24}$$

Table 1 compares the various criteria that confirm the oscillation of the solutions of Eq (2.24). Theorem 2.2 provides the best results for oscillation.

Table 1. Oscillation parameters of Eq (2.24).

	Theorem 1.1	Theorem 3 in [29]	Theorem 3 in [29]	Theorem 2.2
Criteria	$a>5.0000$	$a>7.6719$	$a>4.4113$	$a>1.7197$

3. Conclusions

The precision of the relationships and inequalities utilized has an impact on the investigation of the oscillatory behavior of FDEs. In this paper, we examined the oscillatory behavior of solutions for NDDE (1.1). We presented a novel relationship between u and $\mathcal{Z}$ as an extension of the findings in [35]. We set more than one standard that guarantees the oscillation of all solutions of the equation under study. Through comparisons and examples, we found the following:

– Our theorems extend the results in [20, 21, 23] to the half-linear case as well as to the case of multiple delays.
– Theorem 2.2 does not need monotonic constraints on the delay functions, while previous results required $\sigma'(v) \geq 0$.
– Based on (2.13), our results not only generalize previous results but also improve upon them. This improvement can be seen in Remarks 2.3–2.5.
– The previous results do not take into account the effect of $\tau(v)$, while the new relationship (2.1) makes our criteria affected by $\tau(v)$.

It would be interesting for future studies to obtain an improved relationship between the solution and the corresponding function without having to require that $\psi(u) \geq l > 0$ and $p_0 < 1$. Recently, there are many studies that have developed oscillation criteria for higher-order differential equations, see for example [41–45]. We also propose to extend our results to include the non-canonical case as well as higher-order equations.

Author contributions

O. M. developed the conceptualization and methodology. A. A. wrote the original draft. O. M. and A. A. provided examples, figures, and discussion. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that there are no conflicts of interest in this paper.

References

1. V. Volterra, Sur la theorie mathematique des phenomenes hereditaires, *J. Math. Pure. Appl.*, **7** (1928), 249–298.
2. A. D. Mishkis, *Lineare differentialgleichungen mit nacheilenden argumentom*, Berlin: Verlag, 1955.
3. R. Bellman, J. M. Danskin, *A survey of the mathematical theory of time lag, retarded control, and hereditary processes*, United States: RAND Corporation, 1954.
4. R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Dordrecht: Springer, 2002. https://doi.org/10.1007/978-94-017-2515-6
5. I. Gyori, G. Ladas, *Oscillation theory of delay differential equations with applications*, Oxford University Press, 1991. https://doi.org/10.1093/oso/9780198535829.001.0001
6. L. H. Erbe, Q. Kong, B. G. Zhong, *Oscillation theory for functional differential equations*, Routledge, 1995. https://doi.org/10.1201/9780203744727
7. O. Moaaz, W. Albalawi, Differential equations of the neutral delay type: More efficient conditions for oscillation, *AIMS Math.*, **8** (2023), 12729–12750. https://doi.org/10.3934/math.2023641
8. A. Palanisamy, J. Alzabut, V. Muthulakshmi, S. S. Santra, K. Nonlaopon, Oscillation results for a fractional partial differential system with damping and forcing terms, *AIMS Math.*, **8** (2023), 4261–4279. https://doi.org/10.3934/math.2023212
9. S. S. Santra, A. K. Sethi, O. Moaaz, K. M. Khedher, S. W. Yao, New oscillation theorems for second-order differential equations with canonical and non-canonical operator via riccati transformation, *Mathematics*, **9** (2021), 1111. https://doi.org/10.3390/math9101111
10. J. K. Hale, Functional differential equations, In: *Analytic theory of differential equations*, Berlin: Springer, 1971. https://doi.org/10.1007/BFb0060406
11. C. Huang, B. Liu, C. Qian, J. Cao, Stability on positive pseudo almost periodic solutions of HPDCNNs incorporating D operator, *Math. Comput. Simulat.*, **190** (2021), 1150–1163. https://doi.org/10.1016/j.matcom.2021.06.027
12. B. Liu, Finite-time stability of CNNs with neutral proportional delays and time-varying leakage delays, *Math. Method. Appl. Sci.*, **40** (2017), 167–174. https://doi.org/10.1002/mma.3976
13. C. Huang, B. Liu, H. Yang, J. Cao, Positive almost periodicity on SICNNs incorporating mixed delays and D operator, *Nonlinear Anal. Model.*, **27** (2022) 719–739. https://doi.org/10.15388/namc.2022.27.27417
14. C. G. Philos, Oscillation theorems for linear differential equation of second order, *Arch. Math.*, **53** (1989), 483–492. https://doi.org/10.1007/BF01324723

