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Research article

A study on extended form of multivariable Hermite-Apostol type Frobenius-Euler polynomials via fractional operators

Mohra Zayed[1], Shahid Ahmad Wani[2,*], Georgia Irina Oros[3] and William Ramŕez[4,5,*]

[1] Mathematics Department, College of Science, King Khalid University, Abha, 61413, Saudi Arabia
[2] Symbiosis Institute of Technology, Symbiosis International (Deemed) University (SIU), Pune, Maharashtra, India
[3] Department of Mathematics and Computer Science, Faculty of Informatics and Sciences, University of Oradea, Oradea 410087, Romania
[4] Section of Mathematics International Telematic University Uninettuno, Rome 00186, Italy
[5] Department of Natural and Exact Sciences, Universidad de la Costa, Barranquilla 080002, Colombia

* **Correspondence:** Email: shahidwani177@gmail.com, shahid.wani@sitpune.edu.in, w.ramirezquiroga@students.uninettunouniversity.net.

Abstract: Originally developed within the realm of mathematical physics, integral transformations have transcended their origins and now find wide application across various mathematical domains. Among these applications, the construction and analysis of special polynomials benefit significantly from the elucidation of generating expressions, operational principles, and other distinctive properties. This study delves into a pioneering exploration of an extended lineage of Frobenius-Euler polynomials belonging to the Hermite-Apostol type, incorporating multivariable variables through fractional operators. Motivated by the exigencies of contemporary engineering challenges, the research endeavors to uncover the operational rules and establishing connections inherent within these extended polynomials. In doing so, it seeks to chart a course towards harnessing these mathematical constructs within diverse engineering contexts, where their unique attributes hold the potential for yielding profound insights. The study deduces operational rules for this generalized family, facilitating the establishment of generating connections and the identification of recurrence relations. Furthermore, it showcases compelling applications, demonstrating how these derived polynomials may offer meaningful solutions within specific engineering scenarios.

Keywords: fractional operators; Eulers' integral; multivariable special polynomials; explicit form; operational connection; applications
Mathematics Subject Classification: 33E20, 33C45, 33B10, 33E30, 11T23

1. Introduction and preliminaries

The investigation into the fusion of diverse polynomial types to generate inventive multi-variable generalized polynomials is a contemporary and applied research field. This area of study is particularly relevant due to the notable attributes inherent in these polynomials, which encompass recurrence and explicit relationships, functional and differential equations, summation formulas, symmetric and convolution properties, and determinant representations. The significance of these characteristics extends across various academic domains, making multi-variable hybrid special polynomials a compelling subject of exploration. One crucial aspect of these polynomials is their capacity to establish recurrence and explicit relationships. This means that certain patterns or behaviors repeat, providing researchers with a powerful tool for useful tool for comprehending and forecasting mathematical phenomena. Additionally, the ability to formulate functional and differential equations using these polynomials enhances their utility in solving complex mathematical problems. This feature is especially valuable in applications where dynamic relationships or rates of change need to be modeled and analyzed.

Summation formulas, another key attribute, allow for the concise representation of series or sequences, simplifying complex mathematical expressions. The symmetric and convolution properties of these polynomials add further versatility, enabling researchers to explore various mathematical operations and manipulations. Moreover, the determinant representations of multi-variable hybrid special polynomials open up new possibilities in linear algebra and matrix theory. The link between polynomials and determinants offers a fresh perspective on solving systems of equations and understanding the structural properties of mathematical objects. The practical applications of these polynomials span a wide range of fields. In number theory, they contribute to the study of integers and their properties, while in combinatorics, they find applications in counting and arranging discrete structures. Classical and numerical analysis benefit from the versatility of these polynomials in approximating functions and solving mathematical problems. Theoretical physics, with its intricate mathematical descriptions of the physical world, also stands to gain from the application of multi-variable hybrid special polynomials. These polynomials can provide elegant solutions to complex equations arising in the realm of theoretical physics. Additionally, the field of approximation theory benefits from the adaptability of these polynomials in representing functions with a high degree of accuracy. This is particularly valuable in scenarios where precise mathematical models are essential, such as in engineering, computer science, and data analysis.

In essence, the exploration of multi-variable hybrid special polynomials is not merely an abstract pursuit but holds substantial promise for addressing real-world challenges and advancing our understanding across a spectrum of scientific and mathematical disciplines. As researchers delve deeper into this intricate realm, the potential for practical applications continues to grow, making it a captivating and impactful avenue of study.

Several new categories of hybrid polynomials have been developed to harness their utility and unlock their application potential. This endeavor aims to enrich further the mathematical tools available for addressing complex challenges across a diverse spectrum of pure and applied mathematical disciplines.

Polynomial sequences hold substantial significance across various domains, encompassing fields like applied mathematics, theoretical physics, and approximation theory. Specifically, Bernstein polynomials of order n serve as fundamental building blocks for polynomials with degrees equal to or less than n. Dattoli and their collaborators comprehensively examined Bernstein polynomials, employing operational techniques to delve into their intricacies and properties [1]. In their exploration, they ventured into the realm of Appell sequences, an expansive class encompassing renowned polynomial sequences such as Euler polynomials, Bernoulli or Miller-Lee polynomials.

