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Research article

g-frame generator sets for projective unitary representations

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Abstract: Frames with special structures play a crucial role in various industrial applications, such as medical imaging, quantum communication, recognition and identification software, and so on. In this paper, we will discuss a more general setting, i.e., a g-frame induced by the projective unitary representation. We show some new results on the dilation property for g-frame generator sets for unitary groups and projective unitary representations. In particular, by using complete wandering operators, several properties of g-frame generators for projective unitary representations have been obtained. Moreover, we explore some characterizations of the g-frame generator dual pairs.

Keywords: g-frame generator set; projective unitary representation; dilation; unitary group; dual g-frame generator
Mathematics Subject Classification: 42C15, 42C40, 47D03

1. Introduction

Frames have appeared implicitly in the mathematical literature before they were introduced officially by Duffin and Schaeffer in the context of the non-harmonic Fourier series [1]. Since the celebrated work by Daubechies et al. [2], frame theory has recently become an important tool in many fields such as sampling theory, signal processing and data compression. In recent years, various generalizations of frames have been proposed for different purposes such as frames of subspaces (fusion frames), oblique frames, pseudo-frames, outer frames and g-frames [3–7]. Indeed, all of these generalizations can be regarded as special cases of g-frames [7]. Today, g-frames, with their applications, have been investigated by many researchers [8–12]. As is well known, frames with a special structure, such as Gabor frames and wavelet frames, not only have great variety for use in applications, but also they have undergone extensive theoretical analysis [13–15]. Moreover, the dilation property is very important in frame theory and has attracted much attention from scholars from different related fields. The classical dilation theorem for frames shows that every frame for a Hilbert space can be dilated to be a Riesz basis for a larger space [15]. Motivated by these aspects of frames, in this work, we are interested in the dilations of the more general g-frames that are generated by a unitary group, or by a projective unitary representation. In addition, we would like to work more with dual g-frame generators for projective unitary representations.

Throughout this paper, H and K are two Hilbert spaces over the field of complex numbers and I is the identity operator on H. The notation $B(H, K)$ refers to the space of all bounded linear operators from H into K, and we write $B(H) = B(H, H)$ as the shorthand. Denote by $\{H_i : i \in \mathbb{J}\}$ a sequence of subspaces of K and by $B(H, H_i)$ the collection of all bounded linear operators from H into H_i for every $i \in \mathbb{J}$, where $\mathbb{J}$ is a countable index set. Let $\{\Lambda_i \in B(H, H_i)\}_{i\in\mathbb{J}}$ be a family of operators. If there exist two constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{J}} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in H,$$

we call $\{\Lambda_i \in B(H, H_i)\}_{i\in\mathbb{J}}$ a *g-frame* for H with respect to $\{H_i\}_{i\in\mathbb{J}}$, where A and B are called the lower and upper *frame bounds*, respectively. For simplification, if the spaces are clear, we will just say that $\{\Lambda_i\}_{i\in\mathbb{J}}$ is a g-frame for H in the sequel. If we only have the upper bound, then $\{\Lambda_i\}_{i\in\mathbb{J}}$ is said to be a *g-Bessel sequence* for H. $\{\Lambda_i\}_{i\in\mathbb{J}}$ is called a *tight g-frame* for H if $A = B$, and a *Parseval g-frame* for H provided that $A = B = 1$ [7].

If $\{\Lambda_i\}_{i\in\mathbb{J}}$ is a g-frame for H, then we can define the operator $S : H \to H$ by

$$Sf = \sum_{i\in\mathbb{J}} \Lambda_i^* \Lambda_i f, \quad \forall f \in H,$$

where Λ_i^* is the adjoint operator of Λ_i. Obviously, S is a well-defined, bounded, positive, invertible operator on H. Noted that S is a *g-frame operator* that is associated with $\{\Lambda_i\}_{i\in\mathbb{J}}$. Another fact is that $\{\Lambda_i S^{-1}\}_{i\in\mathbb{J}}$ is also a g-frame for H and $\{\Lambda_i S^{-\frac{1}{2}}\}_{i\in\mathbb{J}}$ is a Parseval g-frame for H (see [7]).

$\{\Lambda_i \in B(H, H_i)\}_{i\in\mathbb{J}}$ is called a *g-orthonormal basis* for H if it satisfies the following:

$$\langle \Lambda_{i_1}^* f_{i_1}, \Lambda_{i_2}^* f_{i_2} \rangle = \delta_{i_1,i_2} \langle f_{i_1}, f_{i_2} \rangle, \; \forall i_1, i_2 \in \mathbb{J}, f_{i_1} \in H_{i_1}, f_{i_2} \in H_{i_2},$$

$$\sum_{i\in\mathbb{J}} \|\Lambda_i f\|^2 = \|f\|^2, \; \forall f \in H,$$

where δ_{i_1,i_2} is the Kronecker delta. Actually, by [16, Corollary 2.13], $\{\Lambda_i\}_{i\in\mathbb{J}}$ is a g-orthonormal basis if and only if $\{\Lambda_i\}_{i\in\mathbb{J}}$ is a g-frame and the first equation holds.

The organization of this article is as follows. In Section 2, the dilation results for g-frame generator sets for unitary group will be given. In Section 3, we focus on the g-frame generator sets for projective unitary representation of countable groups and consider the corresponding dilation property. In Section 4, we give some characterizations of g-frame generators for projective unitary representation in terms of complete wandering operators. Moreover, we study some properties of g-frame generators. In Section 5, we introduce the notion of dual g-frame generators and explore some equivalent characterizations of g-frame generator dual pairs.

2. g-frame generator sets for unitary groups

In this section, we mainly focus on the dilation problem for g-frame generator sets for a countable unitary group, as well as present some existing dilation results.

Recall that a *unitary system* is a subset of unitary operators acting on H which contains the identity operator I in $B(H)$ [17]. Evidently, a unitary group is a special case of unitary system. Two unitary systems $\mathcal{U}_1$ and $\mathcal{U}_2$ acting on Hilbert spaces H_1 and H_2, respectively, are said to be *unitarily equivalent* if there is a unitary operator $T : H_1 \to H_2$ such that $T\mathcal{U}_1T^* = \mathcal{U}_2$.

According to [18], if H_1, H_2 are Hilbert spaces, let $H_1 \odot H_2$ be the algebraic tensor product over $\mathbb{C}$. Denote an inner product on $H_1 \odot H_2$ by

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle = \langle \xi_1, \eta_1\rangle_1\langle \xi_2, \eta_2\rangle_2, \ \forall \xi_1, \eta_1 \in H_1, \xi_2, \eta_2 \in H_2$$

extended by linearity, where $\langle \cdot, \cdot\rangle_i$ is the inner product of H_i. Then the Hilbert space tensor product $H_1 \otimes H_2$ is the completion of $H_1 \odot H_2$. Generally, if $S_1, T_1 \in B(H_1)$ and $S_2, T_2 \in B(H_2)$, we can define $S_1 \otimes S_2 \in B(H_1 \otimes H_2)$ by

$$(S_1 \otimes S_2)(\xi \otimes \eta) = S_1\xi \otimes S_2\eta, \ \ \forall \xi \in H_1, \eta \in H_2.$$

Then $(S_1 \otimes S_2)(T_1 \otimes T_2) = S_1T_1 \otimes S_2T_2, S_1 \otimes (S_2 + T_2) = (S_1 \otimes S_2 + S_1 \otimes T_2)$ and $(S_1 \otimes S_2)^* = S_1^* \otimes S_2^*$.

In what follows, we use $\mathbb{J}$ to denote a countable index set and $\mathcal{I} \subseteq \mathbb{J}$ to denote a finite set.

Definition 2.1. ([19, Definition 2.1]) Let $\mathcal{U}$ be a countable unitary system on H and $\{\Lambda_i \in B(H, K)\}_{i\in\mathbb{J}}$. We say that $\{\Lambda_i\}_{i\in\mathbb{J}}$ is a *g-Bessel sequence* (*g-frame, Parseval g-frame, or tight g-frame*) *generator set* for $\mathcal{U}$ if $\{\Lambda_iU^*\}_{i\in\mathbb{J},U\in\mathcal{U}}$ is a g-Bessel sequence (g-frame, Parseval g-frame, or tight g-frame) for H.

In particular, for $\Lambda \in B(H, K)$, we say that Λ is a *g-Bessel sequence* (*g-frame, Parseval g-frame, or tight g-frame*) *generator* for $\mathcal{U}$ if $\{\Lambda U^*\}_{U\in\mathcal{U}}$ is a g-Bessel sequence (g-frame, Parseval g-frame, or tight g-frame) for H.

Inspired by the above definition, we introduce the following notion.

Definition 2.2. Let $\mathcal{U}$ be a countable unitary system on H and $\{\Lambda_{iV} \in B(H, K)\}_{i\in\mathbb{J},V\in\mathcal{U}}$. $\{\Lambda_{iV}\}_{i\in\mathbb{J},V\in\mathcal{U}}$ is called a *diagonal* (*Parseval or tight*) *g-frame generator set* for $\mathcal{U}$ if $\{\delta_{VU}\Lambda_{iV}U^*\}_{i\in\mathbb{J},V,U\in\mathcal{U}}$ is a (Parseval or tight) g-frame for H, where δ_{VU} is the Kronecker delta.

