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Research article

Ulam-Hyers stability and existence results for a coupled sequential Hilfer-Hadamard-type integrodifferential system

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Abstract: This study aimed to investigate the existence, uniqueness, and Ulam-Hyers stability of solutions in a nonlinear coupled system of Hilfer-Hadamard sequential fractional integrodifferential equations, which were further enhanced by nonlocal coupled Hadamard fractional integrodifferential multipoint boundary conditions. The desired conclusions were obtained by using well-known fixed-point theorems. It was emphasized that the fixed-point technique was useful in determining the existence and uniqueness of solutions to boundary value problems. In addition, we examined the solution's Ulam-Hyers stability for the suggested system. The resulting results were further demonstrated and validated using demonstration instances.

Keywords: coupled integrodifferential system; sequential derivatives; Hadamard integrals; derivatives; Hilfer-Hadamard derivatives; multi-points; existence; uniqueness; Ulam-Hyers stability
Mathematics Subject Classification: 34A08, 34B15, 45G15

Abbreviations

The following abbreviations are used in this manuscript:

BVPs	Boundary Value Problems
HHFDEs	Hilfer-Hadamard Fractional-order Differential Equations
HHFIEs	Hilfer-Hadamard Fractional-order Integrodifferential Equations
HFIs	Hadamard Fractional Integrals
HHFDs	Hilfer-Hadamard Fractional Derivatives
CFDs	Caputo Fractional Derivatives
HFDs	Hilfer Fractional Derivatives
HFDEs	Hilfer Fractional Differential Equations
HFDs	Hadamard Fractional Derivatives (HFDs)
CHFDs	Caputo-Hadamard Fractional Derivatives (CHFDs)

1. Introduction

This study presents and examines a new nonlinear sequential Hilfer-Hadamard fractional-order integrodifferential equations (HHFIEs) with nonlocal coupled multipoint and Hadamard fractional integrodifferential boundary conditions. The formulation of the problem is as follows:

$$\begin{cases} ({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1,\beta_1} + \lambda_1 {}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1-1,\beta_1})\mathcal{S}(\varpi) = \mathcal{F}(\varpi,\mathcal{S}(\varpi),\mathcal{Z}(\varpi),\mathcal{I}^{\mathfrak{p}_1}\mathcal{S}(\varpi),\mathcal{I}^{\mathfrak{p}_2}\mathcal{Z}(\varpi)), \\ ({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2,\beta_2} + \lambda_2 {}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2-1,\beta_2})\mathcal{Z}(\varpi) = \mathcal{G}(\varpi,\mathcal{S}(\varpi),\mathcal{Z}(\varpi),\mathcal{I}^{\mathfrak{q}_1}\mathcal{S}(\varpi),\mathcal{I}^{\mathfrak{q}_2}\mathcal{Z}(\varpi)), \end{cases} \tag{1.1}$$

and it is enhanced by nonlocal coupled multipoint and Hadamard fractional integrodifferential boundary conditions:

$$\begin{cases} \mathcal{S}(1) = 0, \ \ \mathcal{S}(\mathcal{T}) = \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}}\mathcal{Z}(\xi_{\mathfrak{j}}) + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}} {}^{\mathcal{H}}\mathcal{I}^{\phi_{\mathfrak{i}}}\mathcal{Z}(\zeta_{\mathfrak{i}}) + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}^{\mathcal{H}}\mathcal{D}_1^{\omega_{\mathfrak{k}}}\mathcal{Z}(\mu_{\mathfrak{k}}), \\ \mathcal{Z}(1) = 0, \ \ \mathcal{Z}(\mathcal{T}) = \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}}\mathcal{S}(\psi_{\mathfrak{u}}) + \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}} {}^{\mathcal{H}}\mathcal{I}^{\delta_{\mathfrak{v}}}\mathcal{S}(\sigma_{\mathfrak{v}}) + \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}^{\mathcal{H}}\mathcal{D}_1^{\vartheta_{\mathfrak{w}}}\mathcal{S}(\pi_{\mathfrak{w}}). \end{cases} \tag{1.2}$$

