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Research article

On the uniform stability of a thermoelastic Timoshenko system with infinite memory

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Abstract: The present research is aim at investigating a thermoelastic Timoshenko system with an infinite memory term on the shear force while the bending moment is under the influence of a thermoelastic dissipation governed by Fourier's law. We prove that the system's stability holds for a broader class of relaxation functions. Under this class of relaxation functions h at infinity, we establish a relation between the decay rate of the solution and the growth of h at infinity. Moreover, we drop the boundedness assumptions on the history data. We employ Neumann-Dirichlet-Neumann boundary conditions for our result. In comparison to the bulk of results in the literature, which frequently enforce the "equal-wave-speed" constraint, the present result shows that the infinite memory of the beam and the thermal damping are strong enough to guarantee stability without any conditions on the parameters.

Keywords: Timoshenko; infinite memory; thermoelasticity; stability analysis
Mathematics Subject Classification: 35B35, 35B40, 35D30, 35D35, 93D20

1. Introduction

We consider the following thermoelastic Timoshenko system, for any $x \in (0, 1)$ and $t > 0$,

$$\begin{cases} \rho_1\Phi_{tt} - k(\Phi_x + \Psi)_x + \displaystyle\int_0^\infty h(s)(\Phi_x + \Psi)_x(t - s)ds = 0, \\ \rho_2\Psi_{tt} - b\Psi_{xx} + k(\Phi_x + \Psi) - \displaystyle\int_0^\infty h(s)(\Phi_x + \Psi)(t - s)ds + \gamma\Theta_x = 0, \\ \rho_3\Theta_t - \beta\Theta_{xx} + \gamma\Psi_{xt} = 0, \end{cases} \tag{1.1}$$

in which the shear force is affected by the viscoelastic law, while the bending moment is regulated by thermoelastic dissipation. In (1.1) Φ, Ψ, and Θ are functions of x and t. Φ is the transverse displacement, Ψ represents the angle of rotation of the center of mass of an element, and Θ stands for the temperature difference. Also, the positive constants $k, b, \gamma, \beta, \rho_3, \rho_2, \rho_1$ represent shear coefficient, flexural rigidity, adhesive stiffness, diffusivity, capacity, moment of mass inertia, and mass density respectively. The kernel h is a given function to be specified later. We consider (1.1) with the following boundary conditions;

$$\Phi_x(0,t) = \Theta_x(0,t) = \Psi(0,t) = \Phi_x(1,t) = \Theta_x(1,t) = \Psi(1,t) = 0, \forall t \geq 0 \tag{1.2}$$

and data

$$\begin{cases} \Phi(x,-t) = \Phi_0(x,t),\ \Psi(x,-t) = \Psi_0(x,t),\ \Theta(x,0) = \Theta_0(x),\ x \in [0,1], t > 0, \\ \Phi_t(x,0) = \Phi_1(x), \Psi_t(x,0) = \Psi_1(x),\quad x \in [0,1]. \end{cases} \tag{1.3}$$

This work is motivated by the fact that a Timoshenko system decays exponentially without additional restrictions on the parameters ρ_1, ρ_2, b, k, provided the transverse displacement and rotation angle are controlled. It is pertinent to mention that in establishing our result, the famous equal-wave-speed condition $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ is not imposed. Letting $h = 0$ in system (1.1) and removing the temperature, we get the classical Timoshenko system [1, 2],

$$\begin{cases} \rho_1\Phi_{tt} - k(\Phi_x + \Psi)_x = 0, & x \in (0,1),\quad t > 0, \\ \rho_2\Psi_{tt} - b\Psi_{xx} + k(\Phi_x + \Psi) = 0, & x \in (0,1),\quad t > 0. \end{cases} \tag{1.4}$$

In the literature, various forms of damping have been applied to the system (1.4). The existence, uniqueness, and stability of the resulting systems are studied rigorously; see [3] and references therein. The system (1.4) assumes the form below when the bending moment experiences thermoelastic dissipation.

$$\begin{cases} \rho_1\Phi_{tt} - k(\Phi_x + \Psi)_x = 0, & x \in (0,1),\quad t > 0, \\ \rho_2\Psi_{tt} - b\Psi_{xx} + k(\Phi_x + \Psi) + \gamma\Theta_x = 0, & x \in (0,1),\quad t > 0, \\ \rho_3\Theta_t - \beta\Theta_{xx} + \gamma\Psi_{xt}, & x \in (0,1),\quad t > 0. \end{cases} \tag{1.5}$$

In [4], the authors assumed that the speed wave is equal and proved an exponential stability result. Furthermore, considering system (1.4), Rodrigues et al. [5] derived

$$\begin{cases} \rho_1\Phi_{tt} - k(\Phi_x + \Psi)_x + k\displaystyle\int_0^t h(t-s)(\Phi_x + \Psi)_x(x,s)ds = 0, \\ \rho_2\Psi_{tt} - b\Psi_{xx} + k(\Phi_x + \Psi) - k\displaystyle\int_0^t h(t-s)(\Phi_x + \Psi)(x,s)ds = 0, \end{cases} \tag{1.6}$$

and further proved that (1.6) is uniformly stable if, and only if,

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}. \tag{1.7}$$

Many Timoshenko systems similar to (1.5) have been studied in the literature; see [6–8]. All the stability results were based on assumption (1.7). Also, for results related to system (1.6) without imposing condition (1.7), see [9–12]. Regarding the stability results for Timoshenko system with past history (infinite memory), Al-Mahdi et al. [13] considered the following memory-type Timoshenko system

$$\begin{cases} \rho_1\Phi_{tt} - K(\Phi_x + \Psi)_x = 0, \\ \rho_2\Psi_{tt} - b\Psi_{xx} + K(\Phi_x + \Psi) + \int\limits_0^{+\infty} g(s)\Psi_{xx}(t-s)\mathrm{d}s = 0, \end{cases} \tag{1.8}$$

with Dirichlet boundary conditions, where Q is a positive nonincreasing function satisfying, the following condition

$$g'(t) \leq -\xi(t)Q(g(t)), \qquad \forall t \geq 0. \tag{1.9}$$

The authors established some new results under some appropriate conditions on ξ and Q. See also [14–16] for additional results on viscoelastic problems with infinite memory. The following provide the organization of the remainder of our work: We give preliminary resources in Section 2 that will be useful in getting our results. We establish several important lemmas in Section 3. Lastly, in Section 4, we examine how fast the energy associated with (1.1)–(1.3) decays.

2. Preliminaries

Henceforth, the variable C will signify a generic positive constant. We represent the canonical norm in $L^2(0,1)$ by $\|.\|_2$. On the relaxation function h, we consider the following assumptions.