Remark 2.3. Let $\mathcal{I} \subseteq \mathbb{J}$ be a finite set and $\mathcal{U}$ be a countable unitary system. For a Hilbert space K, define an operator

$$L_{iU} : K \to \ell^2(\mathcal{I} \times \mathcal{U}) \otimes K,$$

$$L_{iU}k = e_{iU} \otimes k, \ \ \forall\ k \in K,$$

where $\{e_{iU} : i \in \mathcal{I}, U \in \mathcal{U}\}$ is the standard basis of $\ell^2(\mathcal{I} \times \mathcal{U})$ and $\otimes$ denotes a tensor product. It is easy to check that, for each $a \in \ell^2(\mathcal{I} \times \mathcal{U})$ and $k \in K$,

$$L_{iU}^* : \ell^2(\mathcal{I} \times \mathcal{U}) \otimes K \to K,$$

$$L_{iU}^*(a \otimes k) = \langle a, e_{iU}\rangle k.$$

We see from Lemma 2.9 in [19] that, for a g-frame generator set $\{\Lambda_i \in B(H, K)\}_{i\in\mathcal{I}}$ for $\mathcal{U}$, the *analysis operator* of $\{\Lambda_iU^*\}_{i\in\mathcal{I},U\in\mathcal{U}}$ is defined by

$$\theta : H \to \ell^2(\mathcal{I} \times \mathcal{U}) \otimes K,$$

$$\theta f = \sum_{i\in\mathcal{I},U\in\mathcal{U}} L_{iU}\Lambda_i U^* f,\ \forall f \in H.$$

The operator $S : H \to H$ given by

$$S f = \theta^*\theta f = \sum_{i\in\mathcal{I},U\in\mathcal{U}} U\Lambda_i^*\Lambda_i U^* f,\ \forall f \in H,$$

is called the *g-frame operator* of $\{\Lambda_i U^*\}_{i\in\mathcal{I},U\in\mathcal{U}}$. Particularly, if Λ is a g-frame generator for $\mathcal{U}$, the analysis operator of $\{\Lambda U^*\}_{U\in\mathcal{U}}$ can be defined by

$$\theta' : H \to \ell^2(\mathcal{U}) \otimes K,$$

$$\theta' f = \sum_{U\in\mathcal{U}} e_U \otimes \Lambda U^* f,\ \ \forall\ f \in H,$$

where $\{e_U\}_{U\in\mathcal{U}}$ is the standard basis for $\ell^2(\mathcal{U})$.

Let $\mathcal{U}$ be a unitary group and $\{e_{iU} : i \in \mathcal{I}, U \in \mathcal{U}\}$ be the standard basis of $\ell^2(\mathcal{I} \times \mathcal{U})$. For each $U \in \mathcal{U}$, we define the unitary operator on $\ell^2(\mathcal{I} \times \mathcal{U})$ by $\lambda_U e_{iV} = e_{i(UV)}$, where $i \in \mathcal{I}, V \in \mathcal{U}$.

The following result shows that a g-frame generator set for a unitary group is an image of an orthonormal basis under a positive operator with a suitable spectrum.

Theorem 2.4. *Let $\mathcal{U}$ be a unitary group on H and $\{\Lambda_i \in B(H, K)\}_{i\in\mathcal{I}}$ be a g-frame generator set for $\mathcal{U}$ with the frame bounds A and B. Then,*

(1) *there exists an isometry $\Phi : H \to \ell^2(\mathcal{I}\times\mathcal{U})\otimes K$ such that $\Phi^*(\lambda_U \otimes I_K)\Phi = U$ for all $U \in \mathcal{U}$, where I_K is the identity operator on K;*

(2) *there exists a positive operator $\Xi : \ell^2(\mathcal{I} \times \mathcal{U}) \otimes K \to \Phi(H)$ such that $\{A^{\frac{1}{2}}, B^{\frac{1}{2}}\} \subseteq \sigma(\Xi|_{\Phi(H)}) \subseteq [A^{\frac{1}{2}}, B^{\frac{1}{2}}]$ and, for any $U \in \mathcal{U}$ and $g \in K$,*

$$\Xi(e_{iU} \otimes g) = \Phi U\Lambda_i^* g,$$

where $\sigma(\Xi|_{\Phi(H)})$ denotes the spectrum of the operator $\Xi|_{\Phi(H)}$;

(3) *Ξ^2 is unitarily equivalent to $S \oplus \mathbf{0}$, where S is the g-frame operator of $\{\Lambda_i U^*\}_{i\in\mathcal{I},U\in\mathcal{U}}$ and $\mathbf{0}$ is the zero operator on $\Phi(H)^\perp$.*

Proof. Assume that S is the g-frame operator for $\{\Lambda_i U^*\}_{i\in\mathcal{I},U\in\mathcal{U}}$; then,

$$S f = \sum_{i\in\mathcal{I},U\in\mathcal{U}} U\Lambda_i^*\Lambda_i U^* f,\ \forall f \in H.$$

First, we want to prove that $SU = US$ for each $U \in \mathcal{U}$. In fact, for $f \in H$,

$$\begin{aligned} USf &= U \sum_{i\in\mathcal{I},V\in\mathcal{U}} V\Lambda_i^*\Lambda_i V^* f \\ &= \sum_{i\in\mathcal{I},V\in\mathcal{U}} UV\Lambda_i^*\Lambda_i V^* f \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in\mathcal{I},V\in\mathcal{U}} (UV)\Lambda_i^*\Lambda_i(UV)^*Uf \\
&= SUf.
\end{aligned}
$$

Since S is an invertible positive operator, we have that $S^{-\frac{1}{2}}U = US^{-\frac{1}{2}}$ for each $U \in \mathcal{U}$. Note that $\{\Lambda_i \in B(H,K)\}_{i\in\mathcal{I}}$ is a g-frame generator set for $\mathcal{U}$; we know that $\{\Lambda_i U^* S^{-\frac{1}{2}}\}_{i\in\mathcal{I},U\in\mathcal{U}}$ is a Parseval g-frame (see [7], Remark1). Then, $\{\Lambda_i S^{-\frac{1}{2}}U^*\}_{i\in\mathcal{I},U\in\mathcal{U}}$ is also a Parseval g-frame, that is, $\{\Lambda_i S^{-\frac{1}{2}}\}_{i\in\mathcal{I}}$ is a Parseval g-frame generator set for $\mathcal{U}$. Let Φ be the analysis operator of the Parseval g-frame $\{\Lambda_i S^{-\frac{1}{2}}U^*\}_{i\in\mathcal{I},U\in\mathcal{U}}$. Then,

$$
\Phi : H \to \ell^2(\mathcal{I}\times\mathcal{U})\otimes K,
$$

$$
\Phi f = \sum_{i\in\mathcal{I},U\in\mathcal{U}} L_{iU}\Lambda_i S^{-\frac{1}{2}}U^* f, \ \forall f \in H.
$$

Thus, for each $f \in H$ and $U \in \mathcal{U}$, we have

$$
\begin{aligned}
(\lambda_U \otimes I_K)\Phi f &= (\lambda_U \otimes I_K)\sum_{i\in\mathcal{I},V\in\mathcal{U}} L_{iV}\Lambda_i S^{-\frac{1}{2}}V^* f \\
&= \sum_{i\in\mathcal{I},V\in\mathcal{U}} (\lambda_U \otimes I_K)(e_{iV}\otimes \Lambda_i S^{-\frac{1}{2}}V^* f) \\
&= \sum_{i\in\mathcal{I},V\in\mathcal{U}} (e_{i(UV)}\otimes \Lambda_i S^{-\frac{1}{2}}V^* f) \\
&= \sum_{i\in\mathcal{I},V\in\mathcal{U}} L_{i(UV)}\Lambda_i S^{-\frac{1}{2}}V^* f \\
&= \sum_{i\in\mathcal{I},V\in\mathcal{U}} L_{i(UV)}\Lambda_i S^{-\frac{1}{2}}(UV)^* Uf \\
&= \Phi Uf.
\end{aligned}
$$

Since $\{\Lambda_i S^{-\frac{1}{2}}U^*\}_{i\in\mathcal{I},U\in\mathcal{U}}$ is a Parseval g-frame, it implies that $\Phi^*\Phi$ is the identity on H. This leads to

$$
\Phi^*(\lambda_U \otimes I_K)\Phi = U,
$$

that is, (1) holds.

