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Research article

Analysis of a hybrid fractional coupled system of differential equations in n-dimensional space with linear perturbation and nonlinear boundary conditions

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Abstract: In this paper, we investigated n-dimensional fractional hybrid differential equations (FHDEs) with nonlinear boundary conditions in a nonlinear coupled system. For this purpose, we used Dhage's fixed point theory, and applied the Krasnoselskii-type coupled fixed point theorem to construct existence conditions of the solution of the FHDEs. To illustrated this idea, suitable examples are presented in 3-dimensional space at the end of the paper.

Keywords: n-dimensional nonlinear coupled system; fractional hybrid differential equations; Dhage's fixed point theory
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1. Introduction and motivation

Fractional calculus is the area of mathematics concerned with the integral and derivative of any arbitrary order, whether real or complex. In the sixteenth century, classical calculus began its journey from solving ordinary order differential equations to tackling equations of arbitrary order. Joseph Louis Lagrange (Lacroix) provided the formal definition for the first time. Subsequently, other mathematicians such as Riemann-Liouville, Abel, Grownwald, and L'Hospital made significant contributions to this field [1, 2].

Fractional differential equations (abbreviated as FDEs) provide a more accurate presentation of real-world problems that involve mathematical equations with memory terms. Ordinary calculus fails to clearly explain these memory terms, leading to increased attention and investigation in this area. Fractional calculus finds application across the globe and has been utilized in a wide variety of physical processes across many different scientific disciplines, including natural sciences, engineering, physics, chemistry, biology, and more [3–5]. This area has been explored from various perspectives, such as qualitative, stability, and optimization theory. The mathematical model's inclusion of physical phenomena is guaranteed by the qualitative theory. Over the past thirty years, many mathematicians have struggled to find solutions to fractional differential equations [6, 7]. Most often the iteration and fixed point techniques have been employed in existence theory. However, coupled systems of FDEs find applications in numerous fields, like physics, economics, biology, chemistry, and more [8, 9]. Recently, the presence of solutions for FDE's that involve Caputo derivatives were examined in [10,11]. Authors in [12, 13] discussed the coupled system of FDEs having different boundary conditions using topological degree theory.

Additionally, a fundamental class of FDEs called fractional hybrid differential equations (FHDEs) has been studied by numerous researchers. Due to their perturbative nature, FHDEs are particularly attractive to mathematicians working in the dynamical system. The existence theory for coupled systems of FHDEs can be developed using the fixed point approach and prior estimate methods [14–18].

Various researchers have studied coupled systems of FHDEs using Dhage's fixed point theory [19, 20]. While reviewing the literature, we noted that FHDEs with linear perturbation and nonlinear integral boundary conditions have been studied in up to 2-dimensional or rarely up to 3-dimensional space. Rare articles are available that address the study of FHDEs corresponding to integral boundary conditions in 3-dimensional, and in n-dimensional real space. FHDEs in higher dimension arise in various fields of sciences and engineering due to their ability to model complex systems with multi-scale phenomena, non-local interaction, and memory effects. Using fixed point techniques, Kumam et al. in [17] examined the following CFHDEs:

$$\begin{cases} D^{\rho}\left(\varkappa\left(\bar{\vartheta}\right)-f_1\left(\bar{\vartheta},\varkappa\left(\bar{\vartheta}\right)\right)\right) & = f_2\left(\bar{\vartheta},F\left(\bar{\vartheta}\right),I^{\alpha}F\left(\bar{\vartheta}\right)\right), \\ D^{\rho}\left(F\left(\bar{\vartheta}\right)-f_1\left(\bar{\vartheta},F\left(\bar{\vartheta}\right)\right)\right) & = f_2\left(\bar{\vartheta},\varkappa\left(\bar{\vartheta}\right),I^{\alpha}\varkappa\left(\bar{\vartheta}\right)\right), \end{cases} \tag{1.1}$$

where $\bar{\vartheta} \in [0,l]\,, l > 0,\ \rho \in (n-1, n]$. System (1.1) is subject to the conditions:

$$\begin{aligned} \varkappa(0) &= \delta_1\varkappa(\eta_1), \\ F(0) &= \delta_1 F(\eta_1), \\ \varkappa(1) &= \delta_2\varkappa(\eta_2), \\ F(1) &= \delta_2 F(\eta_2), \\ \varkappa^i(0) = 0 \quad F^i(0) &= 0, \quad \forall\, i = 1, 2, ..., n-1. \end{aligned}$$

In the above system, $\alpha > 0,\ \eta_1, \eta_2 \in (0,1)\ f_1 : [0,l]\times\Re \to \Re$ is continues, $f_1\left(\bar{\vartheta},\varkappa\left(\bar{\vartheta}\right)\right)|_{\bar{\vartheta}=0}= 0$ and $f_2 : [0,l]\times\Re\times\Re \to \Re$, and D^{ρ} is the Caputo fractional derivative (CFD) of order $[\rho]$ ($[\rho]$ is the integer part of ρ). Motivated by the work of Kumam et al., we are fascinated in the presence of a solution to the following n-dimensional FHDEs in a nonlinear coupled system:

$$\begin{cases} D^{\alpha_1}\left(\varkappa_1\left(\bar{\vartheta}\right)-P_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)\right) & = Q_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right), \\ D^{\alpha_2}\left(\varkappa_2\left(\bar{\vartheta}\right)-P_2\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)\right) & = Q_2\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right), \\ D^{\alpha_3}\left(\varkappa_3\left(\bar{\vartheta}\right)-P_3\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)\right) & = Q_3\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right), \\ & \vdots \\ D^{\alpha_n}\left(\varkappa_n\left(\bar{\vartheta}\right)-P_n\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)\right) & = Q_n\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right), \end{cases} \tag{1.2}$$

where $\bar{\vartheta} \in [0,1]$. System (1.2) is subject to the conditions:

$$\begin{cases} \varkappa_1(0) = h_1(\varkappa_1); & \varkappa_1(1) = \frac{1}{\Gamma(\alpha_1)} \int_0^1 (1-\bar{s})^{\alpha_1 - 1} \varphi_1(\bar{s}, \varkappa_1(\bar{s})) \, d\bar{s}, \\ \varkappa_2(0) = h_2(\varkappa_2); & \varkappa_2(1) = \frac{1}{\Gamma(\alpha_2)} \int_0^1 (1-\bar{s})^{\alpha_2 - 1} \varphi_2(\bar{s}, \varkappa_2(\bar{s})) \, d\bar{s}, \\ \varkappa_3(0) = h_3(\varkappa_1); & \varkappa_3(1) = \frac{1}{\Gamma(\alpha_3)} \int_0^1 (1-\bar{s})^{\alpha_3 - 1} \varphi_3(\bar{s}, \varkappa_3(\bar{s})) \, d\bar{s}, \\ & \vdots \\ \varkappa_n(0) = h_n(\varkappa_n); & \varkappa_n(1) = \frac{1}{\Gamma(\alpha_n)} \int_0^1 (1-\bar{s})^{\alpha_n - 1} \varphi_n(\bar{s}, \varkappa_n(\bar{s})) \, d\bar{s}, \end{cases}$$

where D^{α_i} is the CFD, $\alpha_i \in (1,2]$, $P_i : [0,1] \times \Re^n \to \Re$, $h_i : \Re \to \Re$, and $\varphi_i : [0,1] \times \Re \to \Re$ are continuous mappings, and $Q_i : [0,1] \times \Re^n \to \Re$ are continuous or piece-wise continues mappings.

2. Preliminaries

In this paper, $\Psi = \{\psi : \Re^+ \to \Re^+ \text{ such that } \psi(\mathrm{t}) < \mathrm{t} \text{ for } \mathrm{t} > 0\}$, $\mathfrak{C}([0,1] \times \Re, \Re)$, denotes the space of continues mappings, $P_i, Q_i : [0,1] \times \Re^n \to \Re$, $h_i : \Re \to \Re$, and $\varphi_i : [0,1] \times \Re \to \Re$ are mappings having the following properties:

(1) The maps $P_i : [0,1] \times \Re^n \to \Re$, $h_i : \Re \to \Re$, and $\varphi_i : [0,1] \times \Re \to \Re$ are continuous mappings.
(2) The maps $Q_i : [0,1] \times \Re^n \to \Re$ are continuous or piece-wise continuous mappings.

