AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 16250–16259.
DOI: 10.3934/math.2024786
Received: 17 January 2024
Revised: 08 April 2024
Accepted: 08 April 2024
Published: 08 May 2024

Research article

Regularity criteria for the 3D generalized Navier-Stokes equations with nonlinear damping term

Jianlong Wu*

School of Mathematical Science, Zhejiang Normal University, 321004, Jinhua, China

* **Correspondence:** Email: wujianlong@zjnu.edu.cn.

Abstract: This paper was devoted to establishing some regularity criteria for the 3D generalized Navier-Stokes equations with nonlinear damping term. We focused on considering the role of the damping term in regularity criteria and the global existence brought by these criteria.

Keywords: Navier-Stokes equations; nonlinear damping term; regularity criteria
Mathematics Subject Classification: 35B65, 76W05, 35Q35

---

1. Introduction

In this paper, we consider the following Cauchy problem of Navier-Stokes equations with the damping term:

$$u_t + (u\cdot\nabla)u + \nabla\pi + \Lambda^{2\alpha}u + |u|^{\beta-1}u = 0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \tag{1.1}$$

$$\operatorname{div} u = 0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \tag{1.2}$$

$$u(x,0) = u_0, \qquad x\in\mathbb{R}^3, \tag{1.3}$$

where $u = u(x,t) \in \mathbb{R}^3$, $\pi = \pi(x,t) \in \mathbb{R}$ represent the unknown velocity field and the pressure respectively. $\alpha \geq 0, \beta \geq 1$ are real parameters. $\Lambda := (-\Delta)^{\frac{1}{2}}$ is defined in terms of Fourier transform by

$$\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi).$$

Damping originates from the dissipation of energy by resistance, which describes many physical phenomena such as porous media flow, resistance or frictional effects, and some dissipation mechanisms (see [1] and references cited therein). When $\alpha = 1$, Cai and Jiu first proved that there exists a weak solution of (1.1)–(1.3) if $\beta > 1$. Furthermore, if $\beta \geq \frac{7}{2}$, the global existence of the strong solution was established. Later, this result was improved by Zhang, Wu and Lu in [2], where the lower bound of β decreased to 3. Zhou [3] proved the lower bound 3 is critical in some sense. For the general case, it is proved that when $\frac{3}{4} \le \alpha < 1, \beta \ge \frac{2\alpha+5}{4\alpha-2}$ or $1 \le \alpha < \frac{5}{4}, \beta \ge 1 + \frac{10}{4\alpha+1}$, the global existence of the solution was established in [4]. For the asymptotic behavior, one can refer to [5–7] for details.

For the generalized Navier-Stokes equations (our system without damping term) when $\alpha = 1$, there are many regularity criteria to the system (1.1)–(1.3). The classical Prodi-Serrin's-type criteria was given in [8–10], where it was proved that if a weak solution $u \in L^p(0, T; L^q(\mathbb{R}^3))$ with $\frac{2}{p} + \frac{3}{q} = 1$, $q \ge 3$, then the solution is regular and unique. Beirão da Veiga [11] established the analogous result: $\nabla u \in L^p(0, T; L^q(\mathbb{R}^3))$ with $\frac{2}{p} + \frac{3}{q} = 2$, $q \ge \frac{3}{2}$. For the general case, in [12], Jiang and Zhu proved that if $\Lambda^\theta u \in L^p(0, T; L^q(\mathbb{R}^3))$ with $\frac{2\alpha}{p} + \frac{3}{q} \le 2\alpha - 1 + \theta$, $\theta \in [1-\alpha, 1]$, $q > \frac{3}{2\alpha-1+\theta}$, then the solution remains smooth on [0,T]. One can refer to [11, 13, 14] for more classical regularity criteria. For the large time behavior, Jiu and Yu proved the algebraic decay of the solution under specific conditions (see [15]).