Let Φ^* be the synthesis operator of $\{\Lambda_i S^{-\frac{1}{2}}U^*\}_{i\in\mathcal{I},U\in\mathcal{U}}$ defined by

$$
\Phi^* : \ell^2(\mathcal{I}\times\mathcal{U})\otimes K \to H,
$$

$$
\Phi^* = \sum_{i\in\mathcal{I},U\in\mathcal{U}} U(\Lambda_i S^{-\frac{1}{2}})^* L_{iU}^*.
$$

Set $\Xi = \Phi S^{\frac{1}{2}}\Phi^*$. Then for all $U \in \mathcal{U}$ and $g \in K$, one has

$$
\begin{aligned}
\Xi(e_{i'U}\otimes g) &= \Phi S^{\frac{1}{2}}\Phi^*(e_{i'U}\otimes g) \\
&= \Phi S^{\frac{1}{2}}\sum_{i\in\mathcal{I},V\in\mathcal{U}} V(\Lambda_i S^{-\frac{1}{2}})^* L_{iV}^*(e_{i'U}\otimes g) \\
&= \Phi S^{\frac{1}{2}}\sum_{i\in\mathcal{I},V\in\mathcal{U}} V(\Lambda_i S^{-\frac{1}{2}})^* \langle e_{i'U}, e_{iV}\rangle g
\end{aligned}
$$

$$= \Phi S^{\frac{1}{2}} U S^{-\frac{1}{2}} \Lambda_{i'}^* g = \Phi U \Lambda_{i'}^* g.$$

Since any element in $\Phi(H)$ can be expressed as $\sum\limits_{i\in\mathcal{I},U\in\mathcal{U}} (e_{iU} \otimes \Lambda_i S^{-\frac{1}{2}} U^* f)$ for $f \in H$, we have

$$\begin{aligned}
&\|\Xi \sum_{i\in\mathcal{I},U\in\mathcal{U}} (e_{iU} \otimes \Lambda_i S^{-\frac{1}{2}} U^* f)\|^2 \\
= &\ \| \sum_{i\in\mathcal{I},U\in\mathcal{U}} \Xi(e_{iU} \otimes \Lambda_i S^{-\frac{1}{2}} U^* f)\|^2 \\
= &\ \| \sum_{i\in\mathcal{I},U\in\mathcal{U}} \Phi U \Lambda_i^* \Lambda_i S^{-\frac{1}{2}} U^* f\|^2 \\
= &\ \| \sum_{i\in\mathcal{I},U\in\mathcal{U}} U \Lambda_i^* \Lambda_i S^{-\frac{1}{2}} U^* f\|^2 \\
= &\ \| \sum_{i\in\mathcal{I},U\in\mathcal{U}} U \Lambda_i^* \Lambda_i U^* S^{-\frac{1}{2}} f\|^2 \\
= &\ \|S^{\frac{1}{2}} f\|^2 = \langle Sf, f\rangle.
\end{aligned}$$

This implies that

$$A\|f\|^2 \le \|\Xi \sum_{i\in\mathcal{I},U\in\mathcal{U}} (e_{iU} \otimes \Lambda_i S^{-\frac{1}{2}} U^* f)\|^2 \le B\|f\|^2.$$

Since, for any $f \in H$, $\Phi f = \sum\limits_{i\in\mathcal{I},U\in\mathcal{U}} (e_{iU} \otimes \Lambda_i S^{-\frac{1}{2}} U^* f)$ and Φ is an isometric operator, it follows that

$$\|f\|^2 = \| \sum_{i\in\mathcal{I},U\in\mathcal{U}} (e_{iU} \otimes \Lambda_i S^{-\frac{1}{2}} U^* f)\|^2.$$

Let $g' = \sum\limits_{i\in\mathcal{I},U\in\mathcal{U}} (e_{iU} \otimes \Lambda_i S^{-\frac{1}{2}} U^* f)$. It turns out that

$$A\|g'\|^2 \le \|\Xi g'\|^2 \le B\|g'\|^2 \quad \forall g' \in \Phi(H),$$

which proves (2).

Finally, to verify (3), we define $U_0 : H \oplus \Phi(H)^\perp \to \ell^2(\mathcal{I} \times \mathcal{U}) \otimes K$ by

$$U_0 h = \begin{cases} \Phi h, & h \in H, \\ h, & h \in \Phi(H)^\perp. \end{cases}$$

Obviously, U_0 is unitary because Φ is an isometric operator. Since $\Phi^*\Phi$ is the identity on H and $\Xi = \Phi S^{\frac{1}{2}} \Phi^*$, we get that

$$\Xi^2 = \Phi S^{\frac{1}{2}} \Phi^* \Phi S^{\frac{1}{2}} \Phi^* = \Phi S \Phi^*.$$

Then,

$$\Xi^2 U_0 h = \begin{cases} \Xi^2 \Phi h = \Phi S h = U_0 S h, & h \in H, \\ \Xi^2 h = \Phi S \Phi^* h = 0 = U_0 \mathbf{0} h, & h \in \Phi(H)^\perp, \end{cases}$$

Thus, $\Xi^2 = U_0(S \oplus \mathbf{0}) U_0^*$. The proof is complete. □

Corollary 2.5. *Suppose that $\mathcal{U}$ is a unitary group on Hilbert space H and $\{\Lambda_i \in B(H,K)\}_{i\in\mathcal{I}}$ is a g-frame generator set for $\mathcal{U}$ with the frame bounds A and B. Then,*

(1) *there exists a Hilbert space $N \supseteq H$ and a unitary group $\mathcal{V}$ on N such that the restriction map $\mathcal{V} \ni V \to V|_H$ is a group isomorphism of $\mathcal{V}$ onto $\mathcal{U}$;*

(2) *there exists a positive operator $\Xi : N \to H$ such that*

$$\{A^{\frac{1}{2}}, B^{\frac{1}{2}}\} \subset \sigma(\Xi|_H) \subset [A^{\frac{1}{2}}, B^{\frac{1}{2}}].$$

Proof. Let $N = \ell^2(\mathcal{I} \times \mathcal{U}) \otimes K$. By Theorem 2.4(1), we know that $\Phi(H)$ is an invariant subspace of $\lambda_U \otimes I_K$ and Φ is an isometric operator, where Φ is defined as in Theorem 2.4. So, we can embed H into N by identifying H with $\Phi(H)$. Denote $\mathcal{V} = \{\lambda_U \otimes I_K\}_{U\in\mathcal{U}}$. Then, clearly, $\mathcal{V}$ is a unitary group on N. Hence, the restriction map $\mathcal{V} \ni V \to V|_H$ is a group isomorphism of $\mathcal{V}$ onto $\mathcal{U}$. By Theorem 2.4(2), it is easily seen that (2) holds. □

3. g-frame generator sets for projective unitary representations

In this section, we survey the dilation property of g-frames in the context of projective unitary representations. For this purpose, we need to recall a few concepts and notations, which can be found in [20].

A *projective unitary representation* π for a countable group G is a mapping $g \to \pi(g)$ from G into the group $\mathcal{U}(H)$ of all unitary operators on a separable Hilbert space H such that

$$\pi(g)\pi(h) = \mu(g,h)\pi(gh) \quad \text{for all } g,h \in G,$$

where $\mu(g,h)$ is a scalar-valued function on $G \times G$ taking values in the circle group $\mathbb{T}$. The function $\mu(g,h)$ is then called a *multiplier* of π. We also say that π is a *μ-projective unitary representation*. It is clear from the definition that

$$\mu(g_1, g_2g_3)\mu(g_2,g_3) = \mu(g_1g_2, g_3)\mu(g_1,g_2), \quad g_1,g_2,g_3 \in G, \tag{3.1}$$

$$\mu(g,\mathbf{e}) = \mu(\mathbf{e},g) = 1, \quad g \in G, \mathbf{e} \text{ is the group unit of } G. \tag{3.2}$$

Any function $\mu : G \times G \to (T)$ satisfying (3.1) and (3.2) above will be called a *multiplier* for G.

Similar to the group unitary representation case, the left regular projective representation with a multiplier μ for G plays a crucial role here. Let μ be a multiplier for G. For each $g \in G$, we define $\lambda_g : \ell^2(\mathcal{I} \times G) \to \ell^2(\mathcal{I} \times G)$ by

$$\lambda_g e_{ih} = \mu(g,h)e_{i(gh)}, \quad h \in G, \tag{3.3}$$

where $\mathcal{I}$ is a finite set and $\{e_{ih}\}_{i\in\mathcal{I},h\in G}$ is the standard orthonormal basis for $\ell^2(\mathcal{I} \times G)$. Clearly, λ_g is a unitary operator on $\ell^2(\mathcal{I} \times G)$.

