AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 16313–16334.
DOI:10.3934/math.2024790
Received: 15 February 2024
Revised: 12 April 2024
Accepted: 16 April 2024
Published: 09 May 2024

*Research article*

# Solving delay integro-differential inclusions with applications

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**Abstract:** This work primarily delves into three key areas: the presence of mild solutions, exploration of the topological and geometrical makeup of solution sets, and the continuous dependency of solutions on a second-order semilinear integro-differential inclusion. The Bohnenblust-Karlin fixed-point method has been integrated with Grimmer's theory of resolvent operators. Ultimately, the study delves into a mild solution for a partial integro-differential inclusion to showcase the achieved outcomes.

**Keywords:** fixed point technique; semilinear integro-differential inclusion; mild solution; partial integro-differential inclusion
**Mathematics Subject Classification:** 26A33, 26D15, 47H08, 47H10

## 1. Introduction

Due to its numerous applications across several fields, the study of semilinear integro-differential (SID) equations has been a popular area of research in recent years. The writers of [1] have investigated whether, under compact conditions, there are mild solutions to the second-order SID problem. The existence, complete controllability, and approximate controllability of mild solutions incorporating measures of noncompactness have also been discussed by the authors of [2] for the same problem.

For a second-order equation that has been defined as in [3], the writers of both publications employed Grimmer's ROs. For further information on resolvent operators (ROs) and integro-differential systems, we refer interested readers to [1,4,5] and the specified references therein.

The uniqueness property generally fails to hold for ordinary Cauchy problems. The set of solutions is a continuum, which means that it is both closed and continuous, as Kneser [6] proved in 1923. Aronszajn [7] in 1942, proved that the solution set is a continuum, compact, and acyclic in the setting of differential inclusions (DIs). Moreover, he designated an $R$-set as this continuum. For DIs containing upper semi-continuous (USC) convex-valued nonlinearities, a similar conclusion has been obtained and verified by several writers; see [8–10]. As discussed in [1], the topological and geometric qualities of solution sets for impulsive DIs on compact intervals include traits like contractibility, acyclicity being an absolute retract, and being $R_\kappa$-sets.

Numerous physical phenomena, depicted through evolution equations, exhibit a dependency on past occurrences to some degree. Consequently, the theory of fractional integro-differential functional equations has notably progressed in recent years. Specifically, functional differential equations with state-dependent delays emerge frequently in various applications as mathematical models. Certain characteristics of these equations distinctly differ from those with constant or time-dependent delays, prompting extensive research in this area over recent years. For more details, see [11–14].

The qualitative features of the solutions can be significantly influenced by certain impulsive perturbations. Moreover, a great deal of real-world problems frequently involve such sudden behavioral changes. For this reason, a great deal of research has been done on the basic and qualitative theories, as well as the applications of discontinuous impulsive systems and inclusions. The following references provide some great results on impulsive systems, impulsive inclusions, and impulsive control strategies: [15–22].

The fixed-point (FP) technique serves as a powerful tool to address nonlinear engineering problems. By leveraging this method, engineers can effectively tackle complex systems where traditional linear approaches fall short. This technique enables the iterative determination of solutions by identifying points that remain unchanged under a given transformation, proving invaluable for tasks such as optimizing designs, analyzing structural stability, and solving various nonlinear equations that are prevalent in engineering applications. See [23–30] for more different applications of this technique.

In many applications, mathematical models of functional differential equations with state-dependent delays are used. When compared to functional equations with a constant or time-dependent delay, these equations exhibit some characteristics that are significantly different. This is what has spurred the recent intensive study of these equations. Readers with an interest in this topic can view [11,31–40].

In continuation of the previous contributions, this manuscript is devoted to presenting the existence of mild solutions, the structure of solution sets, the continuous dependency on initial constraints, and the selection set for second-order SID equations with infinite delay, expressed in the following manner:

$$\begin{cases} \Im''(\rho) \in \Upsilon(\rho)\Im(\rho) + \mho\left(\rho, \Im_\rho, (\Theta\Im)(\rho)\right) + \beta\int\limits_0^\rho \vartheta(\rho, r)\,\Im(r)dr,\ \beta > 0,\ \text{if}\, \rho \in \Lambda, \\ \Im'(0) = \beta\kappa_0 \in B,\ \Im(\rho) = \beta Z(\rho),\ \text{if}\, \rho \in \mathbb{R}_-, \end{cases} \tag{1.1}$$

where $\Lambda \in [0, \theta]$, $\Upsilon(\rho) : D(\Upsilon(\rho)) \subset B \to B$, $\vartheta(\rho, r)$ denotes linear operators on $B$ with the dense domain $D(\Upsilon(\rho))$, which is independent of $\rho$ and subject to $D(\Upsilon(\rho)) \subset D(\vartheta(\rho, r))$, and $\Theta$ is an operator described as follows:

$$(\Theta\Im)(\rho) = \beta \int_0^{\theta} \Omega(\rho, r, \Im(r)dr,$$

$\mho : \Lambda \times \psi \times B \to P(B)$ is a nonlinear multi-valued function, $Z : \mathbb{R}_- \to B$ is a given function and $(B, \|.\|)$ is a Banach space (BS).

The rest of this paper is organized as follows: In Section 2, we emphasize some definitions and introduce notations relevant to ROs, abstract phase spaces (APSs), and certain features of multi-valued operators (MVOs). These ideas are critical for laying the groundwork for our results. Section 3 investigates the existence of mild solutions to the model (1.1). Sections 4 and 5 build on this foundation by demonstrating the compactness of the solution sets and establishing their classification as $R_\kappa$-sets. In Section 6, we investigate the continuous dependence on initial constraints as well as the selection set of $\mho$. Ultimately, we give an example that serves as a practical application, highlighting the practical significance of our findings in Section 7. Ultimately, in Sections 8 and 9, a conclusion and some abbreviations have been presented.

## 2. Preliminaries

This section introduces a number of notations, definitions, FP theorems, and preparatory facts that will be used throughout the rest of the work.

Assume that $C(\Lambda, B)$ is the BS of a continuous function from $\Lambda$ into $B$.

Consider the system below, which was originally described by Henríquez and Pozo [3]:

$$\begin{cases} \wp''(\rho) \in \Upsilon(\rho)\wp(\rho) + \int\limits_r^{\rho} \vartheta(\rho, l)\, \wp(l) dl, \ r \le \rho \le \theta, \\ \Im(r) = 0 \in B, \ \Im'(\rho) = z \in B, \end{cases} \tag{2.1}$$

for $0 \le r \le \theta$. We set $\phi = \{(\rho, r) : 0 \le r \le \theta\}$. Now, we present several criteria that satisfy the operator $\vartheta$.

(C$_1$) The operator $\vartheta(\rho, r) : D(\Upsilon) \to B$ is linear and bounded for $0 \le r \le \rho \le \theta$ and $\vartheta(.,.)\,\wp$ is continuous for every $\wp \in D(\Upsilon)$. Moreover,

$$\|\vartheta(\rho, r)\,\wp\| \le \lambda \|\wp\|_{D(\Upsilon)},$$

where $\lambda > 0$ is a constant independent of $\rho, r \in \phi$.

(C$_2$) For all $\wp \in D(\Upsilon)$ and all $0 \le r \le \rho_1 \le \rho_2 \le \theta$, there is a constant $M_\vartheta > 0$ such that

$$\|\vartheta(\rho_2, r)\,\wp - \vartheta(\rho_1, r)\,\wp\| \le M_\vartheta |\rho_2 - \rho_1| \|\vartheta\|_{D(\Upsilon)}.$$

(C$_3$) There exists a positive constant $\lambda_1$ such that

$$\left\| \int\limits_{\mu}^{\rho} Q(\rho, r)\,\vartheta(r, \mu)\,\wp dr \right\| \le \lambda_1 \|\vartheta\|, \text{ for all } \wp \in D(\Upsilon).$$

It has been demonstrated that there is an RO $S(\rho, r)_{\rho \ge r}$, associated with model (2.1) under the above criteria.