15. I. V. Kamenev, Oscillation criteria, connected with averaging, for the solutions of second order ordinary differential equations, *Differentsial'nye Uravneniya*, **10** (1974), 246–252.

16. J. Sugie, K. Ishihara, Philos-type oscillation criteria for linear differential equations with impulsive effects, *J. Math. Anal. Appl.*, **470** (2019), 911–930. https://doi.org/10.1016/j.jmaa.2018.10.041

17. Z. Xu, X. Liu, Philos-type oscillation criteria for Emden-Fowler neutral delay differential equations, *J. Comput. Appl. Math.*, **206** (2007), 1116-1126. https://doi.org/10.1016/j.cam.2006.09.012

18. O. Moaaz, E. M. Elabbasy, A. Muhib, Oscillation criteria for even-order neutral differential equations with distributed deviating arguments, *Adv. Differ. Equ.*, **2019** (2019), 297. https://doi.org/10.1186/s13662-019-2240-z

19. J. Manojlović, Oscillation theorems for nonlinear differential equations, *Electron. J. Qual. Theory Differ. Equ.*, **2000** (2000), 1–21. https://doi.org/10.14232/ejqtde.2000.1.1

20. J. Džurina, D. Lacková, Oscillation results for second order nonlinear differential equations, *Centr. Eur. J. Math.*, **2** (2004), 57–66. https://doi.org/10.2478/BF02475950

21. Y. Şahiner, On oscillation of second order neutral type delay differential equations, *Appl. Math. Comput.*, **150** (2004), 697–706. https://doi.org/10.1016/S0096-3003(03)00300-X

22. Z. Xu, P. Weng, Oscillation of second order neutral equations with distributed deviating argument, *J. Comput. Appl. Math.*, **202** (2007), 460–477. https://doi.org/10.1016/j.cam.2006.03.001

23. L. Ye, Z. Xu, Oscillation criteria for second order quasilinear neutral delay differential equations, *Appl. Math. Comput.*, **207** (2009), 388–396. https://doi.org/10.1016/j.amc.2008.10.051

24. Q. Li , R. Wang, F. Chen, T. Li, Oscillation of second-order nonlinear delay differential equations with nonpositive neutral coefficients, *Adv. Differ. Equ.*, **2015** (2015), 35. https://doi.org/10.1186/s13662-015-0377-y

25. R. Arul, V. S. Shobha, Improvement results for oscillatory behavior of second order neutral differential equations with nonpositive neutral term, *J. Adv. Math. Comput. Sci.*, **12** (2015), 1–7. https://doi.org/10.9734/BJMCS/2016/20641

26. S. R. Grace, Oscillatory behavior of second-order nonlinear differential equations with a nonpositive neutral term, *Mediterr. J. Math.*, **14** (2017), 229. https://doi.org/10.1007/s00009-017-1026-3

27. S. R. Grace, J. R. Graef, I. Jadlovská, Oscillatory behavior of second order nonlinear delay differential equations with positive and negative neutral terms. *Differ. Equ. Appl.*, **12** (2020), 201–211. https://doi.org/10.7153/dea-2020-12-13

28. B. Baculíková, B. Sudha, K. Thangavelu, E. Thandapani, Oscillation of second order delay differential equations with nonlinear nonpositive neutral term, *Math. Slovaca*, **72** (2022), 103–110. https://doi.org/10.1515/ms-2022-0007

29. S. Grace, J. Dzurina, I. Jadlovska, T. Li, An improved approach for studying oscillation of second-order neutral delay differential equations, *J. Inequal. Appl.*, **2018** (2018), 193. https://doi.org/10.1186/s13660-018-1767-y

