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Research article

Hardy type identities and inequalities with divergence type operators on smooth metric measure spaces

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Abstract: We gave the Hardy type identities and inequalities for the divergence type operator $L_{f,V}$ on smooth metric measure spaces. Additionally, we improved a Rellich type inequality by using the improved Hardy type inequality. Our results improved and included many previously known results as special cases.

Keywords: Hardy type inequality; Hardy type identity; smooth metric measure space; divergence type operator
Mathematics Subject Classification: 53C21, 26D10

1. Introduction

Hardy type inequalities play crucial roles in analysis, probability, and partial differential equations. We first recall the classical L^2- Hardy inequality:

$$\int_{\mathbb{R}^n} |\nabla\phi|^2\, dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^2} dx,$$

for $n \geq 3$ and $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. The constant $\left(\frac{n-2}{2}\right)^2$ is sharp and is never attained by nontrivial functions. The Rellich inequality is a natural generalization of the above Hardy inequality. In $\mathbb{R}^n$, it reads as follows:

$$\int_{\mathbb{R}^n} |\Delta\phi|^2\, dx \geq \left(\frac{n(n-4)}{4}\right)^2 \int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^4} dx,$$

where $n \geq 5$ and $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$.

In recent years, there have been many results in the literature on the Hardy and Rellich type inequalities in the context of a complete Riemannian manifold. In particular, the following Hardy inequality has been first established on Riemannian manifold (M, g) by Carron [1]:

$$\int_M \rho^{\alpha} |\nabla\phi|^2 \, dv_g \geq \frac{(C+\alpha-1)^2}{4} \int_M \rho^{\alpha-2}\phi^2 dv_g,$$

where $\alpha \in \mathbb{R}$, $C+\alpha-1 > 0$, $\phi \in C_0^{\infty}\left(M \setminus \rho^{-1}\{0\}\right)$. The weight function ρ is nonnegative and it satisfies $|\nabla\rho| = 1$ and $\Delta\rho \geq \frac{C}{\rho}$ in the sense of distribution. Here, dv_g, ∇, and Δ denote the volume element, the gradient, and the Laplace operator on M. Under the same geometric assumptions on the weight function ρ, Kombe and Özaydin [2] established the weighted L^p-Hardy inequality:

$$\int_M \rho^{\alpha} |\nabla\phi|^p \, dv_g \geq \left(\frac{C+1+\alpha-p}{p}\right)^p \int_M \rho^{\alpha-p}\phi^p dv_g,$$

where $1 \leq p < \infty$, $C+1+\alpha-p > 0$, and $\phi \in C_0^{\infty}\left(M \setminus \rho^{-1}\{0\}\right)$. Kombe and Özaydin [3] also proved a new weighted Hardy-Poincaré inequality. They showed that if M is a complete non-compact Riemannian manifold of dimension $n > 1$ and ρ is a nonnegative function on M such that $|\nabla\rho| = 1$ and $\Delta\rho \geq \frac{C}{\rho}$ in the sense of distribution, where $C > 0$, the following inequality holds:

$$\int_M \rho^{\alpha+p} |\nabla\rho \cdot \nabla\phi|^p \, dv_g \geq \left(\frac{C+1+\alpha}{p}\right)^p \int_M \rho^{\alpha} |\phi|^p \, dv_g.$$

Xia [4] proved the following Hardy type inequality on a complete non-compact Riemannian manifold. Let M be an n-dimensional complete non-compact Riemannian manifold, where $n \geq 2$, and let ρ be a nonnegative function on M such that $|\nabla\rho| = 1$ and $\Delta\rho \geq \frac{C}{\rho} + H$ in the sense of distribution, where H is a continuous function on M and C is a constant. The result is the following: For any $p, q \in \mathbb{R}$ with $q > 1 + C$ and any compactly supported smooth function $\phi \in C_0^{\infty}\left(M \setminus \rho^{-1}\{0\}\right)$, the following inequality holds:

$$(q-C-1) \int_M \frac{|\phi|^p}{\rho^q} dv_g \leq |p| \int_M \frac{|\phi|^{p-1}}{\rho^{q-1}} |\nabla\phi| \, dv_g + \int_M \frac{H\,|\phi|^p}{\rho^{q-1}} dv_g.$$

Huang and Ye [5] considered the first order Hardy inequalities using simple identities. This basic setting not only permits to derive quickly many well-known Hardy inequalities with optimal constants, but also supplies improved or new estimates in miscellaneous situations. We also refer the interested reader to [6–16], which are excellent monographs on the topic.

In this paper, we are interested in proving some Hardy type identities and inequalities on the smooth metric measure spaces related to the divergence type operator $L_{f,V}$. Before that, we would like to briefly introduce the smooth metric measure spaces and the divergence type operator $L_{f,V}$.

A smooth metric measure space $(M^n, g, d\mu)$ is a Riemannian manifold (M^n, g) equipped with a conformal Riemannian volume $d\mu = \frac{1}{V}e^{-f}dv_g$, where dv_g denotes the Riemannian volume measure on M, V is a positive twice differentiable function on M, and f is a real-valued smooth function on M.

On a smooth metric measure space $(M^n, g, d\mu)$, we can define the weighted Ricci curvature $\widehat{Ric}_{f,m}^{V}$ [17] given by

$$\widehat{Ric}_{f,m}^{V} = \frac{\Delta_f V}{V} g - \frac{1}{V}\nabla^2 V + Ric_f^m,$$

where

$$\Delta_f \cdot := e^f \operatorname{div}\left(e^{-f}\nabla\cdot\right) = \Delta \cdot -\nabla f \nabla, \tag{1.1}$$

is the f-Laplacian (see [18–21]),

$$Ric_f^m = Ric + \nabla^2 f - \frac{1}{m-n}\nabla f \otimes \nabla f,$$

is the m-Bakry-Émery Ricci curvature, and m is a real constant. Here, ∇, Δ, ∇^2, and div are the gradient, Laplacian, Hessian, and divergence operator on M with respect to g, respectively.

We introduce the divergence type operator $L_{f,V}$ on M as follows (see [17, 22] for detail):

$$L_{f,V}\varphi = e^f \operatorname{div}\left[e^{-f}V^2\nabla\left(\frac{\varphi}{V}\right)\right], \tag{1.2}$$

where φ is a smooth function and V is a positive twice differentiable function defined on M. In general, the divergence type operator $L_{f,V}$ is not self-adjoint with respect to the standard L^2-inner product because of the first order term, but it is self-adjoint with respect to the weighted measure $d\mu = \frac{1}{V}e^{-f}dv_g$. That is, for any smooth functions $u, v \in C_0^\infty(M)$, we have

$$\int_M uL_{f,V}v d\mu = -\int_M V^3\nabla\frac{u}{V}\nabla\frac{v}{V}d\mu = \int_M vL_{f,V}u d\mu. \tag{1.3}$$

We observe that the divergence type operator $L_{f,V}$ encompasses, as very special cases, many differential linear operators. In the case that $V = 1$, i.e., $L_{f,1}\varphi = \Delta_f\varphi$. In the case that f is constant, i.e., $L_{f,V}\varphi = L_V\varphi = V\Delta\varphi - \varphi\Delta V$, we call L_V by the generalized Schrödinger operator. From the viewpoint of geometry, the generalized Schrödinger operator L_V plays an important role in the geometric understanding of the sub-static manifolds; see [23].

Du and Mao [24] proved that some Hardy and Rellich type inequalities on the smooth metric measure spaces related to the f-Laplace. Also, Li, Abolarinwa, Alkhaldi, and Ali [25] generalized some integral inequalities of Hardy type to the setting of the smooth metric measure spaces. These studies are without the curvature conditions. On the other hand, the case of additional curvature conditions has been studied. Kolesnikov and Milman [26] proved the Hardy-Poincaré inequality under the curvature condition $Ric_f^m > 0$ on M:

$$\frac{m}{m-1}\int_M \varphi^2 e^{-f}dv_g \leq \int_M e^{-f}\left\langle\left(Ric_f^m\right)^{-1}\nabla\varphi, \nabla\varphi\right\rangle dv_g,$$

where $\frac{1}{m} \in (-\infty, \frac{1}{n}]$ and $\varphi \in C^1(M)$. Huang and Zhu [22] gave the following Hardy-Poincaré inequality: Let M be a compact Riemannian manifold and $\widehat{Ric}^V_{f,m} > 0$, then

$$\frac{m}{m-1}\int_M V\varphi^2 e^{-f}dv_g \leq \int_M e^{-f}V\left\langle\left(\widehat{Ric}^V_{f,m}\right)^{-1}\nabla\varphi, \nabla\varphi\right\rangle dv_g,$$

where $\varphi \in C^\infty(M)$. Huang and Zhu [22] studied weighted L^2-Hardy-Poincaré inequalities on a smooth metric measure space related to the divergence type operator $L_{f,V}$ under the curvature condition.

