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Research article

The inhomogeneous complex partial differential equations for bi-polyanalytic functions

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Abstract: In this paper, we study a Riemann-Hilbert problem related to inhomogeneous complex partial differential operators of higher order on the unit disk. Applying the Cauchy-Pompeiu formula, we find out the solvable conditions and obtain the representation of the solutions. Then, we investigate the boundary value problems for bi-polyanalytic functions with the Dirichlet and Riemann-Hilbert boundary conditions, obtain the specific solution and the solvable conditions, and extend the conclusion to the corresponding higher-order problems. Therefore, we obtain the solution to the half-Neumann problem of higher order for bi-polyanalytic functions.

Keywords: Riemann-Hilbert problems; complex partial differential equations; Cauchy-Pompeiu formula; bi-polyanalytic functions; Dirichlet problems
Mathematics Subject Classification: 32A30, 30C45

1. Introduction

The Cauchy integral representations play an important role in the function theory of one or several complex variables, among which the Cauchy integral formula and the Cauchy-Pompeiu formula [1] are the two most important categories. The classical Cauchy-Pompeiu formulas [2] in $\mathbb{C}$ are:

$$w(z) = \frac{1}{2\pi i}\int_{\partial G} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_G w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in G, \tag{1.1}$$

$$w(z) = -\frac{1}{2\pi i}\int_{\partial G} w(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_G w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}, \quad z \in G, \tag{1.2}$$

where G is a bounded smooth domain in the complex plane and $w \in C^1(G;\mathbb{C}) \bigcap C(\overline{G};\mathbb{C})$.

Obviously, the classical Cauchy-Pompeiu formulas are closely related to complex partial differential operators. Thus, lots of boundary value problems related to complex partial differential equations were solved with the help of the Cauchy integral formula and the Cauchy-Pompeiu formula in the complex plane $\mathbb{C}$, see [3–6]. By iterating the Cauchy-Pompeiu formula for one variable, the solutions to second order systems in polydomains composed by the Laplace and the Bitsadze operators were obtained [7]. The Riemann-Hilbert problems for generalized analytic vectors and complex elliptic partial differential equations of higher order were investigated in $\mathbb{C}$ [8]. By constructing a weighted Cauchy-type kernel, the Cauchy-Pompeiu integral representation in the constant weights case and orthogonal case were obtained [9]. The classical Cauchy-Pompeiu formula was generalized to different cases for different kinds of functions in $\mathbb{C}$ [7], among which a high order case is:

$$w(z) = \sum_{\mu=0}^{m-1} \frac{1}{2\pi i} \int_{\partial G} \frac{1}{\mu!} \frac{(\overline{z-\zeta})^{\mu}}{\zeta - z} \partial_{\bar{\zeta}}^{\mu} w(\zeta) d\zeta - \frac{1}{\pi} \int_{G} \frac{1}{(m-1)!} \frac{(\overline{z-\zeta})^{m-1}}{\zeta - z} \partial_{\bar{\zeta}}^{m} w(\zeta) d\xi d\eta, \tag{1.3}$$

where $w \in C^m(G;\mathbb{C}) \bigcap C^{m-1}(\overline{G};\mathbb{C})$ and $1 \leq m$, see [10]. The first item in the right hand of (1.3) is the Cauchy integral expression for m-holomorphic functions on D, and the second item in (1.3) is a singular integral operator. Let

$$T_{0,m} f(z) = \frac{-1}{\pi} \int_{D} \frac{1}{(m-1)!} \frac{(\overline{z-\zeta})^{m-1}}{\zeta - z} f(\zeta) d\xi d\eta, \tag{1.4}$$

where $m \geq 1$, $f \in C(\partial G)$ with G being a bounded domain in the complex plane, then

$$\frac{\partial^m T_{0,m} f(z)}{\partial \bar{z}^m} = f(z),$$

see [11]. Therefore, the solution to $\partial_{\bar{z}}^m w(z) = f(z)$ can be expressed as (1.3), and $T_{0,m} f(z)$ provides a particular solution to $\partial_{\bar{z}}^m w(z) = f(z)$. Thus it can be seen that, the Cauchy-Pompeiu formulas provide the integral representations of solutions to some partial differential equations.

The generalizations of the classical Cauchy-Pompeiu formula have contributed to the flourishing development of boundary value problems in $\mathbb{C}$. Many types of boundary value problems for analytic functions or polyanalytic functions have emerged in $\mathbb{C}$. Among them, Riemann boundary value problems and Dirichlet problems are the two major categories. By using the Cauchy-Pompeiu formula and its generalizations, the solvable conditions were found, and the integral expressions of the solutions to some problems were obtained. Some were discussed on different ranges of the unit disk, such as a generalized Cauchy-Riemann equation with super-singular points on a half-plane [12], the variable exponent Riemann boundary value problem for Liapunov open curves [13], Schwarz boundary value problems for polyanalytic equation in a sector ring [14], and so on [15–17]. Some were investigated for different equations, such as bi-harmonic equations with a p-Laplacian [18] and so on [12,19,20], which are different from the partial differential equations in [7].

As we all know, analytic functions are defined by a pair of Cauchy-Riemann equations. Bi-analytic functions [21] arise from the generalized system of Cauchy-Riemann equations

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \theta, & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \omega, \\ (k+1)\frac{\partial \theta}{\partial x} + \frac{\partial \omega}{\partial y} = 0, & (k+1)\frac{\partial \theta}{\partial y} - \frac{\partial \omega}{\partial x} = 0, \end{cases}$$

where $k \in \mathbb{R}$, $k \neq -1$, $\phi(z) = (k+1)\theta - i\omega$ is called the associated function of $f(z) = u + iv$. A special case of bi-analytic functions is

$$\partial_{\bar{z}} f(z) = \frac{\lambda - 1}{4\lambda} \phi(z) + \frac{\lambda + 1}{4\lambda} \overline{\phi(z)}, \quad \partial_{\bar{z}} \phi(z) = 0.$$

Bi-analytic functions are generalizations of analytic functions. They are important to studying elasticity problems and, therefore, have attracted the attention of many scholars. Begehr and Kumar [22] successfully obtained the solution to the Schwarz and Neumann boundary value problems for bi-polyanalytic functions. Lin and Xu [23] investigated the Riemann problem for (λ, k) bi-analytic functions. There are many other excellent conclusions as well [24, 25].

So far, there have been few results for boundary value problems of bi-analytic functions with Riemann-Hilbert boundary conditions. Stimulated by this, we investigate this type of problems for bi-polyanalytic functions [26]

$$\partial_{\bar{z}} f(z) = \frac{\lambda - 1}{4\lambda}\phi(z) + \frac{\lambda + 1}{4\lambda}\overline{\phi(z)}, \quad \partial_{\bar{z}}^n \phi(z) = 0 \ \ (n \geq 1),$$

where $\lambda \in \mathbb{R}\backslash\{-1, 0, 1\}$, which are the extensions of bi-analytic functions.

To solve the boundary value problems for bi-polyanalytic functions, in this paper, we first discuss a Riemann-Hilbert problem related to complex partial differential operators of higher order on the unit disk, and then we investigate the boundary value problems for bi-polyanalytic functions with the Dirichlet and Riemann-Hilbert boundary conditions, the half-Neumann boundary conditions and the mixed boundary conditions. Applying this method, we can also discuss other related systems of complex partial differential equations of higher order for bi-polyanalytic functions.

2. Some lemmas

Let $\Re z$ and $\Im z$ represent the real and imaginary parts of the complex number z, respectively. Let $C^m(G)$ represent the set of functions whose partial derivatives of order m are all continuous within G, and $C(G)$ represent the set of continuous functions on G. To get the main results, we need the following lemmas:

Lemma 2.1. *[2] Let G be a bounded smooth domain in the complex plane, $f \in L_1(G; \mathbb{C})$ and*

$$Tf(z) = \frac{-1}{\pi}\int_G \frac{f(\zeta)}{\zeta - z} d\xi d\eta,$$

then $\partial_{\bar{z}} Tf(z) = f(z)$.

Lemma 2.2. *Let D be the unit disk in $\mathbb{C}$. For $\varphi \in C(\partial D, \mathbb{R})$ and $f_1 \in C^{m-1}(\bar{D}, \mathbb{C})$ $(m \geq 1)$ with $\partial_{\bar{z}} f_{\kappa-1} = f_\kappa$ $(\kappa = 2, \cdots, m)$, $f_m = 0$, let*

$$\begin{aligned} W(z) = &\frac{1}{2\pi i}\int_{\partial D} \varphi(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\partial D}\sum_{\mu=1}^{m-1}\frac{1}{\mu!}\Re[(\overline{z - \zeta})^\mu f_\mu(\zeta)]\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} \\ &+ \frac{1}{2\pi}\int_{\partial D}\sum_{\mu=0}^{m-1}\frac{1}{\mu!}\Im\{[(\overline{z-\zeta})^\mu - (-\bar{\zeta}^\mu)]f_\mu(\zeta)\}\frac{d\zeta}{\zeta}, \end{aligned} \tag{2.1}$$

then $\Re W = \varphi$, $\partial_{\bar{z}}^m W(z) = 0$ and $\partial_{\bar{z}}^s W(z) = f_s(z)$ $(s = 1, 2, \cdots, m - 1)$.

Proof. $\Re W = \varphi$ is due to the property of the Schwarz operator, as $\frac{\zeta+z}{\zeta-z}$ is a pure imaginary number if $z \to \partial D$.