The exploration and thorough examination of novel classes of hybrid special polynomials associated with Appell sequences, as evidenced in sources like [2–7], impart a critical role to these polynomials across a diverse array of disciplines. This impact extends to fields such as engineering, biology, medicine, and the physical sciences. The importance of these hybrid special polynomials is emphasized by their distinctive features, which include their involvement in generating functions, integral representations, series definitions, differential equations, and more. The identification and understanding of innovative families of hybrid special polynomials, particularly those linked to Appell sequences, mark a significant advancement in mathematical research. The references cited, such as [2–8], serve as foundational works that contribute to the knowledge and exploration of these specialized polynomials. The r-parametric forms and certain charecteristics and properties of multivariable special polynomials are explored in [9–14].

In engineering, the application of hybrid special polynomials is noteworthy due to their versatile characteristics. These polynomials play a crucial role in solving differential equations, providing elegant solutions to complex engineering problems. The interplay between these polynomials and generating functions is particularly valuable in engineering applications, where the ability to represent functions concisely is essential. In the realm of biology, the implications of hybrid special polynomials are diverse. Their involvement in series definitions allows for the succinct representation of biological processes and phenomena. This can aid in modeling and understanding complex biological systems, offering insights into patterns and relationships within biological data.

In the physical sciences, the significance of hybrid special polynomials is further accentuated. Their role in differential equations proves instrumental in describing physical phenomena and predicting behavior in various scientific domains. The use of series definitions and integral representations contributes to the development of mathematical models that accurately represent physical processes. The defining characteristics of these hybrid special polynomials, including their involvement in generating functions, integral representations, series definitions, differential equations, and more, underscore their versatility and applicability across diverse disciplines. As researchers delve deeper into the intricacies of these polynomials, new avenues for application and discovery emerge, expanding the influence of these mathematical entities in fields critical to human understanding and progress.

In various technological and scientific fields, challenges are frequently raised in the form of differential equations, with solutions often taking the shape of special functions. Consequently, these hybrid special polynomials prove invaluable in articulating and resolving challenges that arise in the continually evolving landscape of scientific disciplines.

The generating function presented in [15] can be expressed as:

$$e^{h_1 t + h_2 t^2 + h_3 t^3} = \sum_{n=0}^{\infty} \mathfrak{D}_n(h_1, h_2, h_3) \frac{t^n}{n!}. \tag{1.1}$$

This generating function corresponds to the 3-variable Hermite polynomials (3VHP) denoted as $\mathfrak{D}_n(h_1, h_2, h_3)$.

When setting h_3 to zero, the 3VHP reduce to a set of polynomials known as the 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) denoted as $\mathfrak{D}_n(h_1, h_2)$. These 2VHKdFP polynomials are well-documented in [16].

Furthermore, if we set h_3 to zero, h_1 to $2h_1$, and h_2 to -1, the 3VHP transform into the classical Hermite polynomials, represented as $\mathfrak{D}_n(h_1)$, as detailed in [17].

Furthermore, the set of polynomials denoted as $\mathcal{D}_n^{[m]}(h_1, h_2, \cdots, h_m)$, commonly referred to as multivariable Hermite Polynomials (MHP) [18], is defined by the following relation:

$$\exp(h_1\xi + h_2\xi^2 + \cdots + h_m\xi^m) = \sum_{n=0}^{\infty} \mathcal{D}_n^{[m]}(h_1, h_2, \cdots, h_m)\frac{\xi^n}{n!}. \tag{1.2}$$

The operational rule for these polynomials is expressed as:

$$\exp\left(h_2\frac{\partial^2}{\partial^2_{h_1}} + h_3\,\frac{\partial^3}{\partial^3_{h_1}} + \cdots + h_m\frac{\partial^m}{\partial^m_{h_1}}\right)h_1^n = \mathcal{D}_n^{[m]}(h_1, h_2, \cdots, h_m). \tag{1.3}$$

Additionally, these polynomials can be represented in series form as:

$$\mathcal{D}_n^{[m]}(h_1, h_2, \cdots h_m) = n! \sum_{r=0}^{[n/m]} \frac{h_m^r\, \mathcal{D}_{n-mr}^{[m]}(h_1, h_2, \cdots, h_{m-1})}{r!\,(n-mr)!}. \tag{1.4}$$

In the study described in [19], a unified formulation is introduced for a particular group of polynomials referred to as the "Apostol type Frobenius Euler polynomials" (ATFEP). These polynomials are formally represented by the denotation $\mathbf{K}_n(h_1; \lambda; u)$, as defined in [20]. Let us review the generative expression associated with these polynomials, which can be represented as follows:

$$\left(\frac{1-u}{\lambda e^{\xi} - u}\right)e^{h_1\xi} = \sum_{n=0}^{\infty} \mathbf{K}_n(h_1; \lambda; u)\frac{\xi^n}{n!}, \tag{1.5}$$

where, $u \in \mathbb{C}$, $u \neq 1$.