Our paper devotes to considering the role of damping terms in regularity criteria for the system (1.1)–(1.3). We will explain the role of damping term in the following two questions:

(1) When does the dissipative term work better than the damping term?

(2) How does the damping term work?

For the first question, if $\alpha \ge \frac{5}{4}$, the generalized Navier-Stokes equations (our system without damping term) exists a global strong solution $u \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^{1+\alpha}(\mathbb{R}^3))$. Consequently, we only consider the case when $\frac{1}{2} < \alpha < \frac{5}{4}$.

For the second question, we utilize two structures brought by the damping term: $\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2$ (Theorems 1.1 and 1.2, when $1 < \alpha < \frac{5}{4}$) and $\frac{1}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1}$ (Theorems 1.3 and 1.4, when $\frac{1}{2} < \alpha < 1$). Actually, $\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2$ works better than $\frac{1}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1}$, because $\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2$ is a first-order estimate resulting from the damping term while $\frac{1}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1}$ is a zero-order estimate resulting from the damping term. However, because of the technical limitation, we still use $\frac{1}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1}$ when $\frac{1}{2} < \alpha < 1$. Consequently, when $\frac{1}{2} < \alpha < 1$, how to utilize $\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2$ may be an insteresting question.

We give our main theorems as follows.

Theorem 1.1. *When $1 < \alpha < \frac{5}{4}$, $\beta < 1 + \frac{10}{4\alpha+1}$, assume that the initial data $u_0(x) \in H^1(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, and $u(x,t)$ is a local strong solution of the system* (1.1)–(1.3)*. If $u(x,t) \in L^p(0, T; L^q(\mathbb{R}^3))$ with*

$$\frac{2\alpha}{p} + \frac{3}{q} \le \max\{\frac{2(\alpha-1)}{3-\beta}, 2\alpha - 1\}, \quad \min\{\frac{9-3\beta}{2(\alpha-1)}, \frac{3}{2\alpha-1}\} < q \le \infty, \tag{1.4}$$

then, for any $T > 0$, the system (1.1)–(1.3) *has a global strong solution satisfying*

$$u \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^{1+\alpha}(\mathbb{R}^3)) \cap L^{\beta+1}(0, T; L^{\beta+1}(\mathbb{R}^3)).$$

Remark 1.1. *In Theorem 1.1, we roughly combine the regularity criteria brought by the dissipative term and the damping term. In fact, we can verify that if $1 < \alpha < \frac{5}{4}$, $2 + \frac{1}{2\alpha-1} < \beta < 1 + \frac{10}{4\alpha+1}$, then $\frac{2(\alpha-1)}{3-\beta} > 2\alpha - 1$. Consequently,* (1.4) *becomes*

$$\frac{2\alpha}{p} + \frac{3}{q} \le \frac{2(\alpha-1)}{3-\beta}, \quad \frac{9-3\beta}{2(\alpha-1)} < q \le \infty, \tag{1.5}$$

which means that damping the term works better than the dissipative term.

Theorem 1.2. *When* $1 < \alpha < \frac{5}{4}$, $5 - 2\alpha < \beta < 1 + \frac{10}{4\alpha+1}$, *assume that the initial data* $u_0(x) \in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$, *and* $u(x,t)$ *is a local strong solution of the system* (1.1)–(1.3). *If* $\Lambda^\alpha u(x,t) \in L^p(0,T;L^q(\mathbb{R}^3))$ *with*

$$\frac{(3-\beta)\alpha}{p(2\alpha-5+\beta)} + \frac{3}{q} \le \alpha + \frac{3}{2}, \quad \frac{3}{1+\alpha} \le q < \infty,$$

then, for any $T > 0$, *the system* (1.1)–(1.3) *has a global strong solution satisfying*

$$u \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^{1+\alpha}(\mathbb{R}^3)) \cap L^{\beta+1}(0,T;L^{\beta+1}(\mathbb{R}^3)).$$

Remark 1.2. *In Theorems 1.1 and 1.2, we consider the regularity criteria when* $\beta < 1 + \frac{10}{4\alpha+1}$, *because the global existence was established in [4] when* $\beta \ge 1 + \frac{10}{4\alpha+1}$. *If* $\beta \ge 1 + \frac{10}{4\alpha+1}$, *the regularity criteria in Theorem 1.1 is satisfied naturally, so we recover the result in [4] when* $1 < \alpha < \frac{5}{4}$.