Here $\alpha_1,\alpha_2 \in (1,2]$, $\beta_1,\beta_2 \in [0,1]$, $\lambda_1,\lambda_2 \in \mathbb{R}_+$, $\mathcal{T} > 1$, $\eta_{\mathfrak{j}},\theta_{\mathfrak{i}},\lambda_{\mathfrak{k}},\mathcal{P}_{\mathfrak{u}},\mathcal{Q}_{\mathfrak{v}},\mathcal{M}_{\mathfrak{w}} \in \mathbb{R}$, $\xi_{\mathfrak{j}},\zeta_{\mathfrak{i}},\mu_{\mathfrak{k}},\psi_{\mathfrak{u}},\sigma_{\mathfrak{v}},\pi_{\mathfrak{w}} \in (1,\mathcal{T})$, ($\mathfrak{j} = 1,2,...,m$, $\mathfrak{i} = 1,2,...,n$, $\mathfrak{k} = 1,2,...,r$, $\mathfrak{u} = 1,2,...,a$, $\mathfrak{v} = 1,2,...,b$, $\mathfrak{w} = 1,2,...,c$), ${}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\alpha_i,\beta_j}$ denotes the Hilfer-Hadamard fractional derivative (HHFD) operator of order $\alpha_i,\beta_j; i = 1,2. j = 1,2.$ ${}^{\mathcal{H}}\mathcal{I}^{\chi}$ is the Hadamard fractional integral (HFI) of order $\chi \in \{\phi_{\mathfrak{i}},\delta_{\mathfrak{v}} > 0\}$ $\mathfrak{i} = 1,2,...,n$, $\mathfrak{v} = 1,2,...,b$, ${}^{\mathcal{H}}\mathcal{D}_1^{\psi}$ is the Hadamard fractional derivative (HFD) of order $\psi \in \{\omega_{\mathfrak{k}},\vartheta_{\mathfrak{w}}\}$ $\mathfrak{k} = 1,2,...,r$, $\mathfrak{w} = 1,2,...,c$, $\mathcal{F},\mathcal{G} : [1,\mathcal{T}] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous functions. It's important to highlight and that this study makes a significant contribution to the existing literature by addressing a distinct setup involving sequential HHFIEs along with coupled multipoint and Hadamard fractional integrodifferential boundary conditions. The methodology employed in this study involves using the fixed-point approach to establish both existence and uniqueness results for the problems (1.1) and (1.2). The process involves converting the given problem into an equivalent fixed-point problem, followed by the application of Leray-Schauder alternative and Banach's fixed-point theorem to establish existence and uniqueness results, respectively. Additionally, we investigate the Ulam-Hyers stability of the solution for the proposed system. The findings of this study are innovative and contribute to the existing literature on boundary value problems (BVPs) concerning coupled systems of sequential HHFIEs. In recent decades, fractional calculus has gained considerable attention and become a prominent area of study in mathematical analysis. This growth is largely attributed to the extensive use of fractional calculus techniques in developing innovative mathematical models to represent various phenomena in fields such as economics, mechanics, engineering, science, and others. References [1–4] offer examples and comprehensive discussions on this subject.

In the upcoming section, we will provide an overview of relevant scholarly articles pertaining to the discussed problem. Among various fractional derivatives introduced, the Riemann-Liouville and Caputo fractional derivatives (CFDs) have garnered significant attention due to their practical applications. The Hilfer fractional derivative, introduced by Hilfer in [5], incorporates the Riemann-Liouville and CFDs as special cases for certain parameter values. Additional insights into this derivative can be found in [6–13]. References [14–18] offer valuable insights into Hilfer-type initial and BVPs. A recent study [19] explores the Ulam-Hyers stability and existence of a fully coupled system featuring integro-multistrip-multipoint boundary conditions and nonlinear sequential Hilfer fractional differential equations (HFDEs). Furthermore, [20] delves into a hybrid generalized HFDE BVP.

In 1892, Hadamard introduced the HFD, defined by a logarithmic function with an arbitrary exponent in its kernel [21]. Subsequent studies, such as those in [22–26], have explored variations such as HHFDs and Caputo-Hadamard fractional derivatives (CHFDs). Notably, for specific values of β-$\beta = 0$ and $\beta = 1$, respectively—HFDs and CHFDs emerge as particular instances of the HHFD.

Stability analysis has been a prominent field of study for fractional differential equations in the last several decades and has drawn a lot of interest from scholars. Numerous stability models, including Lyapunov, exponential, and Mittag-Leffler stability, have been thoroughly examined in the literature. We suggest reviewing publications [27–31] for historical perspective on Ulam-Hyers stability and current improvements.

The problem of existence and Ulam stability of solutions for the following Hilfer-Hadamard fractional differential equations (HHFDEs) [32] is stated as follows:

$$\begin{cases} \left({}^{H}\mathcal{D}_{1}^{\alpha,\beta}x\right)(t) = f(t,u(t)), \text{ for } t \in \mathcal{J} := [1,\mathcal{T}], \\ \left({}^{H}\mathcal{I}_{1}^{1-\gamma}x\right)(t)\Big|_{t=1} = \varphi, \end{cases} \tag{1.3}$$

where $0 < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha + \beta - \alpha\beta$, $\mathcal{T} > 1$, $\varphi \in \mathbb{R}$, and $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ is a given function. ${}^{H}\mathcal{I}_{1}^{1-\gamma}$ denotes the left-sided mixed Hadamard integral of order $1-\gamma$, and ${}^{H}\mathcal{D}_{1}^{\alpha,\beta}$ is the HHFD of order α and type β, introduced by Hilfer. In [33], existence results were established for an HHFDE with nonlocal integro-multipoint boundary conditions:

$$\begin{cases} {}^{\mathcal{H}\mathcal{H}}\mathcal{D}_{1}^{\alpha,\beta}x(t) = f(t,x(t)), \quad t \in [1,T], \\ x(1) = 0, \quad \sum_{i=1}^{m} \theta_i x(\xi_i) = \lambda^{H}\mathcal{I}^{\delta}x(\eta), \end{cases} \tag{1.4}$$

where $\alpha \in (1,2]$, $\beta \in [0,1]$, $\theta_i, \lambda \in \mathbb{R}$, $\eta, \xi_i \in (1,T)$ $(i = 1,2,...,m)$, ${}^{H}\mathcal{I}^{\delta}$ is the Hadamard fractional integral (HFI) of order $\delta > 0$, and $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Problem (1.4) represents a non-coupled system, in contrast to problems (1.1)-(1.2), which is a coupled system. The systems (1.1)-(1.2) presents nonlocal coupled Hadamard fractional integrodifferential and multipoint boundary conditions, whereas the problem (1.4) involves discrete boundary conditions with HFIs. The authors of [34] established existence results for nonlocal mixed Hilfer-Hadamard fractional BVPs:

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}_1^{\alpha,\beta}x(t) = f(t, x(t)), \quad t \in [1, T], \\ x(1) = 0, \quad x(T) = \sum_{j=1}^{m} \eta_j x(\xi_j) + \sum_{i=1}^{n} \zeta_i {}^{H}\mathcal{I}^{\phi_i} x(\theta_i) + \sum_{k=1}^{r} \lambda_{kH}\mathcal{D}_1^{\omega_k} x(\mu_k), \end{cases} \tag{1.5}$$

where $\alpha \in (1, 2]$, $\beta \in [0, 1]$, $\eta_i, \zeta_i, \lambda_k \in \mathbb{R}$, $\xi_i, \theta_i, \mu_k \in (1, T)$, $(j = 1, 2, ..., m), (i = 1, 2, ..., n), (k = 1, 2, ..., r)$, ${}^{H}\mathcal{I}^{\phi_i}$ is the HFI of order $\phi_i > 0$, ${}_{H}\mathcal{D}_1^{\mu_k}$ is the HFD of order $\mu_k > 0$, and $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Equation (1.5) does not represent a coupled system, unlike Eqs (1.1) and (1.2), which does. In the latter, there are nonlocal coupled Hadamard fractional integrodifferential and multipoint boundary conditions. In contrast, Eq (1.5) involves multipoint boundary conditions comprising HFIs and HFDs. Furthermore, in [35], investigations were conducted on coupled HHFDEs within generalized Banach spaces. The authors of the aforementioned study [36] successfully derived existence results for a coupled system of HHFDEs with nonlocal coupled boundary conditions:

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}_1^{\alpha,\beta}u(t) = f(t, u(t), v(t)), \quad 1 < \alpha \leq 2, \quad \varpi \in [1, \mathcal{T}], \\ {}^{\mathcal{HH}}\mathcal{D}_1^{\gamma,\delta}v(t) = g(t, u(t), v(t)), \quad 1 < \gamma \leq 2, \quad \varpi \in [1, \mathcal{T}], \\ u(1) = 0, \quad {}^{\mathcal{H}}\mathcal{D}_1^{\varsigma}u(T) = \sum_{i=1}^{m} \int_1^T {}^{\mathcal{H}}\mathcal{D}_1^{\varrho_i}u(s)d\mathcal{H}_i(s) + \sum_{i=1}^{n} \int_1^T {}^{\mathcal{H}}\mathcal{D}_1^{\sigma_i}v(s)d\mathcal{K}_i(s), \\ v(1) = 0, \quad {}^{\mathcal{H}}\mathcal{D}_1^{\vartheta}v(T) = \sum_{i=1}^{p} \int_1^T {}^{\mathcal{H}}\mathcal{D}_1^{\eta_i}u(s)d\mathcal{P}_i(s) + \sum_{i=1}^{q} \int_1^T {}^{\mathcal{H}}\mathcal{D}_1^{\theta_i}v(s)d\mathcal{Q}_i(s), \end{cases} \tag{1.6}$$

where $\alpha, \gamma \in (1, 2]$, $\beta, \delta \in [0, 1]$, $\mathcal{T} > 1$, ${}^{\mathcal{HH}}\mathcal{D}^{\alpha,\beta}$, ${}^{\mathcal{HH}}\mathcal{D}_1^{\gamma,\delta}$ denotes the HHFD operator of order $\alpha, \beta, \gamma, \delta$. ${}^{\mathcal{H}}\mathcal{D}_{1^+}^{\chi}$ is the HFD operator of order $\chi \in \{\varsigma, \vartheta, \varrho_i, \eta_i, \sigma_i, \theta_i\}$, $(i = 1, 2, ..., m), (i = 1, 2, ..., n), (i = 1, 2, ..., p), (i = 1, 2, ..., q)$, $f, g : [1, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. In the boundary conditions, Riemann-Stieltjes integrals with $\mathcal{H}_i, \mathcal{K}_i, \mathcal{P}_i, \mathcal{Q}_i$, $(i = 1, 2, ..., m), (i = 1, 2, ..., n), (i = 1, 2, ..., p), (i = 1, 2, ..., q)$, functions of bounded variation. Problem (1.6) involves a coupled system of HHFDEs, while problems (1.1)-(1.2) deals with a coupled system of sequential HHFIEs. In problems (1.1)-(1.2), there is nonlocal coupled multipoint and Hadamard fractional integrodifferential boundary conditions, whereas in problem (1.6), Stieltjes-integral boundary conditions are incorporated, involving HFDs. In problems (1.1)-(1.2), the nonlinearity depends on the unknown function and its fractional integrals at lower orders are included. Conversely, in problem (1.6), the nonlinearity depends on the unknown function, but it does not involve fractional integrals at lower orders. The researchers in [37] performed an examination of a coupled system of HHFDEs with nonlocal coupled HFI boundary conditions:

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\alpha_1,\beta_1}u(t) = \varrho_1(t, u(t), v(t)), \quad 1 < \alpha_1 \leq 2, \quad \varpi \in \mathcal{E} := [1, \mathcal{T}], \\ {}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\alpha_2,\beta_2}v(t) = \varrho_2(t, u(t), v(t)), \quad 2 < \alpha_2 \leq 3, \quad \varpi \in \mathcal{E} := [1, \mathcal{T}], \\ u(1) = 0, \quad u(T) = \lambda_1 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_1}v(\eta_1), \\ v(1) = 0, \quad v(\eta_2) = 0, \quad v(T) = \lambda_2 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_2}u(\eta_3), \quad 1 < \eta_1, \eta_2, \eta_3 < \mathcal{T}, \end{cases} \tag{1.7}$$

where $\alpha_1 \in (1, 2]$, $\alpha_2 \in (2, 3]$, $\beta_1, \beta_2 \in [0, 1]$, $\mathcal{T} > 1$, $\delta_1, \delta_2 > 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$, ${}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\alpha_i,\beta_j}$ denotes the HHFD operator of order $\alpha_i, \beta_j; i = 1, 2. j = 1, 2.$ ${}^{\mathcal{H}}\mathcal{I}_{1^+}^{\chi}$ is the HFI operator of order $\chi \in \{\delta_1, \delta_2\}$, and ϱ_1, ϱ_2 :