Now we recollect some results, facts, and definitions [14–20]:

Definition 2.1. *The Riemenn-Liouville (R-L) integral of order* $[\alpha] : \alpha > 0$ *(*$[\alpha]$ *is the integer part of* α*) of a mapping* $g : \Re^+ \to \Re$ *is defined as:*

$$I^\alpha g\left(\bar{\vartheta}\right) = \frac{1}{\Gamma(\alpha)} \int_0^{\bar{\vartheta}} \left(\bar{\vartheta} - \bar{s}\right)^{\alpha - 1} g(\bar{s}) \, d\bar{s},$$

under the criterion that the right-hand side is defined piece-wise over $\Re^+$.

Definition 2.2. *The CFD of order* $[\alpha] : \alpha > 0$ *(*$[\alpha]$ *is the integer part of* α*) of a mapping* $g : \Re^+ \to \Re$ *is defined as:*

$$D^\alpha g\left(\bar{\vartheta}\right) = \frac{1}{\Gamma(m-\alpha)} \int_0^{\bar{\vartheta}} \left(\bar{\vartheta} - \bar{s}\right)^{m-\alpha-1} g^m(\bar{s}) \, d\bar{s},$$

under the criterion that the right-hand side is defined piece-wise over $\Re^+$*, and* $m = [\alpha] + 1$.

Definition 2.3. *For a mapping* $\varkappa\left(\bar{\vartheta}\right) \in \mathfrak{C}([0,1] \times \Re, \Re)$, *the integral* I^α, $\alpha \in (n-1, n]$ *is defined as:*

$$I^\alpha \left(D^\alpha \varkappa\left(\bar{\vartheta}\right)\right) = a_0 + a_1 \bar{\vartheta} + a_2 \bar{\vartheta}^2 + \ldots + a_{n-1} \bar{\vartheta}^{n-1} + I^\alpha g\left(\bar{\vartheta}\right),$$

where $D^\alpha \varkappa\left(\bar{\vartheta}\right) = g\left(\bar{\vartheta}\right)$.

Definition 2.4. *A mapping* $\mathscr{T} : \mathscr{X} \to \mathscr{X} : \mathscr{X} \in \mathfrak{C}([0,1] \times \Re, \Re)$ *is a contraction on* $\mathscr{X}$ *if there exist* $0 < \alpha < 1$, *which satisfies the following condition for all* $\varkappa, F \in \mathscr{X}$,

$$\|\mathscr{T}(\varkappa) - \mathscr{T}(F)\| \leq \alpha \|\varkappa - F\|.$$

Definition 2.5. *A mapping* $\mathscr{T} : \mathscr{X} \to \mathscr{X} : \mathscr{X} \in \mathfrak{C}([0,1] \times \mathfrak{R}, \mathfrak{R})$ *has a coupled fixed point* $(\varkappa, F)$ *if* $\mathscr{T}(\varkappa, F) = \varkappa$ *and* $\mathscr{T}(F, \varkappa) = F$.

Definition 2.6. *Consider a Banach space* $\mathscr{X}$ *and a subset* $\mathcal{S}$ *of* $\mathscr{X}$, *which is bounded, convex, and closed, and* $\mathfrak{B} : \mathcal{S} \to \mathscr{X}$. *Then* $\mathfrak{B}$ *is completely continuous if it is:*

(1) Continuous on $\mathcal{S}$.
(2) Uniformly bounded on $\mathcal{S}$.
(3) Uniformly continuous on $\mathcal{S}$.

Theorem 2.7. *Consider a Banach space* $\mathscr{X}$ *and a subset* $\mathcal{S}$ *of* $\mathscr{X}$, *which is bounded, convex, and closed, and* $\tilde{\mathcal{S}} = \mathcal{S} \times \mathcal{S}$. *Let* $\mathfrak{A} : \mathscr{X} \to \mathscr{X}$ *and* $\mathfrak{B} : \mathcal{S} \to \mathscr{X}$ *be two operators such that:*

(C_1) *There is a positive constant* $\alpha < 1$, *and* $\psi_{\mathfrak{A}} \in \Psi$ *such that:*

$$\|\mathfrak{A}(\varkappa) - \mathfrak{A}(F)\| \le \alpha \psi_{\mathfrak{A}} \|\varkappa - F\|.$$

(C_2) $\mathfrak{B}$ *is completely continuous on* $\mathcal{S}$.

(C_3) $\varkappa = \mathfrak{A}(\varkappa) + \mathfrak{B}(F) \Longrightarrow \varkappa \in \mathcal{S}$ *for* $\forall F \in \mathcal{S}$.

Then the operator $\mathscr{T}(\varkappa, F) = \mathfrak{A}(\varkappa) + \mathfrak{B}(F)$ *has one or more coupled fixed point(s) in* $\tilde{\mathcal{S}}$.

3. Main results

For the analysis of existence results, we assume that for $\forall\, \varkappa, F \in \mathfrak{C}([0,1] \times \mathfrak{R}^n, \mathfrak{R})$, where $\varkappa(\bar{\vartheta}) = \left(\varkappa_1(\bar{\vartheta}), \varkappa_2(\bar{\vartheta}), ..., \varkappa_n(\bar{\vartheta})\right)$, $F(\bar{\vartheta}) = \left(F_1(\bar{\vartheta}), F_2(\bar{\vartheta}), ..., F_n(\bar{\vartheta})\right)$, $\bar{\vartheta} \in [0,1]$, the following conditions hold:

(A_1): $\left|h(\varkappa_i) - h(\varkappa_j)\right| \le \left|\varkappa_i - \varkappa_j\right|$; $i, j = 1, 2, 3, ...$

(A_2): There are constants $M \ge L > 0$ such that:

$$\left|P\left(\bar{\vartheta}, \varkappa_1(\bar{\vartheta}), \varkappa_2(\bar{\vartheta}), ..., \varkappa_n(\bar{\vartheta})\right) - P\left(\bar{\vartheta}, F_1(\bar{\vartheta}), F_2(\bar{\vartheta}), ..., F_n(\bar{\vartheta})\right)\right| \le \frac{L\left|\varkappa(\bar{\vartheta}) - F(\bar{\vartheta})\right|}{2n\left(M + \left|\varkappa(\bar{\vartheta}) - F(\bar{\vartheta})\right|\right)}.$$

(A_3): There exists a continuous mapping $g(\bar{\vartheta}) \in C([0,1], \mathfrak{R})$ such that:

$$Q\left(\bar{\vartheta}, \varkappa_1(\bar{\vartheta}), \varkappa_2(\bar{\vartheta}), ..., \varkappa_n(\bar{\vartheta})\right) \le g(\bar{\vartheta}).$$

Lemma 3.1. *If* $P_i\left(\bar{\vartheta}, \varkappa_1(\bar{\vartheta}), \varkappa_2(\bar{\vartheta}), ..., \varkappa_n(\bar{\vartheta})\right)|_{\bar{\vartheta}=0} = P_i\left(\bar{\vartheta}, \varkappa_1(\bar{\vartheta}), \varkappa_2(\bar{\vartheta}), ..., \varkappa_n(\bar{\vartheta})\right)|_{\bar{\vartheta}=1} = 0$ *for each* $i = 1, 2, 3...n$, *then the representation of Eq (1.2) in integral form is given by:*

$$\begin{aligned}
\varkappa_i(\bar{\vartheta}) &= P_i\left(\bar{\vartheta}, \varkappa_1(\bar{\vartheta}), \varkappa_2(\bar{\vartheta}), ..., \varkappa_n(\bar{\vartheta})\right) + \left(1 - \bar{\vartheta}\right) h_i(\varkappa_i) + \frac{\bar{\vartheta}}{\Gamma(\alpha_i)} \int_0^1 (1 - \bar{s})^{\alpha_i - 1} \varphi_i(\bar{s}, \varkappa_i(\bar{s}))\, d\bar{s} \\
&\quad - \frac{\bar{\vartheta}}{\Gamma(\alpha_i)} \int_0^1 (1 - \bar{s})^{\alpha_i - 1} Q_i(\bar{s}, \varkappa_1(\bar{s}), \varkappa_2(\bar{s}), ..., \varkappa_n(\bar{s}))\, d\bar{s} \\
&\quad + \frac{1}{\Gamma(\alpha_i)} \int_0^{\bar{\vartheta}} \left(\bar{\vartheta} - \bar{s}\right)^{\alpha_i - 1} Q_i(\bar{s}, \varkappa_1(\bar{s}), \varkappa_2(\bar{s}), ..., \varkappa_n(\bar{s}))\, d\bar{s},
\end{aligned}$$

for each $i = 1, 2, 3, ..., n$.