When we set h_1 to zero in the previous expression, it yields the Apostol-type Frobenius-Euler numbers (ATFEN) denoted as $\mathbf{K}_n(\lambda; u)$ of order β, described as:

$$\left(\frac{1-u}{\lambda e^{\xi} - u}\right) = \sum_{n=0}^{\infty} \mathbf{K}_n(\lambda; u)\frac{\xi^n}{n!}. \tag{1.6}$$

Moreover, when we set u to minus one, the Apostol-type Frobenius-Euler polynomials (ATFEP) become Apostol-Euler polynomials denoted as $\mathfrak{A}_n(h_1; \lambda)$. Additionally, when we set λ to one, the Apostol-Euler polynomials (AEP) transform into Euler polynomials represented as $A_n(h_1)$, as detailed in [21]. Moreover, the ATFEP, when λ equals one, becomes the Frobenius-Euler polynomials denoted as $\mathbb{K}_n(h_1; u)$, as outlined in [22].

Fractional calculus is one of the fields of mathematical analysis that is remarkably growing. Applications are feasible in a variety of disciplines, including probability theory, statistics, economics, physics, biology, or electrochemistry.

The idea of fractional calculus, which entails the extension of integration to non-integer orders, possesses a fascinating historical context. Its roots can be traced back to the early stages of differential calculus, notably in the latter part of 17th century when the distinguished mathematician and philosopher Leibniz, in his rivalry with Newton, first proposed the concept of a fractional derivative with an order of $1/2$. However, it wasn't until Liouville's dedicated and thorough investigations that a comprehensive exploration of this subject occurred, ultimately yielding precise and rigorously conducted research.

The integration of integral transformations and specialized polynomials provides a robust and efficient methodology for dealing with fractional derivatives. This approach has gained considerable prominence and is recognized as a potent tool with widespread applicability across diverse industries. By combining particular polynomials like Chebyshev, Hermite, or Laguerre polynomials with integral transforms like Laplace or Fourier transforms, researchers and practitioners are able to develop efficient methods for deriving solutions for fractional differential equations. Such techniques have demonstrated their efficacy in various sectors, including engineering, finance, signal processing, and physics.

The fusion of integral transforms and specialized polynomials has emerged as a reliable method in the realm of fractional calculus. The interest in this method is underscored by the historical contributions of mathematicians and engineers, as evidenced in the published results of Oldham and Widder [23, 24]. Fractional operators, a long-standing focus of mathematical inquiry, have been effectively addressed using integral transforms, with roots tracing back to the seminal contributions of Riemann and Liouville, as highlighted in academic literature [23, 24].

Notably, the seamless integration of integral transformations and specialized polynomials has been acknowledged as a valuable technique, as demonstrated in works such as [25, 26]. These sources emphasize the importance of this combined approach and provide further insights into its real-world applications and theoretical advancements when handling fractional derivatives. Researchers and practitioners have extensively explored the benefits of this technique, leading to a deeper understanding of fractional calculus and its versatile applications.

In practical terms, this integrated methodology enables researchers and professionals to navigate the complexities of fractional calculus with precision. By employing integral transforms alongside specialized polynomials, such as Laguerre, Hermite, or Chebyshev polynomials, they can develop efficient solutions for fractional differential equations. The versatility of these techniques is exemplified by their successful application in diverse sectors. In engineering, for instance, the integrated approach aids in modeling and solving complex problems. In finance, it facilitates the analysis of fractional processes, while in signal processing, it contributes to the extraction of meaningful information from signals. Furthermore, in physics, this methodology proves valuable in describing and predicting fractional phenomena.

In conclusion, the fusion of integral transformations and specialized polynomials stands as a reliable and effective strategy for addressing fractional derivatives. Its broad applicability across various industries highlights its significance as a powerful tool in the hands of researchers and professionals. This integrated approach not only facilitates the analysis and solution of fractional differential equations but also contributes to advancing our understanding of fractional calculus and its practical implications in diverse fields.

In [25], the authors explored the broader potential of integral transforms. Within their research, they investigate the use of Euler's integral to increase the range of applications for integral transforms beyond their conventional boundaries. The Euler's integral, represented as:

$$q^{-\mu} = \frac{1}{\Gamma(\mu)} \int_0^\infty e^{-q\xi} \xi^{\mu-1} d\xi, \qquad \min\{\mathrm{Re}(\mu), \mathrm{Re}(q)\} > 0, \tag{1.7}$$

provides a thorough basis for enhancing the versatility and effectiveness of integral transformations in a variety of fields. Through the integration of Euler's integral into the integral transform framework, researchers are able to acquire the ability to address a wider range of complex mathematical equations encountered across a range of fields. This enlarged framework provides fresh viewpoints on fractional derivatives and their applications, inspiring creative solutions and methods of problem-solving.

This study highlights the potential for more progress in this area and provides practitioners and researchers with a valuable instrument for dealing with difficult problems involving fractional derivatives in a wider setting.

Furthermore, in the same study [25], it becomes evident that the following axioms hold true for first and second-order derivatives:

$$\left(\beta - \frac{\partial}{\partial h_1}\right)^{-\mu} g(h_1) = \frac{1}{\Gamma(\mu)} \int_0^\infty e^{-\beta\xi} \xi^{\mu-1}\, e^{\xi \frac{\partial}{\partial h_1}} g(h_1) d\xi = \frac{1}{\Gamma(\mu)} \int_0^\infty e^{-\beta\xi} \xi^{\mu-1}\, g(h_1 + \xi) d\xi, \tag{1.8}$$

$$\left(\beta - \frac{\partial^2}{\partial h_1^2}\right)^{-\mu} g(h_1) = \frac{1}{\Gamma(\mu)} \int_0^\infty e^{-\beta\xi} \xi^{\mu-1}\, e^{\xi \frac{\partial^2}{\partial h_1^2}} g(h_1) d\xi, \tag{1.9}$$

these equations hold true for first and second-order derivatives, as demonstrated in their research.