Let

$$\begin{cases} A = \dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\sum_{\mu=1}^{m-1}\frac{1}{\mu!}\Re[(\overline{z-\zeta})^{\mu} f_{\mu}(\zeta)]\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}, \\ B = \dfrac{1}{2\pi}\displaystyle\int_{\partial D}\sum_{\mu=0}^{m-1}\frac{1}{\mu!}\Im\{[(\overline{z-\zeta})^{\mu} - (-\bar{\zeta}^{\mu})]f_{\mu}(\zeta)\}\frac{d\zeta}{\zeta}. \end{cases}$$

Then

$$\begin{cases} A_{\bar{z}} = \displaystyle\sum_{\mu=1}^{m-1}\frac{1}{\mu!}\Big[\frac{1}{2\pi i}\int_{\partial D}\mu(\overline{z-\zeta})^{\mu-1} f_{\mu}(\zeta)\frac{d\zeta}{\zeta-z} - \frac{1}{2\pi i}\int_{\partial D}\mu(\overline{z-\zeta})^{\mu-1} f_{\mu}(\zeta)\frac{d\zeta}{2\zeta}\Big], \\ A^{s}_{\bar{z}} = \displaystyle\sum_{\mu=s}^{m-1}\frac{1}{(\mu-s)!}\Big[\frac{1}{2\pi i}\int_{\partial D}(\overline{z-\zeta})^{\mu-s}\frac{f_{\mu}d\zeta}{\zeta-z} - \frac{1}{2\pi i}\int_{\partial D}(\overline{z-\zeta})^{\mu-s}\frac{f_{\mu}d\zeta}{2\zeta}\Big]\ (s=2,\cdots,m-1), \\ A^{m}_{\bar{z}} = 0, \end{cases} \tag{2.2}$$

and

$$\begin{cases} B_{\bar{z}} = \displaystyle\sum_{\mu=1}^{m-1}\frac{1}{4\pi i\mu!}\int_{\partial D}\mu(\overline{z-\zeta})^{\mu-1} f_{\mu}(\zeta)\frac{d\zeta}{\zeta}, \\ B^{s}_{\bar{z}} = \displaystyle\sum_{\mu=s}^{m-1}\frac{1}{4\pi i(\mu-s)!}\int_{\partial D}(\overline{z-\zeta})^{\mu-s} f_{\mu}(\zeta)\frac{d\zeta}{\zeta}\ \ (s=2,\cdots,m-1), \\ B^{m}_{\bar{z}} = 0. \end{cases} \tag{2.3}$$

By (2.2), (2.3), and the Cauchy-Pompeiu formula (1.1), we get

$$\begin{aligned} W_{\bar{z}} &= A_{\bar{z}} + B_{\bar{z}} = \sum_{\mu=1}^{m-1}\frac{1}{\mu!}\frac{1}{2\pi i}\int_{\partial D}\mu(\overline{z-\zeta})^{\mu-1} f_{\mu}(\zeta)\frac{d\zeta}{\zeta-z} \\ &= \frac{1}{2\pi i}\int_{\partial D}\Big[\sum_{\mu=1}^{m-1}\frac{1}{(\mu-1)!}(\overline{z-\zeta})^{\mu-1} f_{\mu}(\zeta)\Big]\frac{d\zeta}{\zeta-z} + \frac{-1}{\pi}\int_{D}\frac{\partial}{\partial\bar{\zeta}}\Big[\sum_{\mu=1}^{m-1}\frac{1}{(\mu-1)!}(\overline{z-\zeta})^{\mu-1} f_{\mu}(\zeta)\Big]\frac{d\sigma_{\zeta}}{\zeta-z} \\ &= \Big[\sum_{\mu=1}^{m-1}\frac{1}{(\mu-1)!}(\overline{z-\zeta})^{\mu-1} f_{\mu}(\zeta)\Big]_{\zeta=z} = f_1(z). \end{aligned}$$

Similarly,

$$\partial^{s}_{\bar{z}}W(z) = \partial^{s}_{\bar{z}}A + \partial^{s}_{\bar{z}}B = f_s(z)\ \ (s=2,\cdots,m-1),\ \ \partial^{m}_{\bar{z}}W(z) = \partial^{m}_{\bar{z}}A + \partial^{m}_{\bar{z}}B = 0.$$

□

Lemma 2.3. *[11] Let G be a bounded smooth domain in the complex plane, and $w \in C^1(G;\mathbb{C})\bigcap C(\overline{G};\mathbb{C})$, then*

$$\int_G w_{\bar{z}}(z)dxdy = \frac{1}{2i}\int_{\partial G} w(z)dz,\ \int_G w_z(z)dxdy = -\frac{1}{2i}\int_{\partial G} w(z)d\bar{z}.$$

3. PDE of higher order with Riemann-Hilbert condition

Theorem 3.1. *Let D be the unit disk in $\mathbb{C}$. For $\gamma, f_k \in C(\partial D)$ with $\partial_{\bar z} f_k = f_{k+1}$ ($k \geq 1, k \in \mathbb{Z}$), let*

$$T_{0,k} f_k(z) = \frac{-1}{\pi} \int_D \frac{1}{(k-1)!} \frac{(\overline{z-\zeta})^{k-1}}{\zeta - z} f_k(\zeta) d\sigma_\zeta, \tag{3.1}$$

then the problem

$$\begin{cases} \Re[\bar\zeta^p W(\zeta)] = \gamma(\zeta) \quad (\zeta \in \partial D), \\ \frac{\partial^k W(z)}{\partial \bar z^k} = f_k(z) \quad (z \in D) \end{cases}$$

is solvable on D.

(i) In the case of $p \geq 0$, the solution can be expressed as:

$$W(z) = z^p \varphi_1(z) + \sum_{s=0}^{k-1} \sum_{l=0}^{s} \sum_{v=0}^{+\infty} \alpha_{sl(v-p)} z^v \bar z^l + T_{0,k} f_k(z), \tag{3.2}$$

where

$$\begin{aligned} \varphi_1(z) = & \frac{1}{2\pi i} \int_{\partial D} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} - \frac{1}{(k-1)!} \frac{z^p}{\pi} \int_D \overline{f_k(\zeta)} \frac{z(z-\zeta)^{k-1}}{1 - z\bar\zeta} d\sigma_\zeta \\ & + \frac{1}{2\pi i} \int_{\partial D} \sum_{\mu=1}^{k-1} \frac{1}{\mu!} \Re\Big[(\overline{z-\zeta})^\mu \sum_{m=0}^{\mu} C_\mu^m (f_{\mu-m}(\zeta) - T_{0,k-\mu+m} f_k(\zeta)) \partial_{\bar\zeta}^m \bar\zeta^p \Big] \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} \\ & + \frac{1}{2\pi} \int_{\partial D} \sum_{\mu=0}^{k-1} \frac{1}{\mu!} \Im\Big[((\overline{z-\zeta})^\mu - (\overline{-\zeta})^\mu) \sum_{m=0}^{\mu} C_\mu^m (f_{\mu-m}(\zeta) - T_{0,k-\mu+m} f_k(\zeta)) \partial_{\bar\zeta}^m \bar\zeta^p \Big] \frac{d\zeta}{\zeta}, \end{aligned} \tag{3.3}$$

$\alpha_{sl(v-p)}$ are arbitrary complex constants satisfying

$$\begin{cases} \sum_{s=0}^{k-1} \sum_{l=0}^{s} \alpha_{sl(v+l)} = 0 \ (v \geq p+k), \ \ \sum_{s=0}^{k-1} \sum_{l=0}^{s} [\alpha_{sl(v+l)} + \overline{\alpha_{sl(l-v)}}] = 0 \ (-p \leq v \leq p), \\ \sum_{s=0}^{k-1} \sum_{l=0}^{s} \alpha_{sl(p+1+t+l)} + \sum_{s=t+1}^{k-1} \sum_{l=t+1}^{s} \overline{\alpha_{sl(-p-1-t+l)}} = 0 \ \ (t = 0, 1, \cdots, k-2, \ k \geq 2). \end{cases} \tag{3.4}$$

For $k = 1$ the last equation in (3.4) is non-existent.

(ii) In the case of $p < 0$, the solution can be expressed as:

$$W(z) = z^p \varphi_2(z) + \sum_{s=0}^{k-1} \sum_{l=0}^{s} \sum_{v=0}^{+\infty} \alpha_{sl(v-p)} z^v \bar z^l + T_{0,k} f_k(z) \tag{3.5}$$

on the condition that

$$\frac{1}{2\pi i} \int_{\partial D} \{\zeta^{-p} [f_{k-1}(\zeta) - T_{0,1} f_k(\zeta)] + \zeta^p [\overline{f_{k-1}(\zeta) - T_{0,1} f_k(\zeta)}]\} \frac{d\zeta}{\zeta^{l+1}} = 0 \ \ (l = 0, 1, \cdots, -p-1), \tag{3.6}$$

where $\alpha_{sl(v-p)}$ is the same as in (i) and

$$
\begin{aligned}
\varphi_2(z) = & \frac{1}{2\pi i}\int_{\partial D}\gamma\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} - \frac{1}{(k-1)!}\frac{z^p}{\pi}\int_D \overline{f_k(\zeta)}\frac{z(z-\zeta)^{k-1}}{1-z\bar{\zeta}}d\sigma_\zeta \\
& + \frac{1}{2\pi i}\int_{\partial D}\sum_{\mu=1}^{k-1}\frac{1}{\mu!}\Re\{\overline{(z-\zeta)}^{\mu}\zeta^{-p}[f_\mu(\zeta)-T_{0,(k-\mu)}f_k(\zeta)]\}\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} \\
& + \frac{1}{2\pi}\int_{\partial D}\sum_{\mu=0}^{k-1}\frac{1}{\mu!}\Im\{(\overline{(z-\zeta)}^{\mu}-\overline{(-\zeta)}^{\mu})\zeta^{-p}[f_\mu(\zeta)-T_{0,(k-\mu)}f_k(\zeta)]\}\frac{d\zeta}{\zeta}.
\end{aligned}
\tag{3.7}
$$