Theorem 1.3. *When* $\frac{1}{2} < \alpha < 1$, $\beta < \min\{\frac{2\alpha+5}{4\alpha-2}, \frac{3\alpha+2}{\alpha}\}$, *assume that the initial data* $u_0(x) \in H^1(\mathbb{R}^3) \cap L^{\beta+1}(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$, *and* $u(x,t)$ *is a local strong solution of the system* (1.1)–(1.3). *If* $u(x,t) \in L^p(0,T;L^q(\mathbb{R}^3))$ *with*

$$\frac{6\alpha - (2\alpha-1)(\beta+1)}{p} + \frac{3}{q} \le 2\alpha - 1, \ \frac{3}{2\alpha-1} < q \le \frac{6\alpha}{2\alpha-1}, \tag{1.6}$$

then, for any $T > 0$, *the system* (1.1)–(1.3) *has a global strong solution satisfying*

$$u \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^{\alpha+1}(\mathbb{R}^3)) \cap L^\infty(0,T;L^{\beta+1}(\mathbb{R}^3)), \ u_t \in L^2(0,T;L^2(\mathbb{R}^3)).$$

Remark 1.3. *If* $\beta \ge \frac{2\alpha+5}{4\alpha-2}$, *the regularity criteria in Theorem 1.3 is satisfied naturally, so we recover the result in [4] when* $\frac{3}{4} \le \alpha < 1$.

Theorem 1.4. *When* $\frac{1}{2} < \alpha < 1$, $\beta < \min\{\frac{2\alpha+5}{4\alpha-2}, \frac{3\alpha+2}{\alpha}\}$, *assume that the initial data* $u_0(x) \in H^1(\mathbb{R}^3) \cap L^{\beta+1}(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$, *and* $u(x,t)$ *is a local strong solution of the system* (1.1)–(1.3). *If* $\Lambda^\alpha u(x,t) \in L^p(0,T;L^q(\mathbb{R}^3))$ *with*

$$\frac{6\alpha - (2\alpha-1)(\beta+1)}{p} + \frac{3}{q} \le 3\alpha - 1, \ \frac{3}{3\alpha-1} < q \le \frac{6\alpha}{3\alpha-1},$$

then, for any $T > 0$, *the system* (1.1) *has a global strong solution satisfying*

$$u \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^{\alpha+1}(\mathbb{R}^3)) \cap L^\infty(0,T;L^{\beta+1}(\mathbb{R}^3)), \ u_t \in L^2(0,T;L^2(\mathbb{R}^3)).$$

2. Regularity criteria

Proof of the Theorem 1.1. Multiplying (1.1) by $-\Delta u$, after integration by parts and taking the divergence-free property into account, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Lambda^{1+\alpha}u\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2 = \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\, dx.$$

For $\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u dx$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u dx\\
&\leq C\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}\||u|^{\frac{3-\beta}{2}}\Delta u\|_{L^2}\\
&\leq \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+C\|u\|_{L^q}^{3-\beta}\|\Delta u\|^2_{L^{\frac{2q}{q-3+\beta}}}\\
&\leq \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|u\|_{L^q}^{3-\beta}\|\nabla u\|_{L^2}^{2(1-\theta_1)}\|\Lambda^{1+\alpha}u\|_{L^2}^{2\theta_1}\\
&\leq \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2+C\|u\|_{L^q}^{\frac{3-\beta}{1-\theta_1}}\|\nabla u\|_{L^2}^2\\
&\leq \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2+C\|u\|_{L^q}^{\frac{2q\alpha(3-\beta)}{2(\alpha-1)-9+3\beta}}\|\nabla u\|_{L^2}^2,
\end{aligned}
$$