An effective approach to dealing with fractional operators involves harnessing the synergy between exponential operators and well-suited integral representations. Researchers and professionals can efficiently manage fractional operators by capitalizing on the inherent properties of exponential operators while selecting appropriate integral representations. This approach facilitates the exploration of cutting-edge mathematical concepts and streamlines the precise analysis of fractional derivatives. The utilization of exponential operators and specialized integral representations forms a robust foundation for addressing fractional operators, ultimately yielding enhanced methods and solutions in various mathematical and scientific domains.

The natural progression of certain aspects of hybrid special polynomials, achieved by incorporating principles of monomiality, operational rules, and other relevant properties, is both evident and beyond dispute. Monomiality first emerged in 1941 when Steffenson initially proposed the idea of a poweroid [27], which was later improved by Dattoli [2]. These operational approaches remain in active use across various domains, including classical optics, quantum mechanics, and mathematical physics. Consequently, these methods are powerful and efficient research instruments.

Consequently, the fusion of multivariable Hermite polynomials $\mathcal{D}_n^{[m]}(h_1, h_2, \cdots, h_m)$ defined in (1.2) and Apostol-type Frobenius-Euler polynomials [28, 29] defined in (1.5), guided by the principles of monomiality and operational rules, leads to the creation of a novel polynomial entity known as multivariable Hermite-Apostol type Frobenius-Euler polynomials. These polynomials are characterized by the generating relation:

$$\left(\frac{1-u}{\lambda e^{\xi} - u}\right) \exp(h_1\xi + h_2\xi^2 + \cdots + h_m\xi^m) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathbf{K}_n^{[m]}(h_1, h_2, \cdots, h_m; \lambda; u) \frac{\xi^n}{n!}, \tag{1.10}$$

accompanied by the operational rule:

$$\exp\left(h_2 \frac{\partial^2}{\partial {h_1}^2} + h_3 \frac{\partial^3}{\partial {h_1}^3} + \cdots + h_m \frac{\partial^m}{\partial {h_1}^m}\right) \left\{\mathbf{F}_n^{[m]}(h_1; \lambda; u)\right\} = {}_{\mathcal{H}}\mathbf{K}_n^{[m]}(h_1, h_2, \cdots, h_m; \lambda; u). \tag{1.11}$$

The remainder of the article unfolds as follows: The expanded version of multivariable Hermite-Apostol type Frobenius-Euler polynomials (MHATFEP) is unveiled and scrutinized using the monomiality principle and operational methodologies. Section 2 introduces these EMVHATFEP by leveraging generating functions and operational definitions involving fractional operators. Moving on to Section 3, we delve into the quasi-monomial attributes inherent to the EMVHATFEP. Additionally, this section lays out the recurrence relations and summation formulas for these extended polynomials. Section 4 offers practical applications through the examination of specific cases, and finally, the paper concludes in the concluding section.

2. Extended multivariable Hermite Apostol type Frobenius Euler polynomials (EMVHATFEP)

The operational rule and generating function for the EMVHATFEP are the main topics of this section. Fractional operators are used to introduce and study these polynomials. First we derive the operational rule for these polynomials as the operational rule offers a method for performing algebraic operations on the EMVHATFEP. First operational connection is demonstrated by the succeeding result:

Theorem 2.1. *The following operational connection holds for EMVHAFEP ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)$:*

$$\left(\beta - \left(h_2\frac{\partial^2}{\partial h_1^2} + h_3\frac{\partial^3}{\partial h_1^3} + \cdots + h_m\frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu} \mathbf{K}_n(h_1; \lambda; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta). \tag{2.1}$$

Proof. By substituting q with $\beta - \left(h_2\frac{\partial^2}{\partial h_1^2} + h_3\frac{\partial^3}{\partial h_1^3} + \cdots + h_m\frac{\partial^m}{\partial h_1^m}\right)$ in Eq (1.7) of Euler's integral and subsequently applying this modified equation to $\mathbf{K}_n(h_1; \lambda; u)$, we obtain the following result:

$$\begin{aligned}&\left(\beta - \left(h_2\frac{\partial^2}{\partial h_1^2} + h_3\frac{\partial^3}{\partial h_1^3} + \cdots + h_m\frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu} \mathbf{K}_n(h_1; \lambda; u)\\ &\qquad = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\beta\xi}\xi^{\mu-1}\exp\left(h_2\xi\frac{\partial^2}{\partial h_1^2} + h_3\xi\frac{\partial^3}{\partial h_1^3} + \cdots + h_m\xi\frac{\partial^m}{\partial h_1^m}\right)\mathbf{K}_n(h_1; \lambda; u)d\xi.\end{aligned} \tag{2.2}$$

As evident from Eq (1.11), the following result is achieved:

$$\begin{aligned}&\left(\beta - \left(h_2\frac{\partial^2}{\partial h_1^2} + h_3\frac{\partial^3}{\partial h_1^3} + \cdots + h_m\frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu} \mathbf{K}_n(h_1; \lambda; u)\\ &\qquad = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\beta\xi}\xi^{\mu-1}{}_{\mathcal{H}}\mathbf{K}_n(h_1, h_2\xi, h_3\xi, \cdots, h_m\xi; \lambda; u)d\xi.\end{aligned} \tag{2.3}$$

