AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 16486–16505.
DOI: 10.3934/math.2024799
Received: 23 March 2024
Revised: 22 Apirl 2024
Accepted: 25 Apirl 2024
Published: 13 May 2024

Research article

On a Diophantine equation involving fractional powers with primes of special types

Liuying Wu*

School of Mathematical Science, Tongji University, Shanghai 200092, China

* **Correspondence:** Email: liuyingw@tongji.edu.cn.

Abstract: Suppose that N is a sufficiently large real number. In this paper it is proved that for $2 < c < \frac{990}{479}$, the Diophantine equation

$$[p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] + [p_5^c] = N$$

is solvable in primes p_1, p_2, p_3, p_4, p_5 such that each of the numbers $p_i + 2, i = 1, 2, 3, 4, 5$ has at most $\left[\frac{6227}{3960-1916c}\right]$ prime factors.

Keywords: almost prime; diophantine equality; fractional powers; exponential sum
Mathematics Subject Classification: 11L07, 11L20, 11N35, 11N36

1. Introduction

For a fixed integer $k \geq 1$ and sufficiently large integer N, the well-known Waring-Goldbach problem is devoted to investigating the solvability of the following Diophantine equality

$$N = p_1^k + p_2^k + \cdots + p_s^k \tag{1.1}$$

in prime variables $p_1, p_2, \ldots, p_s$. Numerous mathematicians have derived many splendid results in this field. For instance, in 1937, Vinogradov [25] proved that such a representation of the type (1.1) exists for every sufficiently large odd integer N with $k = 1, s = 3$. Later in 1938, based upon Vinogradov's work, Hua [9] showed that (1.1) is solvable for every sufficiently large integer N satisfying that $N \equiv 5 \pmod{24}$ with $k = 2, s = 5$.

In 1933, Segal [21, 22] studied the following anolog of the well-known Waring problem. Suppose that $c > 1$ and $c \notin \mathbb{N}$; there exists a positive integer $s = s(c)$ such that for every sufficiently large natural number N, the equation

$$N = [m_1^c] + [m_2^c] + \cdots + [m_s^c]$$

has a solution with $m_1, m_2, \ldots, m_s$ integers, where $[t]$ denotes the integral part of any $t \in \mathbb{R}$.

To obtain a result that is analogous to the ternary Goldbach problem, in 1995, Laporta and Tolev [13] considered the equation

$$[p_1^c] + [p_2^c] + [p_3^c] = N, \tag{1.2}$$

where p_1, p_2, p_3 are prime numbers, $c \in \mathbb{R}, c > 1, N \in \mathbb{N}$, and $[t]$ denotes the integral part of t. They proved that if $1 < c < \frac{17}{16}$ and N is a sufficiently large integer, then the Eq (1.2) has a solution in prime numbers p_1, p_2, p_3. Later, the upper bound of c was enlarged to

$$\frac{12}{11}, \quad \frac{258}{235}, \quad \frac{137}{119}, \quad \frac{3113}{2703}, \quad \frac{3581}{3106}$$

by Kumchev and Nedeva [12], Zhai and Cao [27], Cai [6], Li and Zhang [15], and Baker [2], successively and respectively.

On the other hand, as an analogue of Hua's theorem on five prime squares, Li and Zhang [14] first studied the solvability of the Diophantine equation

$$[p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] + [p_5^c] = N \tag{1.3}$$

in prime numbers p_1, p_2, p_3, p_4, p_5. They proved that if $1 < c < \frac{4109054}{1999527}, c \neq 2$ and N is a sufficiently large integer, then the Eq (1.3) has a solution in prime numbers p_1, p_2, p_3, p_4, p_5. Later this result was improved by Li [17] who enlarged the upper bound for c to $\frac{408}{197}$, and by Baker [2] who replaced $\frac{408}{197}$ by $\frac{609}{293}$.

For any natural number r, let $\mathscr{P}_r$ denote an almost-prime with at most r prime factors, counted according to multiplicity. There are many papers that are devoted to the study of problems involving primes of a special type. In 1973, Chen [4] established that there exist infinitely many primes p such that $p + 2$ has at most 2 prime factors. In 2000, Tolev [24] proved that for every sufficiently large integer $N \equiv 3 \pmod 6$, the equation

$$p_1 + p_2 + p_3 = N \tag{1.4}$$

has a solution in prime numbers p_1, p_2, p_3 such that $p_1 + 2 \in \mathscr{P}_2, p_2 + 2 \in \mathscr{P}_5, p_3 + 2 \in \mathscr{P}_7$. After that, this result was improved by some mathematicians, and the best result in this field was obtained by Matomäki and Shao [18], who showed that for every sufficiently large integer $N \equiv 3 \pmod 6$ the Eq (1.4) has a solution in prime numbers p_1, p_2, p_3 such that $p_1 + 2, p_2 + 2, p_3 + 2 \in \mathscr{P}_2$.

Bearing in mind the result of [18], it is natural for us to conjecture that if c is close to 1, then the Eq (1.2) is solvable in primes p_1, p_2, p_3 such that $p_i + 2 \in \mathscr{P}_2$. An attempt to establish this kind of the result was first made by Petrov [19], who showed that, for $1 < c < \frac{17}{16}$ and every sufficiently large integer N, the Eq (1.2) is solvable in prime numbers p_1, p_2, p_3 such that each of the numbers $p_i + 2$ has at most $[\frac{95}{17-16c}]$ prime factors, counted according to multiplicity. Recently, Li et al. [16] improved Petrov's result; they extended the range of c to $1 < c < \frac{2173}{1930}$ and reduced the number of prime factors of $p_i + 2, i = 1, 2, 3$ to $[\frac{11387}{4346-3860c}]$.

Referencing Hua's work, Tolev [24] also showed that for every sufficiently large integer $N \equiv 5 \pmod{24}$, the equation

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 = N \tag{1.5}$$

has a solution in prime numbers p_1, p_2, p_3, p_4, p_5 such that $p_1 + 2 \in \mathscr{P}_2, p_2 + 2 \in \mathscr{P}_2, p_3 + 2 \in \mathscr{P}_5, p_4 + 2 \in \mathscr{P}_5$ and $p_5 + 2 \in \mathscr{P}_8$. And later in 2009, Cai and Lu [5] improved Tolev's result by showing that the Eq (1.5) has a solution in prime numbers p_1, p_2, p_3, p_4, p_5 such that $p_1 + 2 \in \mathcal{P}_2, p_2 + 2 \in \mathcal{P}_2, p_3 + 2 \in \mathcal{P}_4, p_4 + 2 \in \mathcal{P}_4$ and $p_5 + 2 \in \mathcal{P}_5$. Motivated by Petrov [19] and Tolev [24], it is reasonable to conjecture that if N is a sufficiently large natural number and c is close to 2, then the Eq (1.3) has a solution in prime numbers p_1, p_2, p_3, p_4, p_5 such that $p_i + 2$ are almost-primes of a certain fixed order.

In this paper, we shall prove the following result.

Theorem 1.1. *Suppose that $2 < c < \frac{990}{479}$ and let N be a sufficiently large natural number. Then the equation (1.3) has a solution in prime numbers p_1, p_2, p_3, p_4, p_5 such that each of the numbers $p_1+2, p_2+2, p_3+2, p_4+2$ and p_5+2 has at most $\left[\frac{6227}{3960-1916c}\right]$ prime factors, counted with the multiplicity.*

2. Preliminaries

Throughout this paper, the letter p, with or without subscript, always stand for prime numbers. We use ε to denote a sufficiently small positive number, and the value of ε may change from statement to statement. As usual, we use $\mu(n), \Lambda(n), \varphi(n)$ and $\tau(n)$ to denote Möbius' function, von Mangolds' function, Euler's function and the Dirichlet divisor function, respectively. We write $f = O(g)$ or, equivalently, $f \ll g$ if $|f| \leq Cg$ for some positive number C. If we have simultaneously, that $A \ll B$ and $B \ll A$, then we shall write $A \asymp B$. Moreover, we shall use (m, n) and $[m, n]$ for the greatest common divisor and the least common multiple of the integers m and n, respectively. And we use $e(\alpha)$ to denote $e^{2\pi i\alpha}$. In addition, we define

$$\begin{aligned} &2 < c < \frac{990}{479}, \qquad X = \left(\frac{N}{3}\right)^{\frac{1}{c}}, \qquad \delta = \frac{990}{479} - c, \qquad \xi = \frac{3c}{2} - \frac{5}{2}, \\ &\eta = \frac{4\delta}{13}, \qquad D = X^{\delta}, \qquad z = X^{\eta}, \qquad \tau = X^{\xi - c}, \qquad P(z) = \prod_{2<p<z} p, \\ &\log \mathbf{p} = \prod_{j=1}^{5} (\log p_j), \qquad \lambda^{\pm}(d) \text{ Rosser's weights of order } D. \end{aligned} \tag{2.1}$$