The primary objectives of the present paper are twofold: First, we want to establish some L^2 Hardy type identities and L^p Hardy type inequalities related to the divergence type operator $L_{f,V}$ on the smooth metric measure spaces without curvature conditions. Second, as application we would like to show some Rellich type inequalities for the divergence type operator $L_{f,V}$.

The remainder of the paper is as follows: In Section 2, we will prove Hardy type identity related to the divergence type operator $L_{f,V}$ and several related corollaries. Then, we will prove some L^p Hardy type inequalities in Section 3. In the last section, we will prove a Rellich type inequality related to the divergence type operator $L_{f,V}$.

2. Hardy type identity and its applications

In this section, we will prove some Hardy type identities related to the divergence type operator $L_{f,V}$ and several related corollaries.

Theorem 2.1. *Let $(M^n, \mathrm{g}, d\mu)$ be an n-dimensional $(n \geq 2)$ complete non-compact smooth metric measure space. Suppose that V is a positive twice differentiable function and f is a real-valued smooth function on M and $W \in C^1(M)$. Then, for any compactly supported smooth functions $\frac{u}{V} \in C_0^1(M)$ and $\frac{\varphi}{V} \in C_0^2(M)$, the following identity holds:*

$$\begin{aligned}\int_M WV^3 \left|\nabla\left(\frac{u}{V}\right)\right|^2 d\mu = &\int_M WV^3 \left|\nabla\left(\frac{u}{V}\right) - \left(\frac{u}{V}\right)\left(\frac{\varphi}{V}\right)^{-1} \nabla\left(\frac{\varphi}{V}\right)\right|^2 d\mu \\ &- \int_M e^f V \left(\frac{u}{V}\right)^2 \left(\frac{\varphi}{V}\right)^{-1} \operatorname{div}\left(e^{-f} V^2 W \nabla\left(\frac{\varphi}{V}\right)\right) d\mu,\end{aligned} \tag{2.1}$$

where $d\mu = \frac{1}{V} e^{-f} dv_g$ and dv_g is the Riemannian volume element related to g.

Proof. As a consequence of the integration by parts (1.3), we have

$$
\begin{aligned}
&\int_M WV^3\left|\nabla\left(\frac{u}{V}\right)-\left(\frac{u}{V}\right)\left(\frac{\varphi}{V}\right)^{-1}\nabla\left(\frac{\varphi}{V}\right)\right|^2 d\mu\\
=&\int_M e^{-f}V^2W\left|\nabla\left(\frac{u}{V}\right)\right|^2 dv_g-2\int_M e^{-f}V^2W\left\langle\nabla\left(\frac{u}{V}\right),\left(\frac{u}{V}\right)\left(\frac{\varphi}{V}\right)^{-1}\nabla\left(\frac{\varphi}{V}\right)\right\rangle dv_g\\
&+\int_M e^{-f}V^2W\left(\frac{u}{V}\right)^2\left(\frac{\varphi}{V}\right)^{-2}\left|\nabla\left(\frac{\varphi}{V}\right)\right|^2 dv_g\\
=&\int_M e^{-f}V^2W\left|\nabla\left(\frac{u}{V}\right)\right|^2 dv_g-\int_M e^{-f}V^2W\left\langle\left(\frac{\varphi}{V}\right)^{-1}\nabla\left(\frac{u}{V}\right)^2,\nabla\left(\frac{\varphi}{V}\right)\right\rangle dv_g\\
&+\int_M e^{-f}V^2W\left(\frac{u}{V}\right)^2\left(\frac{\varphi}{V}\right)^{-2}\left|\nabla\left(\frac{\varphi}{V}\right)\right|^2 dv_g\\
=&\int_M e^{-f}V^2W\left|\nabla\left(\frac{u}{V}\right)\right|^2 dv_g+\int_M\left(\frac{u}{V}\right)^2\operatorname{div}\left(e^{-f}V^2W\left(\frac{\varphi}{V}\right)^{-1}\nabla\left(\frac{\varphi}{V}\right)\right)dv_g\\
&+\int_M e^{-f}V^2W\left(\frac{u}{V}\right)^2\left(\frac{\varphi}{V}\right)^{-2}\left|\nabla\left(\frac{\varphi}{V}\right)\right|^2 dv_g\\
=&\int_M e^{-f}V^2W\left|\nabla\left(\frac{u}{V}\right)\right|^2 dv_g+\int_M\left(\frac{\varphi}{V}\right)^{-1}\left(\frac{u}{V}\right)^2\operatorname{div}\left(e^{-f}V^2W\nabla\left(\frac{\varphi}{V}\right)\right)dv_g\\
&-\int_M\left(\frac{\varphi}{V}\right)^{-2}\left(\frac{u}{V}\right)^2\left\langle\nabla\left(\frac{\varphi}{V}\right),e^{-f}V^2W\nabla\left(\frac{\varphi}{V}\right)\right\rangle dv_g+\int_M e^{-f}V^2W\left(\frac{u}{V}\right)^2\left(\frac{\varphi}{V}\right)^{-2}\left|\nabla\left(\frac{\varphi}{V}\right)\right|^2 dv_g\\
=&\int_M WV^3\left|\nabla\left(\frac{u}{V}\right)\right|^2 d\mu+\int_M e^fV\left(\frac{u}{V}\right)^2\left(\frac{\varphi}{V}\right)^{-1}\operatorname{div}\left(e^{-f}V^2W\nabla\left(\frac{\varphi}{V}\right)\right)d\mu.
\end{aligned}
$$

Then, we can get

$$
\begin{aligned}
\int_M WV^3\left|\nabla\left(\frac{u}{V}\right)\right|^2 d\mu=&\int_M WV^3\left|\nabla\left(\frac{u}{V}\right)-\left(\frac{u}{V}\right)\left(\frac{\varphi}{V}\right)^{-1}\nabla\left(\frac{\varphi}{V}\right)\right|^2 d\mu\\
&-\int_M e^fV\left(\frac{u}{V}\right)^2\left(\frac{\varphi}{V}\right)^{-1}\operatorname{div}\left(e^{-f}V^2W\nabla\left(\frac{\varphi}{V}\right)\right)d\mu.
\end{aligned}
$$

This completes the proof of Theorem 2.1. □

As a special case for $W = 1$ in Theorem 2.1, we have the following.

Corollary 2.2. *Let $(M^n, g, d\mu)$ be an n-dimensional $(n \geq 2)$ complete non-compact smooth metric measure space. Suppose that V is a positive twice differentiable function and f is a real-valued smooth function on M. Then, for any $\frac{u}{V} \in C_0^1(M)$ and $\frac{\varphi}{V} \in C_0^2(M)$, the following identity holds:*

$$
\begin{aligned}
\int_M V^3\left|\nabla\left(\frac{u}{V}\right)\right|^2 d\mu=&\int_M V^3\left|\nabla\left(\frac{u}{V}\right)-\left(\frac{u}{V}\right)\left(\frac{\varphi}{V}\right)^{-1}\nabla\left(\frac{\varphi}{V}\right)\right|^2 d\mu\\
&-\int_M V\left(\frac{u}{V}\right)^2\left(\frac{\varphi}{V}\right)^{-1}\left(L_{f,V}\varphi\right)d\mu,
\end{aligned}
\tag{2.2}
$$

where $d\mu = \frac{1}{V}e^{-f}dv_g$ and dv_g is the Riemannian volume element related to g.

By using Corollary 2.2, we have the following.

Corollary 2.3. *Let $(M^n, g, d\mu)$ be an n-dimensional $(n \geq 2)$ complete non-compact smooth metric measure space. Suppose that V is a positive twice differentiable function and f is a real-valued smooth function on M. Then, for any $\frac{u}{V} \in C_0^1(M)$ and $\frac{\varphi}{V} \in C_0^2(M)$, the following inequality holds:*

$$\int_M V^3 \left|\nabla\left(\frac{u}{V}\right)\right|^2 d\mu \geq -\int_M V\left(\frac{u}{V}\right)^2\left(\frac{\varphi}{V}\right)^{-1}\left(L_{f,V}\varphi\right) d\mu, \tag{2.3}$$

where $d\mu = \frac{1}{V}e^{-f}dv_g$ and dv_g is the Riemannian volume element related to g.