$\mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. In Eq (1.7), we encounter a coupled system of HHFDEs, whereas Eqs (1.1) and (1.2) addresses a coupled system of sequential HHFIEs. In the latter, there are nonlocal coupled Hadamard fractional integrodifferential and multipoint boundary conditions, whereas Eq (1.7) involves multipoint and HFI boundary conditions. In Eq (1.7), solutions are obtained for the coupled system of HHFDEs, while in Eqs (1.1) and (1.2), solutions are derived for the coupled system of sequential HHFIEs. In Eqs (1.1) and (1.2), the nonlinearity depends on the unknown function and its fractional integrals at lower orders are included. Conversely, in Eq (1.7), the nonlinearity depends on the unknown function but does not involve fractional integrals at lower orders. Furthermore, in [38], a two-point BVP for a system of nonlinear sequential HHFDEs was investigated:

$$\begin{cases} ({}^{\mathcal{HH}}\mathcal{D}_1^{\alpha_1,\beta_1} + \lambda_1 {}^{\mathcal{HH}}\mathcal{D}_1^{\alpha_1-1,\beta_1})u(t) = f(t, u(t), v(t)), \ t \in [1, e], \\ ({}^{\mathcal{HH}}\mathcal{D}_1^{\alpha_2,\beta_2} + \lambda_2 {}^{\mathcal{HH}}\mathcal{D}_1^{\alpha_2-1,\beta_2})v(t) = g(t, u(t), v(t)), \ t \in [1, e], \\ u(1) = 0, \ \ u(e) = \mathcal{A}_1, \ \ v(1) = 0, \ \ v(e) = \mathcal{A}_2, \end{cases} \tag{1.8}$$

where $\alpha_1, \alpha_2 \in (1, 2]$, $\beta_1, \beta_2 \in [0, 1]$, $\lambda_1, \lambda_2, \mathcal{A}_1, \mathcal{A}_2 \in \mathbb{R}_+$, $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Equation (1.8) features a two-point boundary condition, while problems (1.1)-(1.2) includes multipoint and Hadamard fractional integrodifferential boundary conditions. In Eqs (1.1) and (1.2), the nonlinearity involves the unknown function and its fractional integrals at lower orders. Conversely, in Eq (1.8), the nonlinearity relies on the unknown function but does not incorporate fractional integrals at lower orders.

The document is organized as follows in the following sections: The fundamental ideas of fractional calculus relevant to this research are introduced in Section 2. An auxiliary lemma addressing the linear versions of problems (1.1) and (1.2) is provided in Section 3. The primary findings are presented in Section 4 along with illustrative examples. Finally, Section 5 provides a few recommendations.

2. Preliminaries

Definition 2.1. *The HFI of order* $\mathfrak{p} > 0$ *for a continuous function* $\mathcal{F} : [a, \infty) \to \mathbb{R}$ *is given by*

$${}^{\mathcal{H}}\mathcal{I}_{a^+}^{\mathfrak{p}}\mathcal{F}(\varpi) = \frac{1}{\Gamma(\mathfrak{p})} \int_a^{\varpi} \Big(\log \frac{\varpi}{\varsigma}\Big)^{\mathfrak{p}-1} \frac{\mathcal{F}(\varsigma)}{\varsigma} d\varsigma, \tag{2.1}$$

where $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. *The HFD of order* $\mathfrak{p} > 0$ *for a function* $\mathcal{F} : [\mathfrak{a}, \infty) \to \mathbb{R}$ *is defined by*

$${}^{\mathcal{H}}\mathcal{D}_{\mathfrak{a}^+}^{\mathfrak{p}}\mathcal{F}(\varpi) = \delta^n({}^{\mathcal{H}}\mathcal{I}_{\mathfrak{a}^+}^{n-\mathfrak{p}}\mathcal{F})(\varpi), \ \ n = [\mathfrak{p}] + 1, \tag{2.2}$$

where $\delta^n = \varpi^n \frac{d^n}{d\omega^n}$ *and* $[\mathfrak{p}]$ *denotes the integer part of the real number* $\mathfrak{p}$.

Lemma 2.3. *If* $\mathfrak{p}, \mathfrak{q} > 0$ *and* $0 < \mathfrak{a} < \mathfrak{b} < \infty$ *then*

$$(1) \quad \left({}^{\mathcal{H}}\mathcal{I}_{a^+}^{\mathfrak{p}}\left(\log \frac{\varpi}{\mathfrak{a}}\right)^{\mathfrak{q}-1}\right)(\mathfrak{x}) = \frac{\Gamma(\mathfrak{q})}{\Gamma(\mathfrak{q}+\mathfrak{p})}\left(\log \frac{\mathfrak{x}}{\mathfrak{a}}\right)^{\mathfrak{q}+\mathfrak{p}-1};$$

$$(2) \quad \left({}^{\mathcal{H}}\mathcal{D}_{a^+}^{\mathfrak{p}}\left(\log \frac{\varpi}{\mathfrak{a}}\right)^{\mathfrak{q}-1}\right)(\mathfrak{x}) = \frac{\Gamma(\mathfrak{q})}{\Gamma(\mathfrak{q}-\mathfrak{p})}\left(\log \frac{\mathfrak{x}}{\mathfrak{a}}\right)^{\mathfrak{q}-\mathfrak{p}-1}.$$