(D_1) h is a positive nonincreasing C^1-function defined from $[0,+\infty)$ to $(0,+\infty)$ and satisfying

$$0 < l = k - \int_0^\infty h(s)ds. \tag{2.1}$$

(D_2) There exist a C^1 linear function Q defined from $[0,+\infty)$ to $[0,+\infty)$ or strictly C^2 convex function on (0, r], satisfying $r \leq h(t_0)$, $\forall t_0 > 0$ with Q(0) = Q′(0) = 0 and a positive nonincreasing function $\xi : [0,+\infty) \to (0,+\infty)$, which is differentiable and satisfies

$$h'(t) \leq -\xi(t)Q\left(h(t)\right), \quad t \geq 0. \tag{2.2}$$

Remark 2.1. *(1) Conditions (D_1) and (D_2) imply that Q is a C^2-convex function which is strictly increasing on $(0,r]$ and satisfies $Q(0) = Q'(0) = 0$. Therefore, there exists $\overline{Q}$, an extension of Q that is increasing strictly and is a strictly convex C^2-function. For instance,*

$$\overline{Q}(s) = \frac{Q''(r)}{2}s^2 + (Q'(r) - Q''(r)r)s + Q(r) - Q'(r)r + \frac{Q''(r)}{2}r^2, \ t > r. \tag{2.3}$$

(2) Since h is continuous, positive and $h(0) > 0$, then for any $t \geq t_0$, $t_0 > 0$, we have

$$\int_0^t h(s)ds \geq \int_0^{t_0} h(s)ds = h_0 > 0. \tag{2.4}$$

Remark 2.2. *[17] Using the strict convexity of Q on $(0,r]$ and the fact that $Q(0) = 0$, then the following inequality holds:*

$$Q(\nu z) \leq \nu Q(z), \ 0 \leq \nu \leq 1 \text{ and } z \in (0,r]. \tag{2.5}$$

Remark 2.3. *We now define four functions that are invaluable in this work. We begin with a decreasing function on* $(0, r]$ *given by*

$$Q_1(t) := \int_t^r \frac{1}{sQ'(s)} ds. \tag{2.6}$$

The remaining three functions Q_2, Q_3, *and* Q_4 *are defined below:*

$$Q_2(t) = tQ'(\varepsilon_0 t), \quad Q_3(t) = t(Q')^{-1}(t), \quad Q_4(t) = (\overline{Q})^*(t), \tag{2.7}$$

which are all convex and increasing on $(0, r]$.

In view of (1.2), integrate $(1.1)_1$ and $(1.1)_3$ over $(0, 1)$ and get

$$\frac{d^2}{dt^2}\int_0^1 \Phi(x,t)dx = 0, \ \ \frac{d}{dt}\int_0^1 \Theta(x,t)dx = 0. \tag{2.8}$$

Solving Eq (2.8) and applying the initial data result to

$$\int_0^1 \Phi(x,t)dx = t\int_0^1 \Phi_1(x)dx + \int_0^1 \Phi_0(x)dx, \quad \int_0^1 \Theta(x,t)dx = \int_0^1 \Theta_0(x)dx. \tag{2.9}$$

Thus, for all $t \geq 0$,

$$\int_0^1 \bar{\Theta}(x,t)dx = 0 \ \text{ and } \ \int_0^1 \bar{\Phi}(\mathrm{x},\mathrm{t})\mathrm{dx} = 0, \tag{2.10}$$

provided

$$\begin{aligned}\bar{\Phi}(x,t) &= \Phi(x,t) - t\int_0^1 \Phi_1(x)dx - \int_0^1 \Phi_0(x)dx,\\ \bar{\Theta}(x,t) &= \Theta(x,t) - \int_0^1 \Theta_0(x)dx.\end{aligned} \tag{2.11}$$

A consequence of Eq (2.10) is that

$$\|\bar{\Theta}\|_2^2 \leq \|\Theta_x\|_2^2, \ \ \|\bar{\Phi}\|_2^2 \leq \|\Phi_x\|_2^2. \tag{2.12}$$

In addition, $(\bar{\Phi}, \Psi, \bar{\Theta})$ satisfies (1.1) with initial data for $\bar{\Phi}$ and $\bar{\Theta}$ given as

$$\begin{aligned}\bar{\Phi}_0(x) &= \Phi_0(x) - \int_0^1 \Phi_0(x)dx,\\ \bar{\Phi}_1(x) &= \Phi_1(x) - \int_0^1 \Phi_1(x)dx,\\ \bar{\Theta}_0(x) &= \Theta_0(x) - \int_0^1 \Theta_0(x)dx.\end{aligned} \tag{2.13}$$

Henceforth, we take $\bar{\Phi} = \Phi$, $\bar{\Theta} = \Theta$. Thus, $\bar{\Phi}_x = \Phi_x$, and $\bar{\Theta}_x = \Theta_x$. For convenience, we write Φ and Θ. The following spaces are required for stating our well-posedness result:

$$\begin{aligned}&L^2_\star = L^2_\star(0,1) = \left\{ w \in L^2(0,1) : \int_0^1 w(x)dx = 0 \right\},\\ &H^1_\star = H^1_\star(0,1) = H^1(0,1) \cap L^2_\star(0,1),\\ &H^2_\star = H^2_\star(0,1) = \left\{ w \in H^2(0,1) : w_x(0) = w_x(1) = 0 \right\}.\end{aligned}$$

The statement of the well-posedness is as follows:

Theorem 2.1. *Suppose* $(\Phi_0, \Phi_1, \Psi_0, \Psi_1, \Theta_0) \in H^1_\star \times L^2_\star \times H^1_0 \times L^2 \times H^1_\star$ *and condition* (D_1) *hold. Then, problem* (1.1)–(1.3) *has a global weak unique solution* (Φ, Ψ, Θ) *such that*

$$\begin{aligned}(\Phi, \Psi) &\in C\left([0, +\infty), H^1_\star \times H^1_0\right) \cap C^1\left([0, +\infty), L^2_\star \times L^2\right), \\ \Theta &\in C\left([0, +\infty), H^1_\star\right) \cap L^2\left([0, +\infty), H^1_\star \cap H^2_\star\right).\end{aligned} \tag{2.14}$$

Additionally, if the initial data

$$(\Phi_0, \Phi_1, \Psi_0, \Psi_1, \Theta_0) \in H^2_\star \cap H^1_\star \times H^1_\star \times H^2 \cap H^1_0 \times H^1_0 \times H^2_\star \cap H^1_\star,$$

then the unique weak solution (Φ, Ψ, Θ) *achieves more regularity as follows:*

$$\begin{aligned}\Phi &\in C\left([0, +\infty), H^2_\star \cap H^1_\star\right) \cap C^1\left([0, +\infty), H^1_\star\right), \\ \Psi &\in C\left([0, +\infty), H^2 \cap H^1_0\right) \cap C^1\left([0, +\infty), H^1_0\right), \\ \Theta &\in C\left((0, +\infty), H^2_\star \cap H^1_\star\right) \cap C^1\left([0, +\infty), H^1_\star\right).\end{aligned}$$

Using Galerkin approximation method [18], Theorem 2.1 can be easily established. We now present some foundational lemmas necessary for this work.