Proof. Applying I^{α_1} on $\varkappa_1\left(\bar{\vartheta}\right)$ of Eq (1.2), we get:

$$\varkappa_1\left(\bar{\vartheta}\right) - P_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right) = a_0 + a_1\bar{\vartheta} + I^{\alpha_1}Q_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right). \tag{3.1}$$

Applying the initial conditions, $\varkappa_1(0) = h_1(\varkappa_1)$ and $P_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)|_{\bar{\vartheta}=0}= 0$, we get:

$$a_0 = h_1(\varkappa_1).$$

Also $\varkappa_1(1) = \frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s}$, and $P_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)|_{\bar{\vartheta}=1}= 0$ gives us

$$\frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s} = a_0 + a_1 + I^{\alpha_1}Q_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)|_{\bar{\vartheta}=1}$$

$a_0 = h_1(\varkappa_1)$, and

$$I^{\alpha_1}Q_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)|_{\bar{\vartheta}=1}= \frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-\bar{s})^{\alpha_1-1} Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}$$

gives us

$$a_1 = \frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s} - h_1(\varkappa_1) - \frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-\bar{s})^{\alpha_1-1} Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}.$$

Putting the values of $a_0, a_1,$ and $I^{\alpha_1}Q_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)|_{\bar{\vartheta}=1}$ in Eq (3.1), we get:

$$\begin{aligned}
&\varkappa_1\left(\bar{\vartheta}\right) - P_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)\\
= \;& h_1(\varkappa_1) + \left(\frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s} - h_1(\varkappa_1)\right.\\
&\left.-\frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-\bar{s})^{\alpha_1-1} Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}\right)\bar{\vartheta} + I^{\alpha_1}Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})),
\end{aligned}$$

which after simplification gives:

$$\begin{aligned}
\varkappa_1\left(\bar{\vartheta}\right) \;=\;& P_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right) + \left(1-\bar{\vartheta}\right)h_1(\varkappa_1) + \frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1 (1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s}\\
&-\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1 (1-\bar{s})^{\alpha_1-1} Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}\\
&+\frac{1}{\Gamma(\alpha_1)}\int_0^{\bar{\vartheta}} \left(\bar{\vartheta}-\bar{s}\right)^{\alpha_1-1} Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}.
\end{aligned}$$

Similarly, we can find $\varkappa_2\left(\bar{\vartheta}\right),\varkappa_3\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)$, where

$$\begin{aligned}
\varkappa_i\left(\bar{\vartheta}\right) \;=\;& P_i\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right) + \left(1-\bar{\vartheta}\right)h_i(\varkappa_i) + \frac{\bar{\vartheta}}{\Gamma(\alpha_i)}\int_0^1 (1-\bar{s})^{\alpha_i-1}\varphi_i(\bar{s},\varkappa_i(\bar{s}))\,d\bar{s}\\
&-\frac{\bar{\vartheta}}{\Gamma(\alpha_i)}\int_0^1 (1-\bar{s})^{\alpha_i-1} Q_i(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}\\
&+\frac{1}{\Gamma(\alpha_i)}\int_0^{\bar{\vartheta}} \left(\bar{\vartheta}-\bar{s}\right)^{\alpha_i-1} Q_i(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s},
\end{aligned}$$

for each $i = 1, 2, 3, ..., n$. This completes the proof. □

Let us define $\mathfrak{A}, \mathfrak{B} : \Re^n \to \Re^n$, by $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, ..., \mathfrak{A}_n)$, and $\mathfrak{B} = (\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, ..., \mathfrak{B}_n)$, where

$$\begin{aligned} \mathfrak{A}_i &= P_i\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) + \left(1 - \bar{\vartheta}\right) h_i\left(\varkappa_i\right), \\ \mathfrak{B}_i &= \frac{\bar{\vartheta}}{\Gamma(\alpha_i)} \int_0^1 (1 - \bar{s})^{\alpha_i - 1} \varphi_i\left(\bar{s}, \varkappa_i(\bar{s})\right) d\bar{s} - \frac{\bar{\vartheta}}{\Gamma(\alpha_i)} \int_0^1 (1 - \bar{s})^{\alpha_i - 1} Q_i\left(\bar{s}, \varkappa_1(\bar{s}), \varkappa_2(\bar{s}), ..., \varkappa_n(\bar{s})\right) d\bar{s} \\ &\quad + \frac{1}{\Gamma(\alpha_i)} \int_0^{\bar{\vartheta}} \left(\bar{\vartheta} - \bar{s}\right)^{\alpha_i - 1} Q_i\left(\bar{s}, \varkappa_1(\bar{s}), \varkappa_2(\bar{s}), ..., \varkappa_n(\bar{s})\right) d\bar{s}, \end{aligned}$$

for each $i = 1, 2, 3, .., n$. Now define $\mathscr{X} : \Re^n \to \Re^n$ by $\mathscr{X} = \mathfrak{A} + \mathfrak{B}$. Then the solution of Eq (1.2) in operator form is given by:

$$\begin{cases} \left(\varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) & = \mathscr{X}\left(\varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) \\ & = \mathfrak{A}\left(\varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) + \mathfrak{B}\left(\varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right). \end{cases} \tag{3.2}$$

Lemma 3.2. *The operator* $\mathfrak{A}$ *satisfies* $\|\mathfrak{A}(\varkappa) - \mathfrak{A}(F)\| \leq \alpha\psi_{\mathfrak{A}}(\|\varkappa - F\|)$, *where* $\alpha < 1$, *and* $\psi_{\mathfrak{A}} \in \Psi$, *for* $\forall \varkappa, F \in \mathfrak{C}([0, 1] \times \Re^n, \Re)$.

Proof. Let $\varkappa\left(\bar{\vartheta}\right) = \left(\varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right)$, and $F\left(\bar{\vartheta}\right) = \left(F_1\left(\bar{\vartheta}\right), F_2\left(\bar{\vartheta}\right), ..., F_n\left(\bar{\vartheta}\right)\right)$. Consider

$$\begin{aligned} & \left|\mathfrak{A}\left(\varkappa\left(\bar{\vartheta}\right)\right) - \mathfrak{A}\left(F\left(\bar{\vartheta}\right)\right)\right| \\ = & \left|(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, ..., \mathfrak{A}_n)\left(\varkappa\left(\bar{\vartheta}\right)\right) - (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, ..., \mathfrak{A}_n)\left(F\left(\bar{\vartheta}\right)\right)\right| \\ = & \left|\mathfrak{A}_1\left(\varkappa\left(\bar{\vartheta}\right)\right) - \mathfrak{A}_1\left(F\left(\bar{\vartheta}\right)\right), \mathfrak{A}_2\left(\varkappa\left(\bar{\vartheta}\right)\right) - \mathfrak{A}_2\left(F\left(\bar{\vartheta}\right)\right), ..., \mathfrak{A}_n\left(\varkappa\left(\bar{\vartheta}\right)\right) - \mathfrak{A}_n\left(F\left(\bar{\vartheta}\right)\right)\right| \\ \leq & \left|\mathfrak{A}_1\left(\varkappa\left(\bar{\vartheta}\right)\right) - \mathfrak{A}_1\left(F\left(\bar{\vartheta}\right)\right)\right| + \left|\mathfrak{A}_2\left(\varkappa\left(\bar{\vartheta}\right)\right) - \mathfrak{A}_2\left(F\left(\bar{\vartheta}\right)\right)\right| + \cdots + \left|\mathfrak{A}_n\left(\varkappa\left(\bar{\vartheta}\right)\right) - \mathfrak{A}_n\left(F\left(\bar{\vartheta}\right)\right)\right|. \end{aligned} \tag{3.3}$$