Lemma 2.1. *Suppose that $D > 4$ is a real number and let $\lambda^{\pm}(d)$ represent the Rosser functions of level D. Then we have the following properties:*

(1) For any positive integer d we have

$$\left|\lambda^{\pm}(d)\right| \leq 1, \quad \lambda^{\pm}(d) = 0 \quad \text{if} \quad d > D \quad \text{or} \quad \mu(d) = 0.$$

(2) If $n \in \mathbb{N}$ then

$$\sum_{d|n} \lambda^{-}(d) \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda^{+}(d). \tag{2.2}$$

(3) If $z \in \mathbb{R}$ and if

$$P(z) = \prod_{2<p<z} p, \quad \mathscr{B} = \prod_{2<p<z} \left(1 - \frac{1}{p}\right), \quad \mathscr{N}^{\pm} = \sum_{d|P(z)} \frac{\lambda^{\pm}(d)}{\varphi(d)}, \quad s_0 = \frac{\log D}{\log z}, \tag{2.3}$$

then we have

$$\mathscr{B} \leq \mathscr{N}^{+} \leq \mathscr{B}\left(F(s_0) + O\left((\log D)^{-1/3}\right)\right),$$

$$\mathscr{B} \geq \mathscr{N}^- \geq \mathscr{B}\left(f(s_0) + O\left((\log D)^{-1/3}\right)\right),$$

where $F(s)$ and $f(s)$ denote the classical functions in the linear sieve theory that are respectively defined by

$$F(s) = \frac{2e^\gamma}{s}\left(1 + \int_2^{s-1} \frac{\log(t-1)}{t}\mathrm{d}t\right), \quad 3 < s \leq 5$$

and

$$f(s) = \frac{2e^\gamma \log(s-1)}{s}, \quad 2 < s \leq 4.$$

Here γ denotes the Euler constant.

Proof. This is a special case of the work by Greaves [8]. □

Lemma 2.2. *Let*

$$\Lambda_i = \sum_{d|(p_i+2,P(z))} \mu(d), \quad \Lambda_i^\pm = \sum_{d|(p_i+2,P(z))} \lambda^\pm(d), \quad i = 1,2,3,4,5.$$

Then we have

$$\begin{aligned}\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5 \geq{}& \Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+\Lambda_4^+\Lambda_5^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^-\Lambda_4^+\Lambda_5^+ \\ &+ \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^-\Lambda_5^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^- - 4\Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+.\end{aligned}$$

Proof. The proof is the same as in Lemma 13 of [3]. □

Lemma 2.3. *Suppose that $f(x) : [a,b] \to \mathbb{R}$ has continuous derivatives of arbitrary order on $[a,b]$, where $1 \leq a < b \leq 2a$. Suppose further that*

$$\left|f^{(j)}(x)\right| \asymp \lambda_1 a^{1-j}, \qquad j \geq 1, \qquad x \in [a,b].$$

Then for any exponential pair (κ, λ), we have

$$\sum_{a<n\leq b} e(f(n)) \ll \lambda_1^\kappa a^\lambda + \lambda_1^{-1}.$$

Proof. See (3.3.4) of [7]. □

Lemma 2.4. *For any complex number z_n, we have*

$$\left|\sum_{a<n\leq b} z_n\right|^2 \leq \left(1 + \frac{b-a}{Q}\right) \sum_{|q|<Q}\left(1 - \frac{|q|}{Q}\right) \sum_{a<n,n+q\leq b} z_{n+q}\overline{z_n},$$

where Q is any positive integer.

Proof. See Lemma 8.17 of [11]. □

Lemma 2.5. *Let t be a non-integer, $\alpha \in (0,1)$ and $H \geq 3$. Then we have*

$$e(-\alpha\{t\}) = \sum_{|h|\leq H} c_h(\alpha)e(ht) + O\left(\min\left(1, \frac{1}{H\|t\|}\right)\right),$$

where

$$c_h(\alpha) = \frac{1 - e(-\alpha)}{2\pi i(h+\alpha)}.$$

Proof. See Lemma 12 of [1]. □

Lemma 2.6. *For any real number θ, we have*

$$\min\left(1, \frac{1}{H\|\theta\|}\right) = \sum_{h=-\infty}^{+\infty} a_h e(h\theta),$$

where

$$a_h \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{h^2}\right).$$

Proof. See (3) of [10]. □

Lemma 2.7. *Let $f(x)$ be a real differentiable function such that $f'(x)$ is monotonic and $f'(x) \geq m > 0$, or $f'(x) \leq -m < 0$, throughout the interval $[a, b]$. Then*

$$\int_a^b e(f(x))\mathrm{d}x \ll \frac{1}{m}.$$

Proof. See Lemma 4.2 of [23]. □

Lemma 2.8. *Suppose that $M > 1, c > 1, c \notin \mathbb{Z}$ and $\gamma > 0$. Let $\mathscr{A}(M; c, \gamma)$ denote the number of solutions of the following inequalities*

$$\left|n_1^c + n_2^c - n_3^c - n_4^c\right| < \gamma, \quad M < n_1, n_2, n_3, n_4 \leq 2M.$$

Then we have

$$\mathscr{A}(M; c, \gamma) \ll \left(\gamma M^{4-c} + M^2\right) M^{\varepsilon}.$$

Proof. See Theorem 2 of [20]. □

3. Beginning of the proof

The central focus of this paper is the study of the sum

$$\Gamma = \sum_{\substack{\frac{X}{2} < p_1, p_2, p_3, p_4, p_5 \leq X \\ [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]=N \\ (p_i+2, P(z))=1 \\ i=1,2,3,4,5}} \log \mathbf{p}.$$

In order to prove Theorem 1.1, we need only to show that $\Gamma > 0$. By the trivial orthogonality relation given by

$$\int_0^1 e(\alpha h)\mathrm{d}\alpha = \begin{cases} 1, & \text{if } h = 0, \\ 0, & \text{otherwise} \end{cases}$$

we can write Γ as

$$\Gamma = \sum_{\substack{\frac{X}{2} < p_1, p_2, p_3, p_4, p_5 \leq X \\ (p_i+2, P(z))=1 \\ i=1,2,3,4,5}} (\log \mathbf{p}) \int_{-\tau}^{1-\tau} e\left(([p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] + [p_5^c] - N)\alpha\right) \mathrm{d}\alpha. \tag{3.1}$$

By the definition of Λ_i in Lemma 2.2, we can see that

$$\Lambda_i = \sum_{d|(p_i+2,P(z))} \mu(d) = \begin{cases} 1, & \text{if } (p_i+2, P(z)) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then by Lemma 2.2 we find that

$$\begin{aligned} \Gamma &= \sum_{\frac{X}{2}<p_1,p_2,p_3,p_4,p_5\le X} (\log \mathbf{p})\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5 \int_{-\tau}^{1-\tau} e\left(([p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]-N)\alpha\right) \mathrm{d}\alpha \\ &\ge \sum_{\frac{X}{2}<p_1,p_2,p_3,p_4,p_5\le X} (\log \mathbf{p}) \int_{-\tau}^{1-\tau} e\left(([p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]-N)\alpha\right) \mathrm{d}\alpha \\ &\quad \times \left(\Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+\Lambda_4^+\Lambda_5^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^-\Lambda_4^+\Lambda_5^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^-\Lambda_5^+ \right. \\ &\quad \left. +\Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^- - 4\Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+\right) \\ &= \Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5-4\Gamma_6, \end{aligned} \tag{3.2}$$

By the symmetric property, we have

$$\begin{aligned} \Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_5 &= \sum_{\frac{X}{2}<p_1,p_2,p_3,p_4,p_5\le X} (\log \mathbf{p})\Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+ \\ &\quad \times \int_{-\tau}^{1-\tau} e\left(([p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]-N)\alpha\right) \mathrm{d}\alpha, \\ \Gamma_6 &= \sum_{\frac{X}{2}<p_1,p_2,p_3,p_4,p_5\le X} (\log \mathbf{p})\Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+ \\ &\quad \times \int_{-\tau}^{1-\tau} e\left(([p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]-N)\alpha\right) \mathrm{d}\alpha. \end{aligned}$$

