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Research article

Asymptotic behavior of some differential inequalities with mixed delays on time scales and their applications

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Abstract: In this paper, we investigate the asymptotic stability of the trajectories governed by some delay differential inequalities on time scales. Based on time scale theory and the fixed-point theorem, some sufficient conditions are obtained for guaranteeing asymptotic stability. It is interesting that the inequalities studied in this paper include the generalized Halanay inequalities. Due to the fact that dynamic systems on a time scale unify discrete and continuous systems, the results of this paper have wider application value. Furthermore, some numerical examples verify the main results.

Keywords: differential inequalities; mixed delays; asymptotic stability; fixed point theorem; time scales
Mathematics Subject Classification: 34A34, 34C11

1. Introduction

The properties of differential inequalities are widely used in the study of dynamical systems and functional differential equations. In 1966, Halanay [1] first proved the following theorem:
Theorem 1.1. Let $z(t)$ be any nonnegative solution of

$$z'(t) \leq -az(t) + b \sup_{t-\tau \leq s \leq t} z(s), \quad t \geq t_0,$$

and $a > b > 0$, then there exist two positive constants $\alpha, \beta > 0$ such that

$$z(t) \leq \alpha e^{-\beta(t-t_0)} \text{ for } t \geq t_0.$$

The above inequality is called the Halanay inequality. Due to wide applications in differential dynamic systems for Halanay inequality, many results have been obtained for Halanay inequality and its generalizations; see [2–10] and related references.

In this paper, we focus on the study of differential inequalities (including Halanay inequalities) with delays on time scales. Let's briefly review the research on the above aspects. B. Ou et al. [11] proved the following theorem:

Theorem 1.2. Let $z(t)$ be any nonnegative solution of

$$z^{\Delta}(t) \leq -a(t)z(t) + b(t) \sup_{t-\tau(t)\leq s\leq t} z(s) + c(t)\int_{0}^{\infty} K(t,s)z(t-s)\Delta s, \quad t \geq t_0,$$
$$z(s) = \phi(s), \quad s \in (-\infty, t_0]_{\mathbb{T}},$$

where $\tau(t),\ a(t),\ b(t)$, and $c(t)$ are rd-continuous and bounded functions, and $K(t,s)$ is nonnegative and continuous. If the following conditions are satisfied:
(1) $\int_0^\infty K(t,s)e_A(t,t-s)\Delta s$ is uniformly bounded for $t \in \mathbb{T}$.
(2) There exist $t_1 > t_0,\ T > 0$ and $\rho > 0$ such that for each $n \in \mathbb{N}$,

$$\int_{t_1+nT}^{t_1+nT+T} \left[a(t) - b^+(t) - c^+(t)\int_0^\infty K(t,s)\Delta s\right] > \rho,$$

where $A = \sup_{t\in\mathbb{T}}\{|a(t)|, |b(t)|, |c(t)|, \frac{a(t)}{1-\mu(t)a(t)}\}$. Then for each $\tau < \frac{1}{A}\ln(1 - B + \frac{\rho}{AT}),\ B = \sup_{t\in\mathbb{T}}\int_0^\infty K(t,s)(e_A(t,t-s) - 1)\Delta s < \frac{\rho}{AT}$, $z(t)$ is exponentially stable, i.e., there exist $\alpha, \beta > 0$ (which may depend on the initial value), and such that

$$z(t) \leq \alpha e_{\ominus\beta}(t,t_0) \text{ for } t \in [t_0, \infty).$$

After that, they generalized the above results to the Halanay inequality on time scales with unbounded coefficients; see [12, 13]. We can find more results for Halanay inequality on time scales in [14, 15]. We have found that the methods used to study the Halanay inequality in existing literatures are mainly mathematical analysis methods, and we only found reference most [2] to study the Halanay inequality using the fixed point theorem. The fixed point theorem is one of the important methods for studying the main branches of mathematical problems, especially in the study of differential equations and dynamical systems. Researchers have obtained a large number of research results using the fixed point theorem, see [16–20]. In this paper, we will consider some delay inequalities by using the fixed point theorem. Our results improve and extend the existing results for Halanay inequality and its generalizations. The major contributions of this work are listed as follows:
(1) Most existing results require the solutions and coefficients of Halanay inequalities to be non-negative; see [3–5, 11, 12]. In this paper, we will remove these limitations.
(2) We develop the research scope of Halanay inequality. Specifically, we study Halanay inequality in more general cases, and the results obtained have wider applicability.
(3) The research methods for the Halanay inequality on time scales are mostly mathematical analysis methods and time scale theory, see [11–14]. The research method of this article is the fixed point theorem. We obtained the properties of delay inequalities under broader conditions.

The contents of this paper are organized as follows: Section 2 gives some preliminaries. Section 3 gives asymptotic behavior for differential inequalities with time-varying delay. Section 4 gives asymptotic behavior for differential inequalities with time-varying delay and distributed delay. In Section 5, some numerical examples are presented to illustrate the validity of the theoretical results. Finally, we conclude this paper.