Theorem 3.1. *Let G be a countable group and π be a projective unitary representation of G on H with a multiplier μ. Assume that $\{\Lambda_i \in B(H,K)\}_{i\in\mathcal{I}}$ is a g-frame generator set for $\{\pi(g)\}_{g\in G}$. Then,*

(1) *there exists a family of operators* $\{C_i \in B(H,K)\}_{i\in\mathcal{I}}$ *such that* $\{C_i\}_{i\in\mathcal{I}}$ *is a Parseval g-frame generator set for* $\{\pi(g)\}_{g\in G}$*;*

(2) *there exists an isometry* $\Phi : H \to \ell^2(\mathcal{I}\times G)\otimes K$ *such that* $\Phi^*(\lambda_g\otimes I_K)\Phi = \pi(g)$ *for all* $g\in G$*;*

(3) *there exists a projective unitary representation* $\Delta(g)$ *of* G *on* $\ell^2(\mathcal{I}\times G)\otimes K$ *and a family of operators* $\{\Lambda'_i\}_{i\in\mathcal{I}}$ *such that* $\{\Lambda'_i\Delta(g)^*\}_{i\in\mathcal{I},g\in G}$ *is a g-orthonormal basis for* $\ell^2(\mathcal{I}\times G)\otimes K$*;*

(4) *there exist* μ*-projective unitary representations* π_1 *and* π_2 *of* G *on a Hilbert space* $\Phi(H)^\perp$ *and* $\Phi(H)$*, respectively, and diagonal Parseval g-frame generator sets* $\{M_{ih}\}_{i\in\mathcal{I},h\in G}$ *and* $\{N_{ih}\}_{i\in\mathcal{I},h\in G}$ *for* $\pi_1(g)$ *and* $\pi_2(g)$*, respectively, such that* $\{\delta_{gh}[M_{ih}\pi_1(g)^*\oplus N_{ih}\pi_2(g)^*]\}_{i\in\mathcal{I},g,h\in G}$ *is a g-orthonormal basis for* $\Phi(H)^\perp\oplus\Phi(H)$*.*

Proof. (1) First, we need to check that $\pi(g)S = S\pi(g)$ holds for all $g\in G$, where S is the g-frame operator for $\{\Lambda_i\pi(g)^*\}_{i\in\mathcal{I},g\in G}$. Indeed, for $g\in G$ and $x\in H$, we have

$$\begin{aligned}
\pi(g)Sx &= \pi(g)\left(\sum_{i\in\mathcal{I},h\in G}\pi(h)\Lambda_i^*\Lambda_i\pi(h)^*x\right)\\
&= \left(\sum_{i\in\mathcal{I},h\in G}\pi(g)\pi(h)\Lambda_i^*\Lambda_i\pi(h)^*x\right)\\
&= \sum_{i\in\mathcal{I},h\in G}\mu(g,h)\pi(gh)\Lambda_i^*\Lambda_i\pi(h)^*x\\
&= \sum_{i\in\mathcal{I},h\in G}\mu(g,h)\pi(gh)\Lambda_i^*\Lambda_i\pi(h)^*\pi(g)^*\pi(g)x\\
&= \sum_{i\in\mathcal{I},h\in G}\mu(g,h)\pi(gh)\Lambda_i^*\Lambda_i(\pi(g)\pi(h))^*\pi(g)x\\
&= \sum_{i\in\mathcal{I},h\in G}\mu(g,h)\overline{\mu(g,h)}\pi(gh)\Lambda_i^*\Lambda_i\pi(gh)^*\pi(g)x\\
&= S\pi(g)x.
\end{aligned}$$

Thus, $\pi(g)S = S\pi(g)$, as claimed. Since S is an invertible positive operator, we know that $S^{-\frac{1}{2}}\pi(g) = \pi(g)S^{-\frac{1}{2}}$ for each $g\in G$. Since $\{\Lambda_i\in B(H,K)\}_{i\in\mathcal{I}}$ is a g-frame generator set for $\{\pi(g)\}_{g\in G}$, it follows that $\{\Lambda_iS^{-\frac{1}{2}}\pi(g)^*\}_{i\in\mathcal{I},g\in G}$ is a Parseval g-frame for H. Let $C_i = \Lambda_iS^{-\frac{1}{2}}$. Then, it can be verified that C_i satisfies the requirements.

(2) Let $\Phi : H\to\ell^2(\mathcal{I}\times G)$ be the analysis operator of the Parseval g-frame $\{C_i\pi(g)^*\}_{i\in\mathcal{I},g\in G}$. Then, for each $x\in H$,

$$\Phi x = \sum_{i\in\mathcal{I},g\in G}(e_{ig}\otimes C_i\pi(g)^*x),$$

where $\{e_{ig}\}_{i\in\mathcal{I},g\in G}$ is the standard orthogonal basis of $\ell^2(\mathcal{I}\times G)$. Obviously, Φ is an isometric operator. Let λ_g be a unitary operator as defined in (3.3). Since $\pi(g)^* = \overline{\mu(g,g^{-1})}\pi(g^{-1})$ and

$$(\lambda_g\otimes I_K)(e_{i(g^{-1}h)}\otimes k) = \mu(g,g^{-1}h)(e_{ih}\otimes k), \tag{3.4}$$

for each $k \in K$, it follows that

$$\begin{aligned}
\Phi\pi(g)x &= \sum_{i\in\mathcal{I},h\in G}(e_{ih}\otimes C_i\pi(h)^*\pi(g)x)\\
&= \sum_{i\in\mathcal{I},h\in G}\{e_{ih}\otimes C_i[\pi(g)^*\pi(h)]^*x\}\\
&= \sum_{i\in\mathcal{I},h\in G}\{e_{ih}\otimes C_i[\overline{\mu(g,g^{-1})}\pi(g^{-1})\pi(h)]^*x\}\\
&= \sum_{i\in\mathcal{I},h\in G}\{e_{ih}\otimes C_i[\overline{\mu(g,g^{-1})}\mu(g^{-1},h)\pi(g^{-1}h)]^*x\}\\
&= \sum_{i\in\mathcal{I},h\in G}\overline{\mu(g,g^{-1}h)\mu(g^{-1},h)}\mu(g,g^{-1})\{(\lambda_g\otimes I_K)(e_{g^{-1}h}\otimes C_i\pi(g^{-1}h)^*x)\}\\
&= (\lambda_g\otimes I_K)\sum_{i\in\mathcal{I},h\in G}(e_{g^{-1}h}\otimes C_i\pi(g^{-1}h)^*x)\\
&= (\lambda_g\otimes I_K)\Phi x,
\end{aligned}$$

for each $x \in H$. In the penultimate step we used (3.1) and (3.2) to eliminate three multiplier terms. Hence,

$$\Phi^*(\lambda_g\otimes I_K)\Phi = \pi(g), \quad \text{for all } g\in G,$$

Thus, (2) is proved.

(3) Define $\Delta(g) = \lambda_g \otimes \pi(g)$. For all $h, g \in G$, we have

$$\begin{aligned}
\Delta(g)\Delta(h) &= [\lambda_g\otimes\pi(g)][\lambda_h\otimes\pi(h)]\\
&= \lambda_g\lambda_h\otimes\pi(g)\pi(h)\\
&= [\mu(g,h)]^2\Delta(gh).
\end{aligned}$$

Denote $\nu(g,h) = [\mu(g,h)]^2$. Then, for any $g_1, g_2, g_3 \in G$,

$$\begin{aligned}
\nu(g_1g_2,g_3)\nu(g_1,g_2) &= [\mu(g_1g_2,g_3)\mu(g_1,g_2)]^2\\
&= [\mu(g_1,g_2g_3)\mu(g_2,g_3)]^2\\
&= \nu(g_1,g_2g_3)\nu(g_2,g_3),
\end{aligned}$$

$$\nu(g,\mathbf{e}) = \nu(\mathbf{e},g) = 1,\ g\in G,\ \mathbf{e} \text{ is the unit of } G,$$

and, consequently, Δ is a ν-projective unitary representation of G.

Let $N = \ell^2(\mathcal{I}\times G)\otimes K$ and $\Lambda_i' = (\lambda_{\mathbf{e}}\otimes\Lambda_i S^{-\frac{1}{2}})$. Then, for all $g, h \in G$, $i, j \in \mathcal{I}$ and $k, k_i, k_j \in K$, we have

$$\begin{aligned}
&\langle\Lambda_i'\Delta(g)^*(e_{ig}\otimes k_i),\Lambda_j'\Delta(h)^*(e_{jg}\otimes k_j)\rangle\\
&= \langle(\lambda_{\mathbf{e}}\otimes\Lambda_i S^{-\frac{1}{2}})(\lambda_g^*\otimes\pi(g)^*)(e_{ig}\otimes k_i),(\lambda_{\mathbf{e}}\otimes\Lambda_j S^{-\frac{1}{2}})(\lambda_h^*\otimes\pi(h)^*)(e_{jh}\otimes k_j)\rangle\\
&= \delta_{i,j}\delta_{g,h}\langle e_{ig}\otimes k_i, e_{jh}\otimes k_j\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i\in\mathcal{I},g\in G} \|\Lambda_i'\Delta(g)^*(e_{ig}\otimes k)\|^2\\
=&\sum_{i\in\mathcal{I},g\in G} \|(\lambda_{\mathbf{e}}\otimes\Lambda_i S^{-\frac{1}{2}})(\lambda_g^*\otimes\pi(g)^*)(e_{ig}\otimes k)\|^2\\
=&\sum_{i\in\mathcal{I},g\in G} \|\lambda_g^* e_{ig}\otimes\Lambda_i S^{-\frac{1}{2}}\pi(g)^* k\|^2\\
=&\sum_{i\in\mathcal{I},g\in G} \|\Lambda_i S^{-\frac{1}{2}}\pi(g)^* k\|^2\\
=&\|e_{ig}\otimes k\|^2.
\end{aligned}$$

Therefore, $\{\Lambda_i'\Delta(g)^*\}_{i\in\mathcal{I},g\in G}$ is a g-orthonormal basis.