30. O. Moaaz, M. Anis, D. Baleanu, A. Muhib, More effective criteria for oscillation of second-order differential equations with neutral arguments, *Mathematics*, **8** (2020), 986. https://doi.org/10.3390/math8060986

31. O. Moaaz, New criteria for oscillation of nonlinear neutral differential equations, *Adv. Differ. Equ.*, **2019** (2019), 484. https://doi.org/10.1186/s13662-019-2418-4

32. O. Moaaz, H. Ramos, J. Awrejcewicz, Second-order Emden-Fowler neutral differential equations: A new precise criterion for oscillation, *Appl. Math. Lett.*, **118** (2021), 107172. https://doi.org/10.1016/j.aml.2021.107172

33. T. S. Hassan, O. Moaaz, A. Nabih, M. B. Mesmouli, A. M. A. El-Sayed, New sufficient condi tions for oscillation of second-order neutral delay differential equations, *Axioms*, **10** (2021), 281. https://doi.org/10.3390/axioms10040281

34. O. Moaaz, A. Muhib, S. Owyed, E. E. Mahmoud, A. Abdelnaser, Second-order neutral differential equations: Improved criteria for testing the oscillation, *J. Math.*, **2021** (2021), 6665103. https://doi.org/10.1155/2021/6665103

35. O. Moaaz, C. Cesarano, B. Almarri, An improved relationship between the solution and its corresponding function in fourth-order neutral differential equations and its applications, *Mathematics*, **11** (2023), 1708. https://doi.org/10.3390/math11071708

36. Q. Feng, B. Zheng, Oscillation criteria for nonlinear third-order delay dynamic equations on time scales involving a super-linear neutral term, *Fractal Fract.*, **8** (2024), 115. https://doi.org/10.3390/fractalfract8020115

37. K. S. Vidhyaa, R. Deepalakshmi, J. R. Graef, E. Thandapani, Oscillatory behavior of semi-canonical third-order delay differental equations with a superlinear neutral term, *Appl. Anal. Discr. Math.*, **00** (2024), 6. https://doi.org/10.2298/AADM210812006V

38. N. Prabaharan, E. Thandapani, E. Tunç, Asymptotic behavior of semi-canonical third-order delay differential equations with a superlinear neutral term, *Palestine J. Math.*, **12** (2023).

39. C. Huang, H. Yang, J. Cao, Weighted pseudo almost periodicity of multi-proportional delayed shunting inhibitory cellular neural networks with D operator, *Discrete Cont. Dyn. S*, **14** (2020), 1259–1272. https://doi.org/10.3934/dcdss.2020372

40. S. Y. Zhang, Q. R. Wang, Oscillation of second-order nonlinear neutral dynamic equations on time scales, *Appl. Math. Comput.*, **216** (2010), 2837–2848. https://doi.org/10.1016/j.amc.2010.03.134

41. J. Alzabut, S. R. Grace, S. S. Santra, G. N. Chhatria, Asymptotic and oscillatory behaviour of third order non-linear differential equations with canonical operator and mixed neutral terms, *Qual. Theory Dyn. Syst.*, **22** (2023), 15. https://doi.org/10.1007/s12346-022-00715-6

42. J. Alzabut, S. R. Grace, G. N. Chhatria, New oscillation results for higher order nonlinear differential equations with a nonlinear neutral terms, *J. Math. Comput. Sci.*, **28** (2023), 294–305. https://doi.org/10.22436/jmcs.028.03.07

43. I. Jadlovská, J. Džurina, J. R. Graef, S. R. Grace, Sharp oscillation theorem for fourth-order linear delay differential equations, *J. Inequal. Appl.*, **2022** (2022), 122. https://doi.org/10.1186/s13660-022-02859-0

44. S. Janaki, V. Ganesan, On the oscillatory behavior of a class of even order nonlinear damped delay differential equations with distributed deviating arguments, *Int. J. Appl. Math.*, **36** (2023), 391. https://doi.org/10.12732/ijam.v36i3.6

45. H. Ramos, O. Moaaz, A. Muhib, J. Awrejcewicz, More effective results for testing oscillation of non-canonical neutral delay differential equations, *Mathematics*, **9** (2021), 1114. https://doi.org/10.3390/math9101114