**Definition 2.1.** [3] A class of a type of bounded linear operator $S(\rho, r)_{\rho\geq r}$ is called an RO for the model (2.1) if it satisfies the following conditions:

($R_1$) The mapping $S : \phi \to \Phi(B)$ is strongly continuous and $\vartheta(\rho, .)\wp$ is continuously differentiable for every $\wp \in B$. Moreover, $S(r,r) = 0$, $\frac{\partial}{\partial\rho}S(\rho, r)\mid_{\rho=r}= I$ and $\frac{\partial}{\partial r}S(\rho, r)\mid_{\rho=r}= -I$ (where $I$ is the identity mapping).

($R_2$) Consider $z \in D(\Upsilon)$. The function $S(., r)z$ is a solution to problem (1.1). This means that

$$\frac{\partial^2}{\partial\rho^2}S(\rho, r)z = \Upsilon(\rho)S(\rho, r)z + \int_r^\rho \vartheta(\rho, l)\, S(l, r)zdl,$$

for all $0 \leq r \leq \rho \leq \theta$.

From condition ($R_1$), there exist positive constants $V_S$ and $\widetilde{V}_S$ such that

$$\|S(\rho, r)\| \leq V_S \text{ and } \left\|\frac{\partial}{\partial r}S(\rho, r)\right\| \leq \widetilde{V}_S,\ (\rho, r) \in \phi.$$

Further, the linear operator

$$\eth(\rho, l)z = \int_l^\rho \vartheta(\rho, r)\, S(r, l)zdr,\ z \in D(\Upsilon),\ 0 \leq r \leq \rho \leq \theta,$$

can be generalized to $B$. This expansion is highly continuous when represented by the comparable notation $\eth(\rho, l)$, noting that $\eth : \phi \to L(B)$ is strongly continuous, and it is confirmed that

$$S(\rho, r)z = S(\rho, r) + \int_l^\rho S\,(\rho, r)\,\eth(r, l)zdr,\ \text{for all } z \in B.$$

Hence, we can say that $S(.)$ is uniformly Lipschitz continuous, i.e., there is a constant $A_S > 0$ such that

$$\|S(\rho + \hbar, l) - S(\rho, r)\| \leq A_S\, |\hbar|\,,\ \text{for all } \rho, \rho + \hbar,\ l \in [0, \theta].$$

In the context of a seminormed linear space $\left(\psi, \|.\|_\psi\right)$ of functions $(-\infty, 0]$ into $\mathbb{R}$, Hale and Kato [41] introduced the following hypotheses:

($X_1$) If $\Im \in C$ and $\Im_0 \in \psi$, then for every $\rho \in \Lambda$ the assumptions below are true:

(i) $\Im_\rho \in \psi$,

(ii) there is a constant $J > 0$ such that $\left|\Im(\rho)\right| \leq J\left\|\Im_\rho\right\|_\psi$,

(iii) there are two functions $K(.)$ and $N(.) : \mathbb{R}_+ \to \mathbb{R}_+$ that are independent of $\Im$ such that $K$ is continuous and bounded, $N$ is locally bounded and

$$\left\|\Im_\rho\right\|_\psi \leq K(\rho)\sup\left\{\left|\Im(r)\right| : 0 \leq r \leq \rho\right\} + N(\rho)\left\|\Im_0\right\|_\psi.$$

($X_2$) $\Im_\rho$ is a $\psi$-valued continuous function on $\mathbb{R}_+$ for the function $\Im \in C$.

($X_3$) The space $\psi$ is complete.

Set $K_* = \sup\{K(\rho) : \rho \in \Lambda\}$, $N_* = \sup\{N(\rho) : \rho \in \Lambda\}$. Define the space

$$C_\varpi = \left\{ W \in C(\mathbb{R}_-, B) : \lim_{p\to-\infty} W(p) \text{ exists in } B \right\},$$

equipped with the norm

$$\|W\|_\varpi = \sup\{|W(p)| : p \le 0\}.$$

Clearly, the hypotheses ($H_1$)–($H_3$) are satisfied in the space $C_\varpi$. So, in what follows, we consider the phase space $\psi = C_\varpi$ and we assume that

$$\mathfrak{R} = C\left(\widetilde{\Lambda}, B\right) = \left\{ \mathfrak{I} : \widetilde{\Lambda} \to B : \mathfrak{I}\big|_{\mathbb{R}_-} \in \psi,\ \mathfrak{I}\,|_\Lambda \in C(\Lambda, B) \right\},$$

such that

$$\|\mathfrak{I}\|_{\mathfrak{R}} = \sup\left\{ |\mathfrak{I}(\rho)| : \rho \in \widetilde{\Lambda} \right\}.$$

Now, we shall discuss some geometric topology concepts. For more information, see [42, 43].

**Definition 2.2.** Let $\beth$ be a BS. A subset $\Upsilon \subset \beth$ is called a retract, if there is a continuous mapping $b : \beth \to \Upsilon$ such that $b(z) = z$ for each $z \in \Upsilon$.

**Definition 2.3.** A subset $\Upsilon \subset \beth$ is called a contractible space if there exist a continuous homotopy $\zeta : \Upsilon \times [0, 1] \to \Upsilon$ and $z_0 \in \Upsilon$ such that

(*) $\zeta(z, 0) = z$, for every $z \in \Upsilon$,
(**) $\zeta(z, 1) = z_0$, for every $z \in \Upsilon$.

It should be noted that any closed convex subset of $\beth$ is contractible.

**Definition 2.4.** Let $\beth \neq \emptyset$. A compact metric space (MS) $\beth$ is called an $R_\kappa$-set if there exists a decreasing sequence of compact contractible MSs $\{\beth_u\}_{u\in\mathbb{N}}$ such that $\beth = \cap_{u=1}^{\infty} \beth_u$.

Clearly, for a compact set, convex sets $\subset \Upsilon R$ $\subset$contractible $\subset R_\kappa$.

**Definition 2.5.** We say that the space $\Upsilon$ is closed cyclic if the following axioms are true:

($\heartsuit_1$) $U_0(\Upsilon) = \mathbb{Q}$, where $\mathbb{Q}$ is a rational number;
($\heartsuit_2$) for every $u > 0$, $U_u(\Upsilon) = 0$, where $U_* = \{U_u\}_{u\ge0}$ is the Čech-homology function with compact carriers and coefficients in $\mathbb{Q}$.

**Lemma 2.6.** *[42] Assume that $\beth$ is a compact MS; then, $\beth$ is a cyclic space, provided that $\beth$ is an $R_\kappa$-set.*

**Theorem 2.7.** *[42] Let $B$ be a normed space, $\beth$ be an MS, and $g_\epsilon : \beth \to B$ be a continuous mapping. Then for every $\epsilon > 0$, there is a locally Lipschitz function $g_\epsilon$ such that*

$$\|g(z) - g_\epsilon(z)\| \le \epsilon, \textit{ for all } z \in \beth.$$

**Definition 2.8.** Let $\beth$ and $\widetilde{\beth}$ be two MSs. A mapping $g : \beth \to \widetilde{\beth}$ is proper if it is continuous and the inverse image of a compact set is compact.

**Lemma 2.9.** *[44] Assume that $(B, \|.\|)$ is a BS and $g : \beth \to B$ is a proper mapping. If for every $\epsilon > 0$, there is a proper mapping $g_\epsilon : \beth \to B$ such that*

1) $\|g(z) - g_\epsilon(z)\| \leq \epsilon$, for all $z \in \beth$;
2) the equation $g_\epsilon(z) = v$ has a unique solution, provided that $\|v\| \leq \epsilon$ for all $v \in B$.

Then, the set $O = g^{-1}(0)$ is $R_\kappa$.