As the special case that $\frac{\varphi}{V} = \left(\frac{\rho}{V}\right)^{\frac{1-C}{2}}$ in the Corollary 2.3, we have the following.

Corollary 2.4. *Let $(M^n, g, d\mu)$ be an n-dimensional $(n \geq 2)$ complete non-compact smooth metric measure space. Suppose that V is a positive twice differentiable function and f is a real-valued smooth function on M. Let ρ be a nonnegative function on M such that $\left|\nabla\left(\frac{\rho}{V}\right)\right| = 1$ and $L_{f,V}\rho \geq C(\frac{\rho}{V})^{-1}V^2$ in the sense of distribution, where C is a constant and $C > 1$. Then, for any $\frac{u}{V} \in C_0^1(M \setminus \left(\frac{\rho}{V}\right)^{-1}\{0\})$, the following inequality holds:*

$$\int_M V^3 \left|\nabla\left(\frac{u}{V}\right)\right|^2 d\mu \geq \frac{(C-1)^2}{4}\int_M V^3\left(\frac{u}{V}\right)^2\left(\frac{\rho}{V}\right)^{-2} d\mu,$$

where $d\mu = \frac{1}{V}e^{-f}dv_g$ and dv_g is the Riemannian volume element related to g.

Proof. Taking $\frac{\varphi}{V} = \left(\frac{\rho}{V}\right)^{\frac{1-C}{2}}$, we have

$$\begin{aligned}
&-\left(\frac{\varphi}{V}\right)^{-1}\left(L_{f,V}\varphi\right) \\
=&-e^f\left(\frac{\rho}{V}\right)^{\frac{C-1}{2}}\operatorname{div}\left(e^{-f}V^2\nabla\left(\frac{\rho}{V}\right)^{\frac{1-C}{2}}\right) \\
=&-\left(\frac{\rho}{V}\right)^{\frac{C-1}{2}}e^f\operatorname{div}\left(\left(\frac{1-C}{2}\right)e^{-f}V^2\left(\frac{\rho}{V}\right)^{\frac{-C-1}{2}}\nabla\left(\frac{\rho}{V}\right)\right) \\
=&-\left(\frac{1-C}{2}\right)\left(\frac{\rho}{V}\right)^{\frac{C-1}{2}}e^f\left\langle\nabla\left(\frac{\rho}{V}\right)^{\frac{-C-1}{2}}, e^{-f}V^2\nabla\left(\frac{\rho}{V}\right)\right\rangle \\
&-\left(\frac{1-C}{2}\right)\left(\frac{\rho}{V}\right)^{\frac{C-1}{2}}\left(\frac{\rho}{V}\right)^{\frac{-C-1}{2}}e^f\operatorname{div}\left(e^{-f}V^2\nabla\left(\frac{\rho}{V}\right)\right) \\
=&\frac{1-C^2}{4}V^2\left(\frac{\rho}{V}\right)^{-2}\left|\nabla\left(\frac{\rho}{V}\right)\right|^2+\frac{C-1}{2}\left(\frac{\rho}{V}\right)^{-1}L_{f,V}\rho.
\end{aligned}$$

Then, we have

$$
\begin{aligned}
&-\left(\frac{\varphi}{V}\right)^{-1}\left(L_{f,V}\varphi\right)\\
=&\frac{1-C^2}{4}V^2\left(\frac{\rho}{V}\right)^{-2}\left|\nabla\left(\frac{\rho}{V}\right)\right|^2+\frac{C-1}{2}\left(\frac{\rho}{V}\right)^{-1}L_{f,V}\rho\\
\geq&\frac{1-C^2}{4}V^2\left(\frac{\rho}{V}\right)^{-2}+\frac{C^2-C}{2}V^2\left(\frac{\rho}{V}\right)^{-2}\\
=&\frac{C^2-2C+1}{4}V^2\left(\frac{\rho}{V}\right)^{-2}\\
=&\frac{(C-1)^2}{4}V^2\left(\frac{\rho}{V}\right)^{-2},
\end{aligned}
$$

where we use the assumption $\left|\nabla\left(\frac{\rho}{V}\right)\right|=1$ and $L_{f,V}\rho\geq C(\frac{\rho}{V})^{-1}V^2$. As a result, we can get

$$
-\left(\frac{\varphi}{V}\right)^{-1}\left(L_{f,V}\varphi\right)\geq\frac{(C-1)^2}{4}V^2\left(\frac{\rho}{V}\right)^{-2}. \tag{2.4}
$$

Substituting (2.4) into (2.3), we can get

$$
\int_M V^3\left|\nabla\left(\frac{u}{V}\right)\right|^2 d\mu\geq\frac{(C-1)^2}{4}\int_M V^3\left(\frac{u}{V}\right)^2\left(\frac{\rho}{V}\right)^{-2}d\mu.
$$

This completes the proof of Corollary 2.4. □

Remark 2.5. In the special case that f is a constant and $V=1$, Corollary 2.4 reduces to Carron's result in [1, Proposition 2.1] .

3. L^p-Hardy type inequality

In this section, we will prove some L^p-Hardy type inequalities on smooth metric measure spaces. Our first result is the following.

Theorem 3.1. *Let $(M^n,g,d\mu)$ be an n-dimensional $(n\geq 2)$ complete non-compact smooth metric measure space. Suppose that V is a positive twice differentiable function and f is a real-valued smooth function on M. Let ρ be a nonnegative function on M such that $\left|\nabla\left(\frac{\rho}{V}\right)\right|=1$ and $L_{f,V}\rho\leq(H+C(\frac{\rho}{V})^{-1})V^2$ in the sense of distribution, where H is a continuous function on M and C is a constant. Then, for any $p,q\in\mathbb{R},q>1+C$, and any compactly supported smooth functions $\frac{\phi}{V}\in C_0^\infty\left(M\setminus\left(\frac{\rho}{V}\right)^{-1}\{0\}\right)$, the following inequality holds:*

$$
\begin{aligned}
(q-C-1)\int_M V^3\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{-q}d\mu\leq&|p|\int_M V^3\left|\frac{\phi}{V}\right|^{p-1}\left(\frac{\rho}{V}\right)^{1-q}\left|\nabla\left(\frac{\phi}{V}\right)\right|d\mu\\
&+\int_M HV^3\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}d\mu.
\end{aligned} \tag{3.1}
$$

Proof. For a vector field X on M, we denote by div X the divergence of X. Note that

$$\begin{aligned}
&\operatorname{div}\left(e^{-f}V^2\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right)\\
=&pe^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left|\frac{\phi}{V}\right|,\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right\rangle+\left|\frac{\phi}{V}\right|^p\operatorname{div}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right)\\
=&pe^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left|\frac{\phi}{V}\right|,\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right\rangle+\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}\operatorname{div}\left(e^{-f}V^2\nabla\left(\frac{\rho}{V}\right)\right)\\
&+(1-q)\,e^{-f}V^2\left|\frac{\phi}{V}\right|^p\left\langle\left(\frac{\rho}{V}\right)^{-q}\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\rho}{V}\right)\right\rangle\\
=&pe^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left|\frac{\phi}{V}\right|,\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right\rangle+e^{-f}\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}\left(L_{f,V}\rho\right)\\
&+(1-q)\,e^{-f}V^2\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{-q}\left|\nabla\left(\frac{\rho}{V}\right)\right|^2.
\end{aligned}$$

By direct computation, we have

$$\begin{aligned}
&\operatorname{div}\left(e^{-f}V^2\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right)\\
=&pe^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left|\frac{\phi}{V}\right|,\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right\rangle+e^{-f}\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}\left(L_{f,V}\rho\right)\\
&+(1-q)\,e^{-f}V^2\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{-q}\left|\nabla\left(\frac{\rho}{V}\right)\right|^2\\
\le&pe^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left|\frac{\phi}{V}\right|,\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right\rangle+e^{-f}V^2\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}\left(H+C\left(\frac{\rho}{V}\right)^{-1}\right)\\
&+(1-q)e^{-f}V^2\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{-q}\\
=&pe^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left|\frac{\phi}{V}\right|,\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right\rangle+(1+C-q)\,e^{-f}V^2\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{-q}\\
&+e^{-f}V^2H\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q},
\end{aligned}\tag{3.2}$$

where we use the assumption $\left|\nabla\left(\frac{\rho}{V}\right)\right|=1$ and $L_{f,V}\rho\le(H+C(\frac{\rho}{V})^{-1})V^2$.