Hence, by (3.1) and (3.2) we obtain

$$\Gamma \ge 5\Gamma_1 - 4\Gamma_6. \tag{3.3}$$

Now define

$$\begin{aligned} L^\pm(\alpha) &= \sum_{\frac{X}{2}<p\le X} (\log p) e\left([p^c]\alpha\right) \sum_{d|(p+2,P(z))} \lambda^\pm(d) \\ &= \sum_{d|P(z)} \lambda^\pm(d) \sum_{\substack{\mu X<p\le X \\ d|p+2}} (\log p) e\left([p^c]\alpha\right). \end{aligned} \tag{3.4}$$

Consider Γ_1 first. By (3.4) we can derive that

$$\Gamma_1 = \int_{-\tau}^{1-\tau} L^-(\alpha)L^+(\alpha)^4 e(-N\alpha)\mathrm{d}\alpha = \Gamma_{11}+\Gamma_{12}, \tag{3.5}$$

where

$$\Gamma_{11} = \int_{-\tau}^{\tau} L^-(\alpha)L^+(\alpha)^4 e(-N\alpha)\mathrm{d}\alpha, \tag{3.6}$$

$$\Gamma_{12} = \int_{\tau}^{1-\tau} L^{-}(\alpha)L^{+}(\alpha)^{4}e(-N\alpha)\mathrm{d}\alpha. \tag{3.7}$$

Similarly, we have

$$\Gamma_{6} = \int_{-\tau}^{\tau} L^{+}(\alpha)^{5}e(-N\alpha)\mathrm{d}\alpha + \int_{\tau}^{1-\tau} L^{+}(\alpha)^{5}e(-N\alpha)\mathrm{d}\alpha =: \Gamma_{61} + \Gamma_{62}. \tag{3.8}$$

Now combining (3.3), (3.5) and (3.8) we get

$$\Gamma \geq 5\Gamma_{11} - 4\Gamma_{61} + (5\Gamma_{12} - 4\Gamma_{62}). \tag{3.9}$$

In the following sections, we shall prove that

$$5\Gamma_{11} - 4\Gamma_{61} \gg \frac{X^{5-c}}{\log^{5} X}, \qquad \Gamma_{12}, \Gamma_{62} \ll X^{5-c-\varepsilon}.$$

4. The integrals Γ_{11} and Γ_{61}

In this section, we will give an asymptotic formula for the integrals Γ_{11} and Γ_{61} defined by (3.6) and (3.8), respectively. We consider the sum

$$L(\alpha) = \sum_{d\leq D} \lambda(d) \sum_{\substack{\frac{X}{2}<p\leq X \\ d|p+2}} (\log p)e([p^{c}]\alpha), \tag{4.1}$$

where $\lambda(d)$ are real numbers satisfying

$$|\lambda(d)| \leq 1, \quad \lambda(d) = 0 \quad \text{if} \quad 2|d \quad \text{or} \quad \mu(d) = 0. \tag{4.2}$$

Furthermore, we define

$$I(\alpha) = \int_{\frac{X}{2}}^{X} e(t^{c}\alpha)\,\mathrm{d}t. \tag{4.3}$$

Lemma 4.1. *Let $L(\alpha)$ and $I(\alpha)$ be defined by* (4.1) *and* (4.3)*, respectively. Suppose that ξ and δ satisfy the following conditions*

$$\xi + 7\delta < 2 \quad \text{and} \quad 3\xi + 6\delta < 2.$$

Then for $|\alpha| \leq \tau$, we have

$$L(\alpha) = \sum_{d\leq D} \frac{\lambda(d)}{\varphi(d)} I(\alpha) + O\left(\frac{X}{\log^{A} X}\right),$$

where $A > 0$ is a sufficiently large constant.

Proof. See Lemma 2.8 in [16]. □

Lemma 4.2. *Let $L(\alpha)$ and $I(\alpha)$ be defined by* (4.1) *and* (4.3)*, respectively. Then we have*

$$\text{(i)} \int_{|\alpha|\leq\tau} |I(\alpha)|^{4}\mathrm{d}\alpha \ll X^{4-c}\log^{4} X,$$

$$\text{(ii)} \int_{|\alpha|\leq\tau} |L(\alpha)|^{4}\mathrm{d}\alpha \ll X^{4-c}\log^{10} X,$$

$$\text{(iii)} \int_{0}^{1} |L(\alpha)|^{4}\,\mathrm{d}\alpha \ll X^{2+3\varepsilon}.$$

Proof. By using the trivial estimate $|I(\alpha)| \ll X$ and $L(x) \ll X\log^2 X$, (i) and (ii) follow from Lemma 2.9 in [16]. For (iii), we have

$$\begin{aligned}\int_0^1 |L(\alpha)|^4\, \mathrm{d}\alpha = &\sum_{\substack{d_i\le D\\ i=1,2,3,4}} \lambda(d_1)\lambda(d_2)\lambda(d_3)\lambda(d_4) \sum_{\substack{\frac{X}{2}<p_1,p_2,p_3,p_4\le X\\ d_i|p_i+2, i=1,2,3,4}} \prod_{i=1}^{4}(\log p_i)\\ &\times \int_0^1 e\left(([p_1^c]+[p_2^c]-[p_3^c]-[p_4^c])\alpha\right)\mathrm{d}\alpha\\ =& \sum_{\substack{\frac{X}{2}<p_1,p_2,p_3,p_4\le X\\ [p_1^c]+[p_2^c]=[p_3^c]+[p_4^c]}} \prod_{i=1}^{4}(\log p_i) \sum_{\substack{d_i\le D, d_i|p_i+2\\ i=1,2,3,4}} \lambda(d_1)\lambda(d_2)\lambda(d_3)\lambda(d_4)\\ \ll& (\log^4 X) \sum_{\substack{\frac{X}{2}<n_1,n_2,n_3,n_4\le X\\ [n_1^c]+[n_2^c]=[n_3^c]+[n_4^c]}} \tau(n_1+2)\tau(n_2+2)\tau(n_3+2)\tau(n_4+2)\\ \ll& X^{\varepsilon}(\log^4 X) \sum_{\substack{\frac{X}{2}<n_1,n_2,n_3,n_4\le X\\ [n_1^c]+[n_2^c]=[n_3^c]+[n_4^c]}} 1 \ll X^{2\varepsilon} \sum_{\substack{\frac{X}{2}<n_1,n_2,n_3,n_4\le X\\ |n_1^c+n_2^c-n_3^c-n_4^c|<4}} 1\\ \ll& X^{3\varepsilon}\left(4X^{4-c}+X^2\right) \ll X^{2+3\varepsilon},\end{aligned}$$

where Lemma 2.8 is applied in the last step. □

Let

$$\mathscr{M}^{\pm}(\alpha) = \sum_{d\le D} \frac{\lambda^{\pm}(d)}{\varphi(d)} I(\alpha) = \mathscr{N}^{\pm} I(\alpha).$$

Then we can easily get the elementary estimate

$$\mathscr{M}^{\pm}(\alpha) \ll |I(\alpha)| \log X. \tag{4.4}$$

By using Lemma 4.2 and (4.4) we find that

$$\begin{aligned}&L^-(\alpha)L^+(\alpha)^4 - \mathscr{M}^-(\alpha)\mathscr{M}^+(\alpha)^4\\ &= \left(L^-(\alpha)-\mathscr{M}^-(\alpha)\right)L^+(\alpha)^4 + \left(L^+(\alpha)-\mathscr{M}^+(\alpha)\right)\mathscr{M}^-(\alpha)L^+(\alpha)^3\\ &\quad + \left(L^+(\alpha)-\mathscr{M}^+(\alpha)\right)\mathscr{M}^-(\alpha)\mathscr{M}^+(\alpha)L^+(\alpha)^2\\ &\quad + \left(L^+(\alpha)-\mathscr{M}^+(\alpha)\right)\mathscr{M}^-(\alpha)\mathscr{M}^+(\alpha)^2L^+(\alpha) + \left(L^+(\alpha)-\mathscr{M}^+(\alpha)\right)\mathscr{M}^-(\alpha)\mathscr{M}^+(\alpha)^3\\ &\ll \frac{X}{\log^A X}\Big(\left|L^+(\alpha)\right|^4 + \left|L^+(\alpha)\right|^3 |I(\alpha)|\log X + \left|L^+(\alpha)\right|^2 |I(\alpha)|^2\log^2 X\\ &\qquad + \left|L^+(\alpha)\right| |I(\alpha)|^3\log^3 X + |I(\alpha)|^4\log^4 X\Big).\end{aligned} \tag{4.5}$$