Now consider,

$$\begin{aligned} & \left|\mathfrak{A}_1\left(\varkappa\left(\bar{\vartheta}\right)\right) - \mathfrak{A}_1\left(F\left(\bar{\vartheta}\right)\right)\right| \\ = & \left|P_1\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) + \left(1 - \bar{\vartheta}\right) h_1(\varkappa_1) - P_1\left(\bar{\vartheta}, F_1\left(\bar{\vartheta}\right), F_2\left(\bar{\vartheta}\right), ..., F_n\left(\bar{\vartheta}\right)\right) - \left(1 - \bar{\vartheta}\right) h_1(F_1)\right| \\ \leq & \left|P_1\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) - P_1\left(\bar{\vartheta}, F_1\left(\bar{\vartheta}\right), F_2\left(\bar{\vartheta}\right), ..., F_n\left(\bar{\vartheta}\right)\right)\right| + \left|\left(1 - \bar{\vartheta}\right) h_1(\varkappa_1) - \left(1 - \bar{\vartheta}\right) h_1(F_1)\right| \\ \leq & \frac{L\left|\varkappa\left(\bar{\vartheta}\right) - F\left(\bar{\vartheta}\right)\right|}{2n\left(M + \left|\varkappa\left(\bar{\vartheta}\right) - F\left(\bar{\vartheta}\right)\right|\right)} + |\varkappa_1 - F_1|. \end{aligned}$$

Similarly, for each $i = 1, 2, 3..., n$,

$$\left|\mathfrak{A}_i\left(\varkappa\left(\bar{\vartheta}\right)\right) - \mathfrak{A}_i\left(F\left(\bar{\vartheta}\right)\right)\right| \leq \frac{L\left|\varkappa\left(\bar{\vartheta}\right) - F\left(\bar{\vartheta}\right)\right|}{2n\left(M + \left|\varkappa\left(\bar{\vartheta}\right) - F\left(\bar{\vartheta}\right)\right|\right)} + |\varkappa_i - F_i|,$$

and hence Eq (3.3) becomes:

$$\left|\mathfrak{A}\left(\varkappa\left(\bar{\vartheta}\right)\right)-\mathfrak{A}\left(F\left(\bar{\vartheta}\right)\right)\right| \leq n\left(\frac{L\left|\varkappa\left(\bar{\vartheta}\right)-F\left(\bar{\vartheta}\right)\right|}{2n\left(M+\left|\varkappa\left(\bar{\vartheta}\right)-F\left(\bar{\vartheta}\right)\right|\right)}\right)+\sum_{i=1}^{n}\left|\varkappa_i-F_i\right|$$
$$=\frac{L\left|\varkappa\left(\bar{\vartheta}\right)-F\left(\bar{\vartheta}\right)\right|}{2\left(M+\left|\varkappa\left(\bar{\vartheta}\right)-F\left(\bar{\vartheta}\right)\right|\right)}+\left|\varkappa-F\right|.$$

Taking $\mathrm{Sup}_{\bar{\vartheta}\in[0,1]}$ on both sides we get,

$$\|\mathfrak{A}(\varkappa)-\mathfrak{A}(F)\| \leq \frac{L\|\varkappa-F\|}{2(M+\|\varkappa-F\|)}+\|\varkappa-F\| = \frac{1}{2}\left(\frac{L\|\varkappa-F\|}{M+\|\varkappa-F\|}+2\|\varkappa-F\|\right).$$

Hence:

$$\|\mathfrak{A}(\varkappa)-\mathfrak{A}(F)\| \leq \alpha\psi_{\mathfrak{A}}(\|\varkappa-F\|),$$

where

$$\alpha=\frac{1}{2} \text{ and } \psi_{\mathfrak{A}}(\|\varkappa-F\|)=\frac{L\|\varkappa-F\|}{M+\|\varkappa-F\|}+2\|\varkappa-F\|.$$

This completes the proof. □

Lemma 3.3. *Let* $\mathcal{S}=\{\varkappa\in\mathfrak{R}^n:\ \|\varkappa\|\leq N\}$, *where*

$$N\geq n\left(L+P_0+\|h\|+\frac{\|\varphi\|}{\Gamma(\alpha+1)}+2\frac{\|g\|}{\Gamma(\alpha+1)}\right)$$

and

$$P_0=max_i\{P_{0i}\},\ \ \|h\|=max_i\{\|h_i\|\},\ \frac{\|\varphi\|}{\Gamma(\alpha+1)}=max_i\left\{\frac{\|\varphi_i\|}{\Gamma(\alpha_i+1)}\right\},$$
$$and\ \frac{\|g\|}{\Gamma(\alpha+1)}=max_i\left\{\frac{\|g_i\|}{\Gamma(\alpha_i+1)}\right\}.$$

Then the operator $\mathfrak{B}$ *is:*

(1) *Continuous on* $\mathcal{S}$.
(2) *Uniformly bounded on* $\mathcal{S}$.
(3) *Uniformly continuous on* $\mathcal{S}$.

Proof. Clearly $\mathcal{S}$ is bounded, convex, and closed.

(1) For continuity, let $\{\varkappa_n\}$ be a sequence in $\mathcal{S}$ that converges to $\varkappa=(\varkappa_1,\varkappa_2,...,\varkappa_n)$ in $\mathcal{S}$. Consider $\varkappa_m=(\varkappa_{1m},\varkappa_{2m},\varkappa_{3m},...,\varkappa_{nm})\in\{\varkappa_n\}$. Consider,

$$\begin{aligned}
&\left|\mathfrak{B}\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-\mathfrak{B}\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|\\
&=\left|(\mathfrak{B}_1,\mathfrak{B}_2,\mathfrak{B}_3,...,\mathfrak{B}_n)\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-(\mathfrak{B}_1,\mathfrak{B}_2,\mathfrak{B}_3,...,\mathfrak{B}_n)\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|\\
&=\left|\left(\mathfrak{B}_1\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-\mathfrak{B}_1\left(\varkappa\left(\bar{\vartheta}\right)\right),\mathfrak{B}_2\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-\mathfrak{B}_2\left(\varkappa\left(\bar{\vartheta}\right)\right),...,\mathfrak{B}_n\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-\mathfrak{B}_n\left(\varkappa\left(\bar{\vartheta}\right)\right)\right)\right|\\
&\leq\left|\mathfrak{B}_1\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-\mathfrak{B}_1\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|+\left|\mathfrak{B}_2\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-\mathfrak{B}_2\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|+\cdots+\left|\mathfrak{B}_n\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-\mathfrak{B}_n\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|.
\end{aligned} \tag{3.4}$$

Now consider,

$$
\begin{aligned}
&\left|\mathfrak{B}_1\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-\mathfrak{B}_1\left(\varkappa\left(\bar{\vartheta}\right)\right)\right| \\
&=\left|\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_{1m}(\bar{s}))\,d\bar{s}-\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}Q_1(\bar{s},\varkappa_{1m}(\bar{s}),\varkappa_{2m}(\bar{s}),...,\varkappa_{nm}(\bar{s}))\,d\bar{s}\right. \\
&+\frac{1}{\Gamma(\alpha_1)}\int_0^{\bar{\vartheta}}\left(\bar{\vartheta}-\bar{s}\right)^{\alpha_1-1}Q_1(\bar{s},\varkappa_{1m}(\bar{s}),\varkappa_{2m}(\bar{s}),...,\varkappa_{nm}(\bar{s}))\,d\bar{s} \\
&-\left(\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s}-\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}\right. \\
&\left.\left.+\frac{1}{\Gamma(\alpha_1)}\int_0^{\bar{\vartheta}}\left(\bar{\vartheta}-\bar{s}\right)^{\alpha_1-1}Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}\right)\right| \\
&\leq\left|\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}(\varphi_1(\bar{s},\varkappa_{1m}(\bar{s}))-\varphi_1(\bar{s},\varkappa_1(\bar{s})))\,d\bar{s}\right| \\
&+\left|\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}(Q_1(\bar{s},\varkappa_{1m}(\bar{s}),\varkappa_{2m}(\bar{s}),...,\varkappa_{nm}(\bar{s}))-Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})))\,d\bar{s}\right| \\
&+\left|\frac{1}{\Gamma(\alpha_1)}\int_0^{\bar{\vartheta}}\left(\bar{\vartheta}-\bar{s}\right)^{\alpha_1-1}(Q_1(\bar{s},\varkappa_{1m}(\bar{s}),\varkappa_{2m}(\bar{s}),...,\varkappa_{nm}(\bar{s}))-Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})))\,d\bar{s}\right| \\
&\leq\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}|\varphi_1(\bar{s},\varkappa_{1m}(\bar{s}))-\varphi_1(\bar{s},\varkappa_1(\bar{s}))|\,d\bar{s} \\
&+\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}|Q_1(\bar{s},\varkappa_{1m}(\bar{s}),\varkappa_{2m}(\bar{s}),...,\varkappa_{nm}(\bar{s}))-Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))|\,d\bar{s} \\
&+\frac{1}{\Gamma(\alpha_1)}\int_0^{\bar{\vartheta}}\left(\bar{\vartheta}-\bar{s}\right)^{\alpha_1-1}|Q_1(\bar{s},\varkappa_{1m}(\bar{s}),\varkappa_{2m}(\bar{s}),...,\varkappa_{nm}(\bar{s}))-Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))|\,d\bar{s},
\end{aligned}
$$