In the rest of paper, we consider following:

$$\begin{aligned} P(\beth) &= \{\Upsilon \subset \beth : \Upsilon \neq \emptyset\}, \\ P^{cl}(\beth) &= \{\Upsilon \subset P(\beth) : \Upsilon \text{ is closed}\}, \\ P^{b}(\beth) &= \{\Upsilon \subset P(\beth) : \Upsilon \text{ is bounded}\}, \\ P^{cv}(\beth) &= \{\Upsilon \subset P(\beth) : \Upsilon \text{ is convex}\}, \\ P^{cp}(\beth) &= \{\Upsilon \subset P(\beth) : \Upsilon \text{ is compact}\}. \end{aligned}$$

Also, $U_{d_*}$ refers to the Pompeiu-Hausdorff MS, which is described as follows:

$$\begin{aligned} U_{d_*} &: P^{cl,b}(\beth) \times P^{cl,b}(\beth) \to \mathbb{R}^+ \\ (\Upsilon, T) &\mapsto U_{d_*}(\Upsilon, T) = \max\{U_*(\Upsilon, T), U_*(T, \Upsilon)\}, \end{aligned}$$

where

$$U_*(\Upsilon, T) = \sup_{c \in \Upsilon} d_*(c, T), \ U_*(T, \Upsilon) = \sup_{t \in T} d_*(t, \Upsilon) \text{ and } d_*(c, T) = \inf_{t \in T} d(c, t).$$

Now, for an MVO $E : \beth \to P\left(\widetilde{\beth}\right)$, the graph $G^r(E)$ of $E$ is defined by

$$G^r(E) = \left\{(z, \mathfrak{I}) : \beth \times \widetilde{\beth} : \mathfrak{I} \in Ez\right\}.$$

$G^r(E)$ is said to be a closed if there exists a sequence $\{(z_u, \mathfrak{I}_u)\}$ in $G^r(E)$ such that $\lim_{u\to\infty}(z_u, \mathfrak{I}_u) = (z, \mathfrak{I}) \in G^r(E)$.

**Definition 2.10.** An MVO $E : \beth \to P^{cl}\left(\widetilde{\beth}\right)$ is called closed if it has a closed graph in $\beth \times \widetilde{\beth}$.

**Lemma 2.11.** *[45] An MVO $E : \beth \to P^{cl}\left(\widetilde{\beth}\right)$ is USC iff it is closed and has a compact range.*

**Definition 2.12.** Let $\varkappa$ be an MVO. Define the set

$$O_{\varkappa,z} = \left\{m \in L^1(\Lambda, \mathbb{R}^+) : m(\rho) \in \varkappa(\rho, z(\rho)), \ \rho, z \in \Lambda\right\}.$$

The set $O_{\varkappa,z}$ is called the selection set and it is convex if and only if $\varkappa$ is convex for all $\rho \in \Lambda$. The set $O_{\varkappa,z}$ is non-empty if $\varkappa$ is a multi-valued $L^1$-Carathéodory for each $z \in C(\Lambda, \mathbb{R}^+)$. This allows us to specify the MVO

$$\begin{aligned} O_\varkappa &: C(\Lambda, \mathbb{R}^+) \to P(C(\Lambda, \mathbb{R}^+)), \\ z &\mapsto O_\varkappa(z) = O_{\varkappa,z}. \end{aligned}$$

The norm of the multi-valued function $\varkappa : \Lambda \times B \to P(B)$ is described as follows:

$$\|\varkappa(\rho, z(\rho))\|_P = \sup\{|m|, m \in \varkappa(\rho, z(\rho))\}.$$

**Lemma 2.13.** *[46] Let* $\varkappa : [c, d] \times B \to P^{cp,cv}(B)$ *be an* $L^1-$*Carathéodory MVO with* $O_{\varkappa,\Im} \neq \emptyset$ *for all* $\Im \in C([c, d], B)$*. Assume also that* $\Phi : L^1([c, d], B) \to C([c, d], B)$ *is a linear continuous mapping. Then, the operator*

$$\begin{aligned} \Phi \circ O_{\varkappa} \quad &: \quad C([c, d], B) \to P^{cp,cv}(C([c, d], B)), \\ \Im \quad &\mapsto \quad (\Phi \circ O_{\varkappa})(\Im) = \Phi(O_{\varkappa,\Im}), \end{aligned}$$

has a closed graph in $C([c, d], B) \times C([c, d], B)$.

**Lemma 2.14.** *[47] For any* $z_0 \in \daleth$, $\limsup_{z \to z_0} E(z) = E(z_0)$, *provided that* $E : \daleth \to P^{cp}(\daleth)$ *is USC.*

**Lemma 2.15.** *[47] Let* $\daleth$ *be a separable BS,* $\{Y_u\}_{u\in\mathbb{N}}$ *be a sequence subset of* $Y$*, and* $Y$ *be compact in* $\daleth$*. Then,*

$$\overline{cs}\left(\limsup_{u\to+\infty} Y_u\right) = \cap_{n>0}\overline{co}\left(\cup_{u\geq n} Y_u\right),$$

where $\overline{co}T$ stands for the closure of the convex hull of $T$.

Further readings and details on MVOs can be found in [42, 46, 48, 49].

**Theorem 2.16.** *[50] Assume that* $J$ *is a closed, convex, and bounded subset of a BS* $\daleth$*. Let* $E : J \to P^{cp,cv}(J)$ *be USC and compact. Then,* $E$ *has a FP.*

**Lemma 2.17.** *[51] Assume that* $z(\rho)$ *and* $\zeta(\rho)$ *are nonnegative continuous functions and*

$$z(\rho) \leq c + \int_{\gamma}^{\rho} \zeta(r) z(r) dr, \ \rho > \gamma, \ c \geq 0.$$

Then,

$$z(\rho) \leq a e^{\left(\int_{\gamma}^{\rho} \zeta(r) dr\right)}, \ \rho > \gamma.$$

## 3. Mild solution to the considered model

In this section, we discuss the existence of a mild solution to the problem (1.1). We begin this section with the definition below.

**Definition 3.1.** We say that the function $\Im \in \Re$ is a mild solution to the model (1.1) if there is a function $g \in O_{\mho,\Im}$ such that

$$\Im(\rho) = \begin{cases} \frac{\partial S(\rho,r)\beta Z(0)}{\partial r} \big|_{r=0} + S(\rho, 0)\beta\kappa_0 + \beta \int_0^{\rho} S(\rho, r) g(r) dr, \ \beta > 0, \ \text{if } \rho \in \Lambda, \\ \beta Z(\rho), \ \text{if } \rho \in \mathbb{R}_-, \end{cases}$$

where $O_{\mho,\Im} = \left\{ m \in L^1(\Lambda, \mathbb{R}^+) : m(\rho) \in \mho\left(\rho, \Im_\rho, (\Theta\Im)(\rho)\right), \ \rho \in \Lambda \right\}$.

Throughout this work, we shall make the following assumptions:

($A_1$) $\mho : \Lambda\times\psi\times B \to P^{cv,cp}(B)$ represents Carathéodory multi-valued functions, and there are constants $\sigma_1, \sigma_2 > 0$ and continuous nondecreasing functions $\tau_1, \tau_2 : \Lambda \to (0, +\infty)$ such that

$$U_{d_*}\left(\mho(\rho, \Im_1, \Im_2), \mho\left(\rho, \widetilde{\Im}_1, \widetilde{\Im}_2\right)\right) \leq \sigma_1\tau_1\left(\left\|\Im_1 - \widetilde{\Im}_1\right\|_\psi\right) + \sigma_2\tau_2\left(\left\|\Im_2 - \widetilde{\Im}_2\right\|_\psi\right),$$

for $\Im_1, \widetilde{\Im}_1 \in \psi$, $\Im_2, \widetilde{\Im}_2 \in B$ with $\|\mho(\rho, 0, 0)\|_P = 0$, $\tau_i(\rho) \leq \rho$, $i = 1, 2$.

($A_2$) For all $\rho \in \Lambda$ and $q_1, q_2 \in (0, +\infty)$, it follows that

$$\daleth(\rho, q_1, q_2) = \left\{ \begin{array}{c} S(\rho, r)\mho(r, \Im_1, \Im_2) : r \in [0, \rho],\ \Im_1 \in \psi,\ \Im_2 \in B, \\ \left\|\Im_1\right\|_\psi \leq q_1 \text{ and } \left\|\Im_2\right\|_\psi \leq q_2, \end{array} \right\}$$

is relatively compact in $B$.