Let

$$\mathcal{J}_\tau = \int_{-\tau}^{\tau} \mathscr{M}^-(\alpha)\mathscr{M}^+(\alpha)^4 e(-N\alpha)\mathrm{d}\alpha. \tag{4.6}$$

Then we can derive from Lemma 4.2, (3.6), (4.5) and (4.6) that

$$\begin{aligned}\Gamma_{11} - \mathcal{J}_\tau &\ll \int_{|\alpha|\le\tau} \left|L^-(\alpha)L^+(\alpha)^4 - \mathscr{M}^-(\alpha)\mathscr{M}^+(\alpha)^4\right| \mathrm{d}\alpha \\ &\ll \frac{X}{\log^{A-4} X}\left(\int_{|\alpha|\le\tau} \left|L^+(\alpha)\right|^4 \mathrm{d}\alpha + \int_{|\alpha|\le\tau} |I(\alpha)|^4 \mathrm{d}\alpha\right) \\ &\ll \frac{X^{5-c}}{\log^{A-14} X}. \end{aligned} \tag{4.7}$$

Define

$$\mathcal{J} = \int_{-\infty}^{+\infty} I(\alpha)^5 e(-N\alpha)\mathrm{d}\alpha. \tag{4.8}$$

Then by the argument used in Lemma 2.13 of [16], we have

$$\mathcal{J} \gg X^{5-c}. \tag{4.9}$$

With the help of Lemma 2.7 we can get that $I(\alpha) \ll |\alpha|^{-1}X^{1-c}$. Hence, from (4.6) and (4.8) we find that

$$\begin{aligned}\left|\mathscr{N}^-\left(\mathscr{N}^+\right)^4\mathcal{J} - \mathcal{J}_\tau\right| &\ll \left(\log^5 X\right)\int_{|\alpha|>\tau} |I(\alpha)|^5 \mathrm{d}\alpha \\ &\ll \left(\log^5 X\right)\int_{|\alpha|>\tau} |\alpha|^{-5}X^{5-5c}\mathrm{d}\alpha \\ &\ll X^{5-c-4\xi}\log^5 X \ll X^{5-c-\varepsilon}. \end{aligned} \tag{4.10}$$

Now combining (4.7) and (4.10) we obtain

$$\Gamma_{11} = \mathscr{N}^-\left(\mathscr{N}^+\right)^4\mathcal{J} + O\left(\frac{X^{5-c}}{\log^{A-14} X}\right). \tag{4.11}$$

Similarly, we can prove that

$$\Gamma_{61} = \left(\mathscr{N}^+\right)^5\mathcal{J} + O\left(\frac{X^{5-c}}{\log^{A-14} X}\right). \tag{4.12}$$

5. The upper bound for the integrals Γ_{12} and Γ_{62}

In this section we shall consider the upper bound for the integrals Γ_{12} and Γ_{62} defined by (3.7) and (3.8), respectively. Define

$$\mathcal{T}(\alpha, X) = \sum_{d\le D}\sum_{\substack{\frac{X}{2}<n\le X \\ d|n+2}} e\left([n^c]\alpha\right).$$

Lemma 5.1. *For $\alpha \in (0, 1)$, we have*

$$\mathcal{T}(\alpha, X) \ll X^{\frac{2c+13}{20}+\varepsilon}D^{\frac{7}{20}} + \frac{\log X}{\alpha X^{c-1}}.$$

Proof. The proof is exactly the same as that of Lemma 2.7 in [16], where the exponential pair $(\kappa,\lambda)=(\frac{1}{9},\frac{13}{18})$ is used. One can see [16] for details. □

Lemma 5.2. *Let $f(n)$ be a complex valued function defined on $n\in(\frac{X}{2},X]$. Then we have*

$$\sum_{\frac{X}{2}<n\le X}\Lambda(n)f(n)=S_1-S_2-S_3,$$

where

$$S_1=\sum_{k\le X^{\frac{1}{3}}}\mu(k)\sum_{\frac{X}{2k}<\ell\le\frac{X}{k}}(\log\ell)f(k\ell),$$
$$S_2=\sum_{k\le X^{\frac{2}{3}}}c(k)\sum_{\frac{X}{2k}<\ell\le\frac{X}{k}}f(k\ell),$$
$$S_3=\sum_{X^{\frac{1}{3}}<k\le X^{\frac{2}{3}}}a(k)\sum_{\frac{X}{2k}<\ell\le\frac{X}{k}}\Lambda(\ell)f(k\ell),$$

and where $a(k),c(k)$ are real numbers satisfying

$$|a(k)|\le\tau(k),\qquad |c(k)|\le\log k.$$

Proof. The proof can be found on page 112 of [26]. □

Lemma 5.3. *Suppose that $2<c<\frac{990}{479}$. Let $\lambda(d)$ be real numbers that satisfy* (4.2) *and $L(\alpha)$ be defined by* (4.1). *Then we have*

$$\sup_{\alpha\in(\tau,1-\tau)}|L(\alpha)|\ll X^{\frac{3}{2}-\frac{c}{4}-\varepsilon}.$$

Proof. From (4.1), it is easy to see that

$$L(\alpha)=L_1(\alpha)+O\left(X^{\frac{1}{2}+\varepsilon}\right),\tag{5.1}$$

where

$$L_1(\alpha)=\sum_{d\le D}\lambda(d)\sum_{\substack{\frac{X}{2}<n\le X\\ d|n+2}}\Lambda(n)e\left([n^c]\alpha\right).$$

Hence by (5.1) we need only to show that the estimation

$$\sup_{\alpha\in(\tau,1-\tau)}|L_1(\alpha)|\ll X^{\frac{3}{2}-\frac{c}{4}-\varepsilon}\tag{5.2}$$

holds for $2<c<\frac{990}{479}$. Let $H=X^{\frac{8}{479}}$. Then, by using Lemma 2.5, we get

$$L_1(\alpha)=\sum_{|h|\le H}c_h(\alpha)\sum_{\frac{X}{2}<n\le X}\Lambda(n)\sum_{\substack{d\le D\\ d|n+2}}\lambda(d)e\left((h+\alpha)n^c\right)$$
$$+O\left((\log X)\sum_{\frac{X}{2}<n\le X}\min\left(1,\frac{1}{H\|n^c\|}\right)\right).\tag{5.3}$$

By Lemmas 2.3 and 2.6 with the exponential pair $(\kappa, \lambda) = (\frac{1}{6}, \frac{2}{3})$, we obtain

$$
\begin{aligned}
&(\log X) \sum_{\frac{X}{2}<n\le X} \min\left(1, \frac{1}{H\|n^c\|}\right)\\
&= (\log X) \sum_{\frac{X}{2}<n\le X} \sum_{k=-\infty}^{+\infty} a_k e(kn^c) \le (\log X) \sum_{k=-\infty}^{+\infty} |a_k| \left| \sum_{\frac{X}{2}<n\le X} e(kn^c)\right|\\
&\ll (\log X)\left(\frac{X\log 2H}{H} + \sum_{1\le k\le H} \frac{1}{k}\left((kX^c)^{\frac{1}{6}} X^{\frac{1}{2}} + \frac{X}{kX^c}\right)\right.\\
&\quad \left. + \sum_{k>H} \frac{H}{k^2}\left((kX^c)^{\frac{1}{6}} X^{\frac{1}{2}} + \frac{X}{kX^c}\right)\right)\\
&\ll (\log X)\left(XH^{-1} + H^{\frac{1}{6}} X^{\frac{c}{6}+\frac{1}{2}} + X^{1-c}\right) \ll X^{\frac{3}{2}-\frac{c}{4}-\varepsilon}. \qquad (5.4)
\end{aligned}
$$

Next we consider the first term on the right-hand side of (5.3). We write it in the following form

$$
S(\alpha) = \sum_{|h|\le H} c_h(\alpha) \sum_{\frac{X}{2}<n\le X} \Lambda(n) f(n), \qquad (5.5)
$$

where

$$
f(n) = \sum_{\substack{d\le D\\ d|n+2}} \lambda(d) e\left((h+\alpha)n^c\right).
$$