which tends to zero when $m\longrightarrow\infty$.
Similarly, for each $i=1,2,3...,n$,

$$\left|\mathfrak{B}_i\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-\mathfrak{B}_i\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|\longrightarrow 0 \text{ as } m\longrightarrow\infty,$$

and hence Eq (3.4) gives:

$$\left|\mathfrak{B}\left(\varkappa_m\left(\bar{\vartheta}\right)\right)-\mathfrak{B}\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|\longrightarrow 0 \text{ as } m\longrightarrow\infty.$$

Hence $\mathfrak{B}$ is continues on $\mathcal{S}$.

(2) For uniform boundedness, consider $\varkappa=(\varkappa_1,\varkappa_2,...,\varkappa_n)\in\mathcal{S}$,

$$
\begin{aligned}
\left|\mathfrak{B}\left(\varkappa\left(\bar{\vartheta}\right)\right)\right| &= \left|(\mathfrak{B}_1,\mathfrak{B}_2,\mathfrak{B}_3,...,\mathfrak{B}_n)\left(\varkappa\left(\bar{\vartheta}\right)\right)\right| \\
&= \left|\mathfrak{B}_1\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|+\left|\mathfrak{B}_2\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|+\left|\mathfrak{B}_3\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|+\cdots+\left|\mathfrak{B}_n\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|.
\end{aligned}
\tag{3.5}
$$

Consider:

$$
\begin{aligned}
\left|\mathfrak{B}_1\left(\varkappa\left(\bar{\vartheta}\right)\right)\right| &= \left|\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,ds\right. \\
&\quad-\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha_1)}\int_0^{\bar{\vartheta}}\left(\bar{\vartheta}-\bar{s}\right)^{\alpha_1-1}Q_1\left(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})\right)d\bar{s}\Bigg)\\
\leq\ &\left|\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s}\right|\\
&+\left|\frac{\bar{\vartheta}}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}Q_1\left(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})\right)d\bar{s}\right|\\
&+\left|\frac{1}{\Gamma(\alpha_1)}\int_0^{\bar{\vartheta}}\left(\bar{\vartheta}-\bar{s}\right)^{\alpha_1-1}Q_1\left(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})\right)d\bar{s}\right|\\
\leq\ &\frac{\|\varphi_1\|}{\Gamma(\alpha_1+1)}+2\frac{\|g_1\|}{\Gamma(\alpha_1+1)}.
\end{aligned}
$$

Similarly, for each $i=1,2,3...,n$,

$$
\left|\mathfrak{B}_i\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|\leq\frac{\|\varphi_i\|}{\Gamma(\alpha_i+1)}+2\frac{\|g_i\|}{\Gamma(\alpha_i+1)}\leq\frac{\|\varphi\|}{\Gamma(\alpha+1)}+2\frac{\|g\|}{\Gamma(\alpha+1)}.
$$

Hence Eq (3.5) gives:

$$
\left|\mathfrak{B}\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|\leq n\left(\frac{\|\varphi\|}{\Gamma(\alpha+1)}+2\frac{\|g\|}{\Gamma(\alpha+1)}\right)\leq N.
$$

Hence $\mathfrak{B}$ is uniformly bounded on $\mathcal{S}$.

(3) For uniform continuity, consider $\varkappa=(\varkappa_1,\varkappa_2,...,\varkappa_n)\in\mathcal{S}$, and $\bar{\vartheta}_1,\bar{\vartheta}_2\in[0,1]$ $\left(\bar{\vartheta}_1<\bar{\vartheta}_2\right)$.

$$
\begin{aligned}
\left|\mathfrak{B}\left(\mathsf{x}\left(\bar{\vartheta}_1\right)\right)-\mathfrak{B}\left(\mathsf{x}\left(\bar{\vartheta}_2\right)\right)\right|&=\left|(\mathfrak{B}_1,\mathfrak{B}_2,\mathfrak{B}_3,...,\mathfrak{B}_n)\left(\varkappa\left(\bar{\vartheta}_1\right)\right)-(\mathfrak{B}_1,\mathfrak{B}_2,\mathfrak{B}_3,...,\mathfrak{B}_n)\left(\varkappa\left(\bar{\vartheta}_2\right)\right)\right|\\
&=\left|\mathfrak{B}_1\left(\varkappa\left(\bar{\vartheta}_1\right)\right)-\mathfrak{B}_1\left(\varkappa\left(\bar{\vartheta}_2\right)\right)\right|+\left|\mathfrak{B}_2\left(\varkappa\left(\bar{\vartheta}_1\right)\right)-\mathfrak{B}_2\left(\varkappa\left(\bar{\vartheta}_2\right)\right)\right|\\
&\quad+\left|\mathfrak{B}_3\left(\varkappa\left(\bar{\vartheta}_1\right)\right)-\mathfrak{B}_3\left(\varkappa\left(\bar{\vartheta}_2\right)\right)\right|+\cdots+\left|\mathfrak{B}_n\left(\varkappa\left(\bar{\vartheta}_1\right)\right)-\mathfrak{B}_n\left(\varkappa\left(\bar{\vartheta}_2\right)\right)\right|. \quad (3.6)
\end{aligned}
$$

Consider:

$$
\begin{aligned}
&\left|\mathfrak{B}_1\left(\left(\varkappa\left(\bar{\vartheta}_1\right)\right)\right)-\mathfrak{B}_1\left(\varkappa\left(\bar{\vartheta}_2\right)\right)\right|\\
&=\left|\frac{\bar{\vartheta}_1}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s}-\frac{\bar{\vartheta}_1}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}Q_1\left(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})\right)d\bar{s}\right.\\
&+\frac{1}{\Gamma(\alpha_1)}\int_0^{\bar{\vartheta}_1}\left(\bar{\vartheta}_1-\bar{s}\right)^{\alpha_1-1}Q_1\left(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})\right)d\bar{s}-\left(\frac{\bar{\vartheta}_2}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s}\right.\\
&-\frac{\bar{\vartheta}_2}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}Q_1\left(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})\right)d\bar{s}\\
&\left.\left.+\frac{1}{\Gamma(\alpha_1)}\int_0^{\bar{\vartheta}_2}\left(\bar{\vartheta}_2-\bar{s}\right)^{\alpha_1-1}Q_1\left(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})\right)d\bar{s}\right)\right|\\
&\leq\left|\frac{\bar{\vartheta}_2-\bar{\vartheta}_1}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s}\right|+\left|\frac{\bar{\vartheta}_2-\bar{\vartheta}_1}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}Q_1\left(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})\right)d\bar{s}\right|\\
&+\left|\frac{1}{\Gamma(\alpha_1)}\left(\int_0^{\bar{\vartheta}_1}\left(\bar{\vartheta}_1-\bar{s}\right)^{\alpha_1-1}-\int_0^{\bar{\vartheta}_2}\left(\bar{\vartheta}_2-\bar{s}\right)^{\alpha_1-1}\right)Q_1\left(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s})\right)d\bar{s}\right|
\end{aligned}
$$