Lemma 2.1. *For any function* $u \in L^2_{loc}([0, +\infty), L^2(0, 1))$*, the following inequalities are true:*

$$\int_0^1 \left(\int_0^\infty h(t-s)(u(t)-u(s))ds\right)^2 dx \le (1-l)(h \diamond u)(t), \tag{2.15}$$

$$\int_0^1 \left(\int_0^x u(y,t)dy\right)^2 dx \le \|w\|_2^2, \tag{2.16}$$

where

$$(h \diamond u)(t) = \int_0^\infty h(s)\|u(t) - u(t-s)\|_2^2 ds.$$

Proof. The result is just a consequence of the Cauchy-Schwarz and Poincaré inequalities. □

Similar to [17], for any $\alpha \in (0, 1)$, take

$$g(t) = \alpha h(t) - h'(t)$$

and

$$D_\alpha = \int_0^{+\infty} \frac{h^2(s)}{\alpha h(s) - h'(s)} ds.$$

Another foundational lemma is as follows:

Lemma 2.2. *Let* (Φ, Ψ, Θ) *be the solution of problem* (1.1)–(1.3)*. Then, for any* $0 < \alpha < 1$*, we have*

$$\int_0^1 \left(\int_0^\infty h(s)\left((\Phi_x + \Psi)(t) - (\Phi_x + \Psi)(t-s)\right) ds\right)^2 dx \le D_\alpha \left(g \diamond (\Phi_x + \Psi)\right)(t), \tag{2.17}$$

where

$$\left(g \diamond (\Phi_x + \Psi)\right)(t) = \int_0^\infty g(s)\|(\Phi_x + \Psi)(t) - (\Phi_x + \Psi)(t-s)\|_2^2 ds.$$

Proof. It follows that

$$
\begin{aligned}
&\int_0^1 \left(\int_0^\infty h(s)\left((\Phi_x+\Psi)(t)-(\Phi_x+\Psi)(t-s)\right)ds\right)^2 dx \\
&= \int_0^1 \left(\int_0^\infty \frac{h(s)}{\sqrt{g(s)}}\sqrt{g(s)}\left((\Phi_x+\Psi)(t)-(\Phi_x+\Psi)(t-s)\right)ds\right)^2 dx \\
&\leq \left(\int_0^{+\infty} \frac{h^2(s)}{g(s)}ds\right)\int_0^1\int_0^\infty g(s)\left((\Phi_x+\Psi)(t)-(\Phi_x+\Psi)(t-s)\right)^2 dsdx \\
&= D_\alpha\left(g\diamond(\Phi_x+\Psi)\right)(t).
\end{aligned}
\tag{2.18}
$$

□

3. Important lemmas

We present additional lemmas that are pivotal for the principal result.

Lemma 3.1. *The energy of the system* (1.1)–(1.3) *satisfies*

$$E(t) = \frac{1}{2}\left(\rho_1\|\Phi_t\|_2^2 + \rho_2\|\Psi_t\|_2^2 + b\|\Psi_x\|_2^2 + \ell\|(\Phi_x+\Psi)\|_2^2\right) + \frac{1}{2}\left(h\diamond(\Phi_x+\Psi)\right)(t) + \frac{\rho_3}{2}\|\Theta\|_2^2, \tag{3.1}$$

where (Φ,Ψ,Θ) *is the solution of* (1.1)–(1.3). *Moreover,* $E(t)$ *satisfies*

$$E'(t) = -\frac{1}{2}h(t)\|\Phi_x+\Psi\|_2^2 + \frac{1}{2}\left(h'\diamond(\Phi_x+\Psi)\right)(t) - \beta\|\Theta_x\|_2^2 \leq 0,\ \forall\, t\geq 0, \tag{3.2}$$

where

$$\left(h\diamond(\Phi_x+\Psi)\right)(t) = \int_0^\infty h(s)\|(\Phi_x+\Psi)(t)-(\Phi_x+\Psi)(t-s)\|_2^2 ds.$$

Proof. Multiply Eq $(1.1)_1$ by Φ_t, Eq $(1.1)_2$ by Ψ_t, and Eq $(1.1)_3$ by Θ. Then, integrate all terms over $(0,1)$ in view of the boundary conditions. Adding the results, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\Phi_t\|_2^2 + k\|\Phi_x+\Psi\|_2^2 + \rho_2\|\Psi_t\|_2^2 + b\|\Psi_x\|_2^2 + \rho_3\|\Theta\|_2^2\right) \\
&\underbrace{-\int_0^1(\Phi_x+\Psi)_t\int_0^\infty h(s)(\Phi_x+\Psi)(x,t-s)dsdx}_{T_1} = -\beta\|\Theta_x\|_2^2.
\end{aligned}
\tag{3.3}
$$

The estimate of the term T_1 is as follows:

$$
\begin{aligned}
T_1 &= \int_0^1(\Phi_x+\Psi)_t\int_0^\infty h(s)\left(\Phi_x+\Psi)(x,t)-(\Phi_x+\Psi)(x,t-s)\right)dsdx \\
&\quad - \int_0^\infty h(s)ds\int_0^1(\Phi_x+\Psi)_t(\Phi_x+\Psi)dx \\
&= \frac{1}{2}\int_0^1\int_0^\infty h(s)\frac{d}{dt}\left((\Phi_x+\Psi)(x,t)-(\Phi_x+\Psi)(x,t-s)\right)^2 dsdx
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^\infty h(s)ds\frac{d}{dt}\|\Phi_x+\Psi\|_2^2\\
&=\frac{1}{2}\frac{d}{dt}\left(h\diamond(\Phi_x+\Psi)\right)(t)-\frac{1}{2}\left(h'\diamond(\Phi_x+\Psi)\right)(t)\\
&-\frac{1}{2}\frac{d}{dt}\left(\int_0^\infty h(s)ds\|\Phi_x+\Psi\|_2^2\right)+\frac{1}{2}h(t)\|\Phi_x+\Psi\|_2^2.
\end{aligned}
\tag{3.4}
$$

The inequality (3.2) is obtained by subbing (3.4) into (3.3). It follows from (3.2) that the energy $E(t)$ is decreasing and bounded above by $E(0)$. □

The next lemma is as follows:

Lemma 3.2. *For all $t \geq 0$, there exists a positive constant M_1 that satisfies*

$$
\int_t^\infty h(s)\|(\Phi_x+\Psi)(t)-(\Phi_x+\Psi)(t-s)\|_2^2 dsdx \leq M_1 h_0(t),\ \forall t\geq 0, \tag{3.5}
$$

with $h_0(t) = \int_0^\infty h(t+s)\left(1+\|(\Phi_x+\Psi)_{0x}(s)\|^2\right)ds$.

Proof.

$$
\begin{aligned}
&\int_0^1\int_t^\infty h(s)\left((\Phi_x+\Psi)(t)-(\Phi_x+\Psi)(t-s)\right)^2 ds\\
&\leq 2\|(\Phi_x+\Psi)(t)\|_2^2\int_t^\infty h(s)ds+2\int_t^\infty h(s)\|(\Phi_x+\Psi)(t-s)\|^2 ds\\
&\leq 2\sup_{s\geq 0}\|(\Phi_x+\Psi)(s)\|_2^2\int_0^\infty h(t+s)ds+2\int_0^\infty h(t+s)\|(\Phi_x+\Psi)(-s)\|_2^2 ds\\
&\leq\left(\frac{4}{\ell}E(s)\right)\int_0^\infty h(t+s)ds+2\int_0^\infty h(t+s)\|(\Phi_x+\Psi)_{0x}(s)\|_2^2 ds\\
&\leq\left(\frac{4}{\ell}E(0)\right)\int_0^\infty h(t+s)ds+2\int_0^\infty h(t+s)\|(\Phi_x+\Psi)_{0x}(s)\|_2^2 ds\\
&\leq M_1\int_0^\infty h(t+s)\left(1+\|(\Phi_x+\Psi)_{0x}(s)\|_2^2\right)ds,
\end{aligned}
\tag{3.6}
$$

where $M_1 = \max\left\{2,\left(\frac{4}{\ell}E(0)\right)\right\}$. □

Lemma 3.3. *Let (Φ,Ψ,Θ) be the unique solution obtained in Theorem 2.1. Then, for any ϵ_1, $\epsilon_2 > 0$, the functional F_1 defined by*

$$
F_1(t)=\rho_3\int_0^1\Psi_t\int_0^x\Theta(y,t)dydx
$$

satisfies (3.7)

$$
\begin{aligned}
F_1'(t)\leq&-\frac{\gamma}{2}\|\Psi_t\|_2^2+\epsilon_1\|\Psi_x\|_2^2+\epsilon_2\|\Phi_x+\Psi\|_2^2+C\left(1+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right)\|\Theta_x\|_2^2\\
&+CD_\alpha\left(g\diamond(\Phi_x+\Psi)\right)(t),\quad\forall\ t\geq 0,
\end{aligned}
\tag{3.7}
$$

where Q and D_α are defined in Lemma 2.2.