By applying Lemma 5.2 we find that

$$
S(\alpha) = S_1 - S_2 - S_3, \qquad (5.6)
$$

where

$$
S_1 = \sum_{|h|\le H} c_h(\alpha) \sum_{k\le X^{\frac{1}{3}}} \mu(k) \sum_{\frac{X}{2k}<\ell\le\frac{X}{k}} (\log \ell) f(k\ell), \qquad (5.7)
$$

$$
S_2 = \sum_{|h|\le H} c_h(\alpha) \sum_{k\le X^{\frac{2}{3}}} c(k) \sum_{\frac{X}{2k}<\ell\le\frac{X}{k}} f(k\ell), \qquad (5.8)
$$

$$
S_3 = \sum_{|h|\le H} c_h(\alpha) \sum_{X^{\frac{1}{3}}<k\le X^{\frac{2}{3}}} a(k) \sum_{\frac{X}{2k}<\ell\le\frac{X}{k}} \Lambda(\ell) f(k\ell), \qquad (5.9)
$$

and $|a(k)| \le \tau(k)$, $|c(k)| \le \log k$. Clearly, by (5.8) we can write S_2 as

$$
S_2 = S_{21} + S_{22}, \qquad (5.10)
$$

where

$$
S_{21} = \sum_{|h|\le H} c_h(\alpha) \sum_{k\le X^{\frac{1}{3}}} c(k) \sum_{\frac{X}{2k}<\ell\le\frac{X}{k}} f(k\ell), \qquad (5.11)
$$

$$S_{22} = \sum_{|h|\le H} c_h(\alpha) \sum_{X^{\frac{1}{3}}<k\le X^{\frac{2}{3}}} c(k) \sum_{\frac{X}{2k}<\ell\le\frac{X}{k}} f(k\ell).$$

Therefore, by (5.6) and (5.10) we have

$$S(\alpha) \ll |S_1| + |S_{21}| + |S_{22}| + |S_3|. \tag{5.12}$$

We consider the sum S_{21} defined by (5.11) first. We change the order of the summation to write it in the following form

$$S_{21} = \sum_{d\le D} \lambda(d) \sum_{|h|\le H} c_h(\alpha) \sum_{k\le X^{\frac{1}{3}}} c(k) \sum_{\substack{\frac{X}{2k}<\ell\le\frac{X}{k} \\ d|k\ell+2}} e\left((h+\alpha)(k\ell)^c\right).$$

Since $\lambda(d) = 0$ for $2|d$, from the condition $d|k\ell+2$ we have that $(k,d) = 1$. Hence there exists an integer ℓ_0 such that $k\ell + 2 \equiv 0 \pmod d$ is equivalent to $\ell \equiv \ell_0 \pmod d$, which means that $\ell = \ell_0 + md$ for some integer m. Therefore, we get

$$S_{21} = \sum_{d\le D} \lambda(d) \sum_{|h|\le H} c_h(\alpha) \sum_{\substack{k\le X^{\frac{1}{3}} \\ (k,d)=1}} c(k) \sum_{\frac{X}{2kd}-\frac{\ell_0}{d}<m\le\frac{X}{kd}-\frac{\ell_0}{d}} e\left((h+\alpha)k^c\left(\ell_0+md\right)^c\right). \tag{5.13}$$

By using Lemma 2.3 with the exponential pair $(\kappa,\lambda) = A^2BA^2B(0,1) = (\frac{1}{20},\frac{33}{40})$ we find that the sum over m in (5.13) is given by

$$\begin{aligned} &\ll \left(|h+\alpha|kdX^{c-1}\right)^{\frac{1}{20}} \left(\frac{X}{kd}\right)^{\frac{33}{40}} + \left(|h+\alpha|kdX^{c-1}\right)^{-1} \\ &\ll |h+\alpha|^{\frac{1}{20}} X^{\frac{c}{20}+\frac{31}{40}} k^{-\frac{31}{40}} d^{-\frac{31}{40}} + |h+\alpha|^{-1} k^{-1} d^{-1} X^{1-c}. \end{aligned} \tag{5.14}$$

Then from (4.2), (5.13) and (5.14) we can obtain

$$S_{21} \ll X^{\varepsilon}\left(X^{\frac{c}{20}+\frac{17}{20}} D^{\frac{9}{40}} H^{\frac{1}{20}} + X^{1-c}\right). \tag{5.15}$$

For the sum S_1 given by (5.7), we can apply partial summation to get rid of the log factor and then proceed as in the same process for S_{21} to get

$$S_1 \ll X^{\varepsilon}\left(X^{\frac{c}{20}+\frac{17}{20}} D^{\frac{9}{40}} H^{\frac{1}{20}} + X^{1-c}\right). \tag{5.16}$$

Now we consider the sum S_3. By a splitting argument, we can decompose S_3 into $O(\log X)$ sums of the following form

$$W(K) = \sum_{|h|\le H} c_h(\alpha) \sum_{K<k<K_1} a(k) \sum_{\frac{X}{2k}<\ell\le\frac{X}{k}} \Lambda(\ell) \sum_{\substack{d\le D \\ d|k\ell+2}} \lambda(d) e\left((h+\alpha)(k\ell)^c\right), \tag{5.17}$$

where

$$K_1 \le 2K, \quad X^{\frac{1}{3}} \le K < K_1 \le X^{\frac{2}{3}}. \tag{5.18}$$

We assume that $K \geq X^{\frac{1}{2}}$ first. It follows from (5.17), (5.18) and Cauchy's inequality that

$$|W(K)|^2 \ll X^{\varepsilon} K \max_{\gamma \in (\tau, H+1)} \sum_{K < k < K_1} \left| \sum_{\frac{X}{2k} < \ell \leq \frac{X}{k}} \Lambda(\ell) \sum_{\substack{d \leq D \\ d | k\ell + 2}} \lambda(d) e\left(\gamma (k\ell)^c\right) \right|^2. \tag{5.19}$$

Suppose that Q is an integer which satisfies

$$1 \leq Q \ll \frac{X}{K}. \tag{5.20}$$

For the inner sum over ℓ in (5.19), by applying Lemma 2.4 we can derive that

$$\begin{aligned} |W(K)|^2 \ll & \frac{X^{1+\varepsilon}}{Q} \max_{\gamma \in (\tau, H+1)} \sum_{K < k < K_1} \sum_{|q| \leq Q} \left(1 - \frac{|q|}{Q}\right) \sum_{\frac{X}{2k} < \ell, \ell + q \leq \frac{X}{k}} \Lambda(\ell) \\ & \times \sum_{\substack{d_1 \leq D \\ d_1 | k\ell + 2}} \lambda(d_1) e\left(-\gamma (k\ell)^c\right) \Lambda(\ell + q) \sum_{\substack{d_2 \leq D \\ d_2 | k(\ell + q) + 2}} \lambda(d_2) e\left(\gamma (k(\ell + q))^c\right) \\ \ll & \frac{X^{1+\varepsilon}}{Q} \max_{\gamma \in (\tau, H+1)} \sum_{d_1 \leq D} \sum_{d_2 \leq D} \lambda(d_1) \lambda(d_2) \sum_{|q| \leq Q} \left(1 - \frac{|q|}{Q}\right) \\ & \times \sum_{\frac{X}{2K_1} < \ell, \ell + q \leq \frac{X}{K}} \Lambda(\ell) \Lambda(\ell + q) \mathscr{S}, \end{aligned} \tag{5.21}$$

where

$$\mathscr{S} = \sum_{\substack{\widetilde{K} < k \leq \widetilde{K_1} \\ d_1 | k\ell + 2 \\ d_2 | k(\ell + q) + 2}} e\left(\gamma k^c \left((\ell + q)^c - \ell^c\right)\right)$$

and

$$\widetilde{K} = \max\left(K, \frac{X}{2\ell}, \frac{X}{2(\ell + q)}\right), \qquad \widetilde{K_1} = \min\left(K_1, \frac{X}{\ell}, \frac{X}{\ell + q}\right).$$

Since $\lambda(d) = 0$ for $2|d$, we can assume that $(d_1 d_2, 2) = 1$. Then it follows from $d_1 | k\ell + 2$ and $d_2 | k(\ell + q) + 2$ that $(d_1, \ell) = (d_2, \ell + q) = 1$. Hence there exists an integer k_0 that is dependent on ℓ, h, d_1, d_2 such that the pair of conditions $k\ell + 2 \equiv 0 \pmod{d_1}$ and $k(\ell + q) + 2 \equiv 0 \pmod{d_2}$ is equivalent to the congruence $k \equiv k_0 \pmod{[d_1, d_2]}$. Thus, we have

$$\mathscr{S} = \sum_{\frac{\widetilde{K} - k_0}{[d_1, d_2]} < m \leq \frac{\widetilde{K_1} - k_0}{[d_1, d_2]}} e(F(m)), \tag{5.22}$$

where

$$F(m) = \gamma \left(k_0 + m[d_1, d_2]\right)^c \left((\ell + q)^c - \ell^c\right). \tag{5.23}$$