Since $q>1+C$ and $\left|\nabla\left(\frac{\rho}{V}\right)\right|=1$, we have

$$
\begin{aligned}
&(q-C-1)\int_M e^{-f}V^2\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{-q}dv_g\\
\le& p\int_M e^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left|\frac{\phi}{V}\right|,\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right\rangle dv_g\\
&+\int_M e^{-f}V^2H\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}dv_g\\
\le&\left|p\int_M e^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left|\frac{\phi}{V}\right|,\left(\frac{\rho}{V}\right)^{1-q}\nabla\left(\frac{\rho}{V}\right)\right\rangle dv_g\right|\\
&+\int_M e^{-f}V^2H\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}dv_g\\
\le&|p|\int_M e^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left(\frac{\rho}{V}\right)^{1-q}\left|\nabla\left(\frac{\phi}{V}\right)\right|\left|\nabla\left(\frac{\rho}{V}\right)\right|dv_g\\
&+\int_M e^{-f}V^2H\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}dv_g\\
=&|p|\int_M e^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left(\frac{\rho}{V}\right)^{1-q}\left|\nabla\left(\frac{\phi}{V}\right)\right|dv_g\\
&+\int_M e^{-f}V^2H\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}dv_g.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
(q-C-1)\int_M V^3\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{-q}d\mu\le&|p|\int_M V^3\left|\frac{\phi}{V}\right|^{p-1}\left(\frac{\rho}{V}\right)^{1-q}\left|\nabla\left(\frac{\phi}{V}\right)\right|d\mu\\
&+\int_M HV^3\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{1-q}d\mu.
\end{aligned}
\tag{3.3}
$$

This completes the proof of Theorem 3.1. □

Remark 3.2. In the special case that $V = 1$ and f is a constant, the inequality (3.1) reduces to Xia's result in [4, Theorem 2.1] .

Theorem 3.3. *Let $(M^n, g, d\mu)$ be an n-dimensional $(n \geq 2)$ complete non-compact smooth metric measure space. Suppose that V is a positive twice differentiable function and f is a real-valued smooth function on M. Let ρ be a nonnegative function on M such that $\left|\nabla\left(\frac{\rho}{V}\right)\right| = 1$ in the sense of distributions. Then, for any $p, q \in \mathbb{R}, 1 < p < +\infty, 0 \leq q \leq p$, and any compactly supported smooth function $\frac{\phi}{V} \in C_0^\infty\left(M \setminus \left(\frac{\rho}{V}\right)^{-1}\{0\}\right)$, we have*

(i) *When $L_{f,V}\rho \leq C\left(\frac{\rho}{V}\right)^{-1}V^2$ in the sense of distributions, where $C < q - 1$ is a constant, the following inequality holds:*

$$
\int_M V^3\left|\frac{\phi}{V}\right|^p\left(\frac{\rho}{V}\right)^{-q}d\mu\le\left(\frac{p}{q-C-1}\right)^q\left(\int_M V^3\left|\frac{\phi}{V}\right|^p d\mu\right)^{\frac{p-q}{p}}\left(\int_M V^3\left|\nabla\left(\frac{\phi}{V}\right)\right|^p d\mu\right)^{\frac{q}{p}}. \tag{3.4}
$$

(ii) *When* $L_{f,V}\rho \geq C\left(\frac{\rho}{V}\right)^{-1} V^2$ *in the sense of distributions, where* $C > q-1$ *is a constant, the following inequality holds:*

$$\int_M V^3 \left|\frac{\phi}{V}\right|^p \left(\frac{\rho}{V}\right)^{-q} d\mu \geq \left(\frac{p}{C+1-q}\right)^q \left(\int_M V^3 \left|\frac{\phi}{V}\right|^p d\mu\right)^{\frac{p-q}{p}} \left(\int_M V^3 \left|\nabla\left(\frac{\phi}{V}\right)\right|^p d\mu\right)^{\frac{q}{p}}. \tag{3.5}$$

Proof. (i) In the special case that $H = 0$ in (3.1), we can get

$$(q-C-1)\int_M V^3 \left|\frac{\phi}{V}\right|^p \left(\frac{\rho}{V}\right)^{-q} d\mu \leq |p| \int_M V^3 \left|\frac{\phi}{V}\right|^{p-1} \left(\frac{\rho}{V}\right)^{1-q} \left|\nabla\left(\frac{\phi}{V}\right)\right| d\mu. \tag{3.6}$$

It follows from the Hölder inequality that

$$\begin{aligned}
&\int_M e^{-f} V^2 \left|\frac{\phi}{V}\right|^{p-1} \left(\frac{\rho}{V}\right)^{1-q} \left|\nabla\left(\frac{\phi}{V}\right)\right| dv_g \\
=& \int_M \left(e^{-f}V^2\right)^{\frac{1}{p}} \left(e^{-f}V^2\right)^{\frac{p-1}{p}} \left|\frac{\phi}{V}\right|^{p-1} \left(\frac{\rho}{V}\right)^{1-q} \left|\nabla\left(\frac{\phi}{V}\right)\right| dv_g \\
\leq& \left(\int_M e^{-f} V^2 \left|\nabla\left(\frac{\phi}{V}\right)\right|^p dv_g\right)^{\frac{1}{p}} \left(\int_M e^{-f} V^2 \left|\frac{\phi}{V}\right|^p \left(\frac{\rho}{V}\right)^{\frac{p(1-q)}{p-1}} dv_g\right)^{\frac{p-1}{p}} \\
\leq& \left(\int_M e^{-f} V^2 \left|\nabla\left(\frac{\phi}{V}\right)\right|^p dv_g\right)^{\frac{1}{p}} \left(\int_M e^{-f} V^2 \left|\frac{\phi}{V}\right|^p \left(\frac{\rho}{V}\right)^{-q} dv_g\right)^{\frac{q-1}{q}} \left(\int_M e^{-f} V^2 \left|\frac{\phi}{V}\right|^p dv_g\right)^{\left(1-\frac{p(q-1)}{q(p-1)}\right)\frac{p-1}{p}}.
\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we can get

$$\begin{aligned}
&(q-C-1)\int_M e^{-f} V^2 \left|\frac{\phi}{V}\right|^p \left(\frac{\rho}{V}\right)^{-q} dv_g \\
\leq& |p| \left(\int_M e^{-f} V^2 \left|\nabla\left(\frac{\phi}{V}\right)\right|^p dv_g\right)^{\frac{1}{p}} \left(\int_M e^{-f} V^2 \left|\frac{\phi}{V}\right|^p \left(\frac{\rho}{V}\right)^{-q} dv_g\right)^{\frac{q-1}{q}} \left(\int_M e^{-f} V^2 \left|\frac{\phi}{V}\right|^p dv_g\right)^{\left(1-\frac{p(q-1)}{q(p-1)}\right)\frac{p-1}{p}} \\
\leq& |p| \left(\int_M e^{-f} V^2 \left|\nabla\left(\frac{\phi}{V}\right)\right|^p dv_g\right)^{\frac{1}{p}} \left(\int_M e^{-f} V^2 \left|\frac{\phi}{V}\right|^p \left(\frac{\rho}{V}\right)^{-q} dv_g\right)^{\frac{q-1}{q}} \left(\int_M e^{-f} V^2 \left|\frac{\phi}{V}\right|^p dv_g\right)^{\frac{p-q}{pq}}.
\end{aligned}$$

Then,

$$\int_M V^3 \left|\frac{\phi}{V}\right|^p \left(\frac{\rho}{V}\right)^{-q} d\mu \leq \left(\frac{p}{q-C-1}\right)^q \left(\int_M V^3 \left|\frac{\phi}{V}\right|^p d\mu\right)^{\frac{p-q}{p}} \left(\int_M V^3 \left|\nabla\left(\frac{\phi}{V}\right)\right|^p d\mu\right)^{\frac{q}{p}}.$$

This completes the proof of (i).

(ii) Similar to the derivation of (3.4) above, the inequality (3.5) can be obtained without any difficulty. This completes the proof of Theorem 3.3.