where

$$\frac{1}{2}-\frac{3-\beta}{2q}=\frac{1}{3}+(\frac{1}{2}-\frac{\alpha}{3})\theta_1+\frac{1-\theta_1}{2},$$

with $\theta_1=\frac{2q+9-3\beta}{2\alpha q}$. The conditions in Theorem 1.1 imply $\theta_1\in[\frac{1}{\alpha},1)$. By direct calculation, we have

$$\frac{3-\beta}{1-\theta_1}=\frac{2q\alpha(3-\beta)}{2(\alpha-1)q-9+3\beta}.$$

Combining the above estimates, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\Lambda^{1+\alpha}u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2\\
&\leq \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2+C\|u\|_{L^q}^{\frac{3-\beta}{1-\theta_1}}\|\nabla u\|_{L^2}^2.
\end{aligned}
$$

A standard Gronwall's inequality shows that

$$\|\nabla u\|_{L^2}^2+\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2)(s)\,ds\leq C(\,t,\|u_0\|_{H^1}).$$

This completes the proof of the Theorem 1.1. □

Proof of the Theorem 1.2. Multiplying (1.1) by $-\Delta u$, after integration by parts and taking the divergence-free property into account, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\Lambda^{1+\alpha}u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2=\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx.$$

For $\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\\
&\leq C\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}\||u|^{\frac{3-\beta}{2}}\Delta u\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + C\|u\|_{L^3}^{3-\beta}\|\Delta u\|_{L^{\frac{6}{\beta}}}^2\\
&\le \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + C\|u\|_{L^2}^{(3-\beta)(1-\theta_2)}\|\Lambda^\alpha u\|_{L^q}^{(3-\beta)\theta_2}\|\nabla u\|_{L^2}^{2(1-\theta_3)}\|\Lambda^{1+\alpha}u\|_{L^2}^{2\theta_3}\\
&\le \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|\Lambda^\alpha u\|_{L^q}^{\frac{(3-\beta)\theta_2}{1-\theta_3}}\|\nabla u\|_{L^2}^2\\
&\le \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|\Lambda^\alpha u\|_{L^q}^{\frac{2(3-\beta)\alpha q}{[(2\alpha+3)q-6][2\alpha-5+\beta]}}\|\nabla u\|_{L^2}^2,
\end{aligned}
$$

where

$$
\begin{cases}
\frac{1}{3} = \theta_2(\frac{1}{q}-\frac{\alpha}{3}) + \frac{1-\theta_2}{2},\\
\frac{\beta}{6} = \frac{1}{3} + \theta_3(\frac{1}{2}-\frac{\alpha}{3}) + \frac{1-\theta_3}{2},
\end{cases}
$$

with $\theta_2 = \frac{q}{(2\alpha+3)q-6}$, $\theta_3 = \frac{5-\beta}{2\alpha}$. The conditions in Theorem 1.2 imply $\theta_2 \in (0,1]$, $\theta_3 \in (\frac{1}{\alpha},1)$. By direct calculation, we have

$$
\frac{(3-\beta)\theta_2}{1-\theta_3} = \frac{2(3-\beta)\alpha q}{[(2\alpha+3)q-6][2\alpha-5+\beta]}.
$$

Combining the above estimates, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Lambda^{1+\alpha}u\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2\\
&\le \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|\Lambda^\alpha u\|_{L^q}^{\frac{2(3-\beta)\alpha q}{[(2\alpha+3)q-6][2\alpha-5+\beta]}}\|\nabla u\|_{L^2}^2.
\end{aligned}
$$

A standard Gronwall's inequality shows that

$$
\|\nabla u\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+1}u\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2)(s)\,ds \le C(t, \|u_0\|_{H^1}).
$$