(4) Let P be the orthogonal projection onto $\Phi(H)$. Then for all $x\in H,\ k\in K,\ g\in G$ and $j\in\mathcal{I}$,

$$\begin{aligned}
\langle\Phi x, P(e_{jg}\otimes k)\rangle &= \langle P\Phi x,(e_{jg}\otimes k)\rangle\\
&= \langle\sum_{i\in\mathcal{I},h\in G}(e_{ih}\otimes C_i\pi(h)^*x), e_{jg}\otimes k\rangle\\
&= \langle C_j\pi(g)^*x,k\rangle\\
&= \langle x,\pi(g)C_j^*k\rangle\\
&= \langle\Phi x,\Phi\pi(g)C_j^*k\rangle.
\end{aligned}$$

Hence, $\{P(e_{jg}\otimes k)-\Phi\pi(g)C_j^*k\}\perp\Phi(H)$. Since $\{P(e_{jg}\otimes k)-\Phi\pi(g)C_j^*k\}\in\Phi(H)$, it implies that

$$P(e_{jg}\otimes k)=\Phi\pi(g)C_j^*k. \tag{3.5}$$

Using (3.4) and (3.5), for $j\in\mathcal{I},\ h\in G$ and $k\in K$, we deduce that

$$\begin{aligned}
&(\lambda_g\otimes I_K)P(e_{jh}\otimes k)\\
=&(\lambda_g\otimes I_K)\Phi\pi(h)C_j^*k\\
=&(\lambda_g\otimes I_K)\sum_{i\in\mathcal{I},v\in G}(e_{iv}\otimes C_i\pi(v)^*\pi(h)C_j^*k)\\
=&\sum_{i\in\mathcal{I},v\in G}\mu(g,v)(e_{i(gv)}\otimes C_i\pi(v)^*\pi(h)C_j^*k))\\
=&\sum_{i\in\mathcal{I},v\in G}\mu(g,v)(e_{i(gv)}\otimes C_i(\pi(g)\pi(v))^*\pi(g)\pi(h)C_j^*k))\\
=&\sum_{i\in\mathcal{I},v\in G}(e_{i(gv)}\otimes C_i\pi(gv)^*\pi(g)\pi(h)C_j^*k))\\
=&\mu(g,h)\Phi\pi(gh)C_j^*k\\
=&\mu(g,h)P(e_{j(gh)}\otimes k)\\
=&P(\lambda_g\otimes I_K)(e_{jh}\otimes k).
\end{aligned}$$

Thus, we get the commutation relation

$$(\lambda_g\otimes I_K)P=P(\lambda_g\otimes I_K),\quad\text{for all } g\in G.$$

Let $M_{ig} = \lambda_{\mathbf{e}} \otimes \Lambda_i S^{-\frac{1}{2}} \pi(g)^*(I - P)$ and $\pi_1(h) = (I - P)(\lambda_h \otimes I_K)$, where $\mathbf{e}$ is the unit of G, $i \in \mathcal{I}$ and $g, h \in G$. Then,

$$\begin{aligned} \delta_{gh} M_{ig} \pi_1(h)^* &= \delta_{gh}(\lambda_{\mathbf{e}} \otimes \Lambda_i S^{-\frac{1}{2}} \pi(g)^*)(I - P)(\lambda_h^* \otimes I_K)(I - P) \\ &= \delta_{gh}(\lambda_h^* \otimes \Lambda_i S^{-\frac{1}{2}} \pi(g)^*)(I - P) \\ &= \{(I - P)(\lambda_g \otimes \pi(g) S^{-\frac{1}{2}} \Lambda_i^*)\}^*. \end{aligned}$$

Since $(I - P)$ is an orthogonal projection, we know that $\{\delta_{gh} M_{ig} \pi_1(h)^*\}_{i \in \mathcal{I}, g, h \in G}$ is a Parseval g-frame for $\Phi(H)^\perp$. For $g, h \in G$, we have

$$\begin{aligned} \pi_1(g)\pi_1(h) = (I - P)(\lambda_g \otimes I_K)(\lambda_h \otimes I_K) = (\lambda_{gh} \otimes I_K)(I - P) \\ = \mu(g, h)(\lambda_{gh} \otimes I_K) = \mu(g, h)\pi_1(gh). \end{aligned}$$

By restricting the domain of $\pi_1(g)$ to $\Phi(H)^\perp$, we can get that π_1 is a μ-projective unitary representation of G on $\Phi(H)^\perp$. For each $i' \in \mathcal{I}$ and $g', h' \in G$, let $N_{i'g'} = \lambda_{\mathbf{e}} \otimes \Lambda_i S^{-\frac{1}{2}} \pi(g')^*$ and $\pi_2(h') = \Phi\pi(h')\Phi^*$. Obviously, π_2 is a μ-projective unitary representation of G on $\Phi(H)$. Moreover, we have

$$\begin{aligned} \delta_{g'h'} N_{i'g'} \pi_2(h')^* &= \delta_{g'h'}[\lambda_{\mathbf{e}} \otimes \Lambda_{i'} S^{-\frac{1}{2}} \pi(g')^*]\Phi\pi(h')\Phi^* \\ &= \delta_{g'h'}[\lambda_{\mathbf{e}} \otimes \Lambda_{i'} S^{-\frac{1}{2}} \pi(g')^*]\Phi\Phi^*(\lambda_{h'} \otimes I_K)\Phi\Phi^* \\ &= \delta_{g'h'}[\lambda_{h'} \otimes \Lambda_{i'} S^{-\frac{1}{2}} \pi(g')^*]P \\ &= [P(\lambda_{g'}^* \otimes \pi(g') S^{-\frac{1}{2}} \Lambda_{i'}^*)]^*. \end{aligned}$$

Hence, $\{\delta_{g'h'} N_{i'g'} \pi_2(h')^*\}_{i \in \mathcal{I}, g, h \in G}$ is a Parseval g-frame for $\Phi(H)$. Thus,

$$\begin{aligned} & \lambda_{\mathbf{e}} \otimes \Lambda_i S^{-\frac{1}{2}} \pi(h)^* \\ = \ & \delta_{gh}[(\lambda_{\mathbf{e}} \otimes \Lambda_i S^{-\frac{1}{2}} \pi(h)^*)P + (\lambda_{\mathbf{e}} \otimes \Lambda_i S^{-\frac{1}{2}} \pi(h)^*)(I - P)] \\ = \ & \delta_{gh}[N_{ih}\pi_2(g)^* \oplus M_{ih}\pi_1(g)^*]. \end{aligned}$$

By (3), it is easy to see that $\{\delta_{gh}[M_{ih}\pi_1(g)^* \oplus N_{ih}\pi_2(g)^*]\}_{i \in \mathcal{I}, g, h \in G}$ is a g-orthonormal basis. This completes the proof. □

4. g-frame generators for projective unitary representations

This section is devoted to investigating the g-frame with the structure of projective unitary representations. We will see that the g-frame generators for a projective unitary representation can be characterized from the perspective of complete wandering operators and operators in the commutant of $\{\pi(g)\}_{g \in G}$.

Definition 4.1. ([21]) Let $\mathcal{U}$ be a unitary system. A *complete wandering operator* for $\mathcal{U}$ is a co-isometry $A \in B(H, K)$ such that $A\mathcal{U}^* = \{A\mathcal{U}^* : U \in \mathcal{U}\}$ is a g-orthonormal basis for H.

Let $\mathcal{Q} \subseteq B(H)$ be a set. The notation $\mathcal{Q}'$ will denote the usual commutant of $\mathcal{Q}$, that is,

$$\mathcal{Q}' = \{T \in B(H) : TQ = QT \text{ for } Q \in \mathcal{Q}\}.$$

It is well known that, if $T \in B(H)$ has closed range, then there exists an operator $T^{\dagger} \in B(H)$ such that

$$N(T^{\dagger}) = R(T)^{\perp}, \ \ R(T^{\dagger}) = N(T)^{\perp} \ \text{ and } \ TT^{\dagger}y = y, \ \ y \in R(T),$$

where $R(T)$ and $N(T)$ denote the range and null space, respectively. This operator is uniquely determined by these properties, and we call it the *pseudo-inverse* of T. Clearly, if T is invertible, then $T^{-1} = T^{\dagger}$.