Proof. (1) For $n = 1$, by Lemma 2.1, $T_{0,k}f_k(z)$ is a particular solution to $\frac{\partial^k W(z)}{\partial \bar{z}^k} = f_k(z)$, then the corresponding general solution is $W(z) = \varphi(z) + T_{0,k}f_k(z)$, where $\varphi(z)$ is a k-holomorphic function with

$$
\Re[\zeta^{-p}\varphi(\zeta)] = \Re[\zeta^{-p}W(\zeta) - \zeta^{-p}T_{0,k}f_k(\zeta)] = \gamma(\zeta) - \Re[\zeta^{-p}T_{0,k}f_k(\zeta)] \doteq \gamma_0(\zeta).
$$

(i) In the case of $p \geq 0 (p \in \mathbb{Z})$:

① Let $\varphi(z) = z^p\varphi_1(z)$, then $\Re[\zeta^{-p}\varphi(\zeta)] = \gamma_0(\zeta) \Leftrightarrow \Re[\varphi_1(\zeta)] = \gamma_0(\zeta)$, and $\varphi(z)$ is k-holomorphic if $\varphi_1(z)$ is k-holomorphic. As

$$
\begin{aligned}
\partial_{\bar{\zeta}}^{\mu}(\bar{\zeta}^p\varphi(\zeta)) &= \sum_{m=0}^{\mu}C_\mu^m\partial_{\bar{\zeta}}^{\mu-m}\varphi(\zeta)\partial_{\bar{\zeta}}^{m}(\bar{\zeta}^p) = \sum_{m=0}^{\mu}C_\mu^m\partial_{\bar{\zeta}}^{\mu-m}[W(\zeta)-T_{0,k}f_k(\zeta)]\partial_{\bar{\zeta}}^{m}(\bar{\zeta}^p) \\
&= \sum_{m=0}^{\mu}C_\mu^m[f_{\mu-m}(\zeta)-T_{0,k-\mu+m}f_k(\zeta)]\partial_{\bar{\zeta}}^{m}(\bar{\zeta}^p),
\end{aligned}
$$

by Lemma 2.2 we get that

$$
\begin{aligned}
\varphi_1(z) = & \frac{1}{2\pi i}\int_{\partial D}\sum_{\mu=0}^{k-1}\frac{1}{\mu!}\Re[\overline{(z-\zeta)}^{\mu}\partial_{\bar{\zeta}}^{\mu}(\bar{\zeta}^p\varphi(\zeta))]\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} \\
& + \frac{1}{2\pi}\int_{\partial D}\sum_{\mu=0}^{k-1}\frac{1}{\mu!}\Im[(\overline{(z-\zeta)}^{\mu}-\overline{(-\zeta)}^{\mu})\partial_{\bar{\zeta}}^{\mu}(\bar{\zeta}^p\varphi(\zeta))]\frac{d\zeta}{\zeta} \\
= & \frac{1}{2\pi i}\int_{\partial D}\gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int_{\partial D}[\zeta^{-p}T_{0,k}f_k(\zeta)+\zeta^p\overline{T_{0,k}f_k(\zeta)}](\frac{1}{\zeta-z}-\frac{1}{2\zeta})d\zeta \\
& + \frac{1}{2\pi i}\int_{\partial D}\sum_{\mu=1}^{k-1}\frac{1}{\mu!}\Re[\overline{(z-\zeta)}^{\mu}\sum_{m=0}^{\mu}C_\mu^m(f_{\mu-m}(\zeta)-T_{0,k-\mu+m}f_k(\zeta))\partial_{\bar{\zeta}}^{m}\bar{\zeta}^p]\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} \\
& + \frac{1}{2\pi}\int_{\partial D}\sum_{\mu=0}^{k-1}\frac{1}{\mu!}\Im\Big[(\overline{(z-\zeta)}^{\mu}-\overline{(-\zeta)}^{\mu})\sum_{m=0}^{\mu}C_\mu^m(f_{\mu-m}(\zeta)-T_{0,k-\mu+m}f_k(\zeta))\partial_{\bar{\zeta}}^{m}\bar{\zeta}^p\Big]\frac{d\zeta}{\zeta}
\end{aligned}
\tag{3.8}
$$

satisfies $\Re[\varphi_1(\zeta)] = \gamma_0(\zeta)$ and $\partial_{\bar{z}}^k\varphi_1(z) = 0$. Furthermore, by (3.1), we obtain that

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{\partial D}\zeta^{-p}T_{0,k}f_k(\zeta)\frac{d\zeta}{\zeta-z} &= \frac{1}{(k-1)!}\frac{-1}{\pi}\int_D f_k(\zeta')\Big[\frac{1}{2\pi i}\int_{\partial D}\zeta^{-p}\frac{(\overline{\zeta-\zeta'})^{k-1}}{\zeta'-\zeta}\frac{d\zeta}{\zeta-z}\Big]d\sigma_{\zeta'} \\
&= \frac{1}{(k-1)!}\frac{-1}{\pi}\int_D f_k(\zeta')\Big[\frac{-1}{2\pi i}\int_{\partial D}\bar{\zeta}^p\frac{(\overline{\zeta-\zeta'})^{k-1}}{\bar{\zeta}\zeta'-1}\frac{d\bar{\zeta}}{z\bar{\zeta}-1}\Big]d\sigma_{\zeta'} = 0,
\end{aligned}
\tag{3.9}
$$

which follows

$$\frac{1}{2\pi i}\int_{\partial D}\zeta^{-p}T_{0,k}f_k(\zeta)\frac{d\zeta}{2\zeta}=0. \tag{3.10}$$

Meanwhile,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial D}\zeta^{p}\overline{T_{0,k}f_k(\zeta)}\frac{d\zeta}{\zeta-z}&=\frac{1}{(k-1)!}\frac{-1}{\pi}\int_{D}\overline{f_k(\zeta')}\Big[\frac{1}{2\pi i}\int_{\partial D}\zeta^{p}\frac{(\zeta-\zeta')^{k-1}}{\bar\zeta'-\bar\zeta}\frac{d\zeta}{\zeta-z}\Big]d\sigma_{\zeta'}\\&=\frac{1}{(k-1)!}\frac{-1}{\pi}\int_{D}\overline{f_k(\zeta')}\frac{z^{p+1}(z-\zeta')^{k-1}}{z\bar\zeta'-1}d\sigma_{\zeta'}\\&=\frac{z^p}{(k-1)!}\frac{-1}{\pi}\int_{D}\overline{f_k(\zeta)}\frac{z(z-\zeta)^{k-1}}{z\bar\zeta-1}d\sigma_{\zeta},\end{aligned} \tag{3.11}$$

which follows

$$\frac{1}{2\pi i}\int_{\partial D}\zeta^{p}\overline{T_{0,k}f_k(\zeta)}\frac{d\zeta}{2\zeta}=0. \tag{3.12}$$

Plugging (3.9)–(3.12) into (3.8), we get (3.3). Therefore $z^p\varphi_1(z)$ is a particular solution to $\Re[\zeta^{-p}\varphi(\zeta)]=\gamma_0(\zeta)$ and $\partial_{\bar z}^k\varphi(z)=0$.

② If $\varphi_0(z)$ is k-holomorphic on D with $\Re[\zeta^{-p}\varphi_0(z)]=0$, then $z^p\varphi_1(z)+\varphi_0(\zeta)$ is the general solution to $\Re[\zeta^{-p}\varphi(\zeta)]=\gamma_0(\zeta)$ and $\partial_{\bar z}^k\varphi(z)=0$. In the following, we seek $\varphi_0(z)$.

As $\varphi_0(z)$ is k-holomorphic, it can be expressed as

$$\varphi_0(z)=\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=0}^{+\infty}C_{slv}z^v\bar z^l\doteq\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=0}^{+\infty}\alpha_{sl(v-p)}z^v\bar z^l, \tag{3.13}$$

where $\alpha_{sl(v-p)}=C_{slv}$ are arbitrary complex constants. Let

$$\alpha_{slv}+\overline{\alpha_{sl(2l-v)}}=A_{slv},\quad \alpha_{slv}=B_{slv},\quad \overline{\alpha_{sl(2l-v)}}=C_{slv}, \tag{3.14}$$