□

Remark 3.4. In the special case that $V = 1$, Theorem 3.3 reduces to the result of Du and Mao in [24, Theorem 2.1].

Then, we prove the following Hardy-Poincaré type inequalities related to the divergence type operator $L_{f,V}$.

Theorem 3.5. *Let $(M^n, g, d\mu)$ be an n-dimensional $(n \geq 2)$ complete non-compact smooth metric measure space. Suppose that V is a positive twice differentiable function and f is a real-valued smooth function on M. Let ρ be a nonnegative function on M such that $\left|\nabla\left(\frac{\rho}{V}\right)\right| = 1$ and $L_{f,V}\rho \geq \left(C\left(\frac{\rho}{V}\right)^{-1} + G\right)V^2$ in the sense of distribution, where $C > 0$ is a constant and G is a continuous function. Then, for any $p, q, \alpha \in \mathbb{R}$, $\varphi \in C_0^\infty\left(M \setminus \left(\frac{\rho}{V}\right)^{-1}\{0\}\right)$, $p \in (1, \infty)$, $A_\alpha = \frac{(C+\alpha+1)}{p}$ with $C + \alpha + 1 > 0$, we have the following inequality:*

$$\int_M V^3\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right), \nabla\left(\frac{\varphi}{V}\right)\right\rangle\right|^p d\mu \geq A_\alpha^p \int_M V^3\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\varphi}{V}\right|^p d\mu + A_\alpha^{p-1}\int_M V^3 G\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^p d\mu. \quad (3.8)$$

Proof. It follows from $\left|\nabla\left(\frac{\rho}{V}\right)\right| = 1$ and $L_{f,V}\rho \geq \left(C\left(\frac{\rho}{V}\right)^{-1} + G\right)V^2$ that

$$\begin{aligned}
& e^f \operatorname{div}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)\nabla\left(\frac{\rho}{V}\right)\right) \\
=& \left\langle\nabla\left(\frac{\rho}{V}\right), V^2\nabla\left(\frac{\rho}{V}\right)\right\rangle + e^f\left(\frac{\rho}{V}\right)\operatorname{div}\left(e^{-f}V^2\nabla\left(\frac{\rho}{V}\right)\right) \\
=& V^2\left|\nabla\left(\frac{\rho}{V}\right)\right|^2 + \left(\frac{\rho}{V}\right)L_{f,V}\rho \\
\geq& V^2\left(1 + C + G\left(\frac{\rho}{V}\right)\right).
\end{aligned} \quad (3.9)$$

Multiplying (3.9) by $\left(\frac{\rho}{V}\right)^\alpha\left|\frac{\varphi}{V}\right|^p$ and integrating both sides over M gives

$$\begin{aligned}
& (1 + C)\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^\alpha\left|\frac{\varphi}{V}\right|^p dv_g + \int_M e^{-f}GV^2\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^p dv_g \\
\leq& \int_M\left(\frac{\rho}{V}\right)^\alpha\left|\frac{\varphi}{V}\right|^p \operatorname{div}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)\nabla\left(\frac{\rho}{V}\right)\right)dv_g \\
=& -\int_M\left\langle\left(\frac{\rho}{V}\right)\nabla\left(\frac{\rho}{V}\right), e^{-f}V^2\nabla\left(\left(\frac{\rho}{V}\right)^\alpha\left|\frac{\varphi}{V}\right|^p\right)\right\rangle dv_g \\
=& -\alpha\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-1}\left|\frac{\varphi}{V}\right|^p\left(\frac{\rho}{V}\right)\left\langle\nabla\left(\frac{\rho}{V}\right), \nabla\left(\frac{\rho}{V}\right)\right\rangle dv_g \\
& - p\int_M e^{-f}V^2\left|\frac{\varphi}{V}\right|^{p-2}\left(\frac{\varphi}{V}\right)\left(\frac{\rho}{V}\right)^\alpha\left(\frac{\rho}{V}\right)\left\langle\nabla\left(\frac{\varphi}{V}\right), \nabla\left(\frac{\rho}{V}\right)\right\rangle dv_g \\
=& -\alpha\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^\alpha\left|\frac{\varphi}{V}\right|^p dv_g - p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^{p-2}\left(\frac{\varphi}{V}\right)\left\langle\nabla\left(\frac{\rho}{V}\right), \nabla\left(\frac{\varphi}{V}\right)\right\rangle dv_g,
\end{aligned} \quad (3.10)$$

which implies

$$\begin{aligned}
& (1 + C + \alpha)\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^\alpha\left|\frac{\varphi}{V}\right|^p dv_g + \int_M e^{-f}V^2 G\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^p dv_g \\
\leq& -p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^{p-2}\left(\frac{\varphi}{V}\right)\left\langle\nabla\left(\frac{\rho}{V}\right), \nabla\left(\frac{\varphi}{V}\right)\right\rangle dv_g.
\end{aligned} \quad (3.11)$$

It follows from the Hölder inequality that

$$\begin{aligned}
&-p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^{p-2}\left(\frac{\varphi}{V}\right)\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle dv_g\\
\leq&\left|p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^{p-2}\left(\frac{\varphi}{V}\right)\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle dv_g\right|\\
\leq& p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^{p-1}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle\right| dv_g\\
=& p\int_M\left(e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\varphi}{V}\right|^{p}\right)^{\frac{p-1}{p}}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle\right|^{p}\right)^{\frac{1}{p}} dv_g\\
\leq& p\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\varphi}{V}\right|^{p}dv_g\right)^{\frac{p-1}{p}}\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle\right|^{p}dv_g\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&-p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^{p-2}\left(\frac{\varphi}{V}\right)\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle dv_g\\
\leq& p\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\varphi}{V}\right|^{p}dv_g\right)^{\frac{p-1}{p}}\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle\right|^{p}dv_g\right)^{\frac{1}{p}}.
\end{aligned}\tag{3.12}$$

Then, we use the Young inequality in (3.12), which is described as follows: Denoting

$$\Phi =: \left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle\right|^{p}dv_g\right)^{\frac{1}{p}} \text{ and } \Psi =: \left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\varphi}{V}\right|^{p}dv_g\right)^{\frac{p-1}{p}},$$

then for any $\epsilon > 0$,

$$\Phi\Psi = \epsilon\Phi\frac{\Psi}{\epsilon} \leq \frac{1}{p}(\epsilon\Phi)^p + \frac{1}{q}\left(\frac{\Psi}{\epsilon}\right)^q \text{ with } q = \frac{p}{p-1} \text{ relating to } p,$$

we can get

$$\begin{aligned}
&\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\varphi}{V}\right|^{p}dv_g\right)^{\frac{p-1}{p}}\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle\right|^{p}dv_g\right)^{\frac{1}{p}}\\
\leq&\frac{(p-1)}{p\epsilon^{\frac{p}{p-1}}}\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\varphi}{V}\right|^{p}dv_g+\frac{1}{p}\epsilon^p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle\right|^{p}dv_g,
\end{aligned}$$

namely,

$$\begin{aligned}
&-p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^{p-2}\left(\frac{\varphi}{V}\right)\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle dv_g\\
\leq&\frac{(p-1)}{\epsilon^{\frac{p}{p-1}}}\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\varphi}{V}\right|^{p}dv_g+\epsilon^p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle\right|^{p}dv_g.
\end{aligned}\tag{3.13}$$

Hence, putting (3.11) and (3.13) together, we can get

$$\begin{aligned}
&\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle\right|^{p}dv_g\\
\geq&\epsilon^{-p}\left(1+C+\alpha-\frac{p-1}{\epsilon^{\frac{p}{p-1}}}\right)\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\varphi}{V}\right|^{p}dv_g+\epsilon^{-p}\int_M e^{-f}V^2G\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^{p}dv_g.
\end{aligned}\tag{3.14}$$

We take $Y(\epsilon)$ to be the function $Y(\epsilon) = \epsilon^{-p}\left(1 + C + \alpha - \frac{p-1}{\epsilon^{\frac{p}{p-1}}}\right)$. By direct computation, we can conclude that $Y(\epsilon)$ reaches its maximum value when $\epsilon = \left(\frac{p}{1+C+\alpha}\right)^{\frac{p-1}{p}}$ (for details, see [25, Theorem 3]).