The new set of polynomials, ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)$, are introduced by utilising the transformation given on the right-hand side of Eq (2.3). These polynomials are recognised as Frobenius-Euler polynomials of the extended Hermite-Apostol type. Consequently, we make the following connection:

$${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta) = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\beta\xi}\xi^{\mu-1}{}_{\mathcal{H}}\mathbf{K}_n(h_1, h_2\xi, h_3\xi, \cdots, h_m\xi; \lambda; u)d\xi. \tag{2.4}$$

Hence, by taking into account expressions (2.3) and (2.4), we confirm the validity of statement (2.1). □

Theorem 2.2. *For the EMVHATFEP, denoted as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)$, the provided generating expression is valid and can be expressed as follows:*

$$\frac{(1-u)\ \exp(h_1 w)}{(\lambda e^w - u)\ (\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)\frac{w^n}{n!}. \tag{2.5}$$

Proof. By multiplying Eq (2.4) by $\frac{w^n}{n!}$ and then summing over all possible values of n, we can deduce the following:

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)\frac{w^n}{n!}\\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(\mu)} \int_0^\infty e^{-\beta\xi}\xi^{\mu-1}{}_{\mathcal{H}}\mathbf{K}_n(h_1, h_2\xi, h_3\xi, \cdots, h_m\xi; \lambda; u)\frac{w^n}{n!}d\xi.\end{aligned}$$

Therefore, considering the expression (1.10) on the right-hand side of the preceding equation, we can determine that:

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)\frac{w^n}{n!}\\ &= \frac{(1-u)\ \exp(h_1 w)}{(\lambda e^w - u)\ \Gamma(\mu)} \int_0^\infty e^{-\left(\beta-(h_2 w^2 + h_3 w^3 + \cdots + h_m w^m)\right)\xi}\ \xi^{\mu-1}d\xi.\end{aligned} \tag{2.6}$$

By examining the integral expression (1.7), we can derive statement (2.5). □

3. Explicit forms and identities

Explicit forms in mathematics and science are crucial for their clarity and directness, revealing underlying structures and aiding interpretation. They simplify calculations, support analytical insights, and facilitate comparisons between objects. Essential for practical applications, they provide efficient models for solving real-world problems, enhancing accessibility and usability in both theoretical research and practical contexts. Engineers, physicists, and practitioners rely on explicit forms to develop computationally efficient mathematical models, driving advancements across various fields of science and engineering.

Continuing, we will now provide the detailed expression for the EMVHATFEP by presenting the following results:

Theorem 3.1. *The EMVHATFEP can be expressed in the following explicit form:*

$${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta) = \sum_{s=0}^{n} \binom{n}{s} \mathbf{K}_s(h_1; \lambda; u)\ {}_{\mu}\mathcal{H}_{n-s}(h_2, h_3, \cdots, h_m; \beta). \tag{3.1}$$

Proof. The generative expression (2.5) can be represented in the following manner:

$$\frac{(1-u)\ \exp(h_1 w)}{(\lambda e^w - u)\ (\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^\mu} = \frac{(1-u)e^{h_1 w}}{(\lambda e^w - u)} \frac{1}{(\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^\mu}. \tag{3.2}$$

This can be further represented as:

$$\sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta) = \sum_{s=0}^{\infty} \mathbf{K}_s(h_1; \lambda; u)\frac{w^s}{s!} \sum_{n=0}^{\infty} {}_{\mu}\mathcal{H}_n(h_2, h_3, \cdots, h_m; \beta)\frac{w^n}{n!}. \tag{3.3}$$

By substituting n with $n-s$ and applying the Cauchy product rule to the right-hand side of the preceding expression, we can derive statement (3.1). □

Theorem 3.2. *The EMVHATFEP adhere to the provided explicit expression:*

$${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta) = \sum_{s=0}^{n} \binom{n}{s} \mathbf{K}_s(\lambda; u)\, {}_{\mu}\mathcal{H}_{n-s}(h_1, h_2, h_3, \cdots, h_m; \beta). \tag{3.4}$$

Proof. The generative expression (2.5) can be represented in the following manner:

$$\frac{(1-u)\ \exp(h_1 w)}{(\lambda e^w - u)\,(\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^\mu} = \frac{(1-u)}{(\lambda e^w - u)} \frac{e^{h_1 w}}{(\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^\mu}. \tag{3.5}$$

This further can be rewritten as

$$\sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta) = \sum_{s=0}^{\infty} \mathbf{K}_s(\lambda; u)\frac{w^s}{s!} \sum_{n=0}^{\infty} {}_{\mu}\mathcal{H}_n(h_1, h_2, h_3, \cdots, h_m; \beta)\frac{w^n}{n!}. \tag{3.6}$$

By substituting n with $n-s$ and applying the Cauchy product rule to the right-hand side of the preceding expression, we can derive statement (3.4). □

Looking ahead, as we examine the generative properties of the EMVHATFEP, denoted as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)$, we can deduce the recurrence relations that govern these polynomials. Recurrence relations represent mathematical formulas that define a multidimensional array or sequence's terms in a recursive manner. They allow us to relate each subsequent term in relation to the ones that precede it. These relations prove particularly valuable when we aim to generate an array or sequence's values in a systematic manner, beginning with one or more initial terms.