then

$$\begin{aligned}0=\Re\{\bar\zeta^p\varphi_0(\zeta)\}&=\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=-p}^{+\infty}(\alpha_{slv}\zeta^{v-l}+\overline{\alpha_{slv}}\zeta^{l-v})\\&=\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=-p}^{p+2l}(\alpha_{slv}+\overline{\alpha_{sl(2l-v)}})\zeta^{v-l}+\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=p+2l+1}^{+\infty}\alpha_{slv}\zeta^{v-l}+\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=-\infty}^{-p-1}\overline{\alpha_{sl(2l-v)}}\zeta^{v-l}\\&=\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=-p}^{p+2l}A_{slv}\zeta^{v-l}+\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=p+2l+1}^{+\infty}B_{slv}\zeta^{v-l}+\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=-\infty}^{-p-1}C_{slv}\zeta^{v-l}\\&=\Big\{\sum_{v=-p}^{p}A_{00v}\zeta^{v}+(\sum_{v=-p}^{p}A_{10v}\zeta^{v}+\sum_{v=-p}^{p+2}A_{11v}\zeta^{v-1})+(\sum_{v=-p}^{p}A_{20v}\zeta^{v}+\sum_{v=-p}^{p+2}A_{21v}\zeta^{v-1}+\sum_{v=-p}^{p+4}A_{22v}\zeta^{v-2})\\&\quad+\cdots+\Big[\sum_{v=-p}^{p}A_{(k-1)0v}\zeta^{v}+\sum_{v=-p}^{p+2}A_{(k-1)1v}\zeta^{v-1}+\cdots+\sum_{v=-p}^{p+2(k-1)}A_{(k-1)(k-1)v}\zeta^{v-(k-1)}\Big]\Big\}\\&\quad+\Big\{\sum_{v=p+1}^{+\infty}B_{00v}\zeta^{v}+(\sum_{v=p+1}^{+\infty}B_{10v}\zeta^{v}+\sum_{v=p+3}^{+\infty}B_{11v}\zeta^{v-1})+(\sum_{v=p+1}^{+\infty}B_{20v}\zeta^{v}+\sum_{v=p+3}^{+\infty}B_{21v}\zeta^{v-1}+\sum_{v=p+5}^{+\infty}B_{22v}\zeta^{v-2})\end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&+\cdots+\Big[\sum_{v=p+1}^{+\infty}B_{(k-1)0v}\zeta^{v}+\sum_{v=p+3}^{+\infty}B_{(k-1)1v}\zeta^{v-1}+\cdots+\sum_{v=p+2k-1}^{+\infty}B_{(k-1)(k-1)v}\zeta^{v-(k-1)}\Big]\Big\}\\
&+\sum_{v=-\infty}^{-p-1}\Big\{C_{00v}\zeta^{v}+(C_{10v}\zeta^{v}+C_{11v}\zeta^{v-1})+(C_{20v}\zeta^{v}+C_{21v}\zeta^{v-1}+C_{22v}\zeta^{v-2})\\
&+\cdots+\Big[C_{(k-1)0v}\zeta^{v}+C_{(k-1)1v}\zeta^{v-1}+\cdots+C_{(k-1)(k-1)v}\zeta^{v-(k-1)}\Big]\Big\}\\
=&\Big\{\sum_{v=-p}^{p}\zeta^{v}\Big[\sum_{l=0}^{k-1}\sum_{s=l}^{k-1}A_{sl(v+l)}\Big]+\zeta^{-p-1}\sum_{l=1}^{k-1}\sum_{s=l}^{k-1}A_{sl(-p-1+l)}+\cdots+\zeta^{-p-k+1}A_{(k-1)(k-l)(-p)}\\
&+\zeta^{p+1}\sum_{l=1}^{k-1}\sum_{s=l}^{k-1}A_{sl(p+1+l)}+\zeta^{p+2}\sum_{l=2}^{k-1}\sum_{s=l}^{k-1}A_{sl(p+2+l)}+\cdots+\zeta^{p+k-1}A_{(k-1)(k-1)(p+2(k-1))}\Big\}\\
&+\Big\{\zeta^{p+1}\sum_{s=0}^{k-1}B_{s0(p+1)}+\zeta^{p+2}\Big[\sum_{s=0}^{k-1}B_{s0(p+2)}+\sum_{s=1}^{k-1}B_{s1(p+3)}\Big]\\
&+\cdots+\zeta^{p+k-1}\sum_{l=0}^{k-2}\sum_{s=l}^{k-1}B_{sl(p+k-1+l)}+\sum_{v=p+k}^{+\infty}\zeta^{v}\Big[\sum_{l=0}^{k-1}\sum_{s=l}^{k-1}B_{sl(v+l)}\Big]\Big\}\\
&+\Big\{\zeta^{-p-1}\sum_{s=0}^{k-1}C_{s0(-p-1)}+\zeta^{-p-2}\Big[\sum_{s=0}^{k-1}C_{s0(-p-2)}+\sum_{s=1}^{k-1}C_{s1(-p-1)}\Big]\\
&+\cdots+\zeta^{-p-k+1}\sum_{l=0}^{k-2}\sum_{s=l}^{k-1}C_{sl(-p-k+1+l)}+\sum_{v=-\infty}^{-p-k}\zeta^{v}\Big[\sum_{l=0}^{k-1}\sum_{s=l}^{k-1}C_{sl(v+l)}\Big]\Big\}\\
=&\sum_{v=-\infty}^{-p-k}\zeta^{v}\Big[\sum_{l=0}^{k-1}\sum_{s=l}^{k-1}C_{sl(v+l)}\Big]+\zeta^{-p-k+1}\Big[\sum_{l=0}^{k-2}\sum_{s=l}^{k-1}C_{sl(-p-k+1+l)}+A_{(k-1)(k-l)(-p)}\Big]+\cdots\\
&+\zeta^{-p-1}\Big[\sum_{s=0}^{k-1}C_{s0(-p-1)}+\sum_{l=1}^{k-1}\sum_{s=l}^{k-1}A_{sl(-p-1+l)}\Big]+\sum_{v=-p}^{p}\zeta^{v}\Big[\sum_{l=0}^{k-1}\sum_{s=l}^{k-1}A_{sl(v+l)}\Big]\\
&+\zeta^{p+1}\Big[\sum_{s=0}^{k-1}B_{s0(p+1)}+\sum_{l=1}^{k-1}\sum_{s=l}^{k-1}A_{sl(p+1+l)}\Big]+\zeta^{p+k-1}\Big[\sum_{l=0}^{k-2}\sum_{s=l}^{k-1}B_{sl(p+k-1+l)}+A_{(k-1)(k-1)(p+2(k-1))}\Big]\\
&+\sum_{v=p+k}^{+\infty}\zeta^{v}\Big[\sum_{l=0}^{k-1}\sum_{s=l}^{k-1}B_{sl(v+l)}\Big],
\end{aligned}
\tag{3.16}
$$

which leads to (3.4) in consideration of (3.14) and

$$
\sum_{l=0}^{k-1}\sum_{s=l}^{k-1}C_{slv}=\sum_{s=0}^{k-1}\sum_{l=0}^{s}C_{slv}.
$$

By ① and ②, (3.2) is the solution to Problem RH.

(ii) In the case of $p<0(p\in\mathbb{Z})$:

① Let $\varphi(z) = z^p\varphi_2(z)$, then $\Re[\zeta^{-p}\varphi(\zeta)] = \gamma_0(\zeta) \Leftrightarrow \Re[\varphi_2(\zeta)] = \gamma_0(\zeta)$. Similar to the discussion of ① in (i), we get a particular solution $\varphi_2(z)$ for $\Re[\varphi_2(\zeta)] = \gamma_0(\zeta)$ where $\varphi_2(z)$ is k holomorphic:

$$\begin{aligned}\varphi_2(z) &= \frac{1}{2\pi i}\int_{\partial D}\gamma_0\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\partial D}\sum_{\mu=1}^{k-1}\frac{1}{\mu!}\Re[\overline{(z-\zeta)}^{\mu}\zeta^{-p}\partial_{\bar\zeta}^{\mu}\varphi(\zeta)]\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\\
&+\frac{1}{2\pi}\int_{\partial D}\sum_{\mu=0}^{k-1}\frac{1}{\mu!}\Im[(\overline{(z-\zeta)}^{\mu}-\overline{(-\zeta)}^{\mu})\zeta^{-p}\partial_{\bar\zeta}^{\mu}\varphi(\zeta)]\frac{d\zeta}{\zeta}\\
&= \frac{1}{2\pi i}\int_{\partial D}\gamma\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int_{\partial D}[\zeta^{-p}T_{0,k}f_k(\zeta) + \zeta^p\overline{T_{0,k}f_k(\zeta)}](\frac{1}{\zeta-z}-\frac{1}{2\zeta})d\zeta\\
&+\frac{1}{2\pi i}\int_{\partial D}\sum_{\mu=1}^{k-1}\frac{1}{\mu!}\Re[\overline{(z-\zeta)}^{\mu}\zeta^{-p}\partial_{\bar\zeta}^{\mu}\varphi(\zeta)]\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\\
&+\frac{1}{2\pi}\int_{\partial D}\sum_{\mu=0}^{k-1}\frac{1}{\mu!}\Im[(\overline{(z-\zeta)}^{\mu}-\overline{(-\zeta)}^{\mu})\zeta^{-p}\partial_{\bar\zeta}^{\mu}\varphi(\zeta)]\frac{d\zeta}{\zeta}.\end{aligned}$$

So we get (3.7) for (3.9)–(3.12) and

$$\partial_{\bar\zeta}^{\mu}\varphi(\zeta) = \partial_{\bar\zeta}^{\mu}[W(\zeta) - T_{0,k}f_k(\zeta)] = \partial_{\bar\zeta}^{\mu}W(\zeta) - \partial_{\bar\zeta}^{\mu}T_{0,k}f_k(\zeta) = f_\mu(\zeta) - T_{0,(k-\mu)}f_k(\zeta).$$

Therefore

$$\varphi(z) = z^p\varphi_2(z) + \sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{\nu=0}^{+\infty}\alpha_{sl(\nu-p)}z^\nu\bar z^l$$

is the solution to $\Re[\zeta^{-p}\varphi(\zeta)] = \gamma_0(\zeta)$ and $\partial_{\bar z}^k\varphi(z) = 0$, where $\alpha_{sl(\nu-p)}$ are arbitrary complex constants satisfying (3.12).