In particular, for $\mathfrak{q} = 1$, *we have* $\left({}^{\mathcal{H}}\mathcal{D}^{\mathfrak{p}}_{a^+}\right)(1) = \frac{1}{\Gamma(1-\mathfrak{p})}\left(\log\frac{\mathfrak{x}}{\mathfrak{a}}\right)^{-\mathfrak{p}} \neq 0, 0 < \mathfrak{p} < 1.$

Definition 2.4. *For* $n-1 < \mathfrak{p} < n$ *and* $0 \le \mathfrak{q} \le 1$, *the HHFD of order* $\mathfrak{p}$ *and* $\mathfrak{q}$ *for* $\mathcal{F} \in \mathcal{L}'(\mathfrak{a}, \mathfrak{b})$ *is defined as*

$$\begin{aligned}({}^{\mathcal{H}\mathcal{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q}}_{\mathfrak{a}^+})(\mathcal{F}(\varpi)) =& ({}^{\mathcal{H}}\mathcal{I}^{\mathfrak{q}(\mathfrak{n}-\mathfrak{p})}_{\mathfrak{a}^+}\delta^{\mathfrak{n}}{}^{\mathcal{H}}\mathcal{I}^{(\mathfrak{n}-\mathfrak{p})(g-\mathfrak{q})}_{\mathfrak{a}^+}\mathcal{F})(\varpi)\\ =& ({}^{\mathcal{H}}\mathcal{I}^{\mathfrak{q}(\mathfrak{n}-\mathfrak{p})}_{\mathfrak{a}^+}\delta^{\mathfrak{n}}{}^{\mathcal{H}}\mathcal{I}^{(\mathfrak{n}-\gamma)}_{\mathfrak{a}^+}\mathcal{F})(\varpi)\\ =& ({}^{\mathcal{H}}\mathcal{I}^{\mathfrak{q}(\mathfrak{n}-\mathfrak{p})}_{\mathfrak{a}^+}\delta^{\mathfrak{n}}{}^{\mathcal{H}}\mathcal{D}^{\gamma}_{\mathfrak{a}^+}\mathcal{F})(\varpi),\ \gamma = \mathfrak{p} + \mathfrak{n}\mathfrak{q} - \mathfrak{p}\mathfrak{q},\end{aligned}$$

where ${}^{\mathcal{H}}\mathcal{I}^{(\cdot)}_{\mathfrak{a}^+}$ *and* ${}^{\mathcal{H}}\mathcal{D}^{(\cdot)}_{\mathfrak{a}^+}$ *are given and defined by (2.1) and (2.2), respectively.*

Theorem 2.5. *If* $\mathcal{F} \in \mathcal{L}^1(\mathfrak{a}, \mathfrak{b}), 0 < \mathfrak{a} < \mathfrak{b} < \infty$, *and* $\left({}^{\mathcal{H}}\mathcal{I}^{n-\gamma}_{a^+}\mathcal{F}\right)(\varpi) \in \mathcal{AC}^n_\delta[\mathfrak{a}, \mathfrak{b}]$, *then*

$$\begin{aligned}{}^{\mathcal{H}}\mathcal{I}^{\mathfrak{p}}_{a^+}\left({}^{\mathcal{H}\mathcal{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q}}_{a^+}\mathcal{F}\right)(\varpi) =& {}^{\mathcal{H}}\mathcal{I}^{\gamma}_{a^+}\left({}^{\mathcal{H}\mathcal{H}}\mathcal{D}^{\gamma}_{a^+}\mathcal{F}\right)(\varpi)\\ =& \mathcal{F}(\varpi) - \sum_{j=o}^{n-1}\frac{(\delta^{(n-j-1)}({}^{\mathcal{H}}\mathcal{I}^{\mathfrak{p}}_{a^+}\mathcal{F}))(\mathfrak{a})}{\Gamma(\gamma-j)}\left(\log\frac{\varpi}{\mathfrak{a}}\right)^{\gamma-j-1},\end{aligned}$$

where $\mathfrak{p} > 0, 0 \le \mathfrak{q} \le 1$ *and* $\gamma = \mathfrak{p} + n\mathfrak{q} - \mathfrak{p}\mathfrak{q}, n = [\mathfrak{p}] + 1$. *Observe that* $\Gamma(\gamma - j)$ *exists for all* $j = 1, 2, \cdots, n-1$ *and* $\gamma \in [\mathfrak{p}, n]$.

We'll utilize established fixed point theorems in Banach spaces to demonstrate the existence and uniqueness of solutions for Hilfer-Hadamard fractional differential systems.