Proof. Start by differentiating F_1, then apply $(1.1)_2$ and $(1.1)_3$. Proceeding with integration by parts in view of (2.10), we arrive at

$$\begin{aligned}F_1'(t) &= -\gamma\|\Psi_t\|_2^2 - \underbrace{\frac{b\rho_3}{\rho_2}\int_0^1 \Psi_x\Theta dx}_{T_2} - \underbrace{\frac{k\rho_3}{\rho_2}\int_0^1(\Phi_x+\Psi)\int_0^x \Theta(y,t)dydx}_{T_3} \\ &\quad \underbrace{-\frac{\rho_3}{\rho_2}\int_0^1\int_0^\infty h(s)\left((\Phi_x+\Psi)(x,t)-(\Phi_x+\Psi)(x,t-s)\right)ds\int_0^x\Theta(y,t)dydx}_{T_4} \\ &\quad + \underbrace{\frac{\rho_3}{\rho_2}\int_0^\infty h(s)ds\int_0^1(\Phi_x+\Psi)\int_0^x\bar{\Theta}(y,t)dydx}_{T_5} + \underbrace{\beta\int_0^1\Psi_t\Theta_x dx}_{T_6} + \frac{\gamma\rho_3}{\rho_2}\|\Theta\|_2^2.\end{aligned} \tag{3.8}$$

Applying (2.10), the Cauchy Schwarz inequality, and progressing similarly as Lemmas 2.1 and 2.2. The following estimates hold:

$$\begin{aligned}&T_2 \le \epsilon_1\|\Psi_x\|_2^2 + \frac{C}{\epsilon_1}\|\Theta_x\|_2^2,\;\; T_3 \le \frac{\epsilon_2}{2}\|\Phi_x+\Psi\|_2^2 + \frac{C}{\epsilon_2}\|\Theta_x\|_2^2, \\ &T_4 \le \frac{CD_\alpha}{2}\left(g\diamond(\Phi_x+\Psi)\right)(t) + \frac{C}{2}\|\Theta_x\|_2^2,\;\; T_5 \le \frac{\epsilon_2}{2}\|\Phi_x+\Psi\|_2^2 + \frac{C}{\epsilon_2}\|\Theta_x\|_2^2, \\ &T_6 \le \frac{\gamma}{2}\|\Psi_t\|_2^2 + \frac{\beta^2}{2\gamma}\|\Theta_x\|_2^2.\end{aligned} \tag{3.9}$$

Subbing (3.9) in (3.8), we arrived at (3.7). □

Lemma 3.4. *Let* (Φ,Ψ,Θ) *be the unique solution obtained in Theorem 2.1. Then, the functional* F_2 *defined by*

$$F_2(t) = -\rho_1\int_0^1(\Phi_x+\Psi)\int_0^x\Phi_t(y,t)dydx$$

satisfies the estimate,

$$F_2'(t) \le -\frac{\ell}{2}\|\Phi_x+\Psi\|_2^2 + C\|\Phi_t\|_2^2 + C\|\Psi_t\|_2^2 + CD_\alpha\left(g\diamond(\Phi_x+\Psi)\right)(t),\;\;\forall\, t\ge 0, \tag{3.10}$$

where Q *and* D_α *are defined in Lemma 2.2.*

Proof. We differentiate F_2 directly to get

$$F_2'(t) = -\rho_1\int_0^1(\Phi_x+\Psi)_t\int_0^x\Phi_t(y,t)dydx - \rho_1\int_0^1(\Phi_x+\Psi)\int_0^x\Phi_{tt}(y,t)dydx.$$

In view of $(1.1)_1$ and applying integration by parts result to

$$\begin{aligned}F_2'(t) &= -\ell\|\Phi_x+\Psi\|_2^2 + \rho_1\|\Phi_t\|_2^2 - \rho_1\int_0^1\Psi_t\int_0^x\Phi_t(y,t)dydx \\ &\quad - \int_0^1(\Phi_x+\Psi)(x,t)\int_0^\infty h(s)\left((\Phi_x+\Psi)(x,t)-(\Phi_x+\Psi)(x,t-s)\right)dsdx.\end{aligned} \tag{3.11}$$

For any $\delta_1 > 0$, we have

$$F_2'(t) \leq -\ell\|\Phi_x+\Psi\|_2^2 + \frac{3\rho_1}{2}\|\bar{\Phi}_t\|_2^2 + \frac{\rho_1}{2}\|\Psi_t\|_2^2 + \delta_1\|\Phi_x+\Psi\|_2^2 + \frac{D_\alpha}{4\delta_1}\left(g \diamond (\Phi_x+\Psi)\right)(t). \tag{3.12}$$

Using (D_1) and choosing $\delta_1 = \dfrac{\ell}{2}$, we get (3.10). □

Lemma 3.5. *Suppose the hypothesis in Lemma 3.4 holds. For any $\epsilon_1 > 0$ and $t_0 > 0$, the functional F_3 defined by*

$$F_3(t) = -\rho_1 \int_0^1 \Phi_t \int_0^x \int_0^\infty h(s)\Big((\Phi_y+\Psi)(y,t) - (\Phi_y+\Psi)(y,t-s)\Big)\,dsdydx$$

satisfies

$$\begin{aligned} F_3'(t) \leq & -\frac{\rho_1(k-\ell)}{2}\|\Phi_t\|_2^2 + C\|\Psi_t\|_2^2 + \epsilon_2\|\Phi_x+\Psi\|_2^2 \\ & + CD_\alpha\left(1+\frac{1}{\epsilon_2}\right)\left(g \diamond (\Phi_x+\Psi)\right)(t), \quad \forall\, t \geq 0, \end{aligned} \tag{3.13}$$

and Q and D_α are defined in Lemma 2.2.