For $q = 0$, by the trivial estimate we have

$$\mathscr{S} \ll \frac{K}{[d_1, d_2]}. \tag{5.24}$$

For the case $q \neq 0$, by (5.23) we get

$$\left|F^{(j)}(m)\right| \asymp |\gamma||q|\ell^{c-1}[d_1, d_2]K^{c-1}\left(\frac{K}{[d_1, d_2]}\right)^{1-j}, \qquad j \geq 1.$$

We apply Lemma 2.3 with the following exponential pair

$$(\kappa, \lambda) = A\left(\frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon\right) = \left(\frac{13}{194} + \varepsilon, \frac{76}{97} + \varepsilon\right)$$

to derive that

$$\begin{aligned}\mathscr{S} &\ll \left(|\gamma||q|\ell^{c-1}[d_1, d_2]K^{c-1}\right)^{\frac{13}{194}+\varepsilon}\left(\frac{K}{[d_1, d_2]}\right)^{\frac{76}{97}+\varepsilon} + \left(|\gamma||q|\ell^{c-1}[d_1, d_2]K^{c-1}\right)^{-1} \\ &\ll X^{\varepsilon}\left(|\gamma|^{\frac{13}{194}}|q|^{\frac{13}{194}}\ell^{\frac{13}{194}(c-1)}[d_1, d_2]^{-\frac{139}{194}}K^{\frac{13}{194}c+\frac{139}{194}} + |\gamma|^{-1}|q|^{-1}\ell^{1-c}[d_1, d_2]^{-1}K^{1-c}\right). \end{aligned} \tag{5.25}$$

Note that

$$\begin{aligned}\sum_{d_1 \leq D}\sum_{d_2 \leq D}\frac{1}{[d_1, d_2]^{\frac{139}{194}}} &= \sum_{r \leq D}\sum_{d_1 \leq D}\sum_{\substack{d_2 \leq D \\ r=(d_1,d_2)}}\left(\frac{r}{d_1 d_2}\right)^{\frac{139}{194}} \\ &\ll \sum_{r \leq D}\sum_{k_1 \leq \frac{D}{r}}\sum_{k_2 \leq \frac{D}{r}}\left(\frac{1}{rk_1k_2}\right)^{\frac{139}{194}} \\ &\ll \sum_{r \leq D} r^{-\frac{139}{194}}\left(\frac{D}{r}\right)^{\frac{55}{97}} \ll D^{\frac{55}{97}},\end{aligned}$$

and

$$\sum_{d_1 \leq D}\sum_{d_2 \leq D}\frac{1}{[d_1, d_2]} \ll (\log D)^3.$$

Then we use the above two estimates, (5.21), (5.24) and (5.25) to get

$$\begin{aligned}|W(K)|^2 \ll{}& \frac{X^{1+\varepsilon}}{Q}\max_{\gamma \in (\tau, H+1)}\sum_{\frac{X}{2K_1} < \ell \leq \frac{X}{K}}\Lambda(\ell)^2\sum_{d_1 \leq D}\sum_{d_2 \leq D}\frac{K}{[d_1, d_2]} \\ &+ \frac{X^{1+2\varepsilon}}{Q}\max_{\gamma \in (\tau, H+1)}\sum_{d_1 \leq D}\sum_{d_2 \leq D}\sum_{0<|q|<Q}\sum_{\frac{X}{2K_1} < \ell, \ell+q \leq \frac{X}{K}}\Lambda(\ell)\Lambda(\ell+q) \\ &\times\left(|\gamma|^{\frac{13}{194}}|q|^{\frac{13}{194}}\ell^{\frac{13}{194}(c-1)}[d_1, d_2]^{-\frac{139}{194}}K^{\frac{13}{194}c+\frac{139}{194}} + |\gamma|^{-1}|q|^{-1}\ell^{1-c}[d_1, d_2]^{-1}K^{1-c}\right) \\ \ll{}& X^{2+\varepsilon}Q^{-1} + \frac{X^{1+2\varepsilon}}{Q}|\gamma_0|^{\frac{13}{194}}K^{\frac{13}{194}c+\frac{139}{194}}\left(\sum_{d_1 \leq D}\sum_{d_2 \leq D}\frac{1}{[d_1, d_2]^{\frac{139}{194}}}\right)\left(\sum_{0<|q|<Q}|q|^{\frac{13}{194}}\right) \\ &\times\left(\sum_{\frac{X}{2K_1} < \ell \leq \frac{X}{K}}\ell^{\frac{13}{194}(c-1)}\right) + \frac{X^{1+2\varepsilon}}{Q}|\gamma_0|^{-1}K^{1-c}\left(\sum_{d_1 \leq D}\sum_{d_2 \leq D}\frac{1}{[d_1, d_2]}\right)\end{aligned}$$

$$\times\left(\sum_{0<|q|<Q}|q|^{-1}\right)\left(\sum_{\frac{X}{2K_1}<\ell\le\frac{X}{K}}\ell^{1-c}\right)$$
$$\ll X^{\varepsilon}\left(X^2Q^{-1}+X^{\frac{13c+375}{194}}Q^{\frac{13}{194}}D^{\frac{55}{97}}|\gamma_0|^{\frac{13}{194}}K^{-\frac{21}{97}}+X|\gamma_0|^{-1}Q^{-1}K^{1-c}\right) \tag{5.26}$$

for some $\gamma_0\in[\tau,H+1]$. We choose

$$Q_0=X^{\frac{13}{207}(1-c)}D^{-\frac{110}{207}}|\gamma_0|^{-\frac{13}{207}}K^{\frac{14}{69}},\qquad Q=\left[\min(Q_0,XK^{-1})\right].$$

Then, it is easy to check that

$$Q^{-1}\asymp Q_0^{-1}+KX^{-1}. \tag{5.27}$$

Substituting (5.27) into (5.26), we obtain

$$\begin{aligned}|W(K)|^2 &\ll X^{\varepsilon}\Big(X^2\left(Q_0^{-1}+KX^{-1}\right)+X^{\frac{13c+375}{194}}Q^{\frac{13}{194}}D^{\frac{55}{97}}|\gamma_0|^{\frac{13}{194}}K^{-\frac{21}{97}}\\ &\quad +X|\gamma_0|^{-1}\left(Q_0^{-1}+KX^{-1}\right)K^{1-c}\Big)\\ &\ll X^{\varepsilon}\left(X^{\frac{13c+380}{207}}D^{\frac{110}{207}}|\gamma_0|^{\frac{13}{207}}+X^{\frac{5}{3}}+X^{\frac{553-181c}{414}}D^{\frac{110}{207}}|\gamma_0|^{-\frac{194}{207}}+X^{1-\frac{c}{2}}|\gamma_0|^{-1}\right),\end{aligned}$$

which implies that

$$|W(K)|\ll X^{\varepsilon}\left(X^{\frac{13c+380}{414}}D^{\frac{55}{207}}|\gamma_0|^{\frac{13}{414}}+X^{\frac{5}{6}}+X^{\frac{553-181c}{828}}D^{\frac{55}{207}}|\gamma_0|^{-\frac{97}{207}}+X^{\frac{1}{2}-\frac{c}{4}}|\gamma_0|^{-\frac{1}{2}}\right). \tag{5.28}$$

When $K<X^{\frac{1}{2}}$, we can represent $W(K)$ as follows:

$$\begin{aligned}W(K)=&\sum_{|h|\le H}c_h(\alpha)\sum_{\frac{X}{2K_1}<\ell\le\frac{X}{K}}\Lambda(\ell)\sum_{\max(K,\frac{X}{2\ell})<k\le\min(K_1,\frac{X}{\ell})}a(k)\\ &\times\sum_{\substack{d\le D\\ d|k\ell+2}}\lambda(d)e\left((h+\alpha)(k\ell)^c\right).\end{aligned}$$

Now we have that $\frac{X}{K}\gg X^{\frac{1}{2}}$, then, we may proceed as in (5.19)–(5.28) but with roles of k and ℓ reversed. Thus we can again derive the estimate (5.28). Consequently, we obtain

$$S_3\ll X^{\varepsilon}\left(X^{\frac{13c+380}{414}}D^{\frac{55}{207}}|\gamma_0|^{\frac{13}{414}}+X^{\frac{5}{6}}+X^{\frac{553-181c}{828}}D^{\frac{55}{207}}|\gamma_0|^{-\frac{97}{207}}+X^{\frac{1}{2}-\frac{c}{4}}|\gamma_0|^{-\frac{1}{2}}\right). \tag{5.29}$$