$$\leq \left|\frac{\bar{\vartheta}_2-\bar{\vartheta}_1}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s}\right|+\left|\frac{\bar{\vartheta}_2-\bar{\vartheta}_1}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}\right|$$
$$+\left|\frac{1}{\Gamma(\alpha_1)}\left(\int_0^{\bar{\vartheta}_1}\left(\left(\bar{\vartheta}_1-\bar{s}\right)^{\alpha_1-1}-\left(\bar{\vartheta}_2-\bar{s}\right)^{\alpha_1-1}\right)-\int_{\bar{\vartheta}_1}^{\bar{\vartheta}_2}\left(\bar{\vartheta}_2-\bar{s}\right)^{\alpha_1-1}\right)Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}\right|$$
$$\leq \left|\frac{\bar{\vartheta}_2-\bar{\vartheta}_1}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}\varphi_1(\bar{s},\varkappa_1(\bar{s}))\,d\bar{s}\right|+\left|\frac{\bar{\vartheta}_2-\bar{\vartheta}_1}{\Gamma(\alpha_1)}\int_0^1(1-\bar{s})^{\alpha_1-1}Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}\right|$$
$$+\left|\frac{1}{\Gamma(\alpha_1)}\left(\int_0^{\bar{\vartheta}_1}\left(\left(\bar{\vartheta}_1-\bar{s}\right)^{\alpha_1-1}-\left(\bar{\vartheta}_2-\bar{s}\right)^{\alpha_1-1}\right)\right)Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}\right|$$
$$+\left|\int_{\bar{\vartheta}_1}^{\bar{\vartheta}_2}\left(\bar{\vartheta}_2-\bar{s}\right)^{\alpha_1-1}Q_1(\bar{s},\varkappa_1(\bar{s}),\varkappa_2(\bar{s}),...,\varkappa_n(\bar{s}))\,d\bar{s}\right|$$
$$\leq \frac{\left(\bar{\vartheta}_2-\bar{\vartheta}_1\right)\|\varphi_1\|}{\Gamma(\alpha_1+1)}+\frac{\|g_1\|}{\Gamma(\alpha_1+1)}\left(\left(\bar{\vartheta}_2^{\alpha_1}-\bar{\vartheta}_1^{\alpha_1}\right)+\left(\bar{\vartheta}_2-\bar{\vartheta}_1\right)\right),$$

which tends to zero when $\bar{\vartheta}_2 \longrightarrow \bar{\vartheta}_1$.
Similarly, for each $i = 1, 2, 3..., n$,

$$\left|\mathfrak{B}_i\left(\left(\varkappa\left(\bar{\vartheta}_1\right)\right)\right)-\mathfrak{B}_i\left(\varkappa\left(\bar{\vartheta}_2\right)\right)\right|\longrightarrow 0 \text{ as } \bar{\vartheta}_2\longrightarrow\bar{\vartheta}_1.$$

Hence Eq (3.6) gives:

$$\left|\mathfrak{B}\left(\left(\varkappa\left(\bar{\vartheta}_1\right)\right)\right)-\mathfrak{B}\left(\varkappa\left(\bar{\vartheta}_2\right)\right)\right|\longrightarrow 0 \text{ as } \bar{\vartheta}_2\longrightarrow\bar{\vartheta}_1.$$

Hence $\mathfrak{B}$ is uniformly continuous on $\mathcal{S}$, and hence the result follows. □

Theorem 3.4. *Suppose $(A_1)-(A_3)$ holds, then there exists a solution to the n-dimensional nonlinear CFHDEs of Eq (1.2).*

Proof. From Eq (3.2), the solution of Eq (1.2) is given by:

$$\begin{cases}\left(\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right) & =\mathscr{X}\left(\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)\\ & =\mathfrak{A}\left(\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right)+\mathfrak{B}\left(\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),...,\varkappa_n\left(\bar{\vartheta}\right)\right).\end{cases}$$

From Lemma 3.2, the operator $\mathfrak{A}$ satisfies

$$\|\mathfrak{A}(\varkappa)-\mathfrak{A}(F)\|\leq\alpha\psi_{\mathfrak{A}}(\|\varkappa-F\|),$$

where $\alpha < 1$, and $\psi_{\mathfrak{A}} \in \Psi$, for $\forall \varkappa, F \in \mathfrak{C}([0,1]\times\mathfrak{R}^n,\mathfrak{R})$. Hence C_1 of Theorem 2.7 is satisfied.
Let $\mathcal{S} = \{\varkappa \in \mathfrak{R}^n : \ \|\varkappa\| \leq N\}$, where

$$N\geq n\left(L+P_0+\|h\|+\frac{\|\varphi\|}{\Gamma(\alpha+1)}+2\frac{\|g\|}{\Gamma(\alpha+1)}\right)\text{ for each } i=1,2,3,...,n,$$

where

$$P_0=\max_i\{P_{0i}\},\ \|h\|=\max_i\{\|h_i\|\},\ \frac{\|\varphi\|}{\Gamma(\alpha+1)}=\max_i\left\{\frac{\|\varphi_i\|}{\Gamma(\alpha_i+1)}\right\},$$
$$\text{and }\frac{\|g\|}{\Gamma(\alpha+1)}=\max_i\left\{\frac{\|g_i\|}{\Gamma(\alpha_i+1)}\right\}.$$

From Lemma 3.3, the operator $\mathfrak{B}$ is:

(1) Continuous on $\mathcal{S}$.
(2) Uniformly bounded on $\mathcal{S}$.
(3) Uniformly continuous on $\mathcal{S}$.

Hence $\mathcal{B}$ is equi-continues and hence is completely continuous on $\mathcal{S}$. Hence, C_2 of Theorem 2.7 is satisfied. To prove C_3 of Theorem 2.7, let $\varkappa = (\varkappa_1, \varkappa_2, ..., \varkappa_n) \in \mathscr{X}$ and $F = (F_1, F_2, ..., F_n) \in \mathcal{S}$ such that $\varkappa = \mathfrak{A}(\varkappa) + \mathfrak{B}(F)$. We need to prove $\varkappa \in \mathcal{S}$. Consider:

$$\begin{aligned}\left|\varkappa\left(\bar{\vartheta}\right)\right| &= \left|\mathfrak{A}(\varkappa(\vartheta)) + \mathfrak{B}\left(F\left(\bar{\vartheta}\right)\right)\right| \leq \left|\mathfrak{A}\left(\varkappa\left(\bar{\vartheta}\right)\right)\right| + \left|\mathfrak{B}\left(F\left(\bar{\vartheta}\right)\right)\right| \\ &= \left|\mathfrak{A}\left(\varkappa\left(\bar{\vartheta}\right)\right)\right| + n\left(\frac{\|\varphi\|}{\Gamma(\alpha+1)} + 2\frac{\|g\|}{\Gamma(\alpha+1)}\right) \text{ (using Lemma (3.3 (ii))).}\end{aligned} \tag{3.7}$$

Consider:

$$\begin{aligned}\left|\mathfrak{A}\left(\varkappa\left(\bar{\vartheta}\right)\right)\right| &= \left|(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, ..., \mathfrak{A}_n)\left(\varkappa\left(\bar{\vartheta}\right)\right)\right| \\ &= \left|\mathfrak{A}_1\left(\varkappa\left(\bar{\vartheta}\right)\right) | + | \mathfrak{A}_2\left(\varkappa\left(\bar{\vartheta}\right)\right) | +...+ | \mathfrak{A}_n\left(\varkappa\left(\bar{\vartheta}\right)\right)\right|.\end{aligned} \tag{3.8}$$

Now consider,

$$\begin{aligned}&\left|\mathfrak{A}_1\left(\varkappa\left(\bar{\vartheta}\right)\right)\right| \\ =&\left|P_1\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) + \left(1-\bar{\vartheta}\right) h_1(\varkappa_1)\right| + \\ \leq&\left|P_1\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right)\right| + \left|\left(1-\bar{\vartheta}\right) h_1(\varkappa_1)\right| \\ =&\left|P_1\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) - P_1\left(\bar{\vartheta}, 0, 0, ..., 0\right) + P_1\left(\bar{\vartheta}, 0, 0, ..., 0\right)\right| + \left|\left(1-\bar{\vartheta}\right) h_1(\varkappa_1)\right| \\ \leq&\left|P_1\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) - P_1\left(\bar{\vartheta}, 0, 0, ..., 0\right)\right| + \left|P_1\left(\bar{\vartheta}, 0, 0, ..., 0\right)\right| + |h_1(\varkappa_1)| \\ \leq&\frac{L\left|x\left(\bar{\vartheta}\right)\right|}{2n\left(M + \left|x\left(\bar{\vartheta}\right)\right|\right)} + |P_{01}| + |h_1(\varkappa_1)| \\ \leq& L + |P_{01}| + |h_1(\varkappa_1)|.\end{aligned}$$