Proof. Direct differentiation of F_3; gives

$$\begin{aligned} F_3'(t) = & \underbrace{-\rho_1 \int_0^1 \Phi_{tt} \int_0^x \int_0^\infty h(s)\Big((\Phi_y+\Psi)(y,t) - (\Phi_y+\Psi)(y,t-s)\Big)\,dsdydx} \\ & \underbrace{-\rho_1 \int_0^1 \Phi_t \int_0^x \int_0^\infty h'(s)(\Big(\Phi_y+\Psi)(y,t) - (\Phi_y+\Psi)(y,t-s)\Big)\,dsdydx}_{T_8} \\ & \underbrace{-\rho_1 \int_0^1 \Phi_t \int_0^x \int_{-\infty}^t h(t-s)(\Phi_y+\Psi)_t(y,t)\,dsdydx}_{T_9}. \end{aligned} \tag{3.14}$$

In a similar approach to obtaining the estimates in (3.9), the following hold:

$$\begin{aligned} T_7 = & \int_0^1 (\Phi_x+\Psi) \int_0^\infty h(s)\left((\Phi_x+\Psi)(t) - (\Phi_x+\Psi)(t-s)\right)dsdx \\ & + \int_0^1 \left(\int_0^\infty h(s)\left((\Phi_x+\Psi)(x,t) - (\Phi_x+\Psi)(x,t-s)\right)ds\right)^2 dx \\ \leq & \ \epsilon_2\|\Phi_x+\Psi\|_2^2 + CD_\alpha\left(1+\frac{1}{\epsilon_2}\right)\left(g \diamond (\Phi_x+\Psi)\right)(t), \forall \epsilon_2 > 0. \end{aligned} \tag{3.15}$$

$$\begin{aligned} T_8 = & -\rho_1 \int_0^1 \Phi_t \int_0^x \int_0^\infty h'(s)\Big((\Phi_y+\Psi)(y,t) - (\Phi_y+\Psi)(y,t-s)\Big)\,dsdydx \\ = & \rho_1 \int_0^1 \Phi_t \int_0^x \int_0^\infty g(s)\Big((\Phi_y+\Psi)(y,t) - (\Phi_y+\Psi)(y,t-s)\Big)\,dsdydx \end{aligned}$$

$$- \rho_1\alpha \int_0^1 \Phi_t \int_0^x \int_0^\infty h(s)\Big((\Phi_y+\Psi)(y,t) - (\Phi_y+\Psi)(y,t-s)\Big) dsdydx$$
$$\leq \frac{\delta_2}{2}\|\Phi_t\|_2^2 + \frac{C(1+D_\alpha)}{\delta_2}\left(g \diamond (\Phi_x+\Psi)\right)(t), \forall \delta_2 > 0. \tag{3.16}$$

To estimate T_9, we recall (2.4) and (2.10). As a result, we get, for any positive δ_2,

$$\begin{aligned}
T_9 &= -\rho_1 \int_0^1 \Phi_t \int_0^x \int_{-\infty}^t h(t-s)(\Phi_y+\Psi)_t(y,t) dsdydx \\
&= -\rho_1 \int_0^\infty h(s)ds \int_0^1 \Phi_t \int_0^x (\Phi_y+\Psi)_t(y,t)dydx \\
&= -\rho_1 \int_0^\infty h(s)ds \int_0^1 \Phi_t \int_0^x \Phi_{yt}(y,t)dydx \\
&\quad - \rho_1 \int_0^\infty h(s)ds \int_0^1 \Phi_t \int_0^x \Psi_t(y,t-s)dydx \\
&= -\rho_1 \int_0^\infty h(s)ds\|\Phi_t\|_2^2 - \rho_1 \int_0^\infty h(s)ds \int_0^1 \Phi_t \int_0^x \Psi_t(y,t)dydx \\
&\leq -\rho_1(k-\ell)\|\Phi_t\|_2^2 + \frac{\delta_2}{2}\|\Phi_t\|_2^2 + \frac{(\rho_1 h_0)^2}{2\delta_2}\|\Psi_t\|_2^2.
\end{aligned} \tag{3.17}$$

Combining (3.15)–(3.17), we obtain

$$\begin{aligned}
F_3'(t) \leq& -\left(\rho_1(k-\ell) - \delta_2\right)\|\Phi_t\|_2^2 + \frac{C}{\delta_2}\|\Psi_t\|_2^2 + \epsilon_2\|\Phi_x+\Psi\|_2^2 \\
&+ CD_\alpha\left(1 + \frac{1}{\epsilon_2} + \frac{1}{\delta_2}\right)\left(g \diamond (\Phi_x+\Psi)\right)(t).
\end{aligned} \tag{3.18}$$

Choosing $\delta_2 = \frac{\rho_1(k-\ell)}{2}$ gives our desired result (3.13). □

Lemma 3.6. *Suppose the hypothesis of Lemma 3.4 is true. Then, the functional F_4 defined by $F_4(t) = \rho_2 \int_0^1 \Psi\Psi_t dx$ satisfies*

$$F_4'(t) \leq -\frac{b}{2}\|\Psi_x\|_2^2 + \rho_2\|\Psi_t\|_2^2 + C\|\Phi_x+\Psi\|_2^2 + CD_\alpha\left(g \diamond (\Phi_x+\Psi)\right)(t) + C\|\Theta_x\|_2^2, \quad \forall\, t \geq 0, \tag{3.19}$$

where Q and D_α are defined in Lemma 2.2.

Proof. We have that

$$\begin{aligned}
F_4'(t) =& \rho_2\|\Psi_t\|_2^2 - b\|\Psi_x\|_2^2 - \underbrace{k\int_0^1 \Psi(\Phi_x+\Psi)dx}_{T_{10}} \\
&+ \underbrace{\int_0^1 \Psi \int_0^\infty h(s)(\Phi_x+\Psi)(x,t-s)dx}_{T_{11}} - \underbrace{\gamma\int_0^1 \Psi\Theta_x dx}_{T_{12}}.
\end{aligned} \tag{3.20}$$

Furthermore, for any $\delta_3 > 0$,

$$\begin{aligned}
T_{10} \leq & \frac{\delta_3}{4}\|\Psi_x\|_2^2 + \frac{C}{\delta_3}\|\Phi_x + \Psi\|_2^2, \\
T_{11} = & \int_0^\infty h(s)ds \int_0^1 \Psi(\Phi_x + \Psi)dx \\
& - \int_0^1 \Psi \int_0^\infty h(s)((\Phi_x + \Psi)(x,t) - (\Phi_x + \Psi)(x, t-s))dx \\
\leq & \frac{\delta_3}{2}\|\Psi_x\|_2^2 + \frac{C}{\delta_3}\|\Phi_x + \Psi\|_2^2 + \frac{CD_\alpha}{\delta_3}\left(g \diamond (\Phi_x + \Psi)\right)(t), \\
T_{12} \leq & \frac{\delta_3}{4}\|\Psi_x\|_2^2 + \frac{C}{\delta_3}\|\Theta_x\|_2^2.
\end{aligned} \tag{3.21}$$

It follows that

$$F_4'(t) \leq -(b-\delta_3)\|\Psi_x\|_2^2 + \rho_2\|\Psi_t\|_2^2 + \frac{C}{\delta_3}\|\Phi_x + \Psi\|_2^2 + \frac{CD_\alpha}{\delta_3}\left(g \diamond (\Phi_x + \Psi)\right)(t) + \frac{C}{\delta_3}\|\Theta_x\|_2^2.$$

Then, choosing $\delta_3 = \dfrac{b}{2}$; gives the desired result (3.19). □

Lemma 3.7. *The functional F_5 defined by*

$$F_5(t) = \int_0^1 \int_0^t J(t-s)(\Phi_x + \Psi)^2(x,s)dsdx, \text{ where } J(t) = \int_t^{+\infty} h(s)ds,$$

satisfies the estimate below:

$$\begin{aligned}
F_5'(t) \leq & -\frac{1}{2}\left(h \diamond (\Phi_x + \Psi)\right)(t) + 3(1-\ell)\|\Phi_x + \Psi\|_2^2 \\
& + \frac{1}{2}\int_0^1 \int_t^{+\infty} h(s)\left(h \diamond (\Phi_x + \Psi)(t) - (\Phi_x + \Psi)(t-s)\right)^2 dsdx.
\end{aligned} \tag{3.22}$$

Proof. The proof has similar steps as in [13]. □

Lemma 3.8. *For suitable choices of $W,\ W_j,\ j = 1,2,3,4,$ the Lyapunov functional*

$$\mathcal{L}(t) = WE(t) + \sum_{j=1}^{4} W_j F_j(t) \tag{3.23}$$

satisfies the estimate

$$d_1 E(t) \leq \mathcal{L}(t) \leq d_2 E(t) \tag{3.24}$$

and

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\lambda\left(\|\Phi_t\|_2^2 + \|\Psi_t\|_2^2 + \|\Psi_x\|_2^2 + \|\Phi_x + \Psi\|_2^2 + \|\Theta_x\|_2^2\right) \\
& + \frac{1}{4}\left(h \diamond (\Phi_x + \Psi)\right)(t),\ \forall\, t \geq 0,
\end{aligned} \tag{3.25}$$

for some $\lambda > 0$ and $d_1,\ d_2 > 0$.