To bound S_{22}, we use the same methodology as for S_3 to derive that

$$S_{22}\ll X^{\varepsilon}\left(X^{\frac{13c+380}{414}}D^{\frac{55}{207}}|\gamma_0|^{\frac{13}{414}}+X^{\frac{5}{6}}+X^{\frac{553-181c}{828}}D^{\frac{55}{207}}|\gamma_0|^{-\frac{97}{207}}+X^{\frac{1}{2}-\frac{c}{4}}|\gamma_0|^{-\frac{1}{2}}\right). \tag{5.30}$$

Now combining (5.12), (5.15), (5.16), (5.29) and (5.30) and from the fact that $\gamma_0\in[\tau,H+1]$, we find that

$$\begin{aligned}S(\alpha)&\ll X^{\varepsilon}\Big(X^{\frac{c}{20}+\frac{17}{20}}D^{\frac{9}{40}}H^{\frac{1}{20}}+X^{1-c}+X^{\frac{13c+380}{414}}D^{\frac{55}{207}}|\gamma_0|^{\frac{13}{414}}+X^{\frac{5}{6}}\\ &\quad +X^{\frac{553-181c}{828}}D^{\frac{55}{207}}|\gamma_0|^{-\frac{97}{207}}+X^{\frac{1}{2}-\frac{c}{4}}|\gamma_0|^{-\frac{1}{2}}\Big)\\ &\ll X^{\varepsilon}\Big(X^{\frac{c}{20}+\frac{8151}{9580}+\frac{9\delta}{40}}+X^{1-c}+X^{\frac{13c}{414}+\frac{10118}{11017}+\frac{55\delta}{207}}+X^{\frac{5}{6}}\quad+X^{\frac{553}{828}+\frac{c}{4}+\frac{55\delta}{207}-\frac{97\xi}{207}}+X^{\frac{1}{2}+\frac{c}{4}-\frac{\xi}{2}}\Big).\end{aligned}$$

Therefore, from condition (2.1) we conclude that if $2 < c < \frac{990}{479}$ then

$$\sup_{\alpha\in(\tau,1-\tau)} |S(\alpha)| \ll X^{\frac{3}{2}-\frac{c}{4}-\varepsilon}. \tag{5.31}$$

With the help of (5.3)–(5.5) and (5.31), we finally obtain that

$$\sup_{\alpha\in(\tau,1-\tau)} |L_1(\alpha)| \ll X^{\frac{3}{2}-\frac{c}{4}-\varepsilon}$$

holds for $2 < c < \frac{990}{479}$, and the proof of Lemma 5.3 is completed. □

Lemma 5.4. *Suppose that* $2 < c < \frac{990}{479}$. *Then we have*

$$\int_\tau^{1-\tau} |L(\alpha)|^5 \, \mathrm{d}\alpha \ll X^{5-c-\varepsilon}.$$

Proof. Let $G(\alpha) = \overline{L(\alpha)}|L(\alpha)|^3$. We have

$$\begin{aligned}
\left|\int_\tau^{1-\tau} |L(\alpha)|^5 \, \mathrm{d}\alpha\right| &= \left|\sum_{d\le D} \lambda(d) \sum_{\substack{\frac{X}{2}<p\le X \\ d|p+2}} (\log p) \int_\tau^{1-\tau} e([p^c]\alpha)G(\alpha)\mathrm{d}\alpha\right| \\
&\le (\log X) \sum_{d\le D} \sum_{\substack{\frac{X}{2}<p\le X \\ d|p+2}} \left|\int_\tau^{1-\tau} e([p^c]\alpha)G(\alpha)\mathrm{d}\alpha\right| \\
&\le (\log X) \sum_{d\le D} \sum_{\substack{\frac{X}{2}<n\le X \\ d|n+2}} \left|\int_\tau^{1-\tau} e([n^c]\alpha)G(\alpha)\mathrm{d}\alpha\right|.
\end{aligned} \tag{5.32}$$

From (5.32) and Cauchy's inequality, we get

$$\begin{aligned}
\left|\int_\tau^{1-\tau} |L(\alpha)|^5 \, \mathrm{d}\alpha\right|^2 &\ll X(\log X)^3 \sum_{d\le D} \sum_{\substack{\frac{X}{2}<n\le X \\ d|n+2}} \left|\int_\tau^{1-\tau} e([n^c]\alpha)G(\alpha)\mathrm{d}\alpha\right|^2 \\
&= X(\log X)^3 \int_\tau^{1-\tau} \overline{G(\beta)}\mathrm{d}\beta \int_\tau^{1-\tau} \mathcal{T}(\alpha-\beta, X)G(\alpha)\mathrm{d}\alpha \\
&\ll X(\log X)^3 \int_\tau^{1-\tau} |G(\beta)|\mathrm{d}\beta \int_\tau^{1-\tau} |\mathcal{T}(\alpha-\beta, X)G(\alpha)| \, \mathrm{d}\alpha.
\end{aligned} \tag{5.33}$$

Now

$$\begin{aligned}
\int_\tau^{1-\tau} |\mathcal{T}(\alpha-\beta, X)G(\alpha)| \, \mathrm{d}\alpha &\ll \int_{\substack{\tau<\alpha<1-\tau \\ |\alpha-\beta|\le X^{-c}}} |\mathcal{T}(\alpha-\beta, X)G(\alpha)| \, \mathrm{d}\alpha \\
&\quad + \int_{\substack{\tau<\alpha<1-\tau \\ |\alpha-\beta|> X^{-c}}} |\mathcal{T}(\alpha-\beta, X)G(\alpha)| \, \mathrm{d}\alpha.
\end{aligned} \tag{5.34}$$

By the trivial bound $\mathcal{T}(\alpha, X) \ll X\log X$ and Lemma 5.3, we have

$$\int\limits_{\substack{\tau<\alpha<1-\tau\\|\alpha-\beta|\le X^{-c}}} |\mathcal{T}(\alpha-\beta, X)G(\alpha)|\,\mathrm{d}\alpha \ll X(\log X)\sup_{\alpha\in(\tau,1-\tau)}|G(\alpha)| \int\limits_{|\alpha-\beta|\le X^{-c}} \mathrm{d}\alpha$$
$$\ll X^{1-c}(\log X)\sup_{\alpha\in(\tau,1-\tau)}|L(\alpha)|^4 \ll X^{7-2c-\varepsilon}. \tag{5.35}$$

From Lemmas 5.1 and 5.3, we obtain

$$\begin{aligned}
&\int\limits_{\substack{\tau<\alpha<1-\tau\\|\alpha-\beta|>X^{-c}}} |\mathcal{T}(\alpha-\beta, X)G(\alpha)|\,\mathrm{d}\alpha\\
&\ll \int\limits_{\substack{\tau<\alpha<1-\tau\\|\alpha-\beta|>X^{-c}}} |L(\alpha)|^4\left(X^{\frac{2c+13}{20}+\varepsilon}D^{\frac{7}{20}} + \frac{X^{1-c}\log X}{|\alpha-\beta|}\right)\mathrm{d}\alpha\\
&\ll X^{\frac{2c+13}{20}+\varepsilon}D^{\frac{7}{20}}\int_0^1 |L(\alpha)|^4\mathrm{d}\alpha + X^{1-c}(\log X)\sup_{\alpha\in(\tau,1-\tau)}|L(\alpha)|^4 \int\limits_{|\alpha-\beta|>X^{-c}} \frac{1}{|\alpha-\beta|}\mathrm{d}\alpha\\
&\ll X^{\frac{2c+53}{20}+\frac{7\delta}{20}+\varepsilon} + X^{7-2c-\varepsilon} \ll X^{7-2c-\varepsilon},
\end{aligned} \tag{5.36}$$

where (iii) of Lemma 4.2 is used. It follows from (5.34)–(5.36) that

$$\int_\tau^{1-\tau} |\mathcal{T}(\alpha-\beta, X)G(\alpha)|\,\mathrm{d}\alpha \ll X^{7-2c-\varepsilon}. \tag{5.37}$$

Combining (5.33), (5.37) and (iii) of Lemma 4.3, we get

$$\left|\int_\tau^{1-\tau} |L(\alpha)|^5\,\mathrm{d}\alpha\right|^2 \ll X(\log X)^3X^{7-2c-\varepsilon}\int_0^1 |L(\alpha)|^4\mathrm{d}\alpha \ll X^{10-2c-\frac{\varepsilon}{2}}. \tag{5.38}$$