Similarly, for each $i = 1, 2, 3..., n$,

$$\left|\mathfrak{A}_i\left(\varkappa\left(\bar{\vartheta}\right)\right)\right| \leq L + |P_{0i}| + |h_i(\varkappa_i)|.$$

Hence, Eq (3.7) gives:

$$\left|\varkappa\left(\bar{\vartheta}\right)\right| \leq nL + \sum_{i=1}^{n} (|P_{0i}| + |h_i(\varkappa_i)|) + n\left(\frac{\|\varphi\|}{\Gamma(\alpha+1)} + 2\frac{\|g\|}{\Gamma(\alpha+1)}\right).$$

Taking $\text{Sup}_{\bar{\vartheta}\in[0,1]}$, we get:

$$\|\varkappa\| \leq n\left(L + P_0 + \|h\| + \frac{\|\varphi\|}{\Gamma(\alpha+1)} + 2\frac{\|g\|}{\Gamma(\alpha+1)}\right) \leq N.$$

Hence, $\varkappa \in \mathcal{S}$. Therefore, by Theorem 2.7, the operator $\mathscr{X}\left(\varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) = \mathfrak{A}\left(\varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right) + \mathfrak{B}\left(\varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), ..., \varkappa_n\left(\bar{\vartheta}\right)\right)$ has a fixed point on $\mathcal{S}$, which is required. □

4. 3-dimensional Euclidean space

To clarify Theorem 3.4, in this section, we construct the following example in 3-dimensional Euclidean space $\Re^3$.

Example 4.1. *Consider the following 3-dimensional FHDE:*

$$\begin{aligned}
D^{\frac{3}{2}}\left(\varkappa_1\left(\bar{\vartheta}\right)\right) - \left(\frac{e^{-\bar{\vartheta}}\left|\varkappa_1\left(\bar{\vartheta}\right) - \varkappa_2\left(\bar{\vartheta}\right) - \varkappa_3\left(\bar{\vartheta}\right)\right|}{15 + \left|\varkappa_1\left(\bar{\vartheta}\right) - \varkappa_2\left(\bar{\vartheta}\right) - \varkappa_3\left(\bar{\vartheta}\right)\right|}\right) &= \frac{\bar{\vartheta}^3}{3} - \left|cos\varkappa_1\left(\bar{\vartheta}\right) - cos\varkappa_2\left(\bar{\vartheta}\right) - cos\varkappa_3\left(\bar{\vartheta}\right)\right|, \\
D^{\frac{3}{2}}\left(\varkappa_2\left(\bar{\vartheta}\right)\right) - \left(\frac{e^{-\bar{\vartheta}}\left|\varkappa_2\left(\bar{\vartheta}\right) - \varkappa_1\left(\bar{\vartheta}\right) - \varkappa_3\left(\bar{\vartheta}\right)\right|}{15 + \left|\varkappa_2\left(\bar{\vartheta}\right) - \varkappa_1\left(\bar{\vartheta}\right) - \varkappa_3\left(\bar{\vartheta}\right)\right|}\right) &= \frac{\bar{\vartheta}^3}{3} - \left|cos\varkappa_2\left(\bar{\vartheta}\right) - cos\varkappa_1\left(\bar{\vartheta}\right) - cos\varkappa_3\left(\bar{\vartheta}\right)\right|, \\
D^{\frac{3}{2}}\left(\varkappa_3\left(\bar{\vartheta}\right)\right) - \left(\frac{e^{-\bar{\vartheta}}\left|\varkappa_3\left(\bar{\vartheta}\right) - \varkappa_1\left(\bar{\vartheta}\right) - \varkappa_2\left(\bar{\vartheta}\right)\right|}{15 + \left|\varkappa_3\left(\bar{\vartheta}\right) - \varkappa_1\left(\bar{\vartheta}\right) - \varkappa_2\left(\bar{\vartheta}\right)\right|}\right) &= \frac{\bar{\vartheta}^3}{3} - \left|cos\varkappa_3\left(\bar{\vartheta}\right) - cos\varkappa_1\left(\bar{\vartheta}\right) - cos\varkappa_2\left(\bar{\vartheta}\right)\right|. \quad (4.1)
\end{aligned}$$

$$\begin{cases}
\varkappa_1(0) = \frac{1}{2}\varkappa_1 & \varkappa_1(1) = \frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_0^1 (1-\bar{s})^{\frac{1}{2}} e^{-\varkappa_1} d\bar{s}, \\
\varkappa_2(0) = \frac{3}{4}\varkappa_2 & \varkappa_2(1) = \frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_0^1 (1-\bar{s})^{\frac{1}{2}} e^{-\varkappa_2} d\bar{s}, \\
\varkappa_3(0) = \frac{1}{3}\varkappa_3 & \varkappa_3(1) = \frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_0^1 (1-\bar{s})^{\frac{1}{2}} e^{-\varkappa_3} d\bar{s}.
\end{cases}$$

Here we have:

$$\begin{aligned}
P_1\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), \varkappa_3\left(\bar{\vartheta}\right)\right) &= \frac{e^{-\bar{\vartheta}}\left|\varkappa_1\left(\bar{\vartheta}\right) - \varkappa_2\left(\bar{\vartheta}\right) - \varkappa_3\left(\bar{\vartheta}\right)\right|}{15 + \left|\varkappa_1\left(\bar{\vartheta}\right) - \varkappa_2\left(\bar{\vartheta}\right) - \varkappa_3\left(\bar{\vartheta}\right)\right|}, \\
P_2\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), \varkappa_3\left(\bar{\vartheta}\right)\right) &= \frac{e^{-\bar{\vartheta}}\left|\varkappa_2\left(\bar{\vartheta}\right) - \varkappa_1\left(\bar{\vartheta}\right) - \varkappa_3\left(\bar{\vartheta}\right)\right|}{15 + \left|\varkappa_2\left(\bar{\vartheta}\right) - \varkappa_1\left(\bar{\vartheta}\right) - \varkappa_3\left(\bar{\vartheta}\right)\right|}, \\
P_3\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), \varkappa_3\left(\bar{\vartheta}\right)\right) &= \frac{e^{-\bar{\vartheta}}\left|\varkappa_3\left(\bar{\vartheta}\right) - \varkappa_1\left(\bar{\vartheta}\right) - \varkappa_2\left(\bar{\vartheta}\right)\right|}{15 + \left|\varkappa_3\left(\bar{\vartheta}\right) - \varkappa_1\left(\bar{\vartheta}\right) - \varkappa_2\left(\bar{\vartheta}\right)\right|}, \\
Q_1\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), \varkappa_3\left(\bar{\vartheta}\right)\right) &= \frac{\bar{\vartheta}^3}{3} - \left|cos\varkappa_1\left(\bar{\vartheta}\right) - cos\varkappa_2\left(\bar{\vartheta}\right) - cos\varkappa_3\left(\bar{\vartheta}\right)\right|, \\
Q_2\left(\bar{\vartheta}, \varkappa_1\left(\bar{\vartheta}\right), \varkappa_2\left(\bar{\vartheta}\right), \varkappa_3\left(\bar{\vartheta}\right)\right) &= \frac{\bar{\vartheta}^3}{3} - \left|cos\varkappa_2\left(\bar{\vartheta}\right) - cos\varkappa_1\left(\bar{\vartheta}\right) - cos\varkappa_3\left(\bar{\vartheta}\right)\right|,
\end{aligned}$$

$$Q_3\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),\varkappa_3\left(\bar{\vartheta}\right)\right) = \frac{\bar{\vartheta}^3}{3}-\left|cos\varkappa_3\left(\bar{\vartheta}\right)-cos\varkappa_1\left(\bar{\vartheta}\right)-cos\varkappa_2\left(\bar{\vartheta}\right)\right|,$$
$$h_1\left(\varkappa_1\right)=\frac{1}{2}\varkappa_1 \quad , \quad \varphi_1\left(\varkappa_1,\bar{s}\right)=\left(1-\bar{s}\right)^{\frac{1}{2}}e^{-\varkappa_1},$$
$$h_2\left(\varkappa_2\right)=\frac{3}{4}\varkappa_2 \quad , \quad \varphi_2\left(\varkappa_2,\bar{s}\right)=\left(1-\bar{s}\right)^{\frac{1}{2}}e^{-\varkappa_2},$$
$$h_3\left(\varkappa_3\right)=\frac{1}{3}\varkappa_3 \quad , \quad \varphi_3\left(\varkappa_3,\bar{s}\right)=\left(1-\bar{s}\right)^{\frac{1}{2}}e^{-\varkappa_3}.$$