This completes the proof of the Theorem 1.2. □

Proof of the Theorem 1.3. Multiplying (1.1) by $-\triangle u$, u_t and adding the two equations, after integration by parts and taking the divergence-free property into account, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\Lambda^\alpha u\|_{L^2}^2 + \frac{1}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1} + \|\Lambda^{1+\alpha}u\|_{L^2}^2 + \|u_t\|_{L^2}^2\\
&+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2\\
&= \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx - \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot u_t\,dx\\
&\le C\|\nabla u\|_{L^3}^3 + C\|u\cdot\nabla u\|_{L^2}^2 + \frac{1}{2}\|u_t\|_{L^2}^2.
\end{aligned}
$$

For $\|\nabla u\|_{L^3}^3$, we have

$$
C\|\nabla u\|_{L^3}^3
$$

$$
\begin{aligned}
&\le C\|u\|_{L^q}^{\delta_1(1-\theta_4)}\|\Lambda^{1+\alpha}u\|_{L^2}^{\delta_1\theta_4}\|u\|_{L^{\beta+1}}^{(3-\delta_1)(1-\theta_5)}\|\Lambda^{1+\alpha}u\|_{L^2}^{(3-\delta_1)\theta_5}\\
&\le \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{2\delta_1(1-\theta_4)}{2-\delta_1\theta_4-(3-\delta_1)\theta_5}}\|u\|_{L^{\beta+1}}^{\frac{2(3-\delta_1)(1-\theta_5)}{2-\delta_1\theta_4-(3-\delta_1)\theta_5}}\\
&\le \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{2\delta_1(1-\theta_4)}{2-\delta_1\theta_4-(3-\delta_1)\theta_5}}\|u\|_{L^{\beta+1}}^{\beta+1}\\
&\le \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{[6\alpha-(2\alpha-1)(\beta+1)]q}{(2\alpha-1)q-3}}\|u\|_{L^{\beta+1}}^{\beta+1},
\end{aligned}
$$

where

$$
\begin{cases}
\frac{1}{3} = \frac{1}{3} + \theta_4(\frac{1}{2} - \frac{1+\alpha}{3}) + \frac{1-\theta_4}{q},\\
\frac{1}{3} = \frac{1}{3} + \theta_5(\frac{1}{2} - \frac{1+\alpha}{3}) + \frac{1-\theta_5}{\beta+1},\\
\frac{2(3-\delta_1)(1-\theta_5)}{2-\delta_1\theta_4-(3-\delta_1)\theta_5} = \beta + 1.
\end{cases}
$$

By directly calculating, we have

$$
\begin{cases}
\theta_4 = \frac{6}{(2\alpha-1)q+6},\\
\theta_5 = \frac{6}{(2\alpha-1)(\beta+1)+6},\\
\delta_1 = \frac{[(2\alpha-1)q+6][6\alpha-(2\alpha-1)(\beta+1)]}{2(\alpha+1)[(2\alpha-1)q+6]-3[(2\alpha-1)(\beta+1)+6]},\\
\frac{2\delta_1(1-\theta_4)}{2-\delta_1\theta_4-(3-\delta_1)\theta_5} = \frac{[6\alpha-(2\alpha-1)(\beta+1)]q}{(2\alpha-1)q-3}.
\end{cases}
$$

The conditions in Theorem 1.3 imply $\theta_4 \in [\frac{1}{1+\alpha}, 1)$, $\theta_5 \in [\frac{1}{1+\alpha}, 1)$, $\delta_1 \in (0, 3)$.