Lemma 2.6. *Let* $\mathcal{H}_1, \mathcal{H}_2 \in C([1, \mathcal{T}], \mathbb{R})$ *such that*

$$\begin{cases}({}^{\mathcal{H}\mathcal{H}}\mathcal{D}^{\alpha_1,\beta_1}_{1_+} + \lambda_1{}^{\mathcal{H}\mathcal{H}}\mathcal{D}^{\alpha_1-1,\beta_1}_{1_+})\mathcal{S}(\varpi) = \mathcal{H}_1(\varpi),\ 1 < \alpha_1 \le 2,\ \varpi \in [1, \mathcal{T}],\\ ({}^{\mathcal{H}\mathcal{H}}\mathcal{D}^{\alpha_2,\beta_2}_{1_+} + \lambda_2{}^{\mathcal{H}\mathcal{H}}\mathcal{D}^{\alpha_2-1,\beta_2}_{1_+})\mathcal{Z}(\varpi) = \mathcal{H}_2(\varpi),\ 1 < \alpha_2 \le 2,\ \varpi \in [1, \mathcal{T}],\end{cases} \tag{2.3}$$

enhanced by the boundary conditions (1.2) if, and only if,

$$\begin{aligned}\mathcal{S}(\varpi) =& \frac{1}{\Delta}\Bigg[\lambda_1\int_1^{\mathcal{T}}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma - \lambda_2\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\int_1^{\xi_{\mathfrak{j}}}\frac{(\mathcal{Z})\varsigma}{\varsigma}d\varsigma - \lambda_2\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}^{\phi_{\mathfrak{i}}}_{1^+}\int_1^{\zeta_{\mathfrak{i}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma\\ &- \lambda_2\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}^{\omega_{\mathfrak{k}}}_{1^+}\int_1^{\mu_{\mathfrak{k}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma + \sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\xi_{\mathfrak{j}}}\left(\log\frac{\xi_{\mathfrak{j}}}{\varsigma}\right)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\\ &+ \sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}^{\phi_{\mathfrak{i}}}_{1^+}\frac{1}{\Gamma(\alpha_2)}\int_1^{\zeta_{\mathfrak{i}}}\left(\log\frac{\zeta_{\mathfrak{i}}}{\varsigma}\right)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma + \sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}^{\omega_{\mathfrak{k}}}_{1^+}\frac{1}{\Gamma(\alpha_2)}\int_1^{\mu_{\mathfrak{k}}}\left(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\right)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\\ &- \frac{1}{\Gamma(\alpha_1)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{\varsigma}\right)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\}\\ &+ \lambda_2\int_1^{\mathcal{T}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma - \sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1\int_1^{\psi_{\mathfrak{u}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma - \sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}^{\delta_{\mathfrak{v}}}_{1^+}\lambda_1\int_1^{\sigma_{\mathfrak{v}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma\end{aligned}$$

$$
\begin{aligned}
&-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1\int_1^{\pi_{\mathfrak{w}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma \\
&+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\psi_{\mathfrak{u}}}\Big(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\sigma_{\mathfrak{v}}}\Big(\log\frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma \\
&+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\pi_{\mathfrak{w}}}\Big(\log\frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma-\frac{1}{\Gamma(\alpha_2)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma \\
&\times\Bigg\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}(\log\xi_{\mathfrak{j}})^{\gamma_2-1}+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log\zeta_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})^{\gamma_2-1}\Bigg\}\Bigg] \\
&-\lambda_1\int_1^{\varpi}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma+\frac{1}{\Gamma(\alpha_1)}\int_1^{\varpi}\left(\log\frac{\varpi}{\varsigma}\right)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma, \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{Z}(\varpi)=&\frac{1}{\Delta}\Bigg[\lambda_2\int_1^{\mathcal{T}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1\int_1^{\psi_{\mathfrak{u}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1\int_1^{\sigma_{\mathfrak{v}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma \\
&-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1\int_1^{\pi_{\mathfrak{w}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\psi_{\mathfrak{u}}}\Big(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma \\
&+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\sigma_{\mathfrak{v}}}\Big(\log\frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma \\
&+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\pi_{\mathfrak{w}}}\Big(\log\frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma \\
&-\frac{1}{\Gamma(\alpha_2)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\Big\{(\log\mathcal{T})^{\gamma_1-1}\Big\}+\lambda_1\int_1^{\mathcal{T}}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma \\
&-\lambda_2\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\int_1^{\xi_{\mathfrak{j}}}\frac{(\mathcal{Z})\varsigma}{\varsigma}d\varsigma-\lambda_2\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\int_1^{\zeta_{\mathfrak{i}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\lambda_2\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\int_1^{\mu_{\mathfrak{k}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma \\
&+\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\xi_{\mathfrak{j}}}\Big(\log\frac{\xi_{\mathfrak{j}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\zeta_{\mathfrak{i}}}\Big(\log\frac{\zeta_{\mathfrak{i}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma \\
&+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\mu_{\mathfrak{k}}}\Big(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma-\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma \\
&\times\Bigg\{\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}(\log\psi_{\mathfrak{u}})^{\gamma_1-1}-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}(\log\sigma_{\mathfrak{v}})^{\gamma_1-1}-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}(\log\pi_{\mathfrak{w}})^{\gamma_1-1}\Bigg\}\Bigg] \qquad (2.5) \\
&-\lambda_2\int_1^{\varpi}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma+\frac{1}{\Gamma(\alpha_2)}\int_1^{\varpi}\left(\log\frac{\varpi}{\varsigma}\right)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma,
\end{aligned}
$$

$$\begin{cases} \Delta = \mathcal{A}_1\mathcal{B}_2 - \mathcal{A}_2\mathcal{B}_1, \\ \mathcal{A}_1 = \left\{(\log \mathcal{T})^{\gamma_1-1}\right\}, \\ \mathcal{B}_2 = \left\{(\log \mathcal{T})^{\gamma_2-1}\right\}, \\ \mathcal{A}_2 = \left\{\sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}}(\log \psi_{\mathfrak{u}})^{\gamma_1-1} - \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}(\log \sigma_{\mathfrak{v}})^{\gamma_1-1} - \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}(\log \pi_{\mathfrak{w}})^{\gamma_1-1}\right\}, \\ \mathcal{B}_1 = \left\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}}(\log \xi_{\mathfrak{j}})^{\gamma_2-1} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log \zeta_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log \mu_{\mathfrak{k}})^{\gamma_2-1}\right\}. \end{cases} \tag{2.6}$$