2. Preliminaries

A time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$. The means for the forward jump operator σ, backward jump operator ρ, regressive rd-continuous functions' set $\mathcal{R}$ and positive regressive rd-continuous functions' set $\mathcal{R}^+$ seen in [21]. The interval $[a,b]_{\mathbb{T}}$ means $[a,b] \cap \mathbb{T}$. The intervals $[a,b)_{\mathbb{T}}$, $(a,b)_{\mathbb{T}}$, and $(a,b]_{\mathbb{T}}$ are defined similarly. $C_{rd}([t_0,\infty)_{\mathbb{T}}$ represents the set of all rd-continuous functions on $[t_0,\infty)_{\mathbb{T}}$. The exponential function on $\mathbb{T}$ is defined by $e_\alpha(t,s) = \exp\left(\int_s^t \xi_{\mu(r)}(\alpha(r))\Delta r\right)$, where

$$\xi_{\mu(r)}(\alpha(r)) = \begin{cases} \frac{1}{\mu(r)} Log(1+\mu(r)\alpha(r)), \ \mu(r) > 0, \\ \alpha(r), \ \ \mu(r) = 0. \end{cases}$$

Lemma 2.1. [21] Let $\alpha, \beta \in \mathcal{R}$. Then

[1] $e_0(t,s) \equiv 1$ and $e_\alpha(t,t) \equiv 1$;
[2] $e_\alpha(\rho(t),s) = (1-\mu(t)\alpha(t))e_\alpha(t,s)$;
[3] $e_\alpha(t,s) = \frac{1}{e_\alpha(s,t)} = e_{\ominus\alpha}(s,t)$, where $\ominus\alpha(t) = -\frac{\alpha(t)}{1+\mu(t)\alpha(t)}$.
[4] $e_\alpha(t,s)e_\alpha(s,r) = e_\alpha(t,r)$;
[5] $e_\alpha(t,s)e_\beta(t,s) = e_{\alpha\oplus\beta}(t,s)$.

Lemma 2.2. [21] Suppose that $y^\Delta = p(t)y + f(t)$ is regressive on a time scale $\mathbb{T}$. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution to the initial value problem

$$y^\Delta = p(t)y + f(t), \ \ y(t_0) = y_0$$

is given by

$$y(t) = e_p(t,t_0)y_0 + \int_{t_0}^t e_p(t,\sigma(\tau))f(\tau)\Delta\tau.$$

Lemma 2.3. [21] Suppose that $y^\Delta = p(t)y + f(t)$ is regressive on a time scale $\mathbb{T}$. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem

$$y^\Delta = -p(t)y^\sigma + f(t), \ \ y(t_0) = y_0$$

is given by

$$y(t) = e_{\ominus p}(t,t_0)y_0 + \int_{t_0}^t e_{\ominus p}(t,\tau)f(\tau)\Delta\tau.$$

Lemma 2.4. [22] For a nonnegative function ρ with $-\rho \in \mathcal{R}^+$, we have

$$1 - \int_s^t \rho(u)\Delta u \le e_{-\rho}(t,s) \le \exp\left\{-\int_s^t \rho(u)\Delta u\right\} \text{ for all } t \ge s.$$

For a nonnegative function ρ with $\rho \in \mathcal{R}^+$, we have

$$1 + \int_s^t \rho(u)\Delta u \le e_\rho(t,s) \le \exp\left\{\int_s^t \rho(u)\Delta u\right\} \text{ for all } t \ge s.$$

Remark 2.1. For $\rho \in \mathcal{R}^+$ and $\rho(r) > 0$ for $r \in [s,t]_{\mathbb{T}}$, we have

$$e_\rho(t,r) \le e_\rho(t,s) \text{ and } e_\rho(a,b) < 1 \; for \; s \le a < b \le t.$$

An additive time scale is a time scale that is closed under addition. There exist many time scales that are not additive; we need the notion of shift operators to avoid additivity assumption on the time scale. In this paper, we will define the delay terms as using shift operators.

Definition 2.1. [23] Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ a fixed number such that there exist operators $\delta_\pm : [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^*$ satisfying the following properties:

(1) The functions $\delta_\pm$ are strictly increasing with respect to their second arguments;

(2) if $(T_1,u), (T_2,u) \in \mathcal{D}_-$ with $T_1 > T_2$, then $\delta_-(T_1,u) < \delta_-(T_2,u)$; if $(T_1,u), (T_2,u) \in \mathcal{D}_+$ with $T_1 > T_2$, then $\delta_+(T_1,u) > \delta_+(T_2,u)$;

(3) if $t \in [t_0,\infty)_{\mathbb{T}}$, then $(t,t_0) \in \mathcal{D}_+$ and $\delta_+(t,t_0) = t$; if $t \in \mathbb{T}^*$, then $(t,t_0) \in \mathcal{D}_+$ and $\delta_+(t,t_0) = t$;

(4) if $(s,t) \in \mathcal{D}_\pm$, then $(s,\delta_\pm(s,t)) \in \mathcal{D}_\mp$ and $\delta_\mp(s,\delta_\pm(s,t)) = t$;

(5) if $(s,t) \in \mathcal{D}_\pm$ and $(s,\delta_\pm(s,t)) \in \mathcal{D}_\mp$, then $(s,\delta_\mp(u,t)) \in \mathcal{D}_\pm$ and $\delta_\mp(u,\delta_\pm(s,t)) = \delta_\pm(s,\delta_\mp(u,t))$.

Then the operators δ_- and δ_+ associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be backward and forward shift operators on the set $\mathbb{T}$, respectively. For more details about shift operators and their applications, see [24–27].

3. Differential inequalities with time-varying delay

Consider the following generalized Halanay's inequality with time-varying delay:

$$\begin{aligned} x^\Delta(t) &\le -a(t)x(t) + b(t) \sup_{0\le s\le\tau(t)} x(\delta_-(s,t)), \quad t \ge t_0, \\ x(s) &= x_0, \quad s \in [\delta_-(\hat{\tau},t_0),t_0]_{\mathbb{T}}, \end{aligned} \tag{3.1}$$

where $\delta_-(s,t)$ is backward shift operator, $t \in \mathbb{T},\ x_0 \in \mathbb{R},\ \tau(t) \ge 0$ is rd-continuous and bounded function with $\tau(t) \le \hat{\tau},\ \hat{\tau}$ is a constant, $a(t)$ and $b(t)$ are rd-continuous on $[t_0,\infty)_{\mathbb{T}}$.