② Secondly, we seek the condition to ensure that $\varphi(z)$ is k-holomorphic from $\varphi_2(z) = z^{-p}\varphi(z)$. As the k-holomorphism of $\varphi_2(z) = z^{-p}\varphi(z)$ is equivalent to the holomorphism of $\partial_{\bar z}^{k-1}\varphi_2(z) = z^{-p}\partial_{\bar z}^{k-1}\varphi(z)$, applying the properties of the Schwarz operator for holomorphic functions and

$$\frac{1}{\zeta-z} = \frac{1}{\zeta}\cdot\frac{1}{1-\frac{z}{\zeta}} = \sum_{l=0}^{\infty}\frac{z^l}{\zeta^{l+1}} \quad (|\frac{z}{\zeta}|<1),$$

we get that

$$\begin{aligned}z^{-p}\partial_{\bar z}^{k-1}\varphi(z) &= \frac{1}{2\pi i}\int_{\partial D}[\zeta^{-p}\partial_{\bar\zeta}^{k-1}\varphi(\zeta) + \zeta^p\overline{\partial_{\bar\zeta}^{k-1}\varphi(\zeta)}]\Big[\frac{2\zeta}{\zeta-z}-1\Big]\frac{d\zeta}{2\zeta}\\
&= \sum_{l=0}^{\infty}\Big\{\frac{1}{2\pi i}\int_{\partial D}[\zeta^{-p}\partial_{\bar\zeta}^{k-1}\varphi(\zeta) + \zeta^p\overline{\partial_{\bar\zeta}^{k-1}\varphi(\zeta)}]\frac{d\zeta}{\zeta^{l+1}}\Big\}z^l\\
&- \frac{1}{2\pi i}\int_{\partial D}[\zeta^{-p}\partial_{\bar\zeta}^{k-1}\varphi(\zeta) + \zeta^p\overline{\partial_{\bar\zeta}^{k-1}\varphi(\zeta)}]\frac{d\zeta}{2\zeta}.\end{aligned}$$

As $p < 0$, then $\varphi(z)$ is k-holomorphic (-i.e., $\partial_{\bar z}^{k-1}\varphi(z)$ is holomorphic) if and only if $z^{-p}\partial_{\bar z}^{k-1}\varphi(z)$ has a zero of order at least $-p$ at $z = 0$. Therefore,

$$\frac{1}{2\pi i}\int_{\partial D}[\zeta^{-p}\partial_{\bar\zeta}^{k-1}\varphi(\zeta) + \zeta^p\overline{\partial_{\bar\zeta}^{k-1}\varphi(\zeta)}]\frac{d\zeta}{\zeta^{l+1}} = 0 \ \ (l = 0, 1, \cdots, -p-1),$$

that is (3.6) as

$$\partial_{\bar{\zeta}}^{k-1}\varphi(\zeta)=\partial_{\bar{\zeta}}^{k-1}[W(\zeta)-T_{0,k}f_k(\zeta)]=\partial_{\bar{\zeta}}^{k-1}W(\zeta)-\partial_{\bar{\zeta}}^{k-1}T_{0,k}f_k(\zeta)=f_{k-1}(\zeta)-T_{0,1}f_k(\zeta).$$

By ① and ②, (3.5) is the solution of Problem RH on the condition of (3.6).

Theorem 3.1 extends the conclusions of Riemann-Hilbert boundary value problems for k-holomorphic functions. Given that $k = 1$ in Theorem 3.1, we can get the following conclusion, which extends the existing results of the corresponding Riemann-Hilbert problems for analytic functions. □

Corollary 3.2. *Let D be the unit disk in $\mathbb{C}$. For $\gamma, f \in C(\partial D)$, the problem*

$$\Re[\bar{\zeta}^p W(\zeta)]=\gamma(\zeta)\ (\zeta\in\partial D),\quad \frac{\partial W(z)}{\partial\bar{z}}=f(z)\ (z\in D)$$

is solvable on D.

(i) In the case of $p \ge 0$, the solution can be expressed as:

$$W(z)=\frac{z^p}{2\pi i}\int_{\partial D}\gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}-\frac{z^{2p+1}}{\pi}\int_D\frac{\overline{f(\zeta)}}{1-z\bar{\zeta}}d\sigma_\zeta+\sum_{v=0}^{2p}\alpha_{v-p}z^v-\frac{1}{\pi}\int_D\frac{f(\zeta)}{\zeta-z}d\sigma_\zeta,$$

where α_{v-p} are arbitrary complex constants satisfying $\alpha_v+\overline{\alpha_{-v}}=0\ (-p\le v\le p)$;

(ii) In the case of $p < 0$, the solution is the same as in (i) on the condition of

$$\frac{1}{2\pi i}\int_{\partial D}\{\gamma(\zeta)-\Re[\zeta^{-p}Tf(\zeta)]\}\frac{d\zeta}{\zeta^{l+1}}=0\ \ (l=0,1,\cdots,-p-1).$$

4. PDE for bi-polyanalytic functions

In this section we discuss several boundary value problems for bi-polyanalytic functions with the Dirichlet, Riemann-Hilbert boundary conditions or the mixed boundary conditions.

Theorem 4.1. *Let D be the unit disk in $\mathbb{C}$, and let $\lambda\in\mathbb{R}\backslash\{-1,0,1\}$, $\varphi,\gamma,f_k\in C(\partial D)$ with $\partial_{\bar{z}}f_k=f_{k+1}$ ($k=1,2,\cdots,n-1,\ n\ge 2$) and $f_n=0$. Then the problem*

$$\begin{cases}\partial_{\bar{z}}f(z)=\frac{\lambda-1}{4\lambda}\phi(z)+\frac{\lambda+1}{4\lambda}\overline{\phi(z)},\quad \partial_{\bar{z}}^n\phi(z)=0,\quad \partial_{\bar{z}}^k\phi(z)=f_k(z)\ \ (z\in D),\\ f(\zeta)=\varphi(\zeta),\quad \Re[\overline{\zeta}^p\phi(\zeta)]=\gamma(\zeta)\ \ (\zeta\in\partial D),\end{cases}\tag{4.1}$$

is solvable on D.

(i) In the case of $p \ge 0$, the solution can be expressed as:

$$\begin{aligned}f(z)=&\frac{1}{2\pi i}\int_{\partial D}\frac{\varphi(\zeta)d\zeta}{\zeta-z}-\frac{1}{\pi}\int_D\Big[\frac{\lambda-1}{4\lambda}(\zeta^p\varphi_1(\zeta)+T_{0,k}f_k(\zeta))+\frac{\lambda+1}{4\lambda}\overline{(\zeta^p\varphi_1(\zeta)+T_{0,k}f_k(\zeta))}\Big]\frac{d\sigma_\zeta}{\zeta-z}\\&-\frac{\lambda-1}{4\lambda}\Big\{\sum_{s=0}^{k-1}\sum_{l=0}^{s}\Big[\sum_{v=0}^{l}\frac{-\alpha_{sl(v-p)}}{l+1}z^v\bar{z}^{l+1}+\sum_{v=l+1}^{+\infty}\frac{\alpha_{sl(v-p)}}{l+1}z^v(z^{-l-1}-\bar{z}^{l+1})\Big]\Big\}\\&-\frac{\lambda+1}{4\lambda}\Big\{\sum_{s=0}^{k-1}\sum_{l=0}^{s}\Big[\sum_{v=0}^{l-1}\frac{\overline{\alpha_{sl(v-p)}}}{v+1}z^l(z^{-v-1}-\bar{z}^{v+1})+\sum_{v=l}^{+\infty}\frac{-\overline{\alpha_{sl(v-p)}}}{v+1}z^l\bar{z}^{v+1}\Big]\Big\}\end{aligned}\tag{4.2}$$

if and only if

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D}\frac{\varphi(\zeta)d\zeta}{1-\bar{z}\zeta}+\frac{1}{\pi}\iint_D\Big[\frac{\lambda-1}{4\lambda}\zeta^p\varphi_1(\zeta)+\frac{\lambda+1}{4\lambda}\bar{\zeta}^p\overline{\varphi_1(\zeta)}\Big]\frac{d\sigma_\zeta}{\bar{z}\zeta-1}+\frac{1}{\pi}\int_D\Big\{\frac{(\overline{z-\zeta})^k f_k(\zeta)}{k!(\bar{z}\zeta-1)}\\
&-\frac{\overline{f_k(\zeta)}}{(k-1)!}\Big[\sum_{s=0}^{k-2}C_{k-1}^s\frac{(\bar{z}\zeta-1)^{k-1-s}-(-1)^{k-1-s}}{(k-1-s)\bar{z}^k}(1-\bar{z}\zeta)^s+\frac{(1-\bar{z}\zeta)^{k-1}\ln(1-\bar{z}\zeta)}{z^k}\Big]\Big\}d\sigma_\zeta \qquad (4.3)\\
&=\frac{\lambda-1}{4\lambda}\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=0}^{l}\alpha_{sl(v-p)}\frac{\bar{z}^{l-v}}{l+1}+\frac{\lambda+1}{4\lambda}\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=l}^{+\infty}\overline{\alpha_{sl(v-p)}}\frac{\bar{z}^{v-l}}{v+1},
\end{aligned}$$

where $T_{0,k}f_k$ and φ_1 are represented by (3.1) and (3.3), respectively, and $\alpha_{sl(v-p)}$ are arbitrary complex constants satisfying (3.4);

(ii) In the case of $p<0$, the solution can be expressed as (4.2) on the condition of (3.6) and (4.3), where $T_{0,k}f_k$ and $\alpha_{sl(v-p)}$ are the same as in (i); however, φ_1 is represented by (3.7).