Proof. From the first equation of (2.3), we have

$$\begin{cases} ({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1,\beta_1} + \lambda_1{}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1-1,\beta_1})\mathcal{S}(\varpi) = \mathcal{H}_1(\varpi), \\ ({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2,\beta_2} + \lambda_2{}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2-1,\beta_2})\mathcal{Z}(\varpi) = \mathcal{H}_2(\varpi). \end{cases} \tag{2.7}$$

Taking both sides of the HFI of order α_1, α_2 (2.7), we obtain

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{I}_{1^+}^{\alpha_1}({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1,\beta_1} + \lambda_1{}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1-1,\beta_1})\mathcal{S}(\varpi) = {}^{\mathcal{HH}}\mathcal{I}_{1^+}^{\alpha_1}\mathcal{H}_1(\varpi), \\ {}^{\mathcal{HH}}\mathcal{I}_{1^+}^{\alpha_2}({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2,\beta_2} + \lambda_2{}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2-1,\beta_2})\mathcal{Z}(\varpi) = {}^{\mathcal{HH}}\mathcal{I}_{1^+}^{\alpha_2}\mathcal{H}_2(\varpi). \end{cases} \tag{2.8}$$

Equation (2.8) can be written as follows:

$$\mathcal{S}(\varpi) = \mathfrak{c}_0(\log \varpi)^{\gamma_1-1} + \mathfrak{c}_1(\log \varpi)^{\gamma_2-2} - \lambda_1 \int_1^{\varpi} \frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma + \frac{1}{\Gamma(\alpha_1)} \int_1^{\varpi} \left(\log \frac{\varpi}{\varsigma}\right)^{\alpha_1-1} \frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma, \tag{2.9}$$

and

$$\mathcal{Z}(\varpi) = \mathfrak{d}_0(\log \varpi)^{\gamma_2-1} + \mathfrak{d}_1(\log \varpi)^{\gamma_2-2} - \lambda_2 \int_1^{\varpi} \frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma + \frac{1}{\Gamma(\alpha_2)} \int_1^{\varpi} \left(\log \frac{\varpi}{\varsigma}\right)^{\alpha_2-1} \frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma, \tag{2.10}$$

where $\mathfrak{c}_0, \mathfrak{d}_0, \mathfrak{c}_1$, and $\mathfrak{d}_1$ are arbitrary constants. Boundary conditions (1.2) combined with (2.9) and (2.10) produce

$$\mathcal{S}(1) = \mathfrak{c}_0(\log 1)^{\gamma_1-1} + \frac{\mathfrak{c}_1}{(\log \varpi)^{2-\gamma_1}} - \lambda_1 \int_1^{\varpi} \frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma + \frac{1}{\Gamma(\alpha_1)} \int_1^{\varpi} \left(\log \frac{\varpi}{\varsigma}\right)^{\alpha_1-1} \frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma = 0, \tag{2.11}$$

$$\mathcal{Z}(1) = \mathfrak{d}_0(\log 1)^{\gamma_2-1} + \frac{\mathfrak{d}_1}{(\log \varpi)^{2-\gamma_2}} - \lambda_2 \int_1^{\varpi} \frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma + \frac{1}{\Gamma(\alpha_2)} \int_1^{\varpi} \left(\log \frac{\varpi}{\varsigma}\right)^{\alpha_2-1} \frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma = 0, \tag{2.12}$$

from which we have $\mathfrak{c}_1 = 0$ and $\mathfrak{d}_1 = 0$. Equations (2.11) and (2.12) can be written as

$$\mathcal{S}(\varpi) = \mathfrak{c}_0(\log \varpi)^{\gamma_1-1} - \lambda_1 \int_1^{\varpi} \frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma + \frac{1}{\Gamma(\alpha_1)} \int_1^{\varpi} \left(\log \frac{\varpi}{\varsigma}\right)^{\alpha_1-1} \frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma, \tag{2.13}$$

$$\mathcal{Z}(\varpi) = \mathfrak{d}_0(\log \varpi)^{\gamma_2-1} - \lambda_2 \int_1^{\varpi} \frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma + \frac{1}{\Gamma(\alpha_2)} \int_1^{\varpi} \left(\log \frac{\varpi}{\varsigma}\right)^{\alpha_2-1} \frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma, \tag{2.14}$$