($A_3$) There is a positive constant $\omega$ and a continuous function $\eta : D_\eta \times B \to B$ such that

$$\left\|\eta(\rho, r, \Im_1) - \eta(\rho, r, \Im_2)\right\| \leq \omega\left\|\Im_1 - \Im_2\right\|,$$

for each $(\rho, r) \in D_\eta$ and $\Im_1, \Im_2 \in B$. Further,

$$\sup_{D_\eta}\{\|\eta(\rho, r, 0)\|\} = \omega^* < +\infty.$$

($A_4$) Under conditions ($C_1$)–($C_3$), there exist positive constants $V_S$ and $\widetilde{V}_S$ such that

$$\|S(\rho, r)\| \leq V_S \text{ and } \left\|\frac{\partial}{\partial r}S(\rho, r)\right\| \leq \widetilde{V}_S,\ (\rho, r) \in \phi.$$

Now, our main theorem in this section is ready to be presented.

**Theorem 3.2.** *The model (1.1) has at least one mild solution, provided that the hypotheses* ($A_1$)–($A_4$) *hold.*

*Proof.* Since the FP technique is a useful tool to investigate the existence of the solution for such systems, we use this technique. We transform problem (1.1) into an FP problem as follows: Define the MVO $\Pi : \Re \to P(\Re)$ by

$$\Pi_\Im = \left\{\chi \in \Re : \chi(\rho) = \left\{ \begin{array}{l} -\frac{\partial S(\rho,r)\beta Z(0)}{\partial r}\mid_{r=0} + S(\rho, 0)\beta\kappa_0 \\ +\beta\int\limits_0^\rho S(\rho, r)g(r)dr,\ \beta > 0,\ \text{if } \rho \in \Lambda, \\ \beta Z(\rho),\ \text{if } \rho \in \mathbb{R}_-, \end{array} \right. \right\}$$

for some $g \in O_{\mho,\Im}$. It is clear that the FP of the operator defined above is equivalent to the solution of system (1.1).

Let the function $z(.) : (-\infty, \theta] \to B$ be described as follows:

$$z(\rho) = \left\{ \begin{array}{l} -\frac{\partial S(\rho,r)\beta Z(0)}{\partial r}\mid_{r=0} + S(\rho, 0)\beta\kappa_0,\ \text{if } \rho \in \Lambda, \\ \beta Z(\rho),\ \text{if } \rho \in \mathbb{R}_-. \end{array} \right.$$

Then, $z_0 = \beta Z$ for any $\beta > 0$, and for all $\varrho \in \mathfrak{R}$ with $\varrho_0 = 0$, we define the function $\widehat{\varrho}$ by

$$\widehat{\varrho}(\rho) = \begin{cases} \varrho(\rho), \text{ if } \rho \in \mathbb{R}^+, \\ 0, \text{ if } \rho \in \mathbb{R}_-. \end{cases}$$

If $\mathfrak{I}$ satisfies the conditions of Definition 3.1, it may be decomposed as $\mathfrak{I}(\rho) = \varrho(\rho) + z(\rho)$, which leads to $\mathfrak{I}_\rho = \varrho_\rho + z_\rho$. Furthermore, the function $\varrho(.)$ satisfies the integral below:

$$\varrho(\rho) = \beta \int_0^\rho S(\rho, r) g(r) dr, \ \rho \in \Lambda,$$

where $g(r) \in \mho(r, \mathfrak{I}_r, (\Theta\mathfrak{I})(r)) = \mho(r, \varrho_r + z_r, \Theta(\varrho + z)(r))$. Set

$$\daleth = \{\varrho \in \mathfrak{R} : \varrho(0) = 0\}.$$

Describe the operator $\widehat{\Pi} : \daleth \to P(\daleth)$ as follows:

$$\widehat{\Pi}_\varrho = \left\{ \widehat{\chi} \in \daleth : \widehat{\chi}(\rho) = \beta \int_0^\rho S(\rho, r) g(r) dr, \ \rho \in \Lambda \right\},$$

where

$$g \in O_{\mho, \widehat{\varrho}+z} = \left\{ m \in L^1(\Lambda, \mathbb{R}^+) : m(\rho) \in \mho\left(\rho, \varrho_\rho + z_\rho, \Theta(\varrho + z)(\rho)\right), \ \rho \in \Lambda \right\}.$$

The existence of an FP for the operator $\Pi$ is obviously equal to the existence of an FP for $\widehat{\Pi}$. As a result, it is sufficient to establish the existence of an FP for $\widehat{\Pi}$. We will show that an operator $\widehat{\Pi}$ satisfies all of the constraints outlined in Theorem 2.16.

Assume that $\phi_\varpi = \{\varrho \in \daleth : \|\varrho\| \le \varpi\}$ with

$$V_S(\sigma_1 \tau_1(\beta \|Z\|_\daleth) + \sigma_2 \tau_2(\alpha^*)) \theta \le \varpi,$$

where $\alpha^*$ is a constant. Clearly, $\phi_\varpi$ is bounded, closed, and convex. Now, the following steps complete the proof:

St. 1: Show that $\widehat{\Pi}(\phi_\varpi) \subset \phi_\varpi$. Indeed, for $\varrho \in \phi_\varpi, \widehat{\chi} \in \widehat{\Pi}_\varrho$ and $\rho \in \Lambda$, using $(A_1)$–$(A_4)$, one has

$$\|\Theta(\varrho + z)(\rho)\| \le \theta\omega\left(\varpi \widetilde{V}_S \beta \|Z_0\| + V_S \|\kappa_0\|\right) + \theta\omega^* = \alpha^*.$$

Then,

$$\left\|\widehat{\chi}(\rho)\right\| \le V_S(\tau_1(\beta \|Z\|_\daleth) \sigma_1 + \tau_2(\alpha^*) \sigma_2) \theta \le \varpi.$$

Hence, $\widehat{\Pi}(\phi_\varpi) \subset \phi_\varpi$. Further, $\widehat{\Pi}(\phi_\varpi)$ is bounded.

St. 2: Claim that $\widehat{\Pi}$ is convex. Assume that $\chi_1, \chi_2 \in \widehat{\Pi}(\varrho)$ for $\varrho \in \phi_\varpi$; then, there exist $g_1, g_2 \in O_{\mho, \widehat{\varrho}+z}$ such that for every $\rho \in \Lambda$, we get

$$\chi_i(\rho) = \beta \int_0^\rho S(\rho, r) g_i(r) dr, \ i = 1, 2.$$

Therefore, for every $\gamma \in [0,1]$ and $\rho \in \Lambda$, we have

$$\gamma\chi_1(\rho) + (1-\gamma)\chi_2(\rho) = \beta \int_0^\rho S(\rho, r)(\gamma g_1(r) + (1-\gamma) g_2(r))\, dr.$$

Since $\mho$ takes convex values, then $O_{\mho,\widehat{\varrho}+z}$ is convex and we can obtain that $(\gamma g_1 + (1-\gamma) g_2) \in O_{\mho,\widehat{\varrho}+z}$. Therefore, $\gamma\chi_1 + (1-\gamma)\chi_2 \in \widehat{\Pi}(\varrho)$. Hence, $\widehat{\Pi}$ is convex.

St. 3: Prove that $\widehat{\Pi}$ is compact. First, we illustrate that $\widehat{\Pi}(\phi_\varpi)$ is equicontinuous. For this, let $\varsigma_1, \varsigma_2 \in \Lambda$, $\varrho \in \phi_\varpi$ and $\widehat{\chi} \in \widehat{\Pi}(\varrho)$; we have

$$\begin{aligned}\left\|\widehat{\chi}(\varsigma_1) - \widehat{\chi}(\varsigma_2)\right\| &\le (\tau_1(\beta \|Z\|_\daleth)\sigma_1 + \tau_2(\alpha^*)\sigma_2) \\ &\quad \times\beta\left(\int_0^{\varsigma_1} \|S(\varsigma_1, r) - S(\varsigma_2, r)\|\, dr + V_S(\varsigma_1 - \varsigma_2)\right) \\ &\to 0 \text{ as } \varsigma_1 \to \varsigma_2.\end{aligned}$$

This illustrates that $\widehat{\Pi}$ maps bounded sets into bounded and equicontinuous sets of $\mathfrak{R}$.