Proof. We get

$$|\mathcal{L}(t) - WE(t)| \leq W_1 |F_1(t)| + W_2 |F_2(t)| + W_3 |F_3(t)| + W_4 |F_4(t)| . \tag{3.26}$$

Applying Young, Poincaré, and Cauchy-Schwarz; inequalities, we obtain

$$\begin{aligned} |\mathcal{L}(t) - WE(t)| \leq & C \left(\|\Phi_t\|_2^2 + \|\Psi_t\|_2^2 + \|\Psi_x\|_2^2 + \|\Phi_x + \Psi\|_2^2 + \|\bar{\Theta}\|_2^2 \right) + C \left(h \diamond (\Phi_x + \Psi) \right)(t) \\ \leq & CE(t). \end{aligned}$$

It follows that

$$(W - C)E(t) \leq \mathcal{L}(t) \leq (W + C)E(t). \tag{3.27}$$

Choosing a large enough W that ensures $(W - C) > 0$ yields (3.24).

Then, using Lemmas 3.1–3.6 and our earlier definition $g = \alpha h - h'$, we get

$$\begin{aligned} \mathcal{L}'(t) \leq & -\left[\frac{\rho_1 (k - \ell) W_3}{2} - W_2 C \right] \|\Phi_t\|_2^2 - \left[\frac{\gamma W_1}{2} - W_2 C - W_3 C - W_4 \rho_2 \right] \|\Psi_t\|_2^2 \\ & - \left[\frac{b W_4}{2} - W_1 \epsilon_1 \right] \|\Psi_x\|_2^2 - \left[\frac{\ell W_2}{2} - \epsilon_2 W_1 - \epsilon_2 W_3 - W_4 C \right] \|\Phi_x + \Psi\|_2^2 \\ & - \left[W\beta - W_1 C \left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \right) - W_4 C \right] \|\Theta_x\|_2^2 + \frac{W\alpha}{2} \left(h \diamond (\Phi_x + \Psi) \right)(t) \\ & - \left[\frac{W}{2} - C D_\alpha \left(W_1 + W_2 + W_3 \left(1 + \frac{1}{\epsilon_2} \right) + W_4 \right) \right] \left(g \diamond (\Phi_x + \Psi) \right)(t), \forall t \geq 0. \end{aligned} \tag{3.28}$$

By choosing

$$\epsilon_1 = \frac{b W_4}{4 W_1}, \quad \epsilon_2 = \frac{\ell W_2}{4 (W_1 + W_3)} \tag{3.29}$$

the inequality (3.28) becomes

$$\begin{aligned} \mathcal{L}'(t) \leq & -\left[\frac{\rho_1 (k - \ell) W_3}{2} - W_2 C \right] \|\bar{\Phi}_t\|_2^2 - \left[\frac{\gamma W_1}{2} - W_2 C - W_3 C - W_4 \rho_2 \right] \|\Psi_t\|_2^2 \\ & - \frac{b W_4}{4} \|\Psi_x\|_2^2 - \left[\frac{\ell W_2}{4} - W_4 C \right] \|\Phi_x + \Psi\|_2^2 + \frac{W\alpha}{2} \left(h \diamond (\Phi_x + \Psi) \right)(t) \\ & - \left[W\beta - W_1 C \left(1 + \frac{4 W_1}{b W_4} + \frac{4 (W_1 + W_3)}{\ell W_2} \right) - W_4 C \right] \|\Theta_x\|_2^2 \\ & - \left[\frac{W}{2} - C D_\alpha \left(W_1 + W_2 + C_{\bar{W}} + W_4 \right) \right] \left(g \diamond (\Phi_x + \Psi) \right)(t), \end{aligned} \tag{3.30}$$

with

$$C_{\bar{W}} = W_3 \left(1 + \frac{4 (W_1 + W_3)}{\ell W_2} \right).$$

At this juncture, we carefully select our constants: First, we choose W_2 big enough so that

$$\frac{\ell W_2}{2} - W_4 C > 0. \tag{3.31}$$

Second, we select W_3 large enough such that

$$\frac{\rho_1(k-\ell)W_3}{2} - W_2C > 0. \tag{3.32}$$

Third, we take W_1 large enough such that

$$\frac{\gamma W_1}{2} - W_2C - W_3C - W_4\rho_2 > 0. \tag{3.33}$$

Now, we see that

$$\frac{\alpha h^2(s)}{g(s)} = \frac{\alpha h^2(s)}{\alpha h(s) - h'(s)} < h(s);$$

therefore, upon applying the dominated convergence theorem, we observe that

$$\alpha D_\alpha = \int_0^{+\infty} \frac{\alpha h^2(s)}{\alpha h(s) - h'(s)} ds \to 0 \ \ as \ \ \alpha \to 0. \tag{3.34}$$

Hence, there exists $\alpha_0 \in (0,1)$ such that if $\alpha < \alpha_0 < 1$, we have

$$\alpha D_\alpha < \frac{1}{2C\left(W_1 + W_2 + W_3\left(1 + \frac{4(W_1+W_3)}{\ell W_2}\right) + W_4\right)}.$$

Lastly, we choose W large enough and take $\alpha = \dfrac{1}{2W}$, so that (3.24) remains valid and

$$W\beta - W_1C\left(1 + \frac{4W_1}{bW_4} + \frac{4(W_1+W_3)}{klW_2}\right) - W_4C > 0 \tag{3.35}$$

and

$$\frac{W}{2} - CD_\alpha\left(W_1 + W_2 + W_3\left(1 + \frac{4(W_1+W_3)}{klW_2}\right) + W_4\right) > 0. \tag{3.36}$$

A combination of (3.29)–(3.36) yields (3.25). □

Lemma 3.9. *Suppose* (D_1) *and* (D_2) *hold. Then, the energy functional satisfies*

$$\int_0^t E(s)ds < \tilde{m}h_1(t), \forall t \in \mathbb{R}^+, \tag{3.37}$$

where $h_1(t) = \left(1 + \int_0^t h_0(s)ds\right)$ *and* h_0 *is defined in Lemma 3.2.*

Proof. To prove the above lemma, we let $F(t) = \mathcal{L}(t) + F_5(t)$. Then, using (3.25) and (3.22), we obtain the following bound for all $t \in \mathbb{R}^+$,

$$F'(t) \leq -ME(t) + \frac{1}{2}\int_0^1 \int_t^{+\infty} h(s)\left((\Phi_x + \Psi)(t) - (\Phi_x + \Psi)(t-s)\right)^2 dsdx, \tag{3.38}$$

where M is some positive constant. As a result, we have

$$
\begin{aligned}
M\int_0^t E(s)ds &\le F(0)-F(t)\\
&\quad+\frac{M_1}{2}\int_0^t\int_0^{+\infty} h(\tau+s)\,(1+|(\Phi_x+\Psi)_{0x}(s)|ds)^2\,d\tau ds \\
&\le F(0)+\frac{M_1}{2}\int_0^t h_0(s)ds.
\end{aligned}
\tag{3.39}
$$