Theorem 3.1. Assume that $x(t)$ satisfies (3.1), $a(t) \ge 0$ with $-a \in \mathcal{R}^+$, and there exists a constant $\gamma_1 > 0$ such that, for $t \ge t_0$,

(i)

$$\sup_{v\in[t_0,t]_{\mathbb{T}}} \int_{t_0}^{v} \exp\left\{ -\int_{\sigma(u)}^{v} a(s)\Delta s \right\} |b(u)|\Delta u \quad \le \gamma_1 < 1;$$

(ii) $\exp\left\{ \int_{t_0}^{t} a(u)\Delta u \right\} \to \infty$ as $t \to \infty$.

Then $x(t) \to 0$ as $t \to \infty$.

Proof. Define the following delay dynamic system:

$$\begin{aligned} x^\Delta(t) &= -a(t)x(t) + b(t) \sup_{0\le s\le\tau(t)} x(\delta_-(s,t)), \quad t \ge t_0, \\ x(s) &= x_0, \quad s \in [\delta_-(\hat{\tau},t_0),t_0]_{\mathbb{T}}. \end{aligned} \tag{3.2}$$

From (3.2) and Lemma 2.2, we obtain

$$x(t) = e_{-a}(t,t_0)x_0 + \int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u) \sup_{0\le s\le\tau(t)} x(\delta_-(s,u))\Delta u. \tag{3.3}$$

Define the space Ω_1 by

$$\Omega_1 = \{x : x \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}),\ x(t) \to 0 \text{ as } t \to \infty\}$$

with the norm $\|x\| = \sup_{t\in[t_0,\infty)_{\mathbb{T}}} |x(t)|$. Then Ω_1 is a Banach space. Define the operator $\Gamma_1 : \Omega_1 \to \Omega_1$ by

$$\begin{aligned}(\Gamma_1 x)(t) &= e_{-a}(t, t_0)x_0 + \int_{t_0}^{t} e_{-a}(t, \sigma(u))b(u) \sup_{0\le s\le\tau(t)} x(\delta_-(s, u))\Delta u, \quad t \ge t_0,\\ (\Gamma_1 x)(s) &= x_0, \quad s \in [\delta_-(\hat{\tau}, t_0), t_0]_{\mathbb{T}}.\end{aligned} \tag{3.4}$$

Obviously, Γ_1 is rd-continuous on $[t_0, \infty)_{\mathbb{T}}$. We first show that $\Gamma_1\Omega_1 \subset \Omega_1$. From Lemma 2.4 and condition (ii), we have

$$|e_{-a}(t, t_0)x_0| \le \exp\Big\{ - \int_{t_0}^{t} a(u)\Delta u\Big\}|x_0| \to 0 \text{ as } t \to \infty. \tag{3.5}$$

Since $x(t) \to 0$ as $t \to \infty$, for any $\varepsilon > 0$, there exists $T_1 > 0$ such that

$$|x(t)| < \varepsilon \quad \text{for } t \ge T_1. \tag{3.6}$$

From Lemma 2.4 and (3.6), we get

$$\begin{aligned}&\Big|\int_{t_0}^{t} e_{-a}(t, \sigma(u))b(u) \sup_{0\le s\le\tau(t)} x(\delta_-(s, u))\Delta u\Big|\\ &= \Big|\int_{t_0}^{T_1} e_{-a}(t, \sigma(u))b(u) \sup_{0\le s\le\tau(t)} x(\delta_-(s, u))\Delta u\\ &+ \int_{T_1}^{t} e_{-a}(t, \sigma(u))b(u) \sup_{0\le s\le\tau(t)} x(\delta_-(s, u))\Delta u\Big|\\ &\le \sup_{0\le s\le\tau(t), t_0\le u\le T_1} |x(\delta_-(s, u))| \int_{t_0}^{T_1} \exp\Big\{ - \int_{\sigma(u)}^{t} a(v)\Delta v\Big\}|b(u)|\Delta u\\ &+ \varepsilon \int_{T_1}^{t} \exp\Big\{ - \int_{\sigma(u)}^{t} a(v)\Delta v\Big\}|b(u)|\Delta u.\end{aligned} \tag{3.7}$$

From (3.7) and condition (ii), there exists $T_2 \ge T_1$, for any $t \ge T_2$ and $\varepsilon > 0$ such that

$$\sup_{0\le s\le\tau(t), t_0\le u\le T_1} |x(\delta_-(s, u))| \int_{t_0}^{T_1} \exp\Big\{ - \int_{\sigma(u)}^{t} a(v)\Delta v\Big\}|b(u)|\Delta u < \varepsilon$$

and

$$\int_{T_1}^{t} \exp\Big\{ - \int_{\sigma(u)}^{t} a(v)\Delta v\Big\}|b(u)|\Delta u < \varepsilon.$$

Thus,

$$\Big|\int_{t_0}^{t} e_{-a}(t, \sigma(u))b(u) \sup_{0\le s\le\tau(t)} x(\delta_-(s, u))\Delta u\Big| < \varepsilon \ as\ t \to \infty. \tag{3.8}$$

Hence, in view of (3.4), (3.5), and (3.8), we obtain that $|(\Gamma_1 x)(t)| \to 0$ as $t \to \infty$ and $\Gamma_1(\Omega_1) \subset \Omega_1$.