Proof. As

$$\partial_{\bar{z}}\Big[\frac{-1}{\pi}\int_D\frac{g(\zeta)}{\zeta-z}d\sigma_\zeta\Big]=g(z),$$

obviously,

$$\frac{-1}{\pi}\int_D\Big[\frac{\lambda-1}{4\lambda}\phi(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\phi(\zeta)}\Big]\frac{d\sigma_\zeta}{\zeta-z}$$

is a special solution to $\partial_{\bar{z}}f(z)=\frac{\lambda-1}{4\lambda}\phi(z)+\frac{\lambda+1}{4\lambda}\overline{\phi(z)}$, then the solution of the problem (4.1) can be represented as

$$f(z)=\psi(z)-\frac{1}{\pi}\int_D\Big[\frac{\lambda-1}{4\lambda}\phi(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\phi(\zeta)}\Big]\frac{d\sigma_\zeta}{\zeta-z}, \qquad (4.4)$$

where $\psi(z)$ is analytic on D to be determined, and $\phi(\zeta)$ is the solution to

$$\Re[\overline{\zeta}^p\phi(\zeta)]=\gamma(\zeta)\ \ (\zeta\in\partial D),\quad \partial_{\bar{z}}^n\phi(z)=0,\quad \partial_{\bar{z}}^k\phi(z)=f_k(z)\ \ (z\in D).$$

Since $\psi(z)$ is analytic, by (4.4),

$$\begin{aligned}
\psi(z)&=\frac{1}{2\pi i}\int_{\partial D}\frac{\psi(\zeta)}{\zeta-z}d\zeta\\
&=\frac{1}{2\pi i}\int_{\partial D}\Big\{\varphi(\zeta)+\frac{1}{\pi}\int_D\Big[\frac{\lambda-1}{4\lambda}\phi(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\phi(\widetilde{\zeta})}\Big]\frac{d\sigma_{\widetilde{\zeta}}}{\widetilde{\zeta}-\zeta}\Big\}\frac{d\zeta}{\zeta-z}\\
&=\frac{1}{2\pi i}\int_{\partial D}\varphi(\zeta)\frac{d\zeta}{\zeta-z}+\frac{1}{\pi}\int_D\Big[\frac{\lambda-1}{4\lambda}\phi(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\phi(\widetilde{\zeta})}\Big]\Big[\frac{1}{2\pi i}\int_{\partial D}\frac{1}{\widetilde{\zeta}-\zeta}\frac{d\zeta}{\zeta-z}\Big]d\sigma_{\widetilde{\zeta}}\\
&=\frac{1}{2\pi i}\int_{\partial D}\frac{\varphi(\zeta)}{\zeta-z}d\zeta
\end{aligned} \qquad (4.5)$$

if and only if

$$\frac{1}{2\pi i}\int_{\partial D}\frac{\bar{z}\psi(\zeta)}{1-\bar{z}\zeta}d\zeta=0\quad (z\in D),$$

that is,

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D}\frac{\bar{z}}{1-\bar{z}\zeta}\Big\{\varphi(\zeta)+\frac{1}{\pi}\int_{D}\Big[\frac{\lambda-1}{4\lambda}\phi(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\phi(\widetilde{\zeta})}\Big]\frac{d\sigma_{\widetilde{\zeta}}}{\widetilde{\zeta}-\zeta}\Big\}d\zeta=0\\
\Leftrightarrow\ &\frac{1}{2\pi i}\int_{\partial D}\frac{\bar{z}\varphi(\zeta)}{1-\bar{z}\zeta}d\zeta+\frac{1}{\pi}\int_{D}\Big[\frac{\lambda-1}{4\lambda}\phi(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\phi(\widetilde{\zeta})}\Big]\Big[\frac{1}{2\pi i}\int_{\partial D}\frac{\bar{z}}{1-\bar{z}\zeta}\frac{d\zeta}{\widetilde{\zeta}-\zeta}\Big]d\sigma_{\widetilde{\zeta}} \qquad (4.6)\\
\Leftrightarrow\ &\frac{1}{2\pi i}\int_{\partial D}\frac{\varphi(\zeta)}{1-\bar{z}\zeta}d\zeta+\frac{1}{\pi}\int_{D}\Big[\frac{\lambda-1}{4\lambda}\phi(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\phi(\zeta)}\Big]\frac{d\sigma_{\zeta}}{\bar{z}\zeta-1}=0.
\end{aligned}
$$

Plugging (4.5) into (4.4),

$$
f(z)=\frac{1}{2\pi i}\int_{\partial D}\frac{\varphi(\zeta)}{\zeta-z}d\zeta-\frac{1}{\pi}\int_{D}\Big[\frac{\lambda-1}{4\lambda}\phi(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\phi(\zeta)}\Big]\frac{d\sigma_{\zeta}}{\zeta-z}. \qquad (4.7)
$$

(i) In the case of $p\geq 0$, by Theorem 3.1,

$$
\phi(z)=z^{p}\varphi_{1}(z)+\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=0}^{+\infty}\alpha_{sl(v-p)}z^{v}\bar{z}^{l}+T_{0,k}f_{k}(z), \qquad (4.8)
$$

where $T_{0,k}f_k$ and φ_1 are represented by (3.1) and (3.3), respectively, and $\alpha_{sl(v-p)}$ are arbitrary complex constants satisfying (3.4).

Applying the Cauchy-Pompeiu formula (1.1),

$$
\begin{aligned}
\frac{1}{\pi}\int_{D}\frac{\zeta^{v}\bar{\zeta}^{l}}{\zeta-z}d\sigma_{\zeta}&=\frac{1}{\pi}\int_{D}\partial_{\bar{\zeta}}\Big(\zeta^{v}\frac{\bar{\zeta}^{l+1}}{l+1}\Big)\frac{d\sigma_{\zeta}}{\zeta-z}=\frac{1}{2\pi i}\int_{\partial D}\zeta^{v}\frac{\bar{\zeta}^{l+1}}{l+1}\frac{d\zeta}{\zeta-z}-z^{v}\frac{\bar{z}^{l+1}}{l+1}\\
&=\frac{1}{l+1}\frac{1}{2\pi i}\int_{\partial D}\frac{\zeta^{v-l-1}}{\zeta-z}d\zeta-z^{v}\frac{\bar{z}^{l+1}}{l+1} \qquad (4.9)\\
&=\begin{cases}\frac{z^{v}}{l+1}(z^{-l-1}-\bar{z}^{l+1}), & v\geq l+1,\\ \frac{-z^{v}}{l+1}\bar{z}^{l+1}, & v<l+1.\end{cases}
\end{aligned}
$$

Similarly,

$$
\frac{1}{\pi}\int_{D}\frac{\bar{\zeta}^{v}\zeta^{l}}{\zeta-z}d\sigma_{\zeta}=\begin{cases}\frac{z^{l}}{v+1}(z^{-v-1}-\bar{z}^{v+1}), & v\leq l-1,\\ \frac{-z^{l}}{v+1}\bar{z}^{v+1}, & v>l-1.\end{cases} \qquad (4.10)
$$

Plugging (4.8)–(4.10) into (4.7), the solution (4.2) follows.

In order to obtain the solvable conditions, the following integrals need to be calculated: First, by Lemma 2.3,

$$
\begin{aligned}
\frac{1}{\pi}\int_{D}\frac{\zeta^{v}\bar{\zeta}^{l}}{\bar{z}\zeta-1}d\sigma_{\zeta}&=\frac{1}{\pi}\int_{D}\partial_{\bar{\zeta}}\Big(\frac{\bar{\zeta}^{l+1}}{l+1}\frac{\zeta^{v}}{\bar{z}\zeta-1}\Big)d\sigma_{\zeta}=\frac{1}{2\pi i}\int_{\partial D}\frac{\bar{\zeta}^{l+1}}{l+1}\frac{\zeta^{v}d\zeta}{\bar{z}\zeta-1}\\
&=\frac{1}{l+1}\frac{1}{2\pi i}\int_{\partial D}\frac{\zeta^{v-l-1}}{\bar{z}\zeta-1}d\zeta \qquad (4.11)\\
&=\begin{cases}0, & v\geq l+1,\\ \frac{-\bar{z}^{l-v}}{l+1}, & v<l+1.\end{cases}
\end{aligned}
$$

Similarly,

$$
\frac{1}{\pi}\int_{D}\frac{\bar{\zeta}^{v}\zeta^{l}}{\bar{z}\zeta-1}d\sigma_{\zeta}=\begin{cases}0, & v\leq l-1,\\ \frac{-\bar{z}^{v-l}}{v+1}, & v>l-1.\end{cases} \qquad (4.12)
$$

Secondly, by the Cauchy-Pompeiu formula (1.1),

$$
\begin{aligned}
\frac{1}{\pi}\int_D T_{0,k}f_k(\zeta)\frac{d\sigma_\zeta}{\bar z\zeta-1} &= \frac{1}{\pi}\int_D\Big[\frac{-1}{\pi}\int_D\frac{1}{(k-1)!}\frac{\overline{(\zeta-\widetilde{\zeta})}^{k-1}}{\widetilde{\zeta}-\zeta}f_k(\widetilde{\zeta})d\sigma_{\widetilde{\zeta}}\Big]\frac{d\sigma_\zeta}{\bar z\zeta-1}\\
&= \frac{-1}{\pi}\int_D\frac{f_k(\widetilde{\zeta})}{(k-1)!}\Big[\frac{1}{\pi}\int_D\frac{\overline{(\zeta-\widetilde{\zeta})}^{k-1}}{\widetilde{\zeta}-\zeta}\frac{d\sigma_\zeta}{\bar z\zeta-1}\Big]d\sigma_{\widetilde{\zeta}}\\
&= \frac{-1}{\pi}\int_D\frac{f_k(\widetilde{\zeta})}{(k-1)!}\Big[\frac{1}{\pi}\int_D\partial_{\bar\zeta}\frac{\overline{(\zeta-\widetilde{\zeta})}^{k}}{k(1-\bar z\zeta)}\frac{d\sigma_\zeta}{\zeta-\widetilde{\zeta}}\Big]d\sigma_{\widetilde{\zeta}}\\
&= \frac{-1}{\pi}\int_D\frac{f_k(\widetilde{\zeta})}{(k-1)!}\Big[\frac{1}{2\pi i}\int_{\partial D}\frac{\overline{(\zeta-\widetilde{\zeta})}^{k}}{k(1-\bar z\zeta)}\frac{d\zeta}{\zeta-\widetilde{\zeta}}-0\Big]d\sigma_{\widetilde{\zeta}}\\
&= \frac{-1}{\pi}\int_D\frac{f_k(\widetilde{\zeta})}{(k-1)!}\Big[\frac{1}{k}\overline{\frac{1}{2\pi i}\int_{\partial D}\frac{(\zeta-\widetilde{\zeta})^{k}d\zeta}{(\zeta-z)(1-\overline{\widetilde{\zeta}}\zeta)}}\Big]d\sigma_{\widetilde{\zeta}}\\
&= \frac{1}{\pi}\int_D\frac{\overline{(z-\zeta)}^{k}f_k(\zeta)}{k!(\bar z\zeta-1)}d\sigma_\zeta.
\end{aligned}
\tag{4.13}
$$