Hence, taking $\tilde{m}=\max\left\{\frac{F(0)}{M},\frac{M_1}{2M}\right\}$, we established (3.37). □

4. Decay result

In this section, we state and prove our decay result. To begin, we define

$$\mu(t):=-\int_0^t h'(s)\int_0^1 |(\Phi_x+\Psi)(t)-(\Phi_x+\Psi)(t-s)|^2dxds\le -c_1E'(t), \tag{4.1}$$

and let

$$\lambda(t):=q(t)\int_0^t\int_0^1 |(\Phi_x+\Psi)(t)-(\Phi_x+\Psi)(t-s)|^2dxds,$$

where

$$\lambda(t)<1,\quad \forall t>0. \tag{4.2}$$

Thus, by (3.37) we have

$$q(t):=\frac{q_0}{h_1(t)}<1 \quad\text{and}\quad 0<q_0<\min\left\{1,(\frac{\ell}{4\tilde{m}})\right\}. \tag{4.3}$$

Lemma 4.1. *Suppose* (D_1) *and* (D_2) *hold, then for all* $t>0$*,*

$$\int_0^t h(s)\int_0^1 |(\Phi_x+\Psi)(t)-(\Phi_x+\Psi)(t-s)|^2dxds\le \frac{1}{q(t)}\overline{Q}^{-1}\left(\frac{q(t)\mu(t)}{\gamma(t)}\right), \tag{4.4}$$

where $\overline{Q}$ *is as defined in Remark 2.3.*

Proof. See [17] for the proof of (4.4). □

Our main result is the theorem below:

Theorem 4.1. *Assume that the conditions* (D_1) *and* (D_2) *hold. Then, we have the following decay result. For all* $0\le s\le t$ *and for strictly positive constant* C*,*

$$E(t)\le\left(\frac{E(0)}{q(s)}\right)Q_2^{-1}\Big[\frac{C+\int_0^t\xi(s)Q_4\left[\frac{c_1}{d}q(s)h_0(s)\right]ds,}{\int_0^t\xi(s)ds}\Big], \tag{4.5}$$

where $q(s), h_0(s), Q_2(s)$*, and* $Q_4(s)$ *are functions defined in Lemma 3.2 and Eq* (2.7)*.*

Proof. For the proof, we first combine Eqs (3.5), (3.25), (4.3), and (4.4). So, for some $M > 0$ and for any $t \geq 0$, we have

$$\mathcal{L}'(t) \leq -ME(t) + \frac{c_1}{q(t)}\overline{Q}^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right) + c_1 h_0(t). \tag{4.6}$$

For $\varepsilon_0 < r$, define a functional $\mathcal{F}$; as

$$\mathcal{F}(t) := \overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)\mathcal{L}(t).$$

Then, $\mathcal{F} \sim E$, and by noting that $\overline{Q}'' \geq 0$, $q' \leq 0$ and $E' \leq 0$, it follows from (4.6) that

$$\begin{aligned}\mathcal{F}'(t) &= \varepsilon_0 \frac{(qE)'(t)}{E(0)}\overline{Q}''\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)\mathcal{L}(t) + \overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)L'(t) \\ &\leq -ME(t)\overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + \frac{c_1}{q(t)}\overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)\overline{Q}^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right) \\ &+ c_1 h_0(t)\overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right).\end{aligned} \tag{4.7}$$

Now, let $\overline{Q}^*$ be the convex conjugate of $\overline{Q}$ [19], then

$$\overline{Q}^*(s) = s(\overline{Q}')^{-1}(s) - \overline{Q}\left[(\overline{Q}')^{-1}(s)\right], \quad \text{if } s \in (0, \overline{Q}'(r)] \tag{4.8}$$

and $\overline{Q}^*$ satisfies the following Young inequality

$$AB \leq \overline{Q}^*(A) + \overline{Q}(B), \quad \text{if } A \in (0, \overline{Q}'(r)],\ B \in (0, r]. \tag{4.9}$$

So, with $A = \overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)$ and $B = \overline{Q}^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right)$, and using (2.2) and (4.7)–(4.9), we get

$$\begin{aligned}\mathcal{F}'(t) &\leq -ME(t)\overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + \frac{c_1}{q(t)}\overline{Q}^*\left(\overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)\right) + c_1\left(\frac{\mu(t)q(t)}{\xi(t)}\right) \\ &+ c_1 h_0(t)\overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) \\ &\leq -ME(t)\overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + c_1\varepsilon_0 \frac{E(t)}{E(0)}\overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + c_1\left(\frac{\mu(t)q(t)}{\xi(t)}\right) \\ &+ c_1 h_0(t)\overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right).\end{aligned} \tag{4.10}$$

For simplicity, let us use $\overline{Q} = Q$. So, multiply (4.10) by $\xi(t)$ and use (4.1) and $\varepsilon_0 \frac{E(t)q(t)}{E(0)} < r$. Also, with $\overline{Q}'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) = Q'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)$, we get

$$\begin{aligned}\xi(t)\mathcal{F}'(t) &\leq -M\xi(t)E(t)Q'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + c_1\xi(t)\varepsilon_0 \frac{E(t)}{E(0)}Q'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) \\ &+ c_1\mu(t)q(t) + c_1 h_0(t)\xi(t)Q'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) \\ &\leq -(ME(0) - c_1\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}Q'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) - c_1 E'(t) \\ &+ c_1\xi(t)h_0(t)Q'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right).\end{aligned}$$

Recalling the definition of Q_2 and choosing ε_0 small enough, we obtain;

$$\begin{aligned}\mathcal{F}_1'(t) &\leq -b\xi(t)\left(\frac{E(t)}{E(0)}\right)Q'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+c_1\xi(t)h_0(t)Q'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\\ &= -b\frac{\xi(t)}{q(t)}Q_2\left(\frac{E(t)q(t)}{E(0)}\right)+c_1\xi(t)h_0(t)Q'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right),\end{aligned} \tag{4.11}$$

for some $b>0$ and for all $t\in\mathbb{R}^+$, where $\mathcal{F}_1=\xi\mathcal{F}+cE\sim E$, satisfying for some $\alpha_1,\alpha_2>0$.

$$\alpha_1\mathcal{F}_1(t)\leq E(t)\leq\alpha_2\mathcal{F}_1(t). \tag{4.12}$$

Moreover, since $Q_2'(t)=Q'(\varepsilon_0 t)+\varepsilon_0 tQ''(\varepsilon_0 t)$, using the strict convexity of Q on $(0,r]$, we find that $Q_2'(t),Q_2(t)>0$ on $(0,1]$. Applying the inequality (4.9) on the the last term in (4.11) with $A=Q'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)$, $B=[\frac{c_1}{d}h_0(t)]$, we obtain

$$\begin{aligned}c_1h_0(t)Q'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)&=\frac{d}{q(t)}\left[\frac{c_1}{d}q(t)h_0(t)\right]\left[Q'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right]\\ &\leq\frac{d}{q(t)}Q_3\Big(Q'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\Big)+\frac{d}{q(t)}Q_3^*\left[\frac{c_1}{d}q(t)h_0(t)\right]\\ &\leq\frac{d}{q(t)}\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\left(Q'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right)\\ &\quad+\frac{d}{q(t)}Q_4\left[\frac{c_1}{d}q(t)h_0(t)\right]\\ &\leq\frac{d}{q(t)}Q_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+\frac{d}{q(t)}Q_4\left[\frac{c_1}{d}q(t)h_0(t)\right].\end{aligned} \tag{4.13}$$