For $x,\ y \in \Omega_1$, from condition (i), we have

$$\begin{aligned}
&\sup_{v\in[t_0,t]_{\mathbb{T}}} \Big|(\Gamma x)(v) - (\Gamma y)(v)\Big| \\
&\le \sup_{v\in[t_0,t]_{\mathbb{T}}} \Big|x(v) - y(v)\Big| \times \sup_{v\in[0,t]_{\mathbb{T}}} \int_{t_0}^{v} e_{-a}(v,\sigma(u))|b(u)|\Delta u \\
&\le \sup_{v\in[t_0,t]_{\mathbb{T}}} \Big|x(v) - y(v)\Big| \times \sup_{v\in[0,t]_{\mathbb{T}}} \int_{t_0}^{v} \exp\Big\{-\int_{\sigma(u)}^{v} a(s)\Delta s\Big\}|b(u)|\Delta u \\
&\le \gamma_1 \sup_{v\in[t_0,t]_{\mathbb{T}}} \Big|x(v) - y(v)\Big|.
\end{aligned}$$

Therefore, we obtain that Γ_1 is a contraction mapping and has a unique fixed point x on Ω_1, which is a solution of (3.2) with the initial condition $x(s) = x_0,\ s \in [\delta_-(\hat{\tau}, t_0), t_0]_{\mathbb{T}}$.

Next, we show that the zero solution of (3.1) is asymptotic stable. If $x(t)$ is a solution of (3.2) with the initial condition $x(s) = x_0,\ s \in [\delta_-(\hat{\tau}, t_0), t_0]_{\mathbb{T}}$. Since $x(t) \in \Omega_1$, then $x(t)$ is bounded on $t \ge t_0$. From (3.5) and (3.8), for any $\varepsilon > 0$, we have

$$\begin{aligned}
|x(t)| &= \Big|e_{-a}(t,t_0)x_0 + \int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u) \sup_{0\le s\le\tau(t)} x(\delta_-(s,u))\Delta u\Big| \\
&\le \Big|e_{-a}(t,t_0)x_0\Big| + \Big|\int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u) \sup_{0\le s\le\tau(t)} x(\delta_-(s,u))\Delta u\Big| \\
&\to 0 \ \ as\ t \to \infty.
\end{aligned}$$

Thus, system (3.2) is asymptotically stable which implies system (3.1) is asymptotically stable. The proof is complete.

Theorem 3.2. Assume that $x(t)$ satisfies (3.1). There exists $f(t) \ge 0$ with $-f \in \mathcal{R}^+$ and there exists constant $\gamma_2 > 0$ such that, for $t \ge t_0$,
(i)

$$\sup_{v\in[t_0,t]_{\mathbb{T}}} \int_{t_0}^{v} \exp\Big\{-\int_{\sigma(u)}^{v} a(s)\Delta s\Big\}\Big(|f(u) - a(u)| + |b(u)|\Big)\Delta u \quad \le \gamma_2 < 1;$$

(ii) $\exp\Big\{\int_{t_0}^{t} f(u)\Delta u\Big\} \to \infty$ as $t \to \infty$.
Then $x(t) \to 0$ as $t \to \infty$.

Proof. From (3.2) and Lemma 2.2, we obtain

$$x(t) = e_{-f}(t,t_0)x_0 + \int_{t_0}^{t} e_{-f}(t,\sigma(u))[f(u) - a(u)]\Delta u + \int_{t_0}^{t} e_{-f}(t,\sigma(u))b(u) \sup_{0\le s\le\tau(t)} x(\delta_-(s,u))\Delta u. \quad (3.9)$$

Define a space Ω_2 by

$$\Omega_2 = \{x : x \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}),\ x(t) \to 0 \text{ as } t \to \infty\}$$

with the norm $\|x\| = \sup_{t\in[t_0,\infty)_{\mathbb{T}}} |x(t)|$. Then Ω_2 is a Banach space. Define the operator $\Gamma_2 : \Omega_1 \to \Omega_2$ by

$$
\begin{aligned}
(\Gamma_2 x)(t) &= e_{-f}(t,t_0)x_0 + \int_{t_0}^{t} e_{-f}(t,\sigma(u))[f(u)-a(u)]\Delta u \\
&\quad + \int_{t_0}^{t} e_{-f}(t,\sigma(u))b(u) \sup_{0\le s\le\tau(t)} x(\delta_-(s,u))\Delta u, \quad t\ge t_0, \\
(\Gamma_2 x)(s) &= x_0, \quad s\in[\delta_-(\hat{\tau},t_0),t_0]_{\mathbb{T}}.
\end{aligned}
\tag{3.10}
$$

Obviously, Γ_2 is rd-continuous on $[t_0,\infty)_{\mathbb{T}}$. Similar to the proofs of (3.5) and (3.8), using Lemma 2.4 and condition (ii), we have

$$
|e_{-f}(t,t_0)x_0| \le \exp\Big\{-\int_{t_0}^{t} f(u)\Delta u\Big\}|x_0| \to 0 \text{ as } t\to\infty \tag{3.11}
$$

and

$$
\begin{aligned}
&\Big|\int_{t_0}^{t} e_{-a}(t,\sigma(u))\Big(f(u)-a(u)\Big)\Delta u\Big| \\
&+\Big|\int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u)\sup_{0\le s\le\tau(t)} x(\delta_-(s,u))\Delta u\Big| < \varepsilon \; as \; t\to\infty.
\end{aligned}
\tag{3.12}
$$

Hence, in view of (3.10)–(3.12), we obtain that $|(\Gamma_2 x)(t)| \to 0$ as $t\to\infty$ and $\Gamma_2(\Omega_2)\subset\Omega_2$.