Now Lemma 5.4 follows from (5.38). □

We are now in a position to estimate Γ_{12} and Γ_{62}. By Hölder's inequality and Lemma 5.4 we find that

$$\begin{aligned}
|\Gamma_{12}| &\ll \int_\tau^{1-\tau} \left|L^-(\alpha)\right|\left|L^+(\alpha)\right|^4\mathrm{d}\alpha\\
&\ll \left(\int_\tau^{1-\tau}\left|L^-(\alpha)\right|^5\mathrm{d}\alpha\right)^{\frac{1}{5}}\left(\int_\tau^{1-\tau}\left|L^+(\alpha)\right|^5\mathrm{d}\alpha\right)^{\frac{4}{5}} \ll X^{5-c-\varepsilon}.
\end{aligned} \tag{5.39}$$

Similarly, for Γ_{62} we have

$$|\Gamma_{62}| \ll \int_\tau^{1-\tau}\left|L^+(\alpha)\right|^5\mathrm{d}\alpha \ll X^{5-c-\varepsilon}. \tag{5.40}$$

6. Proof of Theorem 1.1

Proposition 6.1. *We have*

$$5\Gamma_{11} - 4\Gamma_{61} \gg \frac{X^{5-c}}{\log^5 X}.$$

Proof. It follows from (4.11), (4.12) and Lemma 2.1(3) that

$$\begin{aligned}
5\Gamma_{11} - 4\Gamma_{61} &= (5\mathscr{N}^- - 4\mathscr{N}^+)(\mathscr{N}^+)^4 \mathcal{J} + O\left(\frac{X^{5-c}}{\log^{A-14} X}\right) \\
&\geq \left(5f\left(\frac{\log D}{\log z}\right) - 4F\left(\frac{\log D}{\log z}\right)\right)\left(1 + O(\log^{-1/3} D)\right)\mathscr{B}^5 \mathcal{J} + O\left(\frac{X^{5-c}}{\log^{A-14} X}\right) \\
&= \left(5f\left(\frac{13}{4}\right) - 4F\left(\frac{13}{4}\right)\right)\mathscr{B}^5 \mathcal{J} + O\left(X^{5-c-\varepsilon}\right) \\
&= \frac{40e^\gamma}{13}\left(\log\frac{9}{4} - \frac{4}{5} - \frac{4}{5}\int_2^{\frac{9}{4}} \frac{\log(t-1)}{t}\mathrm{d}t\right)\mathscr{B}^5 \mathcal{J} + O\left(X^{5-c-\varepsilon}\right) \\
&\geq 0.001\mathscr{B}^5 \mathcal{J} + O\left(X^{5-c-\varepsilon}\right) \gg \frac{X^{5-c}}{\log^5 X},
\end{aligned}$$

where the following trivial estimate is used:

$$\mathscr{B} \asymp \frac{1}{\log X}.$$

□

Now according to (3.9), (5.39), (5.40) and Proposition 6.1, we obtain

$$\Gamma \geq (5\Gamma_{11} - 4\Gamma_{61}) + O(|\Gamma_{12}| + |\Gamma_{62}|) \gg \frac{X^{5-c}}{\log^5 X},$$

which implies that $\Gamma > 0$ for a sufficiently large natural number N. Then, (1.3) would have a solution in primes p_1, p_2, p_3, p_4, p_5 satisfying

$$(p_1 + 2, P(z)) = (p_2 + 2, P(z)) = (p_3 + 2, P(z)) = (p_4 + 2, P(z)) = (p_5 + 2, P(z)) = 1. \tag{6.1}$$

Suppose that $p_i + 2$ has l prime factors, counted with multiplicity. From (6.1) and the condition $\frac{X}{2} < p_i \leq X$ we see that

$$X + 2 \geq p_i + 2 \geq z^l = X^{\eta l}.$$

Then, $l \leq \eta^{-1}$. This means that $p_j + 2$ has at most $\left[\frac{6227}{3960-1916c}\right]$ prime factors counted with multiplicity. Now Theorem 1.1 is proved.

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to thank the anonymous referees for many useful comments on the manuscript.

Conflict of interest

The author declares no conflict of interest.

References

1. K. Buriev, Additive problems with prime numbers, Ph.D. thesis, *Moscow University*, 1989.
2. R. Baker, Some Diophantine equations and inequalities with primes, *Funct. Approx. Comment. Math.*, **64** (2021), 203–250. https://doi.org/10.7169/facm/1912
3. J. Brüdern, E. Fouvry, Lagrange's Four Squares Theorem with almost prime variables, *J. Reine Angew. Math.*, **454** (1994), 59–96. https://doi.org/10.1515/crll.1994.454.59
4. J. R. Chen, On the representation of a larger even integer as the sum of a prime and the product of at most two primes, *Sci. Sinica*, **16** (1973), 157–176.
5. Y. C. Cai, M. G. Lu, Additive problems involving primes of special type, *Acta Arith.*, **140** (2009), 189–204. https://doi.org/10.4064/aa140-2-6
6. Y. C. Cai, On a Diophantine equation involving primes, *Ramanujan J.*, **50** (2019), 151–162. https://doi.org/10.1007/s11139-018-0027-6
7. S. W. Graham, G. Kolesnik, *Van der Corput's Method of Exponential Sums*, New York: Cambridge University Press, 1991. https://doi.org/10.1017/cbo9780511661976
8. G. Greaves, *Sieves in Number Theory*, Berlin: Springer, 2001. https://doi.org/10.1007/978-3-662-04658-6
9. L. K. Hua, Some results in additive prime number theory, *Quart. J. Math. Oxford*, **9** (1938), 68–80. https://doi.org/10.1093/qmath/os-9.1.68
10. D. R. Heath-Brown, The Pjateckiĭ-Šapiro prime number theorem, *J. Number Theory*, **16** (1983), 242–266. https://doi.org/10.1016/0022-314X(83)90044-6
11. H. Iwaniec, E. Kowalski, *Analytic Number Theory*, New York: American Mathematical Society, 2004.
12. A. Kumchev, T. Nedeva, On an equation with prime numbers, *Acta Arith.*, **83** (1998), 117–126. https://doi.org/10.4064/aa-83-2-117-126
13. M. Laporta, D. I. Tolev, On an equation with prime numbers, *Mat. Zametki*, **57** (1995), 926–929. https://doi.org/10.1007/bf02304564
14. J. Li, M. Zhang, On a Diophantine inequality with five prime variables, preprint paper, 2018.
15. J. Li, M. Zhang, On a Diophantine equation with three prime variables, *Integers*, **10** (2019), A39.

16. J. Li, F. Xue, M. Zhang, On a ternary Diophantine equality involving fractional powers with prime variables of a special form, *Ramanujan J.*, **58** (2022), 1171–1199. https://doi.org/10.1007/s11139-021-00517-5

17. S. Li, On a Diophantine equation with prime numbers, *Int. J. Number Theory*, **15** (2019), 1601–1616. https://doi.org/10.1142/s1793042119300011

18. K. Matomäki, X. Shao, Vinogradov's three primes theorem with almost twin primes, *Compos. Math.*, **153** (2017), 1220–1256. https://doi.org/10.1112/s0010437x17007072

19. Z. H. Petrov, On an equation involving fractional powers with prime numbers of a special type, *God. Sofiĭ. Univ. "Sv. Kliment Okhridski." Fac. Mat. Inform.*, **104** (2017), 171–183.

20. O. Robert, P. Sargos, Three-dimensional exponential sums with monomials, *J. Reine Angew. Math.*, **591** (2006), 1–20. https://doi.org/10.1515/crelle.2006.012

21. B. I. Segal, A general theorem concerning some properties of an arithmetical function, *C. R. Acad. Sci. URSS*, **3** (1933), 95–98.

22. B. I. Segal, The Waring theorem with fractional and irrational degrees, in Russian, *Trudy Mat. Inst. Stekov*, **5** (1933), 73–86.

23. E. G. Titchmarsh, *The Theory of the Riemann Zeta-Function*, New York: Oxford University Press, 1986.

24. D. I. Tolev, Additive problems with prime numbers of special type, *Acta Arith.*, **96** (2000), 53–88. https://doi.org/10.4064/aa96-1-2

25. I. M. Vinogradov, Representation of an odd number as a sum of three primes, *C. R. (Doklady) Acad. Sci. URSS*, **15** (1937), 291–294.

26. R. C. Vaughan, An elementary method in prime number theory, *Acta Arith.*, **37** (1980), 111–115. https://doi.org/10.4064/aa-37-1-111-115

27. W. G. Zhai, X. D. Cao, A Diophantine equation with prime numbers, in Chinese, *Acta Math. Sin.*, **45** (2002), 443–454.