Now, combining (4.11), (4.13), and taking d small enough, we obtain

$$\begin{aligned}\mathcal{F}_1'(t)&\leq -b\frac{\xi(t)}{q(t)}Q_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+\frac{d\xi(t)}{q(t)}Q_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\\ &\quad+\frac{d\xi(t)}{q(t)}Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right)\\ &\leq -b_1\frac{\xi(t)}{q(t)}Q_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)+\frac{d\xi(t)}{q(t)}Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right),\end{aligned} \tag{4.14}$$

where $b_1=b-d>0$. $(qE)(t)$ is decreasing since $E'<0$ and $q'<0$. Also, since Q_2 is increasing, we have, for $0\leq t\leq T$,

$$Q_2\left(\frac{E(T)q(T)}{E(0)}\right)\leq Q_2\left(\frac{E(t)q(t)}{E(0)}\right). \tag{4.15}$$

Putting (4.14) into (4.15) and multiplying by $q(t)$, we get

$$q(t)\mathcal{F}_1'(t)+b_1\xi(t)Q_2\left(\frac{E(T)q(T)}{E(0)}\right)\leq d\xi(t)Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right), \tag{4.16}$$

since $q'<0$, then for all $0\leq t\leq T$,

$$\Big(q(t)\mathcal{F}_1\Big)'(t)+b_1\xi(t)Q_2\left(\frac{E(T)q(T)}{E(0)}\right)\leq d\xi(t)Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right). \tag{4.17}$$

Integrating (4.17) over $[0, T]$ and using $q(0) = 1$, we get

$$Q_2\left(\frac{E(T)q(T)}{E(0)}\right)\int_0^T \xi(t)dt \leq \frac{\mathcal{F}_1(0)}{k_1} + \int_0^T \xi(t)Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right)dt. \tag{4.18}$$

Thus,

$$Q_2\left(\frac{E(T)q(T)}{E(0)}\right) \leq \left[\frac{\frac{\mathcal{F}_1(0)}{c_1} + \int_0^T \xi(t)Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right)dt,}{\int_0^T \xi(t)dt}\right]. \tag{4.19}$$

Hence,

$$\left(\frac{E(T)q(T)}{E(0)}\right) \leq Q_2^{-1}\left[\frac{\frac{\mathcal{F}_1(0)}{c_1} + \int_0^T \xi(t)Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right)dt,}{\int_0^T \xi(t)dt}\right], \tag{4.20}$$

which yields

$$E(T) \leq \left(\frac{E(0)}{q(T)}\right)Q_2^{-1}\left[\frac{C + \int_0^T \xi(t)Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right)dt,}{\int_0^T \xi(t)dt}\right], \tag{4.21}$$

where $C = \max\{1, \frac{\mathcal{F}_1(0)}{c_1}\}$. □

Example 4.2. *For our example, we let $h(t) = \frac{a}{(1+t)^\nu}$, where $\nu > 1$ and $a \in (0, \nu - 1)$. We take $\xi(t) = \nu a^{\frac{-1}{\nu}}$ and $Q(t) = t^{\frac{\nu+1}{\nu}}$, so $Q'(t) = a_0 t^{\frac{1}{\nu}}$.*

We will discuss two cases:

Case 1: If $m_0 \leq 1 + \|(\Phi_x + \Psi)_{x0}\|^2 \leq m_1$, then, we have the following:

$$\begin{aligned}
&Q_4(t) = a_1 t^{\frac{\nu+1}{\nu}}, \quad Q_2(t) = a_2 t^{\frac{\nu+1}{\nu}},\\
&a_3(1+t)^{-\nu+1} \leq h_0(t) \leq a_4(1+t)^{-\nu+1},\\
&\int_0^T \xi(t)Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right)dt < +\infty,\\
&Q_2^{-1}\left[\frac{C + \int_0^T \xi(t)Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right)dt}{\int_0^T \xi(t)dt}\right] \leq a_5 T^{-(\frac{\nu}{\nu+1})},
\end{aligned} \tag{4.22}$$

$$\frac{q_0}{q(T)} \leq a_6 \begin{cases} 1 + \ln(1+T), & \nu = 2,\\ 2, & \nu > 2,\\ (1+T)^{-\nu+2+r}, & 1 < \nu < 2\,. \end{cases} \tag{4.23}$$

Therefore,

$$E(T) \leq a_7 \begin{cases} \Big(1 + \ln(1+T)\Big)t^{-(\frac{\nu}{\nu+1})}, & \nu = 2,\\ T^{-(\frac{\nu}{\nu+1})}, & \nu > 2,\\ (1+T)^{-(\nu-2+\frac{\nu}{\nu+1})}, & 1 < \nu < 2\,. \end{cases} \tag{4.24}$$

Hence, for $\nu \geq 2$ or $\sqrt{2} < \nu < 2$, we have $\lim_{T\to+\infty} E(T) = 0$.

Case 2: If $m_0(1+t)^r \leq 1 + \|(\Phi_x+\Psi)_{x0}\|^2 \leq m_1(1+t)^r$*, where* $0 < r < \nu - 1$*, then we have the following:*

$$a_3(1+t)^{-\nu+1+r} \leq h_0(t) \leq a_4(1+t)^{-\nu+1+r},$$

$$\int_0^T \xi(t)Q_4\left(\frac{c_1}{d}q(t)h_0(t)\right)dt < +\infty, \tag{4.25}$$

$$\frac{q_0}{q(T)} \leq a_6 \begin{cases} 1+\ln(1+T), & \nu - r = 2, \\ 2, & \nu - r > 2, \\ (1+T)^{-\nu+2+r}, & 1 < \nu - r < 2\,. \end{cases} \tag{4.26}$$

Then

$$E(T) \leq a_7 \begin{cases} \Big(1+\ln(1+T)\Big)t^{-(\frac{\nu}{\nu+1})}, & \nu - r = 2, \\ T^{-(\frac{\nu}{\nu+1})}, & \nu - r > 2, \\ (1+T)^{-(\nu-2-r+\frac{\nu}{\nu+1})}, & 1 < \nu - r < 2\,. \end{cases} \tag{4.27}$$

Thus for $\nu - r \geq 2$ *or* $\frac{1}{2}(r+\sqrt{r^2+4r+8}) < \nu < r+2$*, we have* $\lim_{T\to+\infty} E(T) = 0$.

5. Conclusions

In this work, we have shown that the Timoshenko beam system with infinite memory acting on the shear force and heat conduction governed by Fourier law acting on the bending moment can be stabilized without any additional conditions on coefficients parameters. We believe with some modification, the same result can be achieve if the heat conduction is governed by Maxwell-Cattaneo or Gurtin-Pipkin heat conductions.

Author contributions

Hasan Almutairi: Validation, Investigation, Resources, Funding, Writing–original draft preparation. Soh Edwin Mukiawa: Conceptualization, Formal analysis, Methodology, Writing–original draft preparation, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no potential conflicts of interest.

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