For $x,\ y\in\Omega_2$, from condition (i), we have

$$
\begin{aligned}
&\sup_{v\in[t_0,t]_{\mathbb{T}}}\Big|(\Gamma_2 x)(v)-(\Gamma_2 y)(v)\Big| \\
&\le \sup_{v\in[t_0,t]_{\mathbb{T}}}\Big|x(v)-y(v)\Big| \times \sup_{v\in[0,t]_{\mathbb{T}}}\int_{t_0}^{v} e_{-a}(v,\sigma(u))\Big(|f(u)-a(u)|+|b(u)|\Big)\Delta u \\
&\le \sup_{v\in[t_0,t]_{\mathbb{T}}}\Big|x(v)-y(v)\Big| \times \sup_{v\in[0,t]_{\mathbb{T}}}\int_{t_0}^{v} \exp\Big\{-\int_{\sigma(u)}^{v} a(s)\Delta s\Big\}\Big(|f(u)-a(u)|+|b(u)|\Big)\Delta u \\
&\le \gamma_2 \sup_{v\in[t_0,t]_{\mathbb{T}}}\Big|x(v)-y(v)\Big|.
\end{aligned}
$$

Therefore, we obtain that Γ_2 is a contraction mapping and has a unique fixed point x on Ω_2, which is a solution of (3.2) with the initial condition $x(s) = x_0,\ s\in[\delta_-(\hat{\tau},t_0),t_0]_{\mathbb{T}}$. Since $x(t)\in\Omega_2$, then $x(t)$ is bounded on $t\ge t_0$. From (3.11) and (3.12), for any $\varepsilon>0$, we have

$$
\begin{aligned}
|x(t)| &= \Big|e_{-a}(t,t_0)x_0 + \int_{t_0}^{t} e_{-a}(t,\sigma(u))[f(u)-a(u)]\Delta u + \int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u)\sup_{0\le s\le\tau(t)} x(\delta_-(s,u))\Delta u\Big| \\
&\le \Big|e_{-a}(t,t_0)x_0\Big| + \Big|\int_{t_0}^{t} e_{-a}(t,\sigma(u))[f(u)-a(u)]\Delta u\Big| + \Big|\int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u)\sup_{0\le s\le\tau(t)} x(\delta_-(s,u))\Delta u\Big| \\
&\to 0 \;\; as \; t\to\infty.
\end{aligned}
$$

Thus, system (3.2) is asymptotically stable which implies system (3.1) is asymptotically stable. The proof is complete.

Remark 3.1. Theorem 3.2 removes the condition of non-negativity of coefficient $a(t)$; therefore, the results of Theorem 3.2 improve the corresponding ones of Theorem 3.1.

Consider the following generalized Halanay's inequality with time-varying delay:

$$\begin{aligned} &x^{\Delta}(t) \leq -a(t)x(\sigma(t)) + b(t) \sup_{0\leq s\leq \tau(t)} x(\delta_-(s,t)), \quad t \geq t_0, \\ &x(s) = x_0, \quad s \in [\delta_-(\hat{\tau}, t_0), t_0]_{\mathbb{T}}, \end{aligned} \tag{3.13}$$

where $\delta_-(s,t)$ is backward shift operator, $t \in \mathbb{T},\ x_0 \in \mathbb{R},\ \tau(t) \geq 0$ is rd-continuous and bounded function with $\tau(t) \leq \hat{\tau},\ \hat{\tau}$ is a constant, $a(t)$ and $b(t)$ are rd-continuous on $[t_0,\infty)_{\mathbb{T}}$. Based on Lemma 2.3 and Theorems 3.1 and 3.2, we have the following two corollaries:

Corollary 3.1. Assume that $x(t)$ satisfies (3.13), $a(t) \geq 0$ with $-a \in \mathcal{R}^+$, and there exists a constant $\gamma_3 > 0$ such that, for $t \geq t_0$,

(i)

$$\sup_{v\in[t_0,t]_{\mathbb{T}}} \int_{t_0}^{v} \exp\left\{-\int_u^v a(s)\Delta s\right\}|b(u)|\Delta u \quad \leq \gamma_3 < 1;$$

(ii) $\exp\left\{\int_{t_0}^t a(u)\Delta u\right\} \to \infty$ as $t \to \infty$.

Then $x(t) \to 0$ as $t \to \infty$.

Corollary 3.2. Assume that $x(t)$ satisfies (3.13). There exists $f(t) \geq 0$ with $-f \in \mathcal{R}^+$ and there exists a constant $\gamma_4 > 0$ such that, for $t \geq t_0$,

(i)

$$\sup_{v\in[t_0,t]_{\mathbb{T}}} \int_{t_0}^{v} \exp\left\{-\int_u^v a(s)\Delta s\right\}\Big(|f(u)-a(u)|+|b(u)|\Big)\Delta u \quad \leq \gamma_4 < 1;$$

(ii) $\exp\left\{\int_{t_0}^t f(u)\Delta u\right\} \to \infty$ as $t \to \infty$.

Then $x(t) \to 0$ as $t \to \infty$.

4. Differential inequalities with mixed delays

Consider the following generalization of Halanay's inequality with mixed delays:

$$\begin{aligned} &x^{\Delta}(t) \leq -a(t)x(t) + b(t) \sup_{0\leq s\leq \tau(t)} x(\delta_-(s,t)) + c(t)\int_0^{\infty} K(s)x(\delta_-(s,t))\Delta s, \quad t \geq t_0, \\ &x(s) = x_0, \quad s \in (-\infty, t_0]_{\mathbb{T}}, \end{aligned} \tag{4.1}$$

where $\delta_-(s,t)$ is backward shift operator, $t \in \mathbb{T},\ x_0 \in \mathbb{R},\ \tau(t) \geq 0$ is rd-continuous and bounded function with $\tau(t) \leq \hat{\tau},\ \hat{\tau}$ is a constant, $a(t),\ b(t)$, and $c(t)$ are rd-continuous on $[t_0,\infty)_{\mathbb{T}}$, and $K(t)$ is rd-continuous on $[0,\infty)_{\mathbb{T}}$.