Theorem 4.2. *Suppose that G is a countable group, π is a projective unitary representation of G on H with a multiplier μ, and $T \in B(H, K)$ is a complete wandering operator for $\{\pi(g)\}_{g\in G}$. Then,*

(1) *an operator $A \in B(H, K)$ is a g-frame generator for $\{\pi(g)\}_{g\in G}$ if and only if there exist an isometric operator $\Theta \in \{\pi(g)\}'_{g\in G}$ and an invertible positive operator $E \in \{\pi(g)\}'_{g\in G}$ such that $A = T\Theta E$. In particular, an operator A is a Parseval g-frame generator for $\{\pi(g)\}_{g\in G}$ if and only if there exists an isometric operator $\Theta \in \{\pi(g)\}'_{g\in G}$ such that $A = T\Theta$;*

(2) *if A is a g-frame generator for $\{\pi(g)\}_{g\in G}$, then there exists a Hilbert space M and an invertible positive operator $E \in \{\pi(g)\}'_{g\in G}$ such that $\{AE^{-1}\pi(g)^*\}_{g\in G}$ is a g-orthonormal basis for M;*

(3) *an operator A is a complete wandering operator for $\{\pi(g)\}_{g\in G}$ if and only if there exists a unitary operator U such that $U \in \{\pi(g)\}'_{g\in G}$ and $A = TU$.*

Proof. (1) Assume that A is a g-frame generator for $\{\pi(g)\}_{g\in G}$; then, there exist two constants C, D with $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{g\in G}\|A\pi(g)^*f\|^2 \leq D\|f\|^2, \ \forall f \in H.$$

Let S be the g-frame operator of $\{A\pi(g)^*\}_{g\in G}$. Then $\{A\pi(g)^*S^{-\frac{1}{2}}\}_{g\in G}$ is a Parseval g-frame. Similar to the proof of Theorem 3.1(1), we can get that $S\pi(g) = \pi(g)S$ for all $g \in G$. Hence, $S^{-\frac{1}{2}} \in \{\pi(g)\}'_{g\in G}$. It shows that $AS^{-\frac{1}{2}}$ is also a Parseval g-frame generator for $\{\pi(g)\}_{g\in G}$. Since T is a complete wandering operator for $\{\pi(g)\}_{g\in G}$, it follows that

$$\sum_{g\in G}\|\pi(g)T^*(AS^{-\frac{1}{2}}\pi(g)^*f)\|^2 = \sum_{g\in G}\|AS^{-\frac{1}{2}}\pi(g)^*f\|^2 = \|f\|^2, \ \forall f \in H.$$

Define a bounded linear operator $\phi : H \to H$ by

$$\phi f = \sum_{g\in G}\pi(g)T^*AS^{-\frac{1}{2}}\pi(g)^*f, \ \forall f \in H. \tag{4.1}$$

It is easy to see that ϕ is isometric on H. Next we show that $\phi \in \{\pi(g)\}'_{g\in G}$. For all $h \in G$ and $f \in H$, we have

$$\begin{aligned}\pi(h)\phi f &= \pi(h)\sum_{g\in G}\pi(g)T^*AS^{-\frac{1}{2}}\pi(g)^*f \\ &= \sum_{g\in G}\mu(h,g)\pi(hg)T^*AS^{-\frac{1}{2}}\pi(g)^*\pi(h)^*\pi(h)f\end{aligned}$$

$$\begin{aligned}
&= \sum_{g\in G}\mu(h,g)\pi(hg)T^*AS^{-\frac{1}{2}}[\pi(h)\pi(g)]^*\pi(h)f \\
&= \sum_{g\in G}\pi(hg)T^*AS^{-\frac{1}{2}}\pi(hg)^*\pi(h)f \\
&= \phi\pi(h)f.
\end{aligned}$$

Similarly, we can obtain that $\phi^* \in \{\pi(g)\}'_{g\in G}$. Let P be the orthogonal projection onto $\phi(H)$. Then, for each $f_1,\ f_2 \in H$,

$$\langle \phi f_1, Pf_2\rangle = \langle \phi f_1, f_2\rangle = \langle f_1, \phi^* f_2\rangle = \langle \phi f_1, \phi\phi^* f_2\rangle.$$

Hence, $\phi\phi^* f_2 = Pf_2$. Note that $\phi \in \{\pi(g)\}'_{g\in G}$ and $\phi^* \in \{\pi(g)\}'_{g\in G}$. Thus,

$$P\pi(g) = \pi(g)P,\ \forall g \in G.$$

For each $f \in H,\ k \in K$ and $g \in G$, we have

$$\begin{aligned}
\langle \phi f, P\pi(g)T^*k\rangle &= \langle \phi f, \pi(g)T^*k\rangle \\
&= \langle \sum_{h\in G}\pi(h)T^*AS^{-\frac{1}{2}}\pi(h)^*f, \pi(g)T^*k\rangle \\
&= \langle AS^{-\frac{1}{2}}\pi(g)^*f, k\rangle = \langle f, \pi(g)S^{-\frac{1}{2}}A^*k\rangle \\
&= \langle \phi f, \phi\pi(g)S^{-\frac{1}{2}}A^*k\rangle.
\end{aligned}$$

Therefore, $\phi\pi(g)S^{-\frac{1}{2}}A^*k = P\pi(g)T^*k$. Since $\phi \in \{\pi(g)\}'_{g\in G}$, $P \in \{\pi(g)\}'_{g\in G}$ and $P = \phi\phi^*$, it implies that $AS^{-\frac{1}{2}} = T\phi$. That is,

$$A = T\phi S^{\frac{1}{2}}.$$

Just let $\Theta = \phi$ and $E = S^{\frac{1}{2}}$. Then, the conclusion holds.

Conversely, we first prove that $\pi(g)^*\Theta = \Theta\pi(g)^*,\ \forall g \in G$. Since π is a projective unitary representation of G on H with a multiplier μ, we have

$$\pi(g)\pi(g^{-1}) = \mu(g, g^{-1})\pi(\mathbf{e}) \Longleftrightarrow \pi(g)^* = \overline{\mu(g, g^{-1})}\pi(g^{-1}),\ \forall g \in G.$$

Observing that $\Theta \in \{\pi(g)\}'_{g\in G}$ and $\pi(g^{-1}) \in \{\pi(h)\}_{h\in G}$, we obtain

$$\pi(g)^*\Theta = \Theta\pi(g)^* \ \text{ and } \ \pi(g)^*E = E\pi(g)^*.$$

Thus, for all $f \in H$ and $g \in G$,

$$\begin{aligned}
\sum_{g\in G}\|A\pi(g)^*f\|^2 &= \sum_{g\in G}\|T\Theta E\pi(g)^*f\|^2 \\
&= \sum_{g\in G}\|T\pi(g)^*\Theta Ef\|^2 \\
&= \|\Theta Ef\|^2 = \|Ef\|^2 \\
&\geq \frac{1}{\|E^{-1}\|^2}\|E^{-1}Ef\|^2 = \frac{1}{\|E^{-1}\|^2}\|f\|^2.
\end{aligned}$$

Combining this with the fact that

$$\begin{aligned}\sum_{g\in G}\|A\pi(g)^*f\|^2 &= \sum_{g\in G}\|T\Theta E\pi(g)^*f\|^2 = \sum_{g\in G}\|T\pi(g)^*\Theta Ef\|^2\\ &= \|\Theta Ef\|^2 = \|Ef\|^2\\ &\leq \|E\|^2\|f\|^2,\end{aligned}$$

for each $f \in H$ and $g \in G$, we have

$$\frac{1}{\|E^{-1}\|^2}\|f\|^2 \leq \sum_{g\in G}\|A\pi(g)^*f\|^2 \leq \|E\|^2\|f\|^2,\ \forall f \in H,$$

that is A is a g-frame generator for $\{\pi(g)\}_{g\in G}$.

Specially, if A is a Parseval g-frame generator for $\{\pi(g)\}_{g\in G}$, then its g-frame operator S is an identity operator on H. Analogous to the above proof, it is easy to see that the result is verified.

(2) Since T is a complete wandering operator for $\{\pi(g)\}_{g\in G}$, it follows T^* is an isometry; thus, there exists a bounded operator $T^\dagger$ such that $TT^\dagger = I$. By (1), we know that $\{AS^{-\frac{1}{2}}\pi(g)^*\}$ is a Parseval g-frame, $AS^{-\frac{1}{2}} = T\phi$, and $P = \phi\phi^*$. Hence, for all $g, h \in G$ and $f, g \in K$,

$$\begin{aligned}&\langle \pi(g)(AS^{-\frac{1}{2}})^*f, \pi(h)(AS^{-\frac{1}{2}})^*g\rangle\\ =\ &\langle \pi(g)\phi^*T^*f, \pi(h)\phi^*T^*g\rangle\\ =\ &\langle \pi(g)PT^*f, \pi(h)PT^*g\rangle.\end{aligned}$$

Let $M = \{T^\dagger x : x \in K, T^*x \in P(H)\}$. Obviously, M is a Hilbert space and $M \subseteq H$. Then, for $f_1, f_2 \in T(M)$,

$$\langle \pi(g)PT^*f_1, \pi(h)PT^*f_2\rangle = \langle \pi(g)T^*f_1, \pi(h)T^*f_2\rangle = \delta_{gh}\langle f_1, f_2\rangle.$$

Taking $E = S^{\frac{1}{2}}$, we have that $\langle \pi(g)(AE^{-1})^*f_1, \pi(h)(AE^{-1})^*f_2\rangle = \delta_{gh}\langle f_1, f_2\rangle$ and $\{AE^{-1}\pi(g)^*\}_{g\in G}$ is a Parseval g-frame on M. This shows that $\{AE^{-1}\pi(g)^*\}$ is a g-orthonormal basis for M.