Finally, the required inequality can be determined by substituting $\epsilon = \left(\frac{p}{1+C+\alpha}\right)^{\frac{p-1}{p}}$ into (3.14) as follows:

$$\begin{aligned}\int_M V^3\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\varphi}{V}\right)\right\rangle\right|^p d\mu \geq & A_\alpha^p\int_M V^3\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\varphi}{V}\right|^p d\mu \\ & + A_\alpha^{p-1}\int_M V^3 G\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\frac{\varphi}{V}\right|^p d\mu.\end{aligned}$$

This completes the proof of Theorem 3.5. □

Using Theorem 3.5, we can obtain the following applications:

Theorem 3.6. *Let $(M^n, g, d\mu)$ be an n-dimensional complete noncompact smooth metric measure space. Suppose that V is a positive twice differentiable function and f is a real-valued smooth function on M. Let ρ be a nonnegative function on M such that $\left|\nabla\left(\frac{\rho}{V}\right)\right| = 1$ in the sense of distributions. Then for any $p, q \in \mathbb{R}, 1 < p < +\infty, 0 \leq q \leq p$, and any compactly supported smooth function $\frac{\phi}{V} \in C_0^\infty\left(M \setminus \left(\frac{\rho}{V}\right)^{-1}\{0\}\right)$, we have*

(i) *When $L_{f,V}\rho \leq C\left(\frac{\rho}{V}\right)^{-1}V^2$ in the sense of distributions, where $C > 0$ is a constant and $C + \alpha < -1$, the following inequality holds:*

$$\int_M V^3\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle\right|^p d\mu \geq \left(\frac{|C+\alpha+1|}{p}\right)^p\int_M V^3\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^p d\mu. \tag{3.15}$$

(ii) *When $L_{f,V}\rho \geq C\left(\frac{\rho}{V}\right)^{-1}V^2$ in the sense of distributions, where $C > 0$ is a constant and $C + \alpha > -1$, the following inequality holds:*

$$\int_M V^3\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle\right|^p d\mu \geq \left(\frac{C+\alpha+1}{p}\right)^p\int_M V^3\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^p d\mu. \tag{3.16}$$

Proof. (i) It follows from $\left|\nabla\left(\frac{\rho}{V}\right)\right| = 1$ and $L_{f,V}\rho \geq C\left(\frac{\rho}{V}\right)^{-1}V^2$ that

$$\begin{aligned}&\operatorname{div}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)\nabla\left(\frac{\rho}{V}\right)\right)\\ =&e^{-f}V^2\left|\nabla\left(\frac{\rho}{V}\right)\right|^2 + \left(\frac{\rho}{V}\right)\operatorname{div}\left(e^{-f}V^2\nabla\left(\frac{\rho}{V}\right)\right)\\ \geq&(1+C)\,e^{-f}V^2.\end{aligned}$$

Thus,

$$\operatorname{div}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)\nabla\left(\frac{\rho}{V}\right)\right) \geq (1+C)\,e^{-f}V^2. \tag{3.17}$$

Multiplying (3.17) by $\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}$and integrating over M yields

$$\begin{aligned}
&(1+C)\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}dv_g\\
\leq&\int_M\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}\operatorname{div}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)\nabla\left(\frac{\rho}{V}\right)\right)dv_g\\
=&-\int_M\left\langle\left(\frac{\rho}{V}\right)\nabla\left(\frac{\rho}{V}\right),e^{-f}V^2\nabla\left(\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}\right)\right\rangle dv_g\\
=&-\alpha\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}dv_g-p\int_M e^{-f}V^2\left|\frac{\phi}{V}\right|^{p-2}\left(\frac{\phi}{V}\right)\left(\frac{\rho}{V}\right)^{\alpha+1}\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g.
\end{aligned}\tag{3.18}$$

Since $C+\alpha+1<0$, by using the Hölder inequality, we can infer from (3.18) that

$$\begin{aligned}
&|C+\alpha+1|\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}dv_g\\
\leq&-p\int_M e^{-f}V^2\left|\frac{\phi}{V}\right|^{p-2}\left(\frac{\phi}{V}\right)\left(\frac{\rho}{V}\right)^{\alpha+1}\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g\\
\leq&\left|p\int_M e^{-f}V^2\left|\frac{\phi}{V}\right|^{p-2}\left(\frac{\phi}{V}\right)\left(\frac{\rho}{V}\right)^{\alpha+1}\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g\right|\\
\leq&p\int_M e^{-f}V^2\left|\frac{\phi}{V}\right|^{p-1}\left(\frac{\rho}{V}\right)^{\alpha+1}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle\right|dv_g\\
\leq&p\int_M\left(e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}\right)^{\frac{p-1}{p}}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle\right|^{p}\right)^{\frac{1}{p}}dv_g\\
\leq&p\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}dv_g\right)^{\frac{p-1}{p}}\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle\right|^{p}dv_g\right)^{\frac{1}{p}}.
\end{aligned}\tag{3.19}$$

It follows from the Young inequality that

$$\begin{aligned}
&|C+\alpha+1|\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}dv_g\\
\leq&p\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}dv_g\right)^{\frac{p-1}{p}}\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle\right|^{p}dv_g\right)^{\frac{1}{p}}\\
\leq&(p-1)\left[\epsilon^{-1}\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}dv_g\right)^{\frac{p-1}{p}}\right]^{\frac{p}{p-1}}+\left[\epsilon\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle\right|^{p}dv_g\right)^{\frac{1}{p}}\right]^{p}\\
=&(p-1)\,\epsilon^{\frac{-p}{p-1}}\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}dv_g+\epsilon^{p}\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle\right|^{p}dv_g.
\end{aligned}\tag{3.20}$$

Thus, for any $\epsilon>0$, from (3.20), we have

$$\begin{aligned}
&\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha+p}\left|\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle\right|^{p}dv_g\\
\geq&\epsilon^{-p}\left(|C+\alpha+1|-(p-1)\,\epsilon^{\frac{-p}{p-1}}\right)\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\left|\frac{\phi}{V}\right|^{p}dv_g.
\end{aligned}$$

Taking

$$\epsilon = \left(\frac{p}{|C+\alpha+1|}\right)^{\frac{p-1}{p}},$$

in the above inequality, we can get

$$\int_M V^3 \left(\frac{\rho}{V}\right)^{\alpha+p} \left|\left\langle \nabla\left(\frac{\rho}{V}\right), \nabla\left(\frac{\phi}{V}\right)\right\rangle\right|^p d\mu \geq \left(\frac{|C+\alpha+1|}{p}\right)^p \int_M V^3 \left(\frac{\rho}{V}\right)^{\alpha} \left|\frac{\phi}{V}\right|^p d\mu.$$

(ii) Similar to the proof of (3.15) above, the inequality (3.16) can be obtained without any difficulty. This completes the proof of Theorem 3.6. □

Remark 3.7. In the special case that $V = 1$, Theorem 3.6 reduces to the result of Du and Mao in [24, Theorem 4.1].

Then, we will prove the weighted L^p-Hardy type inequality .

Theorem 3.8. *Let $(M^n, g, d\mu)$ be an n-dimensional $(n \geq 2)$ complete non-compact smooth metric measure space. Suppose that V is a positive twice differentiable function and f is a real-valued smooth function on M. Let ρ be a nonnegative function on M such that $\left(\frac{\rho}{V}\right) L_{f,V}\rho \geq CV^2 \left|\nabla\left(\frac{\rho}{V}\right)\right|^2$ in the sense of distribution, where $C > 0$ is a constant. Then, the following inequality holds for any $\frac{\phi}{V} \in C_0^\infty\left(M \setminus \rho^{-1}\{0\}\right)$:*

$$\begin{aligned}
&\int_M V^3 \left(\frac{\rho}{V}\right)^{\alpha} \left(\sinh^\beta\left(\frac{\rho}{V}\right)\right) \left|\nabla\left(\frac{\rho}{V}\right)\right|^{2-p} \left|\nabla\left(\frac{\phi}{V}\right)\right|^p d\mu \\
\geq &\left(\frac{C+1+\alpha+\beta-p}{p}\right)^p \int_M V^3 \left(\frac{\rho}{V}\right)^{\alpha-p} \left(\sinh^\beta\left(\frac{\rho}{V}\right)\right) \left|\frac{\phi}{V}\right|^p \left|\nabla\left(\frac{\rho}{V}\right)\right|^2 d\mu,
\end{aligned} \tag{3.21}$$

where $p, \alpha, \beta \in \mathbb{R}$, $1 \leq p < \infty$, and $C + 1 + \alpha + \beta - p > 0$.