Next, we claim that $\widehat{\Pi}(\phi_\varpi(\rho)) = \{\chi(\rho) : \chi \in \phi_\varpi\}$ is relatively compact for each $\rho \in \Lambda$. Let $\wp \in \phi_\varpi$, and by using the mean value theorem, we arrive at the conclusion that

$$\widehat{\chi}(\rho) \in \rho\overline{co\,(S(\rho, r) f(r))} \subseteq \rho\overline{\beth(\rho, \beta \|Z\|_\daleth, \alpha^*)}.$$

Using condition $(A_2)$ and Mazur's theorem, we infer that $\widehat{\Pi}(\phi_\varpi(\rho))$ is relatively compact. Consequently, we conclude that $\widehat{\Pi}$ is compact by concurrently considering all of the previously specified properties and applying the Ascoli-Arzelá theorem.

St. 4: Illustrate that $\widehat{\Pi}$ has a closed graph. In this regard, assume that $\varrho_u \to \varrho$, and $\widehat{\chi}_u \to \widehat{\chi}$, as $u \to \infty$ and $\widehat{\chi}_u \in \widehat{\Pi}(\varrho_u)$. We shall show that $\widehat{\chi} \in \widehat{\Pi}(\varrho)$, that is, there exists $g \in O_{\mho,\widehat{\varrho}+z}$ such that

$$\widehat{\chi}(\rho) = \beta \int_0^\rho S(\rho, r) g(r) dr, \text{ for each } \rho \in \Lambda.$$

Now, $\widehat{\chi}_u \in \widehat{\Pi}(\varrho_u)$ means that there exists $g_u \in O_{\mho,\widehat{\varrho}_u+z}$ such that

$$\widehat{\chi}_u(\rho) = \beta \int_0^\rho S(\rho, r) g_u(r) dr, \text{ for each } \rho \in \Lambda.$$

Define the linear continuous operator $\Phi : L^1(\Lambda, B) \to C(\Lambda, B)$ by

$$g \mapsto \Phi(g)(\rho) = \beta \int_0^\rho S(\rho, r) g(r) dr, \ \rho \in \Lambda.$$

According to Lemma 2.13, the operator $\Phi \circ O_{\mho,\widehat{\varrho}_u+z}$ has a closed graph; further, $\widehat{\chi}_u \in \Phi \circ O_{\mho,\widehat{\varrho}_u+z}$, so,

$$\widehat{\chi}(\rho) = \beta \int_0^\rho S(\rho, r) g(r) dr, \ \rho \in \Lambda.$$

Hence, $g \in O_{\mho,\widehat{\varrho}+z}$. Now, using the FP theorem of Bohnenblust-Karlin (Theorem 2.16), we arrive at the conclusion that $\widehat{\Pi}$ has at least one FP $\varrho^*$. Therefore, $\mathfrak{I}^* = \varrho^* + z$ is an FP of the operator $\Pi$, which is a mild solution to the problem (1.1).

□

## 4. The composition of solution sets

In this section, we discuss the structure of solution sets to the problem (1.1).

**Theorem 4.1.** *The solution of the model (1.1) is non-empty and compact, provided that the hypotheses* $(A_1)$–$(A_4)$ *are satisfied.*

*Proof.* Assume that $O(W)$ is a solution set to problem (1.1); then, by Theorem 3.2, $O(W) \neq \emptyset$. It only remains for us to prove that $O(W)$ is a compact set. For this, let the sequence $\{\mathfrak{I}^u\}_{u\in\mathbb{N}} \subset O(W)$; then, there exist sequences denoted by $g_u \in O_{\mho,\mathfrak{I}^u}$ such that

$$\mathfrak{I}^u(\rho) = \begin{cases} -\frac{\partial S(\rho,r)\beta Z(0)}{\partial r}\mid_{r=0} + S(\rho,0)\beta\kappa_0 \\ +\beta\int\limits_0^\rho S(\rho,r)\, g_u(r)dr,\ \beta>0,\ u\in\mathbb{N},\ \rho\in\Lambda, \\ \beta Z(\rho),\ \text{if } \rho\in\mathbb{R}_-. \end{cases}$$

Let $\urcorner_{O(W)} = \{\mathfrak{I}^u \in S(W), u\in\mathbb{N}\}$. Then, from the steps of Theorem 3.2, we have that $\urcorner_{O(W)}$ is bounded and equicontinuous. Hence, $\urcorner_{O(W)}$ is a compact set, that is, a subsequence $\mathfrak{I}^u_j \subset O(W)$ with $\mathfrak{I}^u_j \to \mathfrak{I}$, as $j\to\infty$. In other words, from $(A_1)$, $g_u$ is dominated. Thanks to the Dunford-Pettis theorem, there exists a subsequence, still denoted $g_u$, that converges weakly to some limit $g \in L^1$. Thanks to Mazur's lemma, there exists $t^u_i \geq 0, i = 1,2,...,u$, such that $\sum_{i=1}^u t^u_i = 1$ and the sequence of convex combinations $m_u(.) = \sum_{i=1}^u t^u_i g_i(.)$ converges strongly to $g \in L^1$.

Since $\mho$ takes convex values, by Lemma 2.15, one has

$$\begin{aligned} g(\rho) &= \bigcup_{u\geq 1}\overline{\{m_u(\rho)\}},\ \rho\in\Lambda,\ \text{almost everywhere (a.e.)} \\ &\subset \bigcup_{u\geq 1}\overline{co}\{g_j(\rho),\ j\geq u\} \\ &\subset \bigcup_{u\geq 1}\overline{co}\left\{\bigcup_{j\geq u}\mho\left(\rho,\mathfrak{I}^j_\rho,\Theta\mathfrak{I}^j(\rho)\right)\right\} \\ &= \overline{co}\left(\limsup_{j\to\infty}\mho\left(\rho,\mathfrak{I}^j_\rho,\Theta\mathfrak{I}^j(\rho)\right)\right). \end{aligned}$$

Because $\mho$ is USC with compact values, by Lemma 2.14, we have

$$\limsup_{j\to\infty}\mho\left(\rho,\mathfrak{I}^j_\rho,\Theta\mathfrak{I}^j(\rho)\right) = \mho\left(\rho,\mathfrak{I}_\rho,\Theta\mathfrak{I}(\rho)\right),\ \rho\in\Lambda \text{ a.e.}$$

Since $\mho$ is closed and convex, we get that $g(\rho) \in \mho\left(\rho,\mathfrak{I}_\rho,\Theta\mathfrak{I}(\rho)\right)$.

Finally, we show that $\Im \in O(W)$. Consider the set $\left\{\wp(\rho) : \rho \in \widetilde{\Lambda}\right\}$ such that

$$\wp(\rho) = \begin{cases} -\frac{\partial S(\rho,r)\beta Z(0)}{\partial r} |_{r=0} + S(\rho,0)\beta\kappa_0 \\ +\beta \int_0^\rho S(\rho,r)\, g_u(r)dr,\ \beta > 0,\ u \in \mathbb{N},\ \rho \in \Lambda, \\ \beta Z(\rho),\ \text{if } \rho \in \mathbb{R}_-, \end{cases}$$

for some $g \in O_{\mho,\wp}$. Now, for $\rho \in \mathbb{R}_-$, we have

$$\left\|\Im^u(\rho) - \wp(\rho)\right\| = 0,\ \text{as } u \to +\infty,$$

and for $\rho \in \Lambda$, we get

$$\left\|\Im^u(\rho) - \wp(\rho)\right\| \leq \beta V_S \int_0^\rho \|g_u(r) - g(r)\|\, dr.$$

The Lebesgue-dominated convergence theorem yields

$$\left\|\Im^u(\rho) - \wp(\rho)\right\| \to 0,\ \text{as } u \to +\infty.$$

Thus,

$$\Im(\rho) = \begin{cases} -\frac{\partial S(\rho,r)\beta Z(0)}{\partial r} |_{r=0} + S(\rho,0)\beta\kappa_0 \\ +\beta \int_0^\rho S(\rho,r)\, g_u(r)dr,\ \beta > 0,\ u \in \mathbb{N},\ \rho \in \Lambda, \\ \beta Z(\rho),\ \text{if } \rho \in \mathbb{R}_-. \end{cases}$$

Hence, $\Im \in O(W)$. Therefore, $O(W)$ is compact, and this completes the proof. □

The second part of this section is devoted to establishing that $O(W)$ constitutes an $R_\kappa$-set. As a result of qualifying as an acyclic space, we consider the following second-order single-valued problem:

$$\begin{cases} \Im''(\rho) \in \Upsilon(\rho)\Im(\rho) + g\left(\rho, \Im_\rho, (\Theta\Im)(\rho)\right) + \beta \int_0^\rho \vartheta(\rho,r)\, \Im(r)dr,\ \beta > 0,\ \text{if } \rho \in \Lambda, \\ \Im'(0) = \beta\kappa_0 \in B,\ \Im(\rho) = \beta Z(\rho),\ \text{if } \rho \in \mathbb{R}_-, \end{cases}$$

where $g : \Lambda \times \psi \times B \to P(B)$ is a given function. We need the hypotheses below to complete our desired goal here.