(3) Let ϕ be the operator defined in (4.1). If A is a complete wandering operator for $\{\pi(g)\}_{g\in G}$, we know that the g-frame operator S is an identity operator on H. Then,

$$\phi f = \sum_{g\in G}\pi(g)T^*A\pi(g)^*f,\ \forall f \in H.$$

Write $U = \phi$. By the proof of (1), we have that $U^*U = I$, $U \in \{\pi(g)\}'_{g\in G}$, and $A = TU$. Hence, U is a unitary operator for H since A is a complete wandering operator. The other direction is trivial. □

From the above theorem, we immediately have the following consequence.

Corollary 4.3. *Let $\mathcal{U}$ be a unitary group on the finite dimensional Hilbert spaces denote by H and $T \in B(H, K)$ be a complete wandering operator for $\mathcal{U}$. If $A \in B(H, K)$ is a g-frame generator for $\mathcal{U}$, then there exists an invertible operator E such that $E \in \mathcal{U}'$ and AE^{-1} is a complete wandering operator for $\mathcal{U}$.*

In what follows, we will construct complete wandering operators through the use of g-frame generators.

Theorem 4.4. *Let G be a countable group, π be a projective unitary representation of G on H with a multiplier μ, and $T \in B(H,K)$ be the complete wandering operator for $\{\pi(g)\}_{g\in G}$. If $A \in B(H,K)$ is a g-frame generator for $\{\pi(g)\}_{g\in G}$, then $B \in B(H,K)$ exists as a g-Bessel sequence generator for $\{\pi(g)\}_{g\in G}$ and a Hilbert space N such that $AS^{-\frac{1}{2}} \oplus B$ is a complete wandering operator for $\{\pi(g)\oplus\pi(g)\}_{g\in G}$ on N, where S is the g-frame operator of $\{A\pi(g)^*\}_{g\in G}$.*

Proof. Assume that T is a complete wandering operator for $\{\pi(g)\}_{g\in G}$ and A is a g-frame generator for $\{\pi(g)\}_{g\in G}$. Define $\theta : H \to H$ by

$$\theta f = \sum_{g\in G}\pi(g)T^*AS^{-\frac{1}{2}}\pi(g)^*f,\ \forall f \in H.$$

Then, θ is an isometry. Similar to the proof of Theorem 4.2(1), we obtain that for each $h \in G, \pi(h)\theta = \theta\pi(h)$. Let P be the orthogonal projection from H onto $\theta(H)$. For all $f_1, f_2 \in H$ we have

$$\begin{aligned}
\langle \theta f_1, Pf_2\rangle &= \langle \theta f_1, f_2\rangle = \langle \sum_{g\in G}\pi(g)T^*AS^{-\frac{1}{2}}\pi(g)^*f_1, f_2\rangle \\
&= \langle f_1, \sum_{g\in G}\pi(g)(AS^{-\frac{1}{2}})^*T\pi(g)^*f_2\rangle \\
&= \langle \theta f_1, \theta[\sum_{g\in G}\pi(g)(AS^{-\frac{1}{2}})^*T\pi(g)^*f_2]\rangle.
\end{aligned}$$

Hence, $Pf_2 = \theta[\sum_{g\in G}\pi(g)(AS^{-\frac{1}{2}})^*T\pi(g)^*f_2]$. Therefore, for each $f \in H$ and $h \in G$,

$$\begin{aligned}
\pi(h)Pf &= \pi(h)\theta\sum_{g\in G}\pi(g)(AS^{-\frac{1}{2}})^*T\pi(g)^*f \\
&= \theta\sum_{g\in G}\pi(h)\pi(g)(AS^{-\frac{1}{2}})^*T\pi(g)^*f \\
&= \theta\sum_{g\in G}\pi(hg)(AS^{-\frac{1}{2}})^*T\pi(hg)^*\pi(h)f \\
&= P\pi(h)f.
\end{aligned}$$

Also, we have that $(I-P)\pi(h) = \pi(h)(I-P),\ h \in G$. Set $B = T(I-P) \in B(H,K)$. Then,

$$\begin{aligned}
\sum_{g\in G}\|B\pi(g)^*f\|^2 &= \sum_{g\in G}\|T(I-P)\pi(g)^*f\|^2 \\
&= \sum_{g\in G}\|T\pi(g)^*(I-P)f\|^2 \\
&= \|(I-P)f\|^2 \le \|f\|^2,\ \ f \in H.
\end{aligned}$$

It implies that B is a g-Bessel sequence generator for $\{\pi(g)\}_{g\in G}$.

Denote $N = H \oplus (\theta(H))^\perp$. Next, we show that $AS^{-\frac{1}{2}} \oplus B$ is a Parseval g-frame generator for $\{\pi(g)\oplus\pi(g)\}_{g\in G}$. For any $g \in G, f_1 \in H$ and $f_2 \in (\theta(H))^\perp$, we can get

$$\begin{aligned}
&\sum_{g\in G}\|(AS^{-\frac{1}{2}}\oplus B)(\pi(g)\oplus\pi(g))^*(f_1\oplus f_2)\|^2\\
=\ &\sum_{g\in G}\|AS^{-\frac{1}{2}}\pi(g)^*f_1+B\pi(g)^*f_2\|^2\\
=\ &\sum_{g\in G}\|AS^{-\frac{1}{2}}\pi(g)^*f_1\|^2+\sum_{g\in G}\|B\pi(g)^*f_2\|^2+2Re\sum_{g\in G}\langle AS^{-\frac{1}{2}}\pi(g)^*f_1,B\pi(g)^*f_2\rangle.
\end{aligned}$$

Applying Theorem 4.2, we have that $AS^{-\frac{1}{2}}=T\theta$ and $P=\theta\theta^*$. So,

$$\begin{aligned}
&\sum_{g\in G}\langle AS^{-\frac{1}{2}}\pi(g)^*f_1,B\pi(g)^*f_2\rangle\\
=\ &\sum_{g\in G}\langle T\theta\pi(g)^*f_1,B\pi(g)^*f_2\rangle\\
=\ &\sum_{g\in G}\langle T\pi(g)^*\theta f_1,T\pi(g)^*(I-P)f_2\rangle\\
=\ &\sum_{g\in G}(\langle T\pi(g)^*\theta f_1,T\pi(g)^*f_2\rangle-\langle T\pi(g)^*\theta f_1,T\pi(g)^*Pf_2\rangle)\\
=\ &\langle\theta f_1,f_2\rangle-\langle\theta f_1,\theta\theta^*f_2\rangle\\
=\ &0.
\end{aligned}$$

Then,

$$\begin{aligned}
&\sum_{g\in G}\|(AS^{-\frac{1}{2}}\oplus B)(\pi(g)\oplus\pi(g))^*(f_1\oplus f_2)\|^2\\
=\ &\sum_{g\in G}\|AS^{-\frac{1}{2}}\pi(g)^*f_1\|^2+\sum_{g\in G}\|B\pi(g)^*f_2\|^2\\
=\ &\|f_1\|^2+\|f_2\|^2=\|f_1\oplus f_2\|^2.
\end{aligned}$$

This proves that $\{AS^{-\frac{1}{2}}\pi(g)^*\oplus B\pi(g)^*\}_{g\in G}$ is a Parseval g-frame.
On the other hand, for all $g\in G$, we have

$$\begin{aligned}
&\pi(g)(AS^{-\frac{1}{2}})^*\oplus\pi(g)B^*\\
=\ &\theta^*\pi(g)T^*\oplus\pi(g)(I-P)T^*\\
=\ &\theta^*P\pi(g)T^*\oplus\pi(g)(I-P)T^*\\
=\ &[\theta^*\oplus(I-P)][P\pi(g)T^*\oplus(I-P)\pi(g)T^*].
\end{aligned}$$

Moreover, for any $g,h\in G$ and $k_1,k_2\in K$, we obtain

$$\begin{aligned}
&\langle[\pi(g)(AS^{-\frac{1}{2}})^*\oplus\pi(g)B^*]k_1,[\pi(h)(AS^{-\frac{1}{2}})^*\oplus\pi(h)B^*]k_2\rangle\\
=\ &\langle[\theta^*\oplus(I-P)][P\pi(g)T^*\oplus(I-P)\pi(g)T^*]k_1,\\
&[\theta^*\oplus(I-P)][P\pi(h)T^*\oplus(I-P)\pi(h)T^*]k_2\rangle\\
=\ &\langle[P\pi(g)T^*\oplus(I-P)\pi(g)T^*]k_1,[P\pi(h)T^*\oplus(I-P)\pi(h)T^*]k_2\rangle
\end{aligned}$$

$$\begin{aligned} &= \langle \pi(g)T^*k_1, \pi(h)T^*k_2\rangle \\ &= \delta_{g,h}\langle k_1, k_2\rangle. \end{aligned}$$

In the last step, we applied T as a complete wandering operator for $\{\pi(g)\}_{g\in G}$. Hence, $AS^{-\frac{1}{2}} \oplus B$ is a complete wandering operator for $\{\pi(g) \oplus \pi(g)\}_{g\in G}$ on N. □

Remark 4.5. If H is a finite dimensional Hilbert space, by the proof of Theorem 4.4, we see that θ is a unitary operator. Then, $\theta^*\theta = \theta\theta^* = I$ and $P = I$. Hence, $B = T(P - I) = \mathbf{0}$, which is a trivial result.