In addition, for $z,\widetilde{\zeta}\in D$, setting $\widetilde{z}=1/\bar z$, in view of

$$
\begin{aligned}
&\frac{1}{\pi}\int_D\frac{(\zeta-\widetilde{\zeta})^{k-1}}{\overline{\widetilde{\zeta}-\zeta}}\frac{d\sigma_\zeta}{\bar z\zeta-1} = \frac{-\widetilde{z}}{\pi}\int_D\frac{(\zeta-\widetilde{\zeta})^{k-1}}{\zeta-\widetilde{z}}\frac{d\sigma_\zeta}{\overline{\zeta-\widetilde{\zeta}}}\\
&= \frac{-\widetilde{z}}{\pi}\int_D\Big[\sum_{s=0}^{k-2}C_{k-1}^s(\zeta-\widetilde{z})^{k-2-s}(\widetilde{z}-\widetilde{\zeta})^s+\frac{(\widetilde{z}-\widetilde{\zeta})^{k-1}}{\zeta-\widetilde{z}}\Big]\frac{d\sigma_\zeta}{\overline{\zeta-\widetilde{\zeta}}}\\
&= \frac{-\widetilde{z}}{\pi}\int_D\partial_\zeta\Big[\sum_{s=0}^{k-2}C_{k-1}^s\frac{(\zeta-\widetilde{z})^{k-1-s}}{k-1-s}(\widetilde{z}-\widetilde{\zeta})^s+(\widetilde{z}-\widetilde{\zeta})^{k-1}\ln(\zeta-\widetilde{z})\Big]\frac{d\sigma_\zeta}{\overline{\zeta-\widetilde{\zeta}}}\\
&= \widetilde{z}\Big[\sum_{s=0}^{k-2}C_{k-1}^s\frac{(\zeta-\widetilde{z})^{k-1-s}}{k-1-s}(\widetilde{z}-\widetilde{\zeta})^s+(\widetilde{z}-\widetilde{\zeta})^{k-1}\ln(\zeta-\widetilde{z})\Big]_{\zeta=\widetilde{\zeta}}\\
&\quad+\frac{\widetilde{z}}{2\pi i}\int_{\partial D}\Big[\sum_{s=0}^{k-2}C_{k-1}^s\frac{(\zeta-\widetilde{z})^{k-1-s}}{k-1-s}(\widetilde{z}-\widetilde{\zeta})^s+(\widetilde{z}-\widetilde{\zeta})^{k-1}\ln(\zeta-\widetilde{z})\Big]\frac{d\bar\zeta}{\overline{\zeta-\widetilde{\zeta}}}\\
&= \widetilde{z}\Big[\sum_{s=0}^{k-2}C_{k-1}^s\frac{(\widetilde{\zeta}-\widetilde{z})^{k-1-s}}{k-1-s}(\widetilde{z}-\widetilde{\zeta})^s+(\widetilde{z}-\widetilde{\zeta})^{k-1}\ln(\widetilde{\zeta}-\widetilde{z})\Big]\\
&\quad-\widetilde{z}\Big[\sum_{s=0}^{k-2}C_{k-1}^s\frac{(-\widetilde{z})^{k-1-s}}{k-1-s}(\widetilde{z}-\widetilde{\zeta})^s+(\widetilde{z}-\widetilde{\zeta})^{k-1}\ln(-\widetilde{z})\Big]\\
&= \widetilde{z}\Big[\sum_{s=0}^{k-2}C_{k-1}^s\frac{(\widetilde{\zeta}-\widetilde{z})^{k-1-s}-(-\widetilde{z})^{k-1-s}}{k-1-s}(\widetilde{z}-\widetilde{\zeta})^s+(\widetilde{z}-\widetilde{\zeta})^{k-1}\ln\frac{\widetilde{\zeta}-\widetilde{z}}{-\widetilde{z}}\Big]\\
&= \sum_{s=0}^{k-2}C_{k-1}^s\frac{(\bar z\widetilde{\zeta}-1)^{k-1-s}-(-1)^{k-1-s}}{(k-1-s)\bar z^k}(1-\bar z\widetilde{\zeta})^s+\frac{(1-\bar z\widetilde{\zeta})^{k-1}\ln(1-\bar z\widetilde{\zeta})}{\bar z^k},
\end{aligned}
$$

in which the logarithmic functions take the principal value, therefore,

$$\begin{aligned}\frac{1}{\pi}\int_D \overline{T_{0,k}f_k(\zeta)}\frac{d\sigma_\zeta}{\bar{z}\zeta-1} &= \frac{1}{\pi}\int_D\Big[\frac{-1}{\pi}\int_D\frac{1}{(k-1)!}\frac{(\zeta-\widetilde{\zeta})^{k-1}}{\overline{\widetilde{\zeta}-\zeta}}\overline{f_k(\widetilde{\zeta})}d\sigma_{\widetilde{\zeta}}\Big]\frac{d\sigma_\zeta}{\bar{z}\zeta-1}\\ &= \frac{-1}{\pi}\int_D\frac{\overline{f_k(\widetilde{\zeta})}}{(k-1)!}\Big[\frac{1}{\pi}\int_D\frac{(\zeta-\widetilde{\zeta})^{k-1}}{\overline{\widetilde{\zeta}-\zeta}}\frac{d\sigma_\zeta}{\bar{z}\zeta-1}\Big]d\sigma_{\widetilde{\zeta}}\\ &= \frac{-1}{\pi}\int_D\Big[\sum_{s=0}^{k-2}C_{k-1}^{s}\frac{(\bar{z}\zeta-1)^{k-1-s}-(-1)^{k-1-s}}{(k-1-s)\bar{z}^k}(1-\bar{z}\zeta)^s\\ &\quad+\frac{(1-\bar{z}\zeta)^{k-1}\ln(1-\bar{z}\zeta)}{\bar{z}^k}\Big]\frac{\overline{f_k(\zeta)}d\sigma_\zeta}{(k-1)!}.\end{aligned} \tag{4.14}$$

Plugging (4.11)–(4.14) into (4.6), the condition (4.3) follows.

(ii) In the case of $p<0$, similar to (i), by Theorem 3.1, the result follows.

Given that $k=1$ in Theorem 4.1, we can get the solution to the following boundary value problem for bi-analytic functions. □

Corollary 4.2. *Let D be the unit disk in $\mathbb{C}$, and let $\lambda\in\mathbb{R}\backslash\{-1,0,1\}$. For $\varphi,\gamma\in C(\partial D)$, the problem*

$$\begin{cases}\partial_{\bar{z}}f(z)=\frac{\lambda-1}{4\lambda}\phi(z)+\frac{\lambda+1}{4\lambda}\overline{\phi(z)}, & \partial_{\bar{z}}\phi(z)=0\ \ (z\in D),\\ f(\zeta)=\varphi(\zeta), & \Re[\overline{\zeta}^p\phi(\zeta)]=\gamma(\zeta)\ \ (\zeta\in\partial D)\end{cases}$$

is solvable on D.

(i) In the case of $p\geq 0$, the solution can be expressed as:

$$\begin{aligned}f(z)=&\frac{1}{2\pi i}\int_{\partial D}\frac{\varphi(\zeta)d\zeta}{\zeta-z}-\frac{1}{\pi}\int_D\Big[\frac{\lambda-1}{4\lambda}\zeta^p\varphi_1(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\zeta^p\varphi_1(\zeta)}\Big]\frac{d\sigma_\zeta}{\zeta-z}\\ &+\frac{\lambda-1}{4\lambda}\Big[\alpha_{-p}\bar{z}+\sum_{v=1}^{+\infty}\alpha_{v-p}z^v(\bar{z}-z^{-1})\Big]+\frac{\lambda+1}{4\lambda}\sum_{v=0}^{+\infty}\overline{\alpha_{v-p}}\bar{z}^{v+1}\end{aligned}$$

if and only if

$$\frac{1}{2\pi i}\int_{\partial D}\frac{\varphi(\zeta)d\zeta}{1-\bar{z}\zeta}+\frac{1}{\pi}\int_D\Big[\frac{\lambda-1}{4\lambda}\zeta^p\varphi_1(\zeta)+\frac{\lambda+1}{4\lambda}\bar{\zeta}^p\overline{\varphi_1(\zeta)}\Big]\frac{d\sigma_\zeta}{\bar{z}\zeta-1}=\frac{\lambda-1}{4\lambda}\alpha_{(-p)}+\frac{\lambda+1}{4\lambda}\sum_{v=0}^{+\infty}\overline{\alpha_{v-p}}\bar{z}^v,$$

where

$$\varphi_1(z)=\frac{1}{2\pi i}\int_{\partial D}\gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}$$

and α_{v-p} are arbitrary complex constants satisfying

$$\alpha_v=0\ (v\geq p+1),\quad \alpha_v+\overline{\alpha_{-v}}=0\ (-p\leq v\leq p).$$

(ii) In the case of $p<0$, the solution is the same as in (i) on the condition that

$$\frac{1}{2\pi i}\int_{\partial D}\gamma(\zeta)\frac{d\zeta}{\zeta^{l+1}}=0\ \ (l=0,1,\cdots,-p-1).$$

Applying Corollary 3.2 and Theorem 4.1 we can get the solution to the following boundary value problem of higher order for bi-polyanalytic functions.