($A_1^*$) $g : \Lambda \times \psi \times B \to P^{cv,cp}(B)$ is a Carathéodory function, and there exist constants $\sigma_1^*, \sigma_2^* > 0$, and continuous nondecreasing functions $\widetilde{\tau}_1, \widetilde{\tau}_2 : \Lambda \to (0, +\infty)$ such that

$$\left\|g(\rho, \Im_1, \Im_2) - g\left(\rho, \widetilde{\Im}_1, \widetilde{\Im}_2\right)\right\| \leq \sigma_1^*\widetilde{\tau}_1\left(\left\|\Im_1 - \widetilde{\Im}_1\right\|_\psi\right) + \sigma_2^*\widetilde{\tau}_2\left(\left\|\Im_2 - \widetilde{\Im}_2\right\|_\psi\right),$$

for $\Im_1, \widetilde{\Im}_1 \in \psi$, $\Im_2, \widetilde{\Im}_2 \in B$ with $f(\rho,0,0) = 0, \widetilde{\tau}_i(\rho) \leq \rho$, $i = 1, 2$.

($A_2^*$) For all $\rho \in \Lambda$ and $q_1, q_2 \in (0, +\infty)$, we have that

$$\daleth(\rho, q_1, q_2) = \left\{ \begin{array}{c} S(\rho, r) f(r, \Im_1, \Im_2) : r \in [0, \rho],\ \Im_1 \in \psi,\ \Im_2 \in B, \\ \left\|\Im_1\right\|_\psi \leq q_1 \text{ and } \left\|\Im_2\right\|_\psi \leq q_2, \end{array} \right\}$$

is relatively compact in $B$.

**Theorem 4.2.** *Under the assumptions* $(A_3)$, $(A_4)$, $(A_1^*)$ *and* $(A_4^*)$, *the solution sets denoted by* $O(g, W)$ *of the model (1.1) are* $R_\kappa$*-sets; hence, it is an acyclic space.*

*Proof.* Define the operator $\Pi_0 : \mathfrak{R} \to \mathfrak{R}$ by

$$\Pi_0 \Im(\rho) = \begin{cases} -\frac{\partial S(\rho, r)\beta Z(0)}{\partial r}\big|_{r=0} + S(\rho, 0)\beta\kappa_0 \\ +\beta \int_0^\rho S(\rho, r)\, g(r, \Im_r, (\Theta\Im)(r))dr,\ \beta > 0,\ \text{if } \rho \in \Lambda, \\ \beta Z(\rho),\ \text{if } \rho \in \mathbb{R}_-. \end{cases} \tag{4.1}$$

Based on Theorem 3.2, $O(g, W) = fix(\Pi_0)$ (where $fix(\Pi_0)$ represents the FP of the operator $\Pi_0$) is non-empty. By the same calculations as St. 1 of Theorem 3.2, there exists $l_s = \max\left\{s, \|\beta Z\|_\psi\right\} > 0$ such that for every $\Im \in O(g, W)$,

$$\left\|\Im\right\|_{\mathfrak{R}} \leq l_s.$$

Consider the following altered problem:

$$\begin{cases} \Im''(\rho) \in \Upsilon(\rho)\Im(\rho) + M\left(\rho, \Im_\rho, (\Theta\Im)(\rho)\right) + \beta \int_0^\rho \vartheta(\rho, r)\, \Im(r)dr,\ \beta > 0,\ \text{if } \rho \in \Lambda, \\ \Im'(0) = \beta\kappa_0 \in B,\ \Im(\rho) = \beta Z(\rho),\ \text{if } \rho \in \mathbb{R}_-, \end{cases} \tag{4.2}$$

where the function $M : \Lambda \times \psi \times B \to B$ is defined by

$$M\left(\rho, \Im_\rho, (\Theta\Im)(\rho)\right) = \begin{cases} g\left(\rho, \Im_\rho, (\Theta\Im)(\rho)\right),\ \text{if } \left\|\Im\right\|_{\mathfrak{R}} \leq l_s; \\ g\left(\rho, \frac{l_s \Im_\rho}{\|\Im_\rho\|_\psi}, \frac{\Im_\rho \Theta\Im(\rho)}{\|\Theta\Im(\rho)\|_B}\right),\ \text{if } \left\|\Im\right\|_{\mathfrak{R}} > l_s. \end{cases}$$

From the hypotheses $(A_1^*)$ and $(A_4^*)$, there exists $\widehat{a} \in L^1(\Lambda, \mathbb{R}^+)$ such that

$$\left\|M\left(\rho, \Im_\rho, (\Theta\Im)(\rho)\right)\right\| \leq \widehat{a}(\rho),\ \rho \in \Lambda \text{ a.e.}$$

Hence, $O(g, W) = O(M, W) = Fix\left(\widehat{\Pi}\right)$, where $\widehat{\Pi} : \mathfrak{R} \to \mathfrak{R}$ is described as follows:

$$\widehat{\Pi}\Im(\rho) = \begin{cases} -\frac{\partial S(\rho, r)\beta Z(0)}{\partial r}\big|_{r=0} + S(\rho, 0)\beta\kappa_0 \\ +\beta \int_0^\rho S(\rho, r)\, M(r, \Im_r, (\Theta\Im)(r))dr,\ \beta > 0,\ \text{if } \rho \in \Lambda, \\ \beta Z(\rho),\ \text{if } \rho \in \mathbb{R}_-. \end{cases}$$

Based on the above calculations, we get

$$\left\|\widehat{\Pi}\Im\right\|_{\Re} \leq \widetilde{V}_S \|\beta Z_0\| + V_S \|\beta \kappa_0\| + V_S \left\|\widehat{a}\right\|_{L^1} = \widetilde{l}_s.$$

Hence, $\widehat{\Pi}$ is uniformly bounded. In the same manner as in St. 3 of Theorem 3.2, the compact perturbation of the identity $\widetilde{T}z = z - \widehat{\Pi}z$, which is a proper map, can be defined thanks to $\widehat{\Pi}$, a compact operator. The compactness of $\widehat{\Pi}$ and Theorem 2.7 and the satisfaction of the requirements in Lemma 2.9 are easily shown. As a result, the set $\widetilde{T}^{-1}(0) = O(M, W) = fix\left(\widehat{\Pi}\right)$ qualifies as an $R_\kappa$-set. Moreover, Lemma 2.6 establishes that it has the property of being acyclic. □

## 5. Continuous dependence of the solution according to initial data

In this section, we study the continuous dependence of the solution to the problem (1.1) under the initial data $\kappa_0$ and $\widetilde{\kappa}_0$.