By taking derivatives with respect to $h_1, h_2, h_3, \cdots, h_m$, and β of the generative expression (2.5), we can deduce the following recurrence relations for the multivariable Hermite-Apostol type Frobenius-Euler polynomials (MVHATFEP) ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)$:

$$\begin{aligned}
\frac{\partial}{\partial h_1}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)\Big) =& n\ {}_{\mu\mathcal{H}}\mathbf{K}_{n-1}(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta),\\
\frac{\partial}{\partial h_2}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)\Big) =& \mu\ n(n-1)\ {}_{\mu+1 H}\mathbf{K}_{n-2}(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)),\\
\frac{\partial}{\partial h_3}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)\Big) =& \mu\ n(n-1)(n-2)\ {}_{\mu+1 H}\mathbf{K}_{n-3}(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta),\\
&\vdots\\
\frac{\partial}{\partial h_m}\Big({}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta)\Big) =& \mu\ n(n-1)(n-2)\cdots(n-m+1)\\
& {}_{\mu+1 H}\mathbf{K}_{n-m}(h_1, h_2, h_3, \cdots, h_m; \lambda; u; \beta),
\end{aligned}$$

$$\frac{\partial}{\partial \beta}\Big({}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) = -\,\mu\ {}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta). \tag{3.7}$$

Upon examining the aforementioned relations, the following expressions are validated:

$$\begin{aligned}\frac{\partial}{\partial h_2}\Big({}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= -\frac{\partial^3}{\partial h_1^2\partial\beta}\,{}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta),\\ \frac{\partial}{\partial h_3}\Big({}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= -\frac{\partial^4}{\partial h_1^3\partial\beta}\,{}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta),\\ &\vdots\\ \frac{\partial}{\partial h_m}\Big({}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\Big) &= -\frac{\partial^{m+1}}{\partial h_1^m\partial\beta}\,{}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta).\end{aligned} \tag{3.8}$$

The operational framework established in Theorem 2.1 can be extended to various identities associated with Frobenius-Euler polynomials, which have been extensively studied to derive the EMVHATFEP denoted as ${}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)$. To accomplish this, we perform the subsequent operation using the operator (O) defined as $\left(\beta-\left(h_2\frac{\partial^2}{\partial h_1^2}+h_3\frac{\partial^3}{\partial h_1^3}+\cdots+h_m\frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu}$ on identities that involve Frobenius-Euler polynomials $\mathbf{K}_n(h_1;u)$ [30]:

$$u\mathbf{K}_n(h_1;u^{-1})+\mathbf{K}_n(h_1;u) = (1+u)\sum_{k=0}^{n}\binom{n}{k}\mathbf{K}_{n-k}(u^{-1})\mathbf{K}_k(h_1;u), \tag{3.9}$$

$$\frac{1}{n+1}\mathbf{K}_k(h_1,u)+\mathbf{K}_{n-k}(h_1,u) = \sum_{k=0}^{n-1}\frac{\binom{n}{k}}{n-k+1}\sum_{l=k}^{n}\left((-u)\mathbf{K}_{l-k}(u)\mathbf{K}_{n-l}(u)\ +2u\mathbf{K}_{n-k}(u)\right)\mathbf{K}_k(h_1,u)\mathbf{K}_n(h_1,u), \tag{3.10}$$

$$\mathbf{K}_n(h_1,u) = \sum_{k=0}^{n}\binom{n}{k}\mathbf{K}_{n-k}(u)\mathbf{K}_k(h_1,u), \qquad (n\in\mathbb{Z}_+). \tag{3.11}$$

The EMVHATFEP, denoted as ${}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)$, are derived by applying the operator (O) to both sides of the preceding equations:

$$\begin{aligned}u\ {}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u^{-1};\beta)&+{}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta)\\ &= (1+u)\sum_{k=0}^{n}\binom{n}{k}\mathbf{K}_{n-k}(u^{-1})\ {}_{\mu}\mathcal{H}\mathbf{K}_k(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta),\end{aligned} \tag{3.12}$$

$$\begin{aligned}\frac{1}{n+1}\ {}_{\mu}\mathcal{H}\mathbf{K}_k(h_1,h_2,h_3,\cdots,h_m;u;\beta)+{}_{\mu}\mathcal{H}\mathbf{K}_{n-k}(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta) = \sum_{k=0}^{n-1}\frac{\binom{n}{k}}{n-k+1}\sum_{l=k}^{n}((-u)\\ \mathbf{K}_{n-l}(u)\ \mathbf{K}_{l-k}(u)+2u\mathbf{K}_{n-k}(u))\ {}_{\mu}\mathcal{H}\mathbf{K}_k(h_1,h_2,h_3,\cdots,h_m;u;\beta){}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta),\end{aligned} \tag{3.13}$$

$${}_{\mu}\mathcal{H}\mathbf{K}_n(h_1,h_2,h_3,\cdots,h_m;u;\beta) = \sum_{k=0}^{n}\binom{n}{k}\mathbf{K}_{n-k}(u)\ {}_{\mu}\mathcal{H}\mathbf{K}_k(h_1,h_2,h_3,\cdots,h_m;\lambda;u;\beta), \qquad (n\in\mathbb{Z}_+). \tag{3.14}$$