To check the assumptions $(A_1)-(A_3)$, *we first consider:*

$$\begin{aligned}\left|h_1\left(\varkappa_1\right)-h_2\left(\varkappa_2\right)\right| &= \left|\frac{1}{2}\varkappa_1-\frac{3}{4}\varkappa_2\right| \\ &\leq \frac{1}{2}\left|\varkappa_1-\varkappa_2\right| \\ &\leq \left|\varkappa_1-\varkappa_2\right|.\end{aligned}$$

Similarly for each $i,j=1,2,3,$

$$\left|h_i\left(\varkappa_1\right)-h_j\left(\varkappa_2\right)\right| \leq \left|\varkappa_1-\varkappa_2\right|.$$

For $M=15$ *and* $L=13<M$, *we have,*

$$\begin{aligned}&\left|P_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),\varkappa_3\left(\bar{\vartheta}\right)\right)-P_2\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),\varkappa_3\left(\bar{\vartheta}\right)\right)\right| \\
&=\left|\frac{e^{-\bar{\vartheta}}\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)-\varkappa_3\left(\bar{\vartheta}\right)\right|}{15+\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)-\varkappa_3\left(\bar{\vartheta}\right)\right|}-\frac{e^{-\bar{\vartheta}}\left|\varkappa_2\left(\bar{\vartheta}\right)-\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_3\left(\bar{\vartheta}\right)\right|}{15+\left|\varkappa_2\left(\bar{\vartheta}\right)-\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_3\left(\bar{\vartheta}\right)\right|}\right| \\
&\leq\left|\frac{e^{-\bar{\vartheta}}\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)-\varkappa_3\left(\bar{\vartheta}\right)\right|}{15+\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|+\left|\varkappa_3\left(\bar{\vartheta}\right)\right|}-\frac{e^{-\bar{\vartheta}}\left|\varkappa_2\left(\bar{\vartheta}\right)-\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_3\left(\bar{\vartheta}\right)\right|}{15+\left|\varkappa_2\left(\bar{\vartheta}\right)-\varkappa_1\left(\bar{\vartheta}\right)\right|+\left|\varkappa_3\left(\bar{\vartheta}\right)\right|}\right| \\
&\leq\frac{\left|e^{-\bar{\vartheta}}\right|2\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|}{15+\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|+\left|\varkappa_3\left(\bar{\vartheta}\right)\right|} \\
&\leq\frac{2\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|}{15+\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|} \\
&\leq\frac{13\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|}{6\left(15+\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|\right)} \\
&=\frac{L\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|}{2\left(3\right)\left(M+\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|\right)}.\end{aligned}$$

Similarly for each $i,j=1,2,3,$

$$\left|P_i\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),\varkappa_3\left(\bar{\vartheta}\right)\right)-P_j\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),\varkappa_3\left(\bar{\vartheta}\right)\right)\right| \leq \frac{L\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|}{2\left(3\right)\left(M+\left|\varkappa_1\left(\bar{\vartheta}\right)-\varkappa_2\left(\bar{\vartheta}\right)\right|\right)}.$$

Now consider,

$$Q_1\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),\varkappa_3\left(\bar{\vartheta}\right)\right)=\frac{\bar{\vartheta}^3}{3}-\left|cos\varkappa_1\left(\bar{\vartheta}\right)-cos\varkappa_2\left(\bar{\vartheta}\right)-cos\varkappa_3\left(\bar{\vartheta}\right)\right|\leq\frac{\bar{\vartheta}^3}{3}.$$

Similarly for each $i=1,2,3$, *there exist* $g_i\left(\bar{\vartheta}\right)\in\mathfrak{C}\left([0,1],\Re\right)$

$$Q_i\left(\bar{\vartheta},\varkappa_1\left(\bar{\vartheta}\right),\varkappa_2\left(\bar{\vartheta}\right),\varkappa_3\left(\bar{\vartheta}\right)\right)\leq g_i\left(\bar{\vartheta}\right).$$

Finally, take $P_0=0$ *and* $L=13$, *for each* $i=1,2,3$,

$$\frac{\left\|g_i\left(\bar{\vartheta}\right)\right\|}{\Gamma\left(\frac{3}{2}+1\right)}=\frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^1\left|\frac{\bar{\vartheta}^3}{3}\right|d\bar{\vartheta}=\frac{1}{1.3293}\left(\frac{1}{12}\right)=0.06268,$$

$$\Gamma\left(\frac{5}{2}\right)=\Gamma\left(1+\frac{3}{2}\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{2}\Gamma\left(1+\frac{1}{2}\right)=\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}=1.3293,$$

$$\frac{\|\varphi_1\|}{\Gamma\left(\frac{3}{2}+1\right)}=\frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^1(1-\bar{s})^{\frac{1}{2}}e^{-\varkappa_1}d\bar{s}=\frac{1}{1.3293}(0.66668)=0.5015,$$

$$\frac{\|\varphi_2\|}{\Gamma\left(\frac{3}{2}+1\right)}=\frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^1(1-\bar{s})^{\frac{1}{2}}e^{-\varkappa_2}d\bar{s}=\frac{1}{1.3293}(0.66668)=0.5015,$$

$$\frac{\|\varphi_3\|}{\Gamma\left(\frac{3}{2}+1\right)}=\frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^1(1-\bar{s})^{\frac{1}{2}}e^{-\varkappa_3}d\bar{s}=\frac{1}{1.3293}(0.66668)=0.5015,$$

and

$$\|h_1(\varkappa_1)\|=\int_0^1\left|\frac{1}{2}\varkappa_1\right|d\varkappa_1=\frac{1}{4},$$

$$\|h_2(\varkappa_2)\|=\int_0^1\left|\frac{3}{4}\varkappa_2\right|d\varkappa_2=\frac{3}{8},$$

$$\|h_3(\varkappa_3)\|=\int_0^1\left|\frac{1}{3}\varkappa_3\right|d\varkappa_3=\frac{1}{6}.$$

So $\|h\|=\frac{3}{8},\frac{\|\varphi\|}{\Gamma(\alpha+1)}=0.5015,$ *and* $\frac{\|g\|}{\Gamma(\alpha+1)}=0.06268.$

$$n\left(L+P_0+\|h\|+\frac{\|\varphi\|}{\Gamma(\alpha+1)}+2\frac{\|g\|}{\Gamma(\alpha+1)}\right)=3(13+0+0.5015+0.06268)=40.69254.$$

Hence $N\geq 41$. *All of the assumptions from* $(A_1)-(A_3)$ *hold, hence by Theorem 3.4, we come to an end that problem (4.1) possesses a solution.*

5. Conclusions

We have successfully investigated an n-dimensional FHDE with nonlinear boundary conditions in a nonlinear coupled system. We utilized Dhage's fixed point theory and applied the Krasnoselskii-type coupled fixed point theorem to establish conditions adequate for the existence of solutions to our problem. To illustrate our idea, we provided a suitable example in 3-dimensional space.

Author contributions

Salma Noor: Investigation, Methodology, Writing–original draft; Aman Ullah: Conceptualization, Project administration, Supervision, Validation; Anwar Ali: Formal Analysis, Visualization, Writing–review & editing; Saud Fahad Aldosary: Data curation, Funding acquisition, Resources, Software. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there exist no conflicts of interest regarding this research work.

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