**Theorem 5.1.** *The solution to the problem (1.1) depends continuously on the initial conditions, provided that the hypotheses listed in Theorem 3.2 are satisfied.*

*Proof.* Consider that $\kappa_0, \kappa_0^*, \widetilde{\kappa}_0, \widetilde{\kappa_0^*} \in B$. From Theorem 3.2, there exist $\Im\left(., \kappa_0, \widetilde{\kappa}_0\right)$, and $\Im^*\left(., \kappa_0^*, \widetilde{\kappa_0^*}\right) \in \Re$ such that, for some $g \in O_{\mho,\Im}$,

$$\Im(\rho) = \begin{cases} -\frac{\partial S(\rho,r)\beta\widetilde{\kappa}_0}{\partial r}\,|_{r=0} + S(\rho,0)\beta\kappa_0 \\ +\beta\int\limits_0^\rho S(\rho,r)\,g(r)dr,\ \beta > 0,\ \text{if } \rho \in \Lambda, \\ \beta Z(\rho),\ \text{if } \rho \in \mathbb{R}_-, \end{cases}$$

and

$$\Im^*(\rho) = \begin{cases} -\frac{\partial S(\rho,r)\beta\widetilde{\kappa_0^*}}{\partial r}\,|_{r=0} + S(\rho,0)\beta\kappa_0^* \\ +\beta\int\limits_0^\rho S(\rho,r)\,g(r)dr,\ \beta > 0,\ \text{if } \rho \in \Lambda, \\ \beta Z(\rho),\ \text{if } \rho \in \mathbb{R}_-. \end{cases}$$

Then for $\ell(\rho) = \left\|\Im(\rho) - \Im^*(\rho)\right\|$, one has

$$\begin{aligned} \ell(\rho) &\leq V_S\left\|\kappa_0 - \kappa_0^*\right\| + \widetilde{V}_S\left\|\widetilde{\kappa}_0 - \widetilde{\kappa_0^*}\right\| \\ &\quad + V_S\int\limits_0^\rho\left[\sigma_1\tau_1\left(\left\|\Im_r - \Im_r^*\right\|_\psi\right) + \sigma_2\tau_2\left(\left\|\Im_r - \Im_r^*\right\|_\psi\right)\right]dr \\ &\leq V_S\left\|\kappa_0 - \kappa_0^*\right\| + \widetilde{V}_S\left\|\widetilde{\kappa}_0 - \widetilde{\kappa_0^*}\right\| \\ &\quad + V_S\int\limits_0^\rho\left[\sigma_1\tau_1\left(K(r)\sup\left\|\Im(r) - \Im^*(r)\right\| : 0 \leq r \leq \rho\right) + \sigma_2\tau_2\left(\left\|\Im(r) - \Im^*(r)\right\|\right)\right]dr \\ &\leq V_S\left\|\kappa_0 - \kappa_0^*\right\| + \widetilde{V}_S\left\|\widetilde{\kappa}_0 - \widetilde{\kappa_0^*}\right\| + V_S\int\limits_0^\rho(\sigma_1 K_* + \sigma_2) + \sup_{0\leq r\leq\rho}\ell(r)\,dr. \end{aligned}$$

Applying Lemma 2.17, we have

$$\left\|\mathfrak{I}-\mathfrak{I}^*\right\|_{\mathfrak{R}} \leq \left(V_S \left\|\kappa_0-\kappa_0^*\right\| + \widetilde{V}_S \left\|\widetilde{\kappa_0}-\widetilde{\kappa_0^*}\right\|\right) e^{(\sigma_1 K_* + \sigma_2)\theta}.$$

Hence,

$$\left\|\mathfrak{I}-\mathfrak{I}^*\right\|_{\mathfrak{R}} \to 0, \text{ as } (\kappa_0, \widetilde{\kappa_0}) \to \left(\kappa_0^*, \widetilde{\kappa_0^*}\right).$$

Therefore, we can assert that the mild solutions to the problem (1.1) exhibit continuous dependence on the initial conditions. □

## 6. Continuous dependence of the solution on a selection set

In this section, we discuss the continuous dependence of the solution of the problem (1.1) under the selection of set $O_{\mho,\mathfrak{I}}$.

**Definition 6.1.** If the following condition holds: for all $\epsilon > 0$, there exists $\xi > 0$ such that for $g, g^* \in O_{\mho,\mathfrak{I}}$, $\|g - g^*\| < \xi$, we have that $\left\|\mathfrak{I}-\mathfrak{I}^*\right\|_{\mathfrak{R}} \leq \epsilon$. Then, the solution of the problem (1.1) depends continuously on the selections of set $O_{\mho,\mathfrak{I}}$.

**Theorem 6.2.** *The solution of the problem (1.1) depends continuously on the selections of set $O_{\mho,\mathfrak{I}}$, provided that the hypotheses listed in Theorems 3.2 are satisfied.*

*Proof.* Consider that $g, g^* \in O_{\mho,\mathfrak{I}}$. Thanks to Theorem 3.2, there exist $\mathfrak{I}, \mathfrak{I}^* \in \mathfrak{R}$ such that

$$\mathfrak{I}(\rho) = \begin{cases} -\frac{\partial S(\rho,r)\beta\widetilde{\kappa_0}}{\partial r}\big|_{r=0} + S(\rho,0)\beta\kappa_0 \\ +\beta\int\limits_0^\rho S(\rho,r)\, g_{\mathfrak{I}}(r)dr,\ \beta>0,\ \text{if } \rho\in\Lambda, \\ \beta Z(\rho),\ \text{if } \rho\in\mathbb{R}_-, \end{cases}$$

and

$$\mathfrak{I}^*(\rho) = \begin{cases} -\frac{\partial S(\rho,r)\beta\widetilde{\kappa_0}}{\partial r}\big|_{r=0} + S(\rho,0)\beta\kappa_0 \\ +\beta\int\limits_0^\rho S(\rho,r)\, g^*_{\mathfrak{I}^*}(r)dr,\ \beta>0,\ \text{if } \rho\in\Lambda, \\ \beta Z(\rho),\ \text{if } \rho\in\mathbb{R}_-. \end{cases}$$

Assume that for $\xi > 0$, $\|g - g^*\| < \xi$. Then for $\widetilde{\ell}(\rho) = \left\|\mathfrak{I}(\rho) - \mathfrak{I}^*(\rho)\right\|$, one can write

$$\begin{aligned} \widetilde{\ell}(\rho) &\leq \int\limits_0^\rho \left\|S(\rho,r)\, g_{\mathfrak{I}}(r) - S(\rho,r)\, g^*_{\mathfrak{I}^*}(r)\right\| dr \\ &\leq V_S \int\limits_0^\rho \left(\|g_{\mathfrak{I}}(r) - g_{\mathfrak{I}^*}(r)\| + \left\|g_{\mathfrak{I}^*}(r) - g^*_{\mathfrak{I}^*}(r)\right\|\right) dr \\ &\leq V_S \int\limits_0^\rho \left[\xi + \sigma_1\tau_1\left(K(r)\sup\left\{\widetilde{\ell}(r) : 0 \leq r \leq \rho\right\}\right) + \sigma_2\tau_2\left(\widetilde{\ell}(r)\right)\right] dr \end{aligned}$$

$$\leq \quad V_S\theta\xi + V_S \int_0^{\rho} (\sigma_1 K_* + \sigma_2) + \sup_{0\leq r\leq\rho} \widetilde{\ell}(r)\, dr.$$

Utilizing Lemma 2.17, we get

$$\left\|\mathfrak{I} - \mathfrak{I}^*\right\|_{\mathfrak{R}} \leq \xi V_S \theta e^{(\sigma_1 K_* + \sigma_2)\theta} = \epsilon(\xi).$$

This implies that the mild solution of the problem (1.1) depends continuously on the set $O_{\mho,\mathfrak{I}}$ for all selections of $\mho$. □