4. Applications

In this section, we will derive particular special cases of the EMVHATFEP and establish their corresponding outcomes:

Corollary 4.1. *The EMVHATFEP can be transformed into the extended multivariable Hermite Frobenius-Euler polynomials by setting $\lambda = 1$. Thus, by inserting $\lambda = 1$ into the left side of Eq* (2.1)*, we obtain the following operational relationship with the EMHFEP that result represented on the right side as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; u; \beta)$*:*

$$\left(\beta - \left(h_2 \frac{\partial^2}{\partial h_1^2} + h_3 \frac{\partial^3}{\partial h_1^3} + \cdots + h_m \frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu} \mathbf{K}_n(h_1; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; u; \beta). \tag{4.1}$$

Corollary 4.2. *The EMVHATFEP can be reduced to the extended multivariable Hermite-Euler polynomials by setting $\lambda = 1$ and $u = -1$. Thus, by substituting $\lambda = 1$, $u = -1$ in the left side of Eq* (2.1)*, the following operational relationship is established with the resulting EMHEP in the right side, represented as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \beta)$*:*

$$\left(\beta - \left(h_2 \frac{\partial^2}{\partial h_1^2} + h_3 \frac{\partial^3}{\partial h_1^3} + \cdots + h_m \frac{\partial^m}{\partial h_1^m}\right)\right)^{-\mu} \mathbf{K}_n(h_1) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \beta). \tag{4.2}$$

Corollary 4.3. *By setting $m = 2$, the EMVHATFEP can be reduced to the extended 2-VHAFEP. Consequently, $m = 2$ is substituted in the left side of Eq* (2.1) *to generate the operational relationship that follows, with the resulting extended 2-VHAFEP being represented in the right side as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta)$*:*

$$\left(\beta - \left(h_2 \frac{\partial^2}{\partial h_1^2}\right)\right)^{-\mu} \mathbf{K}_n(h_1; \lambda; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta). \tag{4.3}$$

Corollary 4.4. *If $\lambda = 1$ and $m = 2$ are set, the EMVHATFEP can be reduced to the extended 2-variable Hermite Frobenius-Euler polynomials. As a result, we create the operational relationship shown below by changing $\lambda = 1$ and $m = 2$ in the left side of Eq* (2.1) *with the extended 2-variable Hermite Frobenius-Euler polynomials that result represented in the right side as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; u; \beta)$*:*

$$\left(\beta - \left(h_2 \frac{\partial^2}{\partial h_1^2}\right)\right)^{-\mu} \mathbf{K}_n(h_1; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; u; \beta). \tag{4.4}$$

Corollary 4.5. *If we set $m = 2$, $h_1 = 2h_1$, and $h_2 = -1$, we may reduce the complexity of the EMVHATFEP to the extended Hermite-Apostol type Frobenius-Euler polynomials. Thus, by inserting $m = 2$, $h_1 = 2h_1$, and $h_2 = -1$ into the left side of Eq* (2.1)*, we create the subsequent operational relationship and express the resulting EHAFEP in the right side as* ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta)$*:*

$$\left(\beta - \left(-\frac{\partial^2}{\partial h_1^2}\right)\right)^{-\mu} \mathbf{K}_n(h_1; \lambda; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, \lambda; u; \beta). \tag{4.5}$$

Corollary 4.6. *By setting $m = 2$, $\lambda = 1$, $h_1 = 2h_1$, and $h_2 = -1$, the EMVHATFEP can be reduced to the extended Hermite Frobenius-Euler polynomials. Thus, by inserting $m = 2$, $\lambda = 1$, $h_1 = 2h_1$, and $h_2 = -1$ into the left side of Eq* (2.1)*, we create the subsequent operational relationship and express the resulting EHFEP in the right side as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1; u; \beta)$:*

$$\left(\beta - \left(-\frac{\partial^2}{\partial h_1^2}\right)\right)^{-\mu} \mathbf{K}_n(h_1; u) = {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1; u; \beta). \tag{4.6}$$

Corollary 4.7. *If $\lambda = 1$, the EMVHATFEP can be reduced to the extended multivariable Hermite Frobenius-Euler polynomials. As a result, we create the generating expression that follows by using $\lambda = 1$ in the left side of Eq* (2.5)*, with the EMHFEP that arise represented in the right side as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; u; \beta)$:*

$$\frac{(1-u)\ \exp(h_1 w)}{(e^w - u)\ (\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; u; \beta) \frac{w^n}{n!}. \tag{4.7}$$

Corollary 4.8. *If $\lambda = 1$ and $u = -1$ are set, the EMVHATFEP can be reduced to the extended multivariable Hermite-Euler polynomials. Thus, we create the generating expression that follows by substituting $\lambda = 1$ and $u = -1$ in the left side of Eq* (2.5)*, with the EMHEP that develop represented in the right side as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \beta)$:*

$$\frac{(2)\ \exp(h_1 w)}{(e^w + 1)\ (\beta - (h_2 w^2 + h_3 w^3 + \cdots + h_m w^m))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2, h_3, \cdots, h_m; \beta) \frac{w^n}{n!}. \tag{4.8}$$