Proof. By direct computation, we have

$$\begin{aligned}
&\int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-p} \sinh^\beta\left(\frac{\rho}{V}\right) \left|\frac{\phi}{V}\right|^p \left|\nabla\left(\frac{\rho}{V}\right)\right|^2 dv_g \\
\leq &\frac{1}{C} \int_M e^{-f} \left(\frac{\rho}{V}\right)^{\alpha-p+1} \sinh^\beta\left(\frac{\rho}{V}\right) \left|\frac{\phi}{V}\right|^p \left(L_{f,V}\rho\right) dv_g \\
= &\frac{1}{C} \int_M \left(\frac{\rho}{V}\right)^{\alpha-p+1} \left|\frac{\phi}{V}\right|^p \sinh^\beta\left(\frac{\rho}{V}\right) \operatorname{div}\left(e^{-f} V^2 \nabla\left(\frac{\rho}{V}\right)\right) dv_g \\
= &-\frac{1}{C} \int_M \left\langle \nabla\left(\sinh^\beta\left(\frac{\rho}{V}\right) \left|\frac{\phi}{V}\right|^p \left(\frac{\rho}{V}\right)^{\alpha-p+1}\right), e^{-f} V^2 \nabla\left(\frac{\rho}{V}\right)\right\rangle dv_g \\
= &-\int_M \frac{\left(\alpha - p + 1 + \beta\left(\frac{\rho}{V}\right)\coth\left(\frac{\rho}{V}\right)\right)}{C} e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-p} \sinh^\beta\left(\frac{\rho}{V}\right) \left|\frac{\phi}{V}\right|^p \left|\nabla\left(\frac{\rho}{V}\right)\right|^2 dv_g \\
&-\frac{p}{C} \int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-p+1} \sinh^\beta\left(\frac{\rho}{V}\right) \left|\frac{\phi}{V}\right|^{p-2} \left(\frac{\phi}{V}\right) \left\langle \nabla\left(\frac{\rho}{V}\right), \nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g,
\end{aligned} \tag{3.22}$$

where we have used the assumption $\left(\frac{\rho}{V}\right) L_{f,V}\rho \geq CV^2 \left|\nabla\left(\frac{\rho}{V}\right)\right|^2$ in the first line of the inequality.

Observing that $\left(\frac{\rho}{V}\right)\coth\left(\frac{\rho}{V}\right) \geq 1$, we can rewrite the inequality (3.22) as

$$\begin{aligned}&\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-p}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^p\left|\nabla\left(\frac{\rho}{V}\right)\right|^2 dv_g\\ \leq&-\frac{(\alpha-p+1+\beta)}{C}\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-p}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^p\left|\nabla\left(\frac{\rho}{V}\right)\right|^2 dv_g\\ &-\frac{p}{C}\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-p+1}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^{p-2}\left(\frac{\phi}{V}\right)\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g.\end{aligned}$$

Then,

$$\begin{aligned}&(C+\alpha-p+1+\beta)\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-p}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^p\left|\nabla\left(\frac{\rho}{V}\right)\right|^2 dv_g\\ \leq& p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-p+1}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g.\end{aligned}\tag{3.23}$$

It follows from the Hölder inequality that

$$\begin{aligned}&(C+\alpha-p+1+\beta)\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-p}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^p\left|\nabla\left(\frac{\rho}{V}\right)\right|^2 dv_g\\ \leq& p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-p+1}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g\\ \leq&\left|p\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-p+1}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^{p-1}\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g\right|\\ \leq& p\int_M\left(e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-p}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^p\left|\nabla\left(\frac{\rho}{V}\right)\right|^2\right)^{\frac{p-1}{p}}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\nabla\left(\frac{\rho}{V}\right)\right|^{2-p}\left|\nabla\left(\frac{\phi}{V}\right)\right|^p\right)^{\frac{1}{p}} dv_g\\ \leq& p\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-p}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^p\left|\nabla\left(\frac{\rho}{V}\right)\right|^2 dv_g\right)^{\frac{p-1}{p}}\left(\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\nabla\left(\frac{\rho}{V}\right)\right|^{2-p}\left|\nabla\left(\frac{\phi}{V}\right)\right|^p dv_g\right)^{\frac{1}{p}}.\end{aligned}\tag{3.24}$$

namely,

$$\begin{aligned}&\int_M V^3\left(\frac{\rho}{V}\right)^{\alpha}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\nabla\left(\frac{\rho}{V}\right)\right|^{2-p}\left|\nabla\left(\frac{\phi}{V}\right)\right|^p d\mu\\ \geq&\left(\frac{C+1+\alpha+\beta-p}{p}\right)^p\int_M V^3\left(\frac{\rho}{V}\right)^{\alpha-p}\sinh^\beta\left(\frac{\rho}{V}\right)\left|\frac{\phi}{V}\right|^p\left|\nabla\left(\frac{\rho}{V}\right)\right|^2 d\mu.\end{aligned}$$

This completes the proof of Theorem 3.8. □

4. Rellich type inequality

In this section, by applying Theorem 3.8 of Section 3, we can give the following Rellich type inequality.

Theorem 4.1. *Let $(M^n, g, d\mu)$ be an n-dimensional $(n \geq 2)$ complete non-compact smooth metric measure space. Suppose that V is a positive twice differentiable function and f is a real-valued smooth function on M. Let ρ be a nonnegative function on M such that $\left|\nabla\left(\frac{\rho}{V}\right)\right| = 1$ and $\left(\frac{\rho}{V}\right)L_{f,V}\rho \geq CV^2$ in* *the sense of distributions, where* $C > 0$ *is a constant and* $\max\{3-C, (7-C)/3\} \le \alpha \le 2$. *Then, the following inequality holds for* $\frac{\phi}{V} \in C_0^\infty\left(M \setminus \left(\frac{\rho}{V}\right)^{-1}\{0\}\right)$:

$$\int_M V^3 \left(\frac{\rho}{V}\right)^{\alpha} \left|L_{f,V}\phi\right|^2 d\mu \ge \left(\frac{C+1-\alpha}{2}\right)^2 \int_M V^3 \left(\frac{\rho}{V}\right)^{\alpha-2} \left|\nabla\left(\frac{\phi}{V}\right)\right|^2 d\mu, \tag{4.1}$$

where $d\mu = \frac{1}{V}e^{-f}dv_g$ *and* dv_g *is the Riemannian volume element related to* g.

Proof. As a consequence of integration by parts (1.3), we compute and estimate the righthand side:

$$\begin{aligned}
&\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-2}\left|\nabla\left(\frac{\phi}{V}\right)\right|^2 dv_g\\
=&\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-2}\left\langle\nabla\left(\frac{\phi}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g\\
=&\int_M \left\langle\nabla\left(\frac{\phi}{V}\right), e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-2}\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g\\
=&-\int_M \left(\frac{\phi}{V}\right)\operatorname{div}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-2}\nabla\left(\frac{\phi}{V}\right)\right) dv_g\\
=&-\int_M \left(\frac{\phi}{V}\right)\left(e^{-f}V^2\left\langle\nabla\left(\frac{\rho}{V}\right)^{\alpha-2},\nabla\left(\frac{\phi}{V}\right)\right\rangle+\left(\frac{\rho}{V}\right)^{\alpha-2}\operatorname{div}\left(e^{-f}V^2\nabla\left(\frac{\phi}{V}\right)\right)\right) dv_g\\
=&-\int_M e^{-f}V^2\left(\frac{\phi}{V}\right)\left\langle\nabla\left(\frac{\rho}{V}\right)^{\alpha-2},\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g-\int_M e^{-f}\left(\frac{\rho}{V}\right)^{\alpha-2}\left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g\\
=&-(\alpha-2)\int_M e^{-f}V^2\left(\frac{\phi}{V}\right)\left(\frac{\rho}{V}\right)^{\alpha-3}\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)\right\rangle dv_g-\int_M e^{-f}\left(\frac{\rho}{V}\right)^{\alpha-2}\left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g\\
=&-\frac{(\alpha-2)}{2}\int_M e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-3}\left\langle\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)^2\right\rangle dv_g-\int_M e^{-f}\left(\frac{\rho}{V}\right)^{\alpha-2}\left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g\\
=&-\frac{(\alpha-2)}{2}\int_M \left(\frac{\rho}{V}\right)^{\alpha-3}\left\langle e^{-f}V^2\nabla\left(\frac{\rho}{V}\right),\nabla\left(\frac{\phi}{V}\right)^2\right\rangle dv_g-\int_M e^{-f}\left(\frac{\rho}{V}\right)^{\alpha-2}\left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g\\
=&\frac{(\alpha-2)}{2}\int_M \left(\frac{\phi}{V}\right)^2\operatorname{div}\left(e^{-f}V^2\left(\frac{\rho}{V}\right)^{\alpha-3}\nabla\left(\frac{\rho}{V}\right)\right) dv_g-\int_M e^{-f}\left(\frac{\rho}{V}\right)^{\alpha-2}\left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g.
\end{aligned} \tag{4.2}$$