For $\|u \cdot \nabla u\|_{L^2}^2$, we have

$$
\begin{aligned}
&C\|u \cdot \nabla u\|_{L^2}^2\\
&\le C\|u\|_{L^q}^{\delta_2}\|u\|_{L^{\beta+1}}^{2-\delta_2}\|\nabla u\|_{L^{\frac{2}{1-\frac{\delta_2}{q}-\frac{2-\delta_2}{\beta+1}}}}\\
&\le C\|u\|_{L^q}^{\delta_2}\|u\|_{L^{\beta+1}}^{2-\delta_2}\|u\|_{L^{\beta+1}}^{2(1-\theta_6)}\|\Lambda^{1+\alpha}u\|_{L^2}^{2\theta_6}\\
&\le \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{\delta_2}{1-\theta_6}}\|u\|_{L^{\beta+1}}^{\frac{2-\delta_2}{1-\theta_6}}\|u\|_{L^{\beta+1}}^2\\
&\le \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{\delta_2}{1-\theta_6}}\|u\|_{L^{\beta+1}}^{\beta+1}\\
&= \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{(3\alpha+2-\alpha\beta)q}{\alpha q-3}}\|u\|_{L^{\beta+1}}^{\beta+1}\\
&\le \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C(\|u\|_{L^q}^{\frac{[6\alpha-(2\alpha-1)(\beta+1)]q}{(2\alpha-1)q-3}} + 1)\|u\|_{L^{\beta+1}}^{\beta+1},
\end{aligned}
$$

where

$$
\begin{cases}
\frac{1}{2} - \frac{\delta_2}{2q} - \frac{2-\delta_2}{2(\beta+1)} = \frac{1}{3} + \theta_6(\frac{1}{2} - \frac{1+\alpha}{3}) + \frac{1-\theta_6}{\beta+1},\\
\frac{2-\delta_2}{1-\theta_6} = \beta - 1.
\end{cases}
$$

By direct calculation, we have

$$
\begin{cases}
\theta_6 = \frac{2q+9-3\beta}{2(\alpha+1)q+3-3\beta},\\
\frac{\delta_2}{1-\theta_6} = \frac{2}{1-\theta_6} + 1 - \beta = \frac{(3\alpha+2-\alpha\beta)q}{\alpha q-3}.
\end{cases}
$$

The conditions in Theorem 1.3 imply $\theta_6 \in [\frac{1}{1+\alpha}, 1)$.

Combining the above estimates, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\Lambda^\alpha u\|_{L^2}^2 + \frac{1}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1} + \|\Lambda^{1+\alpha}u\|_{L^2}^2 + \|u_t\|_{L^2}^2 \\ &+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2 \\ &\le \frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C(\|u\|_{L^q}^{\frac{[6\alpha-(2\alpha-1)(\beta+1)]q}{(2\alpha-1)q-3}} + 1)\|u\|_{L^{\beta+1}}^{\beta+1}.\end{aligned}$$

A standard Gronwall's inequality shows that

$$\begin{aligned}&\|\nabla u\|_{L^2}^2 + \|u\|_{L^{\beta+1}}^{\beta+1} + \|\Lambda^\alpha u\|_{L^2}^2 + \int_0^t (\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \|\Lambda^{1+\alpha}u\|_{L^2}^2 + \|u_t\|_{L^2}^2)(\tau)d\tau \\ \le\ & C(\,t, \|u_0\|_{H^1}, \|u_0\|_{L^{\beta+1}}).\end{aligned}$$

This completes the proof of the Theorem 1.3. □

Proof of the Theorem 1.4. Multiplying (1.1) by $-\triangle u$, u_t and adding the two equations, after integration by parts and taking the divergence-free property into account, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\Lambda^\alpha u\|_{L^2}^2 + \frac{1}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1} + \|\Lambda^{1+\alpha}u\|_{L^2}^2 + \|u_t\|_{L^2}^2 \\ &+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx - \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot u_t\,dx \\ &\le C\|\nabla u\|_{L^3}^3 + C\|u\cdot\nabla u\|_{L^2}^2 + \frac{1}{2}\|u_t\|_{L^2}^2.\end{aligned}$$