Theorem 4.1. Assume that $x(t)$ satisfies (4.1), $a(t) \geq 0$ with $-a \in \mathcal{R}^+$, $\int_0^\infty |K(s)|\Delta s < \infty$ and there exists a constant $\gamma_5 > 0$ such that, for $t \geq t_0$,

(i)

$$\sup_{v\in[t_0,t]_{\mathbb{T}}} \int_{t_0}^{v} \exp\left\{-\int_{\sigma(u)}^v a(s)\Delta s\right\}\Big(|b(u)+|c(u)|\int_0^{\infty}|K(s)|\Delta s\Big)\Delta u \quad \leq \gamma_5 < 1;$$

(ii) $\exp\left\{\int_{t_0}^{t} a(u)\Delta u\right\} \to \infty$ as $t \to \infty$.
Then $x(t) \to 0$ as $t \to \infty$.
Proof. Define the following delay dynamic system:

$$\begin{aligned} &x^{\Delta}(t) = -a(t)x(t) + b(t) \sup_{0\le s\le \tau(t)} x(\delta_-(s,t)) + c(t)\int_0^{\infty} K(s)x(\delta_-(s,t))\Delta s, \quad t \ge t_0, \\ &x(s) = x_0, \quad s \in (-\infty, t_0]_{\mathbb{T}}. \end{aligned} \tag{4.2}$$

From (4.2) and Lemma 2.2, we obtain

$$\begin{aligned} x(t) = {} & e_{-a}(t,t_0)x_0 + \int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u) \sup_{0\le s\le \tau(t)} x(\delta_-(s,u))\Delta u \\ & + \int_{t_0}^{t} e_{-a}(t,\sigma(u))c(u)\int_0^{\infty} K(s)x(\delta_-(s,u))\Delta s\Delta u. \end{aligned} \tag{4.3}$$

Define a space Ω_3 by

$$\Omega_3 = \{x : x \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}),\ x(t) \to 0 \text{ as } t \to \infty\}$$

with the norm $\|x\| = \sup_{t\in[t_0,\infty)_{\mathbb{T}}|x(t)|}$. Then Ω_3 is a Banach space. Define the operator $\Gamma_3 : \Omega_3 \to \Omega_3$ by

$$\begin{aligned} (\Gamma_3 x)(t) = {} & e_{-a}(t,t_0)x_0 + \int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u) \sup_{0\le s\le \tau(t)} x(\delta_-(s,u))\Delta u \\ & + \int_{t_0}^{t} e_{-a}(t,\sigma(u))c(u)\int_0^{\infty} K(s)x(\delta_-(s,u))\Delta s\Delta u, \quad t \ge t_0, \\ (\Gamma_3 x)(s) = {} & x_0, \quad s \in (-\infty, t_0]_{\mathbb{T}}. \end{aligned} \tag{4.4}$$

Obviously, Γ_3 is rd-continuous on $[t_0,\infty)_{\mathbb{T}}$. Similar to the proofs of (3.5) and (3.8), using Lemma 2.4 and condition (ii), we have

$$|e_{-a}(t,t_0)x_0| \le \exp\left\{-\int_{t_0}^{t} a(u)\Delta u\right\}|x_0| \to 0 \text{ as } t \to \infty \tag{4.5}$$

and

$$\begin{aligned} & \left|\int_{t_0}^{t} e_{-a}(t,\sigma(u))c(u)\int_0^{\infty} K(s)x(\delta_-(s,u))\Delta s\Delta u\right| \\ & + \left|\int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u) \sup_{0\le s\le \tau(t)} x(\delta_-(s,u))\Delta u\right| < \varepsilon \ as\ t \to \infty. \end{aligned} \tag{4.6}$$

Hence, in view of (4.4)–(4.6), we obtain that $|(\Gamma_3 x)(t)| \to 0$ as $t \to \infty$ and $\Gamma_3(\Omega_3) \subset \Omega_3$.

For $x,\ y \in \Omega_2$, from condition (i), we have

$$\begin{aligned} & \sup_{v\in[t_0,t]_{\mathbb{T}}} \Big|(\Gamma_2 x)(v) - (\Gamma_2 y)(v)\Big| \\ & \le \sup_{v\in[t_0,t]_{\mathbb{T}}} \Big|x(v) - y(v)\Big| \times \sup_{v\in[0,t]_{\mathbb{T}}} \int_{t_0}^{v} e_{-a}(v,\sigma(u))\Big(|b(u)| + |c(u)|\int_0^{\infty} |K(s)|\Delta s\Big)\Delta u \\ & \le \sup_{v\in[t_0,t]_{\mathbb{T}}} \Big|x(v) - y(v)\Big| \times \sup_{v\in[0,t]_{\mathbb{T}}} \int_{t_0}^{v} \exp\left\{-\int_{\sigma(u)}^{v} a(s)\Delta s\right\}\Big(|b(u)| + |c(u)|\int_0^{\infty} |K(s)|\Delta s\Big)\Delta u \\ & \le \gamma_5 \sup_{v\in[t_0,t]_{\mathbb{T}}} \Big|x(v) - y(v)\Big|. \end{aligned}$$