It follows from $\left|\nabla\left(\frac{\rho}{V}\right)\right| = 1$ and $\left(\frac{\rho}{V}\right)L_{f,V}\rho \ge CV^2$ that

$$
\begin{aligned}
&\int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-2} \left|\nabla\left(\frac{\phi}{V}\right)\right|^2 dv_g \\
=&\frac{(\alpha-2)}{2} \int_M \left(\frac{\phi}{V}\right)^2 \operatorname{div}\left(e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-3} \nabla\left(\frac{\rho}{V}\right)\right) dv_g - \int_M e^{-f} \left(\frac{\rho}{V}\right)^{\alpha-2} \left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g \\
=&-\frac{(2-\alpha)}{2} \int_M \left((\alpha-3)\, e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-4} \left(\frac{\phi}{V}\right)^2 \left|\nabla\left(\frac{\rho}{V}\right)\right|^2 + \left(\frac{\rho}{V}\right)^{\alpha-3} \left(\frac{\phi}{V}\right)^2 \operatorname{div}\left(e^{-f} V^2 \nabla\left(\frac{\rho}{V}\right)\right)\right) dv_g \\
&- \int_M e^{-f} \left(\frac{\rho}{V}\right)^{\alpha-2} \left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g \\
=&-\frac{(2-\alpha)}{2} \int_M \left((\alpha-3)\, e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-4} \left(\frac{\phi}{V}\right)^2 + e^{-f} \left(\frac{\rho}{V}\right)^{\alpha-3} \left(\frac{\phi}{V}\right)^2 \left(L_{f,V}\rho\right)\right) dv_g \\
&- \int_M e^{-f} \left(\frac{\rho}{V}\right)^{\alpha-2} \left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g \\
\leq&-\frac{(2-\alpha)(C+\alpha-3)}{2} \int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-4} \left(\frac{\phi}{V}\right)^2 dv_g - \int_M e^{-f} \left(\frac{\rho}{V}\right)^{\alpha-2} \left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-2} \left|\nabla\left(\frac{\phi}{V}\right)\right|^2 dv_g \\
\leq&-\frac{(2-\alpha)(C+\alpha-3)}{2} \int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-4} \left(\frac{\phi}{V}\right)^2 dv_g - \int_M e^{-f} \left(\frac{\rho}{V}\right)^{\alpha-2} \left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g.
\end{aligned} \tag{4.3}
$$

Applying the Hölder inequality to the above, we have

$$
\begin{aligned}
&- \int_M e^{-f} \left(\frac{\rho}{V}\right)^{\alpha-2} \left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g \\
\leq&\left|\int_M e^{-f} \left(\frac{\rho}{V}\right)^{\alpha-2} \left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g\right| \\
\leq&\left(\int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha} \left|L_{f,V}\phi\right|^2 dv_g\right)^{\frac{1}{2}} \left(\int_M e^{-f} V^{-2} \left(\frac{\phi}{V}\right)^2 \left(\frac{\rho}{V}\right)^{\alpha-4} dv_g\right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&- \int_M e^{-f} \left(\frac{\rho}{V}\right)^{\alpha-2} \left(\frac{\phi}{V}\right)\left(L_{f,V}\phi\right) dv_g \\
\leq&\left(\int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha} \left|L_{f,V}\phi\right|^2 dv_g\right)^{\frac{1}{2}} \left(\int_M e^{-f} V^{-2} \left(\frac{\phi}{V}\right)^2 \left(\frac{\rho}{V}\right)^{\alpha-4} dv_g\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.4}
$$

Then, substituting (4.4) into (4.3),

$$
\begin{aligned}
&\left(\int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha} \left|L_{f,V}\phi\right|^2 dv_g\right)^{\frac{1}{2}} \left(\int_M e^{-f} V^{-2} \left(\frac{\phi}{V}\right)^2 \left(\frac{\rho}{V}\right)^{\alpha-4} dv_g\right)^{\frac{1}{2}} \\
\geq&\int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-2} \left|\nabla\left(\frac{\phi}{V}\right)\right|^2 dv_g + \frac{(2-\alpha)(C+\alpha-3)}{2} \int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-4} \left(\frac{\phi}{V}\right)^2 dv_g.
\end{aligned}
$$

We denote

$$A = \int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-2} \left|\nabla\left(\frac{\phi}{V}\right)\right|^2 dv_g,$$
$$B = \int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-4} \left(\frac{\phi}{V}\right)^2 dv_g,$$
$$D = \int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha} \left|L_{f,V}\phi\right|^2 dv_g,$$

and then can get

$$\sqrt{D} \geq \sqrt{A}\left(\sqrt{\frac{A}{B}} + \frac{(2-\alpha)(C+\alpha-3)}{2}\sqrt{\frac{B}{A}}\right).$$

Denoting further $S = \frac{A}{B}$,

$$\sqrt{D} \geq \sqrt{A}\left(\sqrt{S} + \frac{(2-\alpha)(C+\alpha-3)}{2}\sqrt{\frac{1}{S}}\right) := \sqrt{A} f(S).$$

Now, we compute a lower bound for the function $f(S) = \sqrt{S} + ((2-\alpha)(C+\alpha-3)/2)\sqrt{1/S}$. To begin, we use Theorem 3.8 $\left(\text{in the special case that } \beta = 0, p = 2, \left|\nabla\left(\frac{\rho}{V}\right)\right| = 1\right)$ and then can get

$$\begin{aligned} A &= \int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-2} \left|\nabla\left(\frac{\phi}{V}\right)\right|^2 dv_g \\ &\geq \left(\frac{C+\alpha-3}{2}\right)^2 \int_M e^{-f} V^2 \left(\frac{\rho}{V}\right)^{\alpha-4} \left(\frac{\phi}{V}\right)^2 dv_g \\ &= \left(\frac{C+\alpha-3}{2}\right)^2 B. \end{aligned}$$

We have the following condition on S:

$$S = \frac{A}{B} \geq \left(\frac{C+\alpha-3}{2}\right)^2.$$

Next, the function $f(S)$ is increasing for $S \in (0, +\infty)$ and it attains its minimum at $S = (2-\alpha)(C+\alpha-3)/2$. However, $(2-\alpha)(C+\alpha-3)/2 \leq ((C+\alpha-3)/2)^2$ when $\alpha \geq (7-C)/3$, so $f(S)$ attains its minimum at $S = ((C+\alpha-3)/2)^2$, and this minimum is equal to $((C+1-\alpha)/2)^2$. Finally, we obtain the following inequality:

$$\begin{aligned} D &= \int_M V^3 \left(\frac{\rho}{V}\right)^{\alpha} \left|L_{f,V}\phi\right|^2 d\mu \\ &\geq \left(\frac{C+1-\alpha}{2}\right)^2 A \\ &= \left(\frac{C+1-\alpha}{2}\right)^2 \int_M V^3 \left(\frac{\rho}{V}\right)^{\alpha-2} \left|\nabla\left(\frac{\phi}{V}\right)\right|^2 d\mu. \end{aligned}$$

This completes the proof of Theorem 4.1. □

5. Conclusions

In this paper, we have established some Hardy type identities and inequalities for the divergence type operator $L_{f,V}$ on smooth metric measure spaces. First, we have established some L^2 Hardy type identities. As their corollary, we have obtained a L^2 Hardy type inequality. Second, we have established some L^p Hardy type inequalities. As their corollary, we have obtained a L^p Rellich type inequality. From the proof of the above results, we see that our method does not work for us to obtain the sharp constants. Hence, we shall further pursue sharp Hardy type inequalities for the divergence type operator $L_{f,V}$ on smooth metric measure spaces in the subsequent papers.

Author contributions

Pengyan Wang: Methodology, writing-original draft; Jiahao Wang: validation, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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