## 7. Application to partial integro-differential inclusion

In this section, we strengthen and enhance the results obtained by studying the existence of a mild solution to the following partial integro-differential inclusion (PIDI):

$$\begin{cases} \frac{\partial^2\beta\kappa(\rho,z)}{\partial\rho^2} - \frac{\partial^2\beta\kappa(\rho,z)}{\partial z^2} + \frac{\beta}{\hbar_2}\int_0^{\rho} \sin\left(\hbar_1(\rho - r)\right) \frac{\partial^2\kappa(\rho,z)}{\partial z^2} dr \\ -\beta\widehat{\alpha}(\rho)\kappa(\rho,z) \in M\left(\rho, \beta\kappa(\rho,z)\right), \text{ if } \rho \in [0,1] \text{ and } z \in (0,2\pi), \\ \kappa(\rho,0) = \kappa(\rho,1) = 0, \text{ if } \rho \in [0,1], \\ \frac{\partial\beta\kappa(\rho,z)}{\partial\rho}\,|_{\rho=0} = \beta\kappa_1(z),\ \kappa(\rho,z) = \beta Z(\rho,z) \text{ if } \rho \in \mathbb{R}_- \text{ and } z \in (0,2\pi), \end{cases} \tag{7.1}$$

where $\widehat{\alpha} : [0,1] \to \mathbb{R}$, $\hbar_2 > 1$, $\hbar_1, \beta > 0$, and the multi-valued mapping $M$ is described as follows:

$$\begin{aligned} M\left(\rho, \beta\kappa(\rho,z)\right) &= \left[0, \int_0^{\rho} \frac{\beta e^{12t}\left\|\kappa(\rho+t,z)\right\|_{L^2}}{64\left(\rho^4 + 4(\rho+t)^3 + 4\rho^2 + 1\right)} dt - \frac{1}{693(\rho+1)^4} \right. \\ &\quad \left. + \int_0^1 \frac{5\sin(2\rho)\, e^{-\rho^5}(1 + \beta\kappa(r,z))}{555\left(1 + 2\rho^3 + r^3\right) e^{15\rho}} dr \right]. \end{aligned}$$

Consider

$$B = H = L^2(0,\pi) = \left\{ q : (0,\pi) \to \mathbb{R} \text{ such that } \int_0^{\pi} |q(z)|^2\, dz < \infty \right\},$$

to be the Hilbert space with the scalar product $\langle q, r\rangle = \int_0^{\pi} q(z)r(z)dz$ and the norm

$$\|q\|_2 = \left( \int_0^{\pi} |q(z)|^2\, dz \right)^{\frac{1}{2}}.$$

Additionally, consider the APS $\psi$ is a bounded uniformly continuous function from $\mathbb{R}_-$ onto $H$ equipped with the norm $\|\tau\|_{\psi} = \sup_{-\infty<t\leq 0} \{\|\tau(t)\|_{L^2}, \tau \in \psi\}$. It is widely known that $\psi$ satisfies the assumptions $(X_1)$ and $(X_2)$ with $Y = 1$ and $K(\rho) = N(\rho) = 1$; see [52]. We define the operator $\widetilde{\Upsilon}$ induced on $H$ as follows:

$$\widetilde{\Upsilon}\wp = \wp'' \text{ and } D(\widetilde{\Upsilon}) = \left\{\wp \in H^2(0,2\pi) : \wp(0) = \wp(2\pi) = 0\right\}.$$

Then, $\widetilde{\Upsilon}$ is the infinitesimal generator of a cos function of operators $(C_0(\rho))_{\rho\in\mathbb{R}}$ on $H$, which is associated with the sin function $(O_0(\rho))_{\rho\in\mathbb{R}}$.

Let $\Upsilon(\rho)\wp = \widetilde{\Upsilon}\wp + \widehat{\alpha}(\rho)\wp$ on $D(\Upsilon)$. Obviously, $\Upsilon(\rho)$ is a closed linear operator. Hence, $\Upsilon(\rho)$ generates $(O(\rho,r))_{(\rho,r)\in\phi}$ such that $O(\rho,r)$ is self-adjoint and compact for all $(\rho,r)\in\phi = \{(\rho,r) : 0\le r\le\rho\le 1\}$; see [3].

Describe the mapping $\Psi(\rho,r) : D(\Upsilon)\subset H\to H$ as follows:

$$\Psi(\rho,r)\wp = \Delta(\rho,r)\widetilde{\Upsilon}\wp,\ \text{for } 0\le r\le\rho\le 1,\ \wp\in D(\Upsilon),$$

where

$$\Delta(\rho,r) = \frac{\beta}{\hbar_2}\sin\left(\hbar_1(\rho-r)\right),\ \beta>0,\ 0\le r\le\rho\le 1.$$

Hence,

$$|\Delta(\rho_2,r)-\Delta(\rho_1,r)| \le \frac{\beta\hbar_1}{\hbar_2}|\cos\hbar_1||\rho_2-\rho_1|,$$

and

$$|\Delta(\rho,r)| \le \frac{\beta}{\hbar_2}|\sin\hbar_1|.$$

Then, the axioms $(C_1)$–$(C_3)$ are true with $\lambda=\lambda_1=\frac{\beta}{\hbar_2}|\sin\hbar_1|$ and $M_\vartheta=\frac{\beta\hbar_1}{\hbar_2}|\cos\hbar_1|$. This suggests that an RO exists, and that it is a compact operator. The monographs [3,53] contain additional information on these facts.

Set $\kappa(\rho)(z)=\kappa(\rho,z)$ for $\rho\in[0,1]$ and define

$$\begin{aligned}\mho(\rho,\Im_1,\Im_2)(z) &= \left[0, \int_0^\rho \frac{\beta e^{12t}\left\|\Im_1(\rho+t,z)\right\|_{L^2}}{64\left(\rho^4+4(\rho+t)^3+4\rho^2+1\right)}dt - \frac{1}{693(\rho+1)^4}\right.\\ &\quad \left. + \int_0^1 \frac{\sin(2\rho)\,\Im_2(\rho)(z)}{e^{15\rho}}dr\right],\end{aligned}$$

$$\Im_2(\rho)(z) = \Theta(\Im_1)(z) = \int_0^1 \frac{5e^{-\rho^5}(1+\beta\Im_1(r,z)}{555(1+2\rho^3+r^3)}dr.$$

These concepts allow us to represent the system (7.1) in its abstract form, i.e., (1.1).

Now, for $\rho\in[0,1]$ and $g(\rho)\in\mho\left(\rho,\mu_\rho,\widetilde{\mu}(\rho)\right)(z)$, we get

$$\|g(\rho)\| \le \frac{1}{693(\rho+1)^4}(1+\|\Sigma\|_\psi)+\sin(2\rho)\,e^{-15\rho}\left\|\widetilde{\Sigma}(\rho)\right\|.$$

So, $\tau_{i+1}(\rho)=\rho+i$, $i=1,2$, denotes continuous nondecreasing functions, and we have

$$\sigma_1=\beta\int_0^\rho \frac{e^{12t}}{64\left(\rho^4+4(\rho+t)^3+4\rho^2+1\right)}dt>0 \text{ and } \sigma_2=\int_0^1 \frac{e^{-\rho^5}\sin(2\rho)}{111e^{15\rho}(1+2\rho^3+r^3)}dr>0,$$

for $\rho \in [0,1]$. Finally, for $\eta$ of condition $(A_3)$, one can write

$$\|\eta(\rho, r, \mu_1) - \eta(\rho, r, \mu_2)\|_2 \leq \frac{1}{111} \|\mu_1 - \mu_2\|_2 .$$

Therefore, all assumptions of Theorem 3.2 are satisfied. Hence, the problem (7.1) has at least one mild solution.

## 8. Conclusions

The necessity to incorporate delays into models representing real-world phenomena dates back to the early twentieth century, as certain dynamics are influenced by historical conditions within populations. For instance, when studying a group comprising only women of childbearing age, the time span from birth to reproductive involvement significantly impacts the population's evolution due to maturation delays. Integro-differential equations, serving as approximations to partial differential equations, play a vital role in the simulation of continuous phenomena across various fields, like population modeling, ecology, fluid dynamics, and aerodynamics. This paper contributes to this domain, focusing on three key aspects: establishing the existence of mild solutions, analyzing the topological and geometric structures of solution sets, and exploring the continuous reliance of solutions on second-order SID inclusions. The Bohnenblust-Karlin FP technique has been applied alongside Grimmer's theory of ROs to facilitate this examination, leading to a detailed exploration of a mild solution for a PIDI to illustrate the obtained results.

## 9. Abbreviations

This section includes a table containing acronyms for regularly used terms and phrases to assist the comprehension and study of our content.

| | |
|---|---|
| SID→semilinear integro-differential | FP→fixed point |
| USC→upper semi-continuous | AR→absolute retract |
| APS→abstract phase space | MVO→multi-valued operator |
| BS→Banach space | MS→metric space |
| PIDI→partial integro-differential inclusion | |

## Author contributions

All authors contributed equally and significantly in the writing of this article. All authors have read and approved the final version of the manuscript for publication.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflicts of interest

The authors declare that they have no conflicts of interest.

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