For $\|\nabla u\|_{L^3}^3$, we have

$$\begin{aligned}&C\|\nabla u\|_{L^3}^3 \\ &\le C\|\Lambda^\alpha u\|_{L^q}^{\delta_3(1-\theta_7)}\|\Lambda^{1+\alpha}u\|_{L^2}^{\delta_3\theta_7}\|u\|_{L^{\beta+1}}^{(3-\delta_3)(1-\theta_5)}\|\Lambda^{1+\alpha}u\|_{L^2}^{(3-\delta_3)\theta_5} \\ &\le \frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|\Lambda^\alpha u\|_{L^q}^{\frac{2\delta_3(1-\theta_7)}{2-\delta_3\theta_7-(3-\delta_3)\theta_5}}\|u\|_{L^{\beta+1}}^{\frac{2(3-\delta_3)(1-\theta_5)}{2-\delta_3\theta_7-(3-\delta_3)\theta_5}} \\ &\le \frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|\Lambda^\alpha u\|_{L^q}^{\frac{2\delta_3(1-\theta_7)}{2-\delta_3\theta_7-(3-\delta_3)\theta_5}}\|u\|_{L^{\beta+1}}^{\beta+1} \\ &\le \frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|\Lambda^\alpha u\|_{L^q}^{\frac{[6\alpha-(2\alpha-1)(\beta+1)]q}{(3\alpha-1)q-3}}\|u\|_{L^{\beta+1}}^{\beta+1},\end{aligned}$$

where

$$\begin{cases}\frac{1}{3} = \frac{1-\alpha}{3} + \theta_7(\frac{1}{2}-\frac{1}{3}) + \frac{1-\theta_7}{q}, \\ \frac{1}{3} = \frac{1}{3} + \theta_5(\frac{1}{2}-\frac{1+\alpha}{3}) + \frac{1-\theta_5}{\beta+1}, \\ \frac{2(3-\delta_3)(1-\theta_5)}{2-\delta_3\theta_7-(3-\delta_3)\theta_5} = \beta+1.\end{cases}$$

By direct calculation, we have

$$\begin{cases}\theta_7 = \frac{6-2\alpha q}{6-q},\\ \theta_5 = \frac{6}{(2\alpha-1)(\beta+1)+6},\\ \delta_3 = \frac{(6-q)[6\alpha-(2\alpha-1)(\beta+1)]}{2(\alpha+1)(6-q)-(3-\alpha q)[(2\alpha-1)(\beta+1)+6]}.\end{cases}$$

The conditions in Theorem 1.3 imply $\theta_7 \in [1-\alpha, 1)$, $\theta_5 \in [\frac{1}{1+\alpha}, 1)$, $\delta_3 \in (0,3)$.

We can estimate $\|u\cdot\nabla u\|_{L^2}^2$ similarily.

Combining the above estimates, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\Lambda^\alpha u\|_{L^2}^2 + \frac{1}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1} + \|\Lambda^{1+\alpha}u\|_{L^2}^2 + \|u_t\|_{L^2}^2\\ &+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2\\ &\le \frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|\Lambda^\alpha u\|_{L^q}^{\frac{[6\alpha-(2\alpha-1)(\beta+1)]q}{(3\alpha-1)q-3}}\|u\|_{L^{\beta+1}}^{\beta+1}.\end{aligned}$$

A standard Gronwall's inequality shows that

$$\begin{aligned}&\|\nabla u\|_{L^2}^2 + \|u\|_{L^{\beta+1}}^{\beta+1} + \|\Lambda^\alpha u\|_{L^2}^2 + \int_0^t (\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \|\Lambda^{1+\alpha}u\|_{L^2}^2 + \|u_t\|_{L^2}^2)(\tau)d\tau\\ \le\ & C(t, \|u_0\|_{H^1}, \|u_0\|_{L^{\beta+1}}).\end{aligned}$$

This completes the proof of the Theorem 1.4. □

3. Conclusions

In this paper, we have established some regularity criteria for the 3D generalized Navier-Stokes equations with nonlinear damping term. First, we consider the case where the dissipative term is superior to the damping term, which corresponds to when the damping term works. Second, in Remark 1.1, we show that the damping term works better than the dissipative term. Furthermore, we have presented that the damping term has different effects in different cases, which shows the balance and the interaction between the dissipative term and the damping term as well as the role of the damping term in regularity criteria. In fact, considering how the damping term works and the interaction between the dissipative term and the damping term is the main idea of this paper.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (Grant No.12071439) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LY19A010016).