Therefore, we obtain that Γ_3 is a contraction mapping and has a unique fixed point x on Ω_3, which is a solution of (4.2) with the initial condition $x(s) = x_0,\ s \in (-\infty, t_0]_{\mathbb{T}}$. Since $x(t) \in \Omega_3$, then $x(t)$ is bounded on $t \geq t_0$. From (4.5) and (4.6), for any $\varepsilon > 0$, we have

$$\begin{aligned}
|x(t)| &= \Big| e_{-a}(t,t_0)x_0 + \int_{t_0}^{t} e_{-a}(t,\sigma(u))c(u)\int_0^\infty K(s)x(\delta_-(s,u))\Delta s\Delta u \\
&\quad + \int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u)\sup_{0\leq s\leq\tau(t)} x(\delta_-(s,u))\Delta u\Big| \\
&\leq \Big| e_{-a}(t,t_0)x_0\Big| + \Big|\int_{t_0}^{t} e_{-a}(t,\sigma(u))c(u)\int_0^\infty K(s)x(\delta_-(s,u))\Delta s\Delta u\Big| \\
&\quad + \Big|\int_{t_0}^{t} e_{-a}(t,\sigma(u))b(u)\sup_{0\leq s\leq\tau(t)} x(\delta_-(s,u))\Delta u\Big| \\
&\to 0 \ \ as\ t \to \infty.
\end{aligned}$$

Thus, system (4.2) is asymptotically stable which implies system (4.1) is asymptotically stable. The proof is complete.

In order to remove the nonnegative limitation of coefficient $a(t)$ in Theorem 4.1, we provide the following theorem:

Theorem 4.2. Assume that $x(t)$ satisfies (4.1). There exists $f(t) \geq 0$ with $-f \in \mathcal{R}^+$ and there exists constant $\gamma_6 > 0$ such that, for $t \geq t_0$,

(i)

$$\begin{aligned}
&\sup_{v\in[t_0,t]_{\mathbb{T}}}\int_{t_0}^{v}\exp\Big\{-\int_{\sigma(u)}^{v} a(s)\Delta s\Big\}\Big(|f(u)-a(u)| + |b(u)| + |c(u)|\int_0^\infty |K(s)|\Delta s\Big)\Delta u \\
&\leq \gamma_6 < 1;
\end{aligned}$$

(ii) $\exp\Big\{\int_{t_0}^{t} f(u)\Delta u\Big\} \to \infty$ as $t \to \infty$.

Then $x(t) \to 0$ as $t \to \infty$.

The proof of Theorem 4.2 is the same as that of Theorem 3.2; we omit it. Furthermore, consider the following generalized Halanay's inequality with mixed delays:

$$\begin{aligned}
&x^{\Delta}(t) \leq -a(t)x(\sigma(t)) + b(t)\sup_{0\leq s\leq\tau(t)} x(\delta_-(s,t)) + c(t)\int_0^\infty K(s)x(\delta_-(s,t))\Delta s, \quad t \geq t_0, \\
&x(s) = x_0, \quad s \in (-\infty, t_0]_{\mathbb{T}},
\end{aligned} \tag{4.7}$$

where $\delta_-(s,t)$ is backward shift operator, $t \in \mathbb{T},\ x_0 \in \mathbb{R},\ \tau(t) \geq 0$ is rd-continuous and bounded function with $\tau(t) \leq \hat{\tau},\ \hat{\tau}$ is a constant, $a(t),\ b(t)$, and $c(t)$ are rd-continuous on $[t_0,\infty)_{\mathbb{T}}$, and $K(t)$ is rd-continuous on $[0,\infty)_{\mathbb{T}}$. Based on Lemma 2.3 and Theorems 4.1 and 4.2, we have the following two corollaries.

Corollary 4.1. Assume that $x(t)$ satisfies (4.7), $a(t) \geq 0$ with $-a \in \mathcal{R}^+$, and there exists a constant $\gamma_7 > 0$ such that, for $t \geq t_0$,

(i)

$$\sup_{v\in[t_0,t]_{\mathbb{T}}}\int_{t_0}^{v}\exp\Big\{-\int_{u}^{v} a(s)\Delta s\Big\}\Big(|b(u) + |c(u)|\int_0^\infty |K(s)|\Delta s\Big)\Delta u \quad \leq \gamma_7 < 1;$$

(ii) $\exp\left\{\int_{t_0}^{t} a(u)\Delta u\right\} \to \infty$ as $t \to \infty$.
Then $x(t) \to 0$ as $t \to \infty$.
Corollary 4.2. Assume that $x(t)$ satisfies (4.7). There exists $f(t) \geq 0$ with $-f \in \mathcal{R}^+$ and there exists constant $\gamma_8 > 0$ such that, for $t \geq t_0$,
(i)

$$\sup_{v\in[t_0,t]_{\mathbb{T}}} \int_{t_0}^{v} \exp\left\{-\int_{u}^{v} a(s)\Delta s\right\}\Big(|f(u)-a(u)|+|b(u)|+|c(u)|\int_{0}^{\infty}|K(s)|\Delta s\Big)\Delta u \leq \gamma_8 < 1;$$

(ii) $\exp\left\{\int_{t_0}^{t} f(u)\Delta u\right\} \to \infty$ as $t \to \infty$.
Then $x(t) \to 0$ as $t \to \infty$.
Remark 4.1. In this paper, we mainly use the Banach contraction mapping principle to study the asymptotic stability of the trajectories governed by some delay differential inequalities on time scales. In fact, we can use Schauder's fixed point theorem to establish the existence of at least one solution for the considered systems (see [28]); we can also use Leggett Williams fixed point theorem to investigate the existence of three solutions to considered systems, see [29]. We hope that more results from systems (3.1) and (4.1) can be obtained in future work.
Remark 4.2. We give the advantages of this paper as follows:
(1) Since many time scales that are not additive, we define the delay terms as using shift operators.
(2) We extend the research scope and develop research methods for Halanay inequality.
(3) The research method of this article can study various types of Halanay inequalities, such as Halanay inequality with impulsive terms and Halanay inequality with stochastic terms.