For example, let $g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, g_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and $G = \{g_1, g_2\} \subseteq B(\mathbb{C}^2)$. Let $H = \mathbb{C}^2, K = span\{e_{g_1}\}$, and the projective unitary representation $\pi(g_i)e_{g_j} = e_{g_i g_j}$, where $i, j = 1, 2$ and e_{g_i} takes the value of 1 at i and zero elsewhere. Clearly, for any $f \in H$, $Tf = \langle f, e_{g_1}\rangle e_{g_1}$ is a complete wandering operator for $\pi(G)$, $Af = \langle f, e_{g_2}\rangle e_{g_1}$ is a Parseval g-frame generator of $\pi(G)$, and $S = I$. Then, for any $f \in H$, we have that $\theta f = \langle f, e_{g_1}\rangle e_{g_2} + \langle f, e_{g_2}\rangle e_{g_1}$. It follows that $P = \theta\theta^* = I$ and $B = T(I - P) = \mathbf{0}$. Hence, $A \oplus \mathbf{0}$ is a complete wandering operator for $\{\pi(g) \oplus \pi(g)\}_{g\in G}$.

5. Dual g-frame generators for projective unitary representations

As we know, one of the essential applications of frames is that they lead to expansions of vectors in the underlying Hilbert space in terms of the frame elements. Dual frames play a key role in this decomposition. So, in this section, we mainly consider the dual g-frame generators for projective unitary representations and give some of their characterizations. We first review the definition of dual g-frames.

From [9, Definition 3.1], a g-Bessel sequence $\{\Gamma_j \in B(H, K)\}_{j\in\mathbb{J}}$ is called a *dual g-frame* for a g-Bessel sequence $\{\Lambda_j \in B(H, K)\}_{j\in\mathbb{J}}$ if

$$f = \sum_{j\in\mathbb{J}} \Lambda_j^* \Gamma_j f, \ \forall f \in H.$$

If S is the g-frame operator for $\{\Lambda_j \in B(H, K)\}_{j\in\mathbb{J}}$, then $\{\Lambda_j S^{-1}\}_{j\in\mathbb{J}}$ is a dual for $\{\Lambda_j\}_{j\in\mathbb{J}}$ and is called the canonical dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$.

For our purpose, and motivated by the above definition, we introduce the following concept.

Definition 5.1. Let G be a countable group and π be a projective unitary representation of G on H with a multiplier μ. Two g-Bessel sequence generators $A, B \in B(H, K)$ for $\{\pi(g)\}_{g\in G}$ are called dual g-frame generators if

$$f = \sum_{g\in G} \pi(g)A^*B\pi(g)^* f, \ \forall f \in H.$$

Let θ_1 and θ_2 be the analysis operators for the g-Bessel sequences $\{A\pi(g)^*\}_{g\in G}$ and $\{B\pi(g)^*\}_{g\in G}$, respectively. Then, for all $f \in H$, we have

$$\begin{aligned} \|f\|^4 &= |\langle f, f\rangle|^2 = |\langle \sum_{g\in G} \pi(g)A^*B\pi(g)^* f, f\rangle|^2 \\ &= |\langle \theta_1 f, \theta_2 f\rangle|^2 \le \|\theta_1 f\|^2 \|\theta_2 f\|^2 \le \|\theta_2\|^2 \|f\|^2 \|\theta_1 f\|^2 \\ &= \|\theta_2\|^2 \|f\|^2 \sum_{g\in G} \|A\pi(g)^* f\|^2, \end{aligned}$$

that is $\sum \|A\pi(g)^*\|^2 \geq \frac{1}{\|\theta_2\|^2}\|f\|^2$. This shows that A is a g-frame generator for $\{\pi(g)\}_{g\in G}$. Similarly, B is a g-frame generator for $\{\pi(g)\}_{g\in G}$. Hence, we also say that A and B form a *g-frame generator dual pair* for $\{\pi(g)\}_{g\in G}$. Moreover, if S is the g-frame operator for the g-frame $\{A\pi(g)^*\}_{g\in G}$, then $\{AS^{-1}\pi(g)^*\}_{g\in G}$ is a dual g-frame for $\{A\pi(g)^*\}_{g\in G}$. We call AS^{-1} the canonical dual g-frame generator of $\{A\pi(g)^*\}_{g\in G}$.

Analogous to [9, Lemma 3.2], we give some elementary characterizations of duals in terms of the analysis operators.

Proposition 5.2. *Let A and B be two g-frame generators for $\{\pi(g)\}_{g\in G}$ with analysis operators θ_1 and θ_2, respectively. Then, the following statements are equivalent:*

(1) *A and B are g-frame generator dual pairs.*

(2) *$\theta_2^*\theta_1 = I$.*

(3) *$\theta_1^*\theta_2 = I$.*

Proof. (1) $\Rightarrow$ (2) Since, for any $f \in H$,

$$\theta_1 f = \sum_{g\in G} e_g \otimes A\pi(g)^* f \text{ and } \theta_2 f = \sum_{h\in G} e_h \otimes B\pi(h)^* f,$$

where $\{e_g\}_{g\in G}$ is the orthonormal basis for $\ell^2(G)$, it follows that

$$\theta_2^*\theta_1 f = \sum_{g\in G} \pi(g)B^*A\pi(g)^* f = f.$$

Hence, $\theta_2^*\theta_1 = I$.

(2)$\Rightarrow$(3) and (3)$\Rightarrow$(1) are obvious. □

Theorem 5.3. *Let G be a countable group, π be a projective unitary representation of G on H with a multiplier μ and $T \in B(H, K)$ be a complete wandering operator for $\{\pi(g)\}_{g\in G}$. Then, $A, B \in B(H, K)$ are g-frame generator dual pairs for $\{\pi(g)\}_{g\in G}$ if and only if there exist isometric operators Θ_1, $\Theta_2 \in \{\pi(g)\}'_{g\in G}$ and invertible positive operators E_1, $E_2 \in \{\pi(g)\}'_{g\in G}$ such that $A = T\Theta_1 E_1$, $B = T\Theta_2 E_2$, and $(\Theta_1 E_1)^*(\Theta_2 E_2) = I$.*

Proof. Suppose that A and B are g-frame generator dual pairs for $\{\pi(g)\}_{g\in G}$. By Theorem 4.2, we know that there exist isometric operators $\Theta_1, \Theta_2 \in \{\pi(g)\}'_{g\in G}$ and invertible positive operators $E_1, E_2 \in \{\pi(g)\}'_{g\in G}$ such that $A = T\Theta_1 E_1$ and $B = T\Theta_2 E_2$. The hypothesis that A and B are g-frame generator dual pairs implies that

$$\sum_{g\in G} \pi(g)A^*B\pi(g)^* f = \sum_{g\in G} \pi(g)(T\Theta_1 E_1)^* T\Theta_2 E_2 \pi(g)^* f = f, \ \forall f \in H. \tag{5.1}$$

Since $\Theta_1, \Theta_2 \in \{\pi(g)\}'_{g\in G}, E_1, E_2 \in \{\pi(g)\}'_{g\in G}$, it follows that $\Theta_i\pi(g)^*\pi(g) = \pi(g)^*\Theta_i\pi(g), i = 1, 2$. So,

$$\Theta_i\pi(g)^* = \pi(g)^*\Theta_i \text{ and } \Theta_i^*\pi(g) = \pi(g)\Theta_i^*, i = 1, 2.$$

Also, we have

$$E_i\pi(g)^* = \pi(g)^*E_i \text{ and } E_i^*\pi(g) = \pi(g)E_i^*, \ i = 1, 2.$$

Therefore, (5.1) becomes

$$\begin{aligned}\sum_{g\in G}\pi(g)A^*B\pi(g)^*f &= (\Theta_1E_1)^*\sum_{g\in G}\pi(g)T^*T\pi(g)^*(\Theta_2E_2)f\\ &= (\Theta_1E_1)^*(\Theta_2E_2)f = f,\end{aligned}$$

which means that $(\Theta_1E_1)^*(\Theta_2E_2) = I$.

The other direction is clear. □

6. Conclusions

In this paper, we extend the dilation theorem to g-frames with some additional structure and give some characterizations of g-frame generators for projective unitary representation in terms of complete wandering operators. Moreover, we introduce the notion of dual g-frame generators and obtain some characterizations of the g-frame generator dual pairs.

Author contributions

Aifang Liu: Conceiving and refining the ideas of the study, Formal analysis, Writing-review and editing; Jian Wu: Writing-original draft preparation, Formal analysis. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

We declare that there are no conflicts of interest.

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