Corollary 4.3. *Let D be the unit disk in* $\mathbb{C}$*, and let* $\lambda \in \mathbb{R}\backslash\{-1, 0, 1\}$*,* $\varphi, \widetilde{\gamma}, \gamma, f_k \in C(\partial D)$ *with* $\partial_{\bar{z}} f_k = f_{k+1}$ *(* $k = 1, 2, \cdots, n-1,\ n \geq 2$*) and* $f_n = 0$*. Then the problem*

$$\begin{cases} \partial_{\bar{z}}^2 W(z) = \frac{\lambda-1}{4\lambda}\phi(z) + \frac{\lambda+1}{4\lambda}\overline{\phi(z)}, \quad \partial_{\bar{z}}^n \phi(z) = 0, \quad \partial_{\bar{z}}^k \phi(z) = f_k(z) \ \ (z \in D), \\ \partial_{\bar{\zeta}} W(\zeta) = \varphi(\zeta), \quad \Re[\overline{\zeta}^q W(\zeta)] = \widetilde{\gamma}(\zeta), \quad \Re[\overline{\zeta}^p \phi(\zeta)] = \gamma(\zeta) \ \ (\zeta \in \partial D) \end{cases}$$

is solvable on D.

(i) In the case of $q \geq 0$*, the solution can be expressed as:*

$$W(z) = \frac{z^q}{2\pi i}\int_{\partial D} \widetilde{\gamma}(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} - \frac{z^{2q+1}}{\pi}\int_D \frac{\overline{f(\zeta)}}{1 - z\bar{\zeta}} d\sigma_\zeta + \sum_{v=0}^{2q} \alpha_{v-q} z^v - \frac{1}{\pi}\int_D \frac{f(\zeta)}{\zeta - z} d\sigma_\zeta,$$

where α_{v-q} *are arbitrary complex constants satisfying* $\alpha_v + \overline{\alpha_{-v}} = 0$ $(-q \leq v \leq q)$ *and* $f(z)$ *is the solution of the problem (4.1) in Theorem 4.1;*

(ii) In the case of $q < 0$*, the solution is the same as in (i) on the condition of*

$$\frac{1}{2\pi i}\int_{\partial D} \{\gamma(\zeta) - \Re[\zeta^{-q} T f(\zeta)]\}\frac{d\zeta}{\zeta^{l+1}} = 0 \ \ (l = 0, 1, \cdots, -q-1).$$

ies

Remark 4.4. *Applying Corollaries 3.2 and 4.3 we can get the solution of the problem:*

$$\begin{cases} \partial_{\bar{z}}^3 W(z) = \frac{\lambda-1}{4\lambda}\phi(z) + \frac{\lambda+1}{4\lambda}\overline{\phi(z)}, \quad \partial_{\bar{z}}^n \phi(z) = 0, \quad \partial_{\bar{z}}^k \phi(z) = f_k(z) \ \ (z \in D), \\ \partial_{\bar{\zeta}}^2 W(\zeta) = \varphi(\zeta), \quad \Re[\overline{\zeta}^p \phi(\zeta)] = \gamma(\zeta) \ \ (\zeta \in \partial D), \\ \Re[\overline{\zeta}^{q_1} W(\zeta)] = \widetilde{\gamma_1}(\zeta), \quad \Re[\overline{\zeta}^{q_2} W(\zeta)] = \widetilde{\gamma_2}(\zeta) \ \ (\zeta \in \partial D), \end{cases}$$

where $\lambda \in \mathbb{R}\backslash\{-1, 0, 1\}$*,* $\varphi, \widetilde{\gamma_1}, \widetilde{\gamma_2}, \gamma, f_k \in C(\partial D)$ *with* $\partial_{\bar{z}} f_k = f_{k+1}$ *(* $k = 1, 2, \cdots, n-1,\ n \geq 2$*) and* $f_n = 0$*. Similarly, the corresponding higher-order problems can be solved.*

Applying Theorem 4.1 and the results for the half-Neumann-n problem in [27], we can draw the following conclusion:

Corollary 4.5. *Let D be the unit disk in* $\mathbb{C}$*, and let* $\lambda \in \mathbb{R}\backslash\{-1, 0, 1\}$*,* $c_s \in \mathbb{C}$*,* $\gamma_s \in C(\partial D)$ *(*$0 \leq s \leq m-2,\ m \geq 2, m \in \mathbb{N}$*),* $\varphi, \gamma, f_k \in C(\partial D)$ *with* $\partial_{\bar{z}} f_k = f_{k+1}$ *(* $k = 1, 2, \cdots, n-1,\ n \geq 2$*) and* $f_n = 0$*. Then the problem*

$$\begin{cases} \partial_{\bar{z}}^m W(z) = \frac{\lambda-1}{4\lambda}\phi(z) + \frac{\lambda+1}{4\lambda}\overline{\phi(z)}, \quad \partial_{\bar{z}}^n \phi(z) = 0, \quad \partial_{\bar{z}}^k \phi(z) = f_k(z) \ \ (z \in D), \\ \partial_{\bar{\zeta}}^{m-1} W(\zeta) = \varphi(\zeta), \quad \Re[\overline{\zeta}^p \phi(\zeta)] = \gamma(\zeta) \ \ (\zeta \in \partial D), \\ \zeta \partial_{\bar{\zeta}}^s W_\zeta(\zeta) = \gamma_s(\zeta) \ \ (\zeta \in \partial D), \quad \partial_{\bar{z}}^s W(0) = c_s \ \ (0 \leq s \leq m-2,\ m \geq 2, m \in \mathbb{N}) \end{cases}$$

is solvable on D, and the solution is given by

$$\begin{aligned} W(z) = \sum_{s=0}^{m-2} c_s \bar{z}^s + \sum_{s=0}^{m-2} \frac{(-1)^{s+1}}{2\pi i}\int_{\partial D} \gamma_s(\zeta)\Big[\frac{(1-|z|^2)^s}{z^s}\log(1 - z\bar{\zeta}) \\ + \sum_{r=0}^{s-1} \frac{\bar{\zeta}^{s-m}}{(s-m)z^r}\sum_{l=0}^{r} C_s^l(-|z|^2)^l\Big] d\zeta + (-1)^{m-1}\frac{z}{\pi}\int_D \frac{f(\zeta)}{\zeta(\zeta - z)}\frac{(\overline{\zeta - z})^{m-2}}{(m-2)!} d\sigma_\zeta \end{aligned}$$

if and only if

$$\frac{1}{2\pi i}\int_{\partial D}\frac{\gamma_{m-2}(\zeta)}{(1-\bar{z}\zeta)\zeta}d\zeta+\frac{\bar{z}}{\pi}\int_{D}\frac{f(\zeta)}{(1-\bar{z}\zeta)^2}d\sigma_\zeta=0$$

and

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\partial D}\frac{\gamma_{m-1-s}(\zeta)}{(1-\bar{z}\zeta)\zeta}d\zeta+\frac{\bar{z}}{2\pi i}\int_{\partial D}\frac{\gamma_{m-s}(\zeta)}{\zeta}\log(1-\bar{z}\zeta)d\zeta\\=&\sum_{r=2}^{s-1}\frac{(-1)^r}{r!(r-1)}\frac{\bar{z}}{2\pi i}\int_{\partial D}\frac{\gamma_{m-1-s+r}(\zeta)}{\zeta}[(\overline{\zeta-z})^{r-1}-(-\bar{z})^{r-1}]d\zeta\\&+\frac{(-1)^s}{(s-1)!}\frac{\bar{z}}{\pi}\int_{D}f(\zeta)\frac{(\overline{\zeta-z})^{s-1}}{(1-\bar{z}\zeta)^2}d\sigma_\zeta,\ 2\le s\le m-1,\end{aligned}$$

where $f(z)$ is the solution of the problem (4.1) in Theorem 4.1.

Remark 4.6. *By Theorem 4.1 and the mixed boundary value problems with combinations of Schwarz, Dirichlet, and Neumann conditions in [27], we can discuss the corresponding mixed boundary value problems for bi-polyanalytic functions, for example,*

$$\begin{cases}\partial_{\bar{z}}^m W(z)=\frac{\lambda-1}{4\lambda}\phi(z)+\frac{\lambda+1}{4\lambda}\overline{\phi(z)}, & \partial_{\bar{z}}^n\phi(z)=0,\quad \partial_{\bar{z}}^k\phi(z)=f_k(z)\ (z\in D),\\ \partial_{\bar{\zeta}}^{m-1}W(\zeta)=\varphi(\zeta), & \Re[\overline{\zeta}^p\phi(\zeta)]=\gamma(\zeta)\ (\zeta\in\partial D),\\ \zeta\partial_{\bar{\zeta}}^s W_\zeta(\zeta)=\gamma_s(\zeta)\ (\zeta\in\partial D), & \partial_{\bar{z}}^s W(0)=c_s\ (0\le s\le m-3,\ m\ge 3, m\in\mathbb{N}),\\ \partial_{\bar{\zeta}}^{m-2}W(\zeta)=\gamma_{m-2}(\zeta)\ (\zeta\in\partial D).\end{cases}$$

5. Conclusions

By using the Cauchy-Pompeiu formula in the complex plane, we first discuss a Riemann-Hilbert boundary value problem of higher order on the unit disk D in $\mathbb{C}$ and obtain the expression of the solution under different solvable conditions. On this basis, we get the specific solutions to the boundary value problems for bi-polyanalytic functions with the Dirichlet and Riemann-Hilbert boundary conditions and the corresponding higher-order problems. Therefore, we obtain the solution to the half-Neumann problem of higher order for bi-polyanalytic functions. The conclusions provide a favorable method for discussing other boundary value problems of bi-polyanalytic functions, such as mixed boundary value problems and the related systems of complex partial differential equations of higher order, and also provide a solid basis for future research on bi-polyanalytic functions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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