Conflict of interest

The author declares no conflict of interest.

References

1. X. J. Cai, Q. S. Jiu, Weak and strong solutions for incompressible Navier-Stokes equations with Damping, *J. Math. Anal. Appl.*, **343** (2008), 799–809. https://doi.org/10.1016/j.jmaa.2008.01.041

2. Z. J. Zhang, X. L. Wu, M. Lu, On the uniqueness of strong solution to the incompressible Navier-Stokes equations with damping, *J. Math. Anal. Appl.*, **377** (2011), 414–419. https://doi.org/10.1016/j.jmaa.2010.11.019

3. Y. Zhou, Regularity and uniqueness for the 3D incompressible Navier-Stokes equations with damping, *Appl. Math. Lett.*, **25** (2012), 1822–1825. https://doi.org/10.1016/j.aml.2012.02.029

4. Z. Jiang, J. Wu, X. Chen, Y. Zhou, Regularity and uniqueness for the 3D incompressible Navier-Stokes equations with damping, *In Press.*

5. Y. Jia, X. W. Zhang, B. Q. Dong, The asymptotic behavior of solutions to three-dimensional Navier-Stokes equations with nonlinear damping, *Nonlinear Anal. Real*, **12** (2011), 1736–1747. https://doi.org/10.1016/j.nonrwa.2010.11.006

6. Z. H. Jiang, M. X. Zhu, The large time behavior of solutions to 3D Navier-Stokes equations with nonlinear damping, *Math. Method. Appl. Sci.*, **35** (2012), 97–102. https://doi.org/10.1002/mma.1540

7. Z. H. Jiang, Asymptotic behavior of strong solutions to the 3D Navier-Stokes equations with a nonlinear damping term, *Nonlinear Anal.*, **75** (2012), 5002–5009. https://doi.org/10.1016/j.na.2012.04.014

8. G. Prodi, Un teorema di unicità per le equazioni di Navier-Stokes, *Ann. Mat. Pur. Appl.*, **48** (1959), 173–182. https://doi.org/10.1007/BF02410664

9. J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations, *Arch. Ration Mech. An.*, **9** (1962), 187–195. https://doi.org/10.1007/BF00253344

10. L. Escauriaza, G. Seregin, V. Šverák, Backward uniqueness for parabolic equations, *Arch. Ration Mech. An.*, **169** (2003), 147–157. https://doi.org/10.1007/s00205-003-0263-8

11. H. B. Veiga, A new regularity class for the Navier-Stokes equations in Rn, *Chinese. Ann. Math. B*, **16** (1995), 407–412.

12. Z. H. Jiang, M. X. Zhu, Asymptotic behavior, regularity criterion and global existence for the generalized Navier-Stokes equations, *Bull. Malays. Math. Sci. Soc.*, **42** (2019), 1085–1100. https://doi.org/10.1007/s40840-017-0531-7

13. J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.*, **63** (1934), 193–248. https://doi.org/10.1007/BF02547354

14. J. T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Commun. Math. Phys.*, **94** (1984), 61–66. http://projecteuclid.org/euclid.cmp/1103941230

15. Q. S. Jiu, H. Yu, Decay of solutions to the three-dimensional generalized Navier-Stokes equations, *Asymptotic Anal.*, **94** (2015), 105–124. https://doi.org/10.3233/ASY-151307