5. Examples

Example 5.1. When $\mathbb{T} = \mathbb{Z}$, consider the following system:

$$\Delta x(k) \leq -a(k)x(k) + b(k) \sup_{0\leq s\leq\tau(k)} x(k-s), \quad k \geq 0,\ k \in \mathbb{Z}, \tag{5.1}$$

where

$$\Delta x(k) = x(k+1) - x(k),\ a(k) = 1 - 0.5\sin(2k+1.5),\ b(k) = 0.2^{k+1}, \tau(k) = 3 - 0.5\cos k.$$

Choosing $\gamma_1 = 0.26 < 1$, we have

$$\sup_{v\in[0,t]_{\mathbb{Z}}} \int_{0}^{v} \exp\left\{-\int_{\sigma(u)}^{v} a(s)\Delta s\right\}|b(u)|\Delta u \quad < 0.26 < 1$$

and $\exp\left\{\int_{0}^{t} a(u)\Delta u\right\} \to \infty$ as $t \to \infty$.
One can see that all conditions of Theorem 3.1 hold. Hence, system (5.1) is asymptotically stable. Figure 1 shows the trajectory of the solution to the system (5.1).
Example 5.2. When $\mathbb{T} = \mathbb{Z}$, consider the following system:

$$\Delta x(k) \leq -a(k)x(k) + b(k) \sup_{0\leq s\leq\tau(k)} x(k-s) + c(k)\sum_{i=0}^{\infty} K(i)x(k-i), \quad k \geq 0,\ k \in \mathbb{Z}, \tag{5.2}$$

where

$$\Delta x(k) = x(k+1) - x(k),\ a(k) = 3 - 0.2\sin(3k+0.5),\ b(k) = 0.4^{k+1},$$
$$c(k) = \left(\frac{1}{16}\right)^{k+1},\ \tau(k) = 4 - \cos k,\ K(i) = 0.6^{i+1}.$$

Choosing $\gamma_2 = 0.86$, we have

$$\sup_{v\in[0,t]_{\mathbb{Z}}} \int_0^v \exp\left\{-\int_{\sigma(u)}^v a(s)\Delta s\right\}\left(|b(u) + |c(u)| \int_0^\infty |K(s)|\Delta s\right)\Delta u \quad < 0.86 < 1$$

and $\exp\left\{\int_0^t a(u)\Delta u\right\} \to \infty$ as $t \to \infty$.

One can see that all conditions of Theorem 4.1 hold. Hence, system (5.2) is asymptotically stable. Figure 2 shows the trajectory of the solution to the system (5.2).

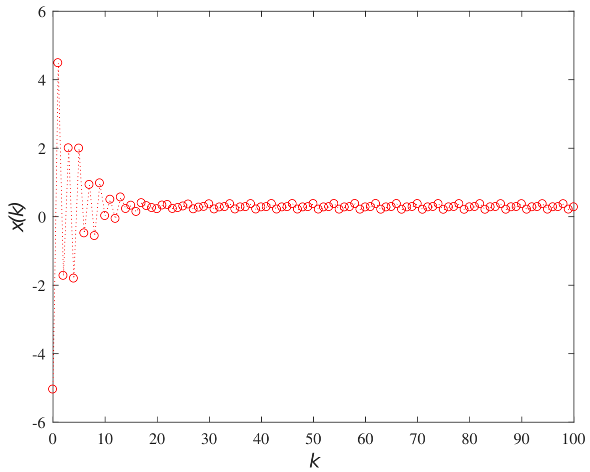

Figure 1. The state's trajectory of the system (5.1).

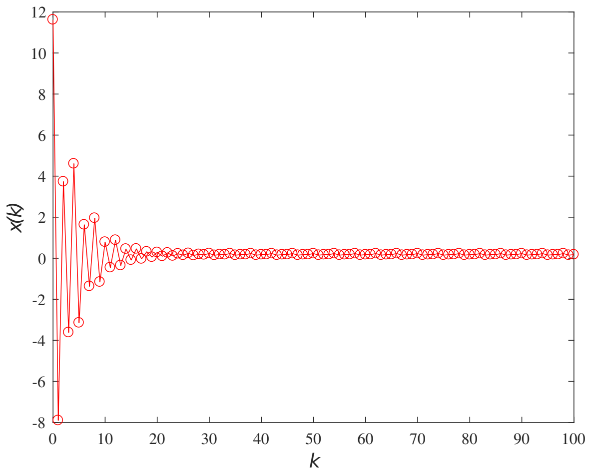

Figure 2. The state's trajectory of the system (5.2).

6. Conclusions

In this work, some novel asymptotical stability results for delay differential inequalities on time scales have been derived by using time scale theory and the fixed point theorem. Our results do not require the system coefficients to be non-negative but extend the corresponding results of [7–9]. It should be pointed out that the use of the fixed point theorem makes the proof process easier to understand. At last, two examples with numerical simulations have been presented to illustrate the effectiveness of our results. In the future, we will study delay differential inequalities with a neutral-type operator on time scales.

Author contributions

Bingxian Wang: Writing-original draft preparation, Writing-review and editing; Mei Xu: Formal analysis, Methodology. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors confirm that they have no conflict of interest in this paper.

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