Corollary 4.9. *If $m = 2$, the EMVHATFEP can be reduced to the extended 2-VHAFEP. Thus, we create the generating expression that follows by changing $m = 2$ in the left side of Eq* (2.5)*, with the extended 2-VHAFEP that results represented in the right side as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta)$:*

$$\frac{(1-u)\ \exp(h_1 w)}{(\lambda e^w - u)\ (\beta - (h_2 w^2))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta) \frac{w^n}{n!}. \tag{4.9}$$

Corollary 4.10. *If $\lambda = 1$ and $m = 2$ are set, the EMVHATFEP can be reduced to the extended 2-variable Hermite Frobenius-Euler polynomials. As a result, we create the generating expression that follows by setting $\lambda = 1$ and $m = 2$ in the left side of Eq* (2.5)*, with the extended 2-variable Hermite Frobenius-Euler polynomials that result represented in the right side as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; u; \beta)$:*

$$\frac{(1-u)\ \exp(h_1 w)}{(e^w - u)\ (\beta - (h_2 w^2))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; u; \beta) \frac{w^n}{n!}. \tag{4.10}$$

Corollary 4.11. *If $m = 2$, $h_1 = 2h_1$, and $h_2 = -1$ are specified, the EMVHATFEP can be reduced to the extended Hermite-Apostol type Frobenius-Euler polynomials. As a result, we construct the generating expression that follows by replacing $m = 2$, $h_1 = 2h_1$, and $h_2 = -1$ in the left side of Eq* (2.5)*, with the extended 2-variable Hermite Frobenius-Euler polynomials that occur represented in the right side as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1, h_2; \lambda; u; \beta)$:*

$$\frac{(1-u)\ \exp(2h_1 w)}{(\lambda e^w - u)\ (\beta - (-w^2))^\mu} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1; \lambda; u; \beta) \frac{w^n}{n!}. \tag{4.11}$$

Corollary 4.12. *By setting $m = 2$, $\lambda = 1$, $h_1 = 2h_1$, and $h_2 = -1$, the EMVHATFEP can be reduced to the extended Hermite Frobenius-Euler polynomials. Thus, by inserting $m = 2$, $\lambda = 1$, $h_1 = 2h_1$, and $h_2 = -1$ into the left side of Eq* (2.5)*, we create the subsequent operational relationship and express the resulting EHFEP on the right side as ${}_{\mu\mathcal{H}}\mathbf{K}_n(h_1; u; \beta)$:*

$$\frac{(1-u)\ \exp(2h_1 w)}{(e^w - u)\ (\beta - (-w^2))^{\mu}} = \sum_{n=0}^{\infty} {}_{\mu\mathcal{H}}\mathbf{K}_n(h_1; u; \beta)\frac{w^n}{n!}. \tag{4.12}$$

5. Conclusions

Multivariable special polynomials play an indispensable role in mathematical analysis, encompassing the examination of functions, limits, continuity, and calculus in multiple variables. These polynomials serve as a versatile framework for expressing and scrutinizing multivariable functions, allowing mathematicians to delve into their properties, encompassing characteristics like differentiability, integrability, and convergence.

This study introduces and investigates the multivariable Hermite-Apostol type Frobenius-Euler polynomials, employing the monomiality principle and operational techniques. Section 2 unfolds the extended polynomials, which are derived via generating functions and operational definitions utilizing fractional operators, resulting in the proof of several critical results. In Section 3, we delve into the quasi-monomial properties of these polynomials, simultaneously establishing recurrence relations and summation formulae. This research article significantly contributes to our comprehension of the multivariable Hermite-Apostol type Frobenius-Euler polynomials and their prospective applications within mathematical and scientific realms.

The multivariable Hermite Apostol type Frobenius-Euler polynomials provide a stable platform for future inquiry, allowing for the analysis of numerous algebraic and analytical features, including differential equations and orthogonality. These adaptable polynomials have several uses in the domains of engineering, physics, statistical physics, quantum mechanics, and mathematical physics. The robustness of this method is strengthened by the recurrence relations and the generating functions that are established regarding such polynomials. This leads to fresh insights into the properties of these polynomials and their possible uses in physics and related fields.

When it comes to developing new families of special functions and identifying characteristics associated with both common and generalised special functions, operational procedures prove to be extremely effective tools. Solving partial differential equation families, such the D'Alembert and Heat forms, becomes simpler when these techniques are applied. When the concept of monomiality is combined with the method presented in this article, a wide range of physical problems involving various types of partial differential equations can be analysed.

In future research projects, the factorization method can be used to investigate families of differential equations related to these polynomials. Integral equations could similarly be investigated employing this method. Moreover, future studies might delve into extended forms of these polynomials through the utilization of fractional operators.

Author contributions

Conceptualization, M.Z., G.I.O, S.A.W. and W.R.; Data curation, G.I.O, M.Z.; Formal analysis, G.I.O; Funding acquisition, G.I.O and S.A.W.; Investigation, M.Z., G.I.O , W.R.and S.A.W.; Methodology, S.A.W.; Project administration, G.I.O , W.R.and S.A.W.; Resources, M.Z.; Software, S.A.W.; Supervision, G.I.O, W.R. and S.A.W.; Validation, M.Z. and G.I.O; Visualization, M.Z.; Writing original draft, S.A.W., W.R., G.I.O and M.Z.; Writing review & editing, G.I.O. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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