from which we have

$$
\begin{aligned}
\mathfrak{c}_0 =&\frac{1}{\Delta}\Bigg[\lambda_1\int_1^{\mathcal{T}}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma-\lambda_2\sum_{\mathrm{j}=1}^{\mathfrak{m}}\eta_{\mathrm{j}}\int_1^{\xi_{\mathrm{j}}}\frac{(\mathcal{Z})\varsigma}{\varsigma}d\varsigma\\
&-\lambda_2\sum_{\mathrm{i}=1}^{\mathfrak{n}}\theta_{\mathrm{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathrm{i}}}\int_1^{\zeta_{\mathrm{i}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\lambda_2\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\int_1^{\mu_{\mathfrak{k}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathrm{j}=1}^{\mathfrak{m}}\eta_{\mathrm{j}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\xi_{\mathrm{j}}}\Big(\log\frac{\xi_{\mathrm{j}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma+\sum_{\mathrm{i}=1}^{\mathfrak{n}}\theta_{\mathrm{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathrm{i}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\zeta_{\mathrm{i}}}\Big(\log\frac{\zeta_{\mathrm{i}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\mu_{\mathfrak{k}}}\Big(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma-\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\}\\
&+\lambda_2\int_1^{\mathcal{T}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1\int_1^{\psi_{\mathfrak{u}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1\int_1^{\sigma_{\mathfrak{v}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma\\
&-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1\int_1^{\pi_{\mathfrak{w}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\psi_{\mathfrak{u}}}\Big(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\sigma_{\mathfrak{v}}}\Big(\log\frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\pi_{\mathfrak{w}}}\Big(\log\frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\alpha_2)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\Big\{\sum_{\mathrm{j}=1}^{\mathfrak{m}}\eta_{\mathrm{j}}(\log\xi_{\mathrm{j}})^{\gamma_2-1}\\
&+\sum_{\mathrm{i}=1}^{\mathfrak{n}}\theta_{\mathrm{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathrm{i}}}(\log\zeta_{\mathrm{i}})^{\gamma_2-1}+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})^{\gamma_2-1}\Big\}\Bigg],
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
\mathfrak{d}_0 =&\frac{1}{\Delta}\Bigg[\lambda_2\int_1^{\mathcal{T}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1\int_1^{\psi_{\mathfrak{u}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1\int_1^{\sigma_{\mathfrak{v}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma\\
&-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1\int_1^{\pi_{\mathfrak{w}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\psi_{\mathfrak{u}}}\Big(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\sigma_{\mathfrak{v}}}\Big(\log\frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\pi_{\mathfrak{w}}}\Big(\log\frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\alpha_2)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\Big\{(\log\mathcal{T})^{\gamma_1-1}\Big\}+\lambda_1\int_1^{\mathcal{T}}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma\\
&-\lambda_2\sum_{\mathrm{j}=1}^{\mathfrak{m}}\eta_{\mathrm{j}}\int_1^{\xi_{\mathrm{j}}}\frac{(\mathcal{Z})\varsigma}{\varsigma}d\varsigma-\lambda_2\sum_{\mathrm{i}=1}^{\mathfrak{n}}\theta_{\mathrm{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathrm{i}}}\int_1^{\zeta_{\mathrm{i}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\lambda_2\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\int_1^{\mu_{\mathfrak{k}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathrm{j}=1}^{\mathfrak{m}}\eta_{\mathrm{j}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\xi_{\mathrm{j}}}\Big(\log\frac{\xi_{\mathrm{j}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma+\sum_{\mathrm{i}=1}^{\mathfrak{n}}\theta_{\mathrm{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathrm{i}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\zeta_{\mathrm{i}}}\Big(\log\frac{\zeta_{\mathrm{i}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma
\end{aligned}
$$

$$+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\mu_{\mathfrak{k}}}\Big(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma-\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma$$
$$\Big\{\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}(\log\psi_{\mathfrak{u}})^{\gamma_1-1}-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}(\log\sigma_{\mathfrak{v}})^{\gamma_1-1}-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}(\log\pi_{\mathfrak{w}})^{\gamma_1-1}\Big\}\Big]. \tag{2.16}$$

Substitute the values of $\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{d}_0$, and $\mathfrak{d}_1$ in (2.9) and (2.10), and we get solutions (2.4) and (2.5). The converse follows by direct computation. This completes the proof. □

3. Main results

Let us introduce the Banach space $\mathcal{E} = \mathcal{T}([1,\mathcal{T}],\mathbb{R})$ endowed with the norm defined by $\|\mathcal{S}\| := \sup_{\varpi\in[1,\mathcal{T}]} |\mathcal{S}(\varpi)|$. Thus, the product space $(\mathcal{E}\times\mathcal{E}, \|\cdot\|_{\mathcal{E}\times\mathcal{E}})$ equipped with the norm $\|\mathcal{S},\mathcal{Z}\|_{\mathcal{E}\times\mathcal{E}} = \|\mathcal{S}\| + \|\mathcal{Z}\|$ for $(\mathcal{S}\times\mathcal{Z}) \in \mathcal{E}\times\mathcal{E}$ is also a Banach space.

In view of Lemma 2.6, we define as operator $\Upsilon : \mathcal{E}\times\mathcal{E} \to \mathcal{E}\times\mathcal{E}$ by

$$\Upsilon(\mathcal{S},\mathcal{Z})(\varpi) = (\Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi), \Upsilon_2(\mathcal{S},\mathcal{Z})(\varpi)), \tag{3.1}$$

where

$$\begin{aligned}\Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi) =&\frac{1}{\Delta}\Big[\Big\{\lambda_1\int_1^{\mathcal{T}}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma-\lambda_2\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\int_1^{\xi_{\mathfrak{j}}}\frac{(\mathcal{Z})\varsigma}{\varsigma}d\varsigma-\lambda_2\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\int_1^{\zeta_{\mathfrak{i}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma\\&-\lambda_2\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\int_1^{\mu_{\mathfrak{k}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma+\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\xi_{\mathfrak{j}}}\Big(\log\frac{\xi_{\mathfrak{j}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\\&+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\zeta_{\mathfrak{i}}}\Big(\log\frac{\zeta_{\mathfrak{i}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\\&+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\mu_{\mathfrak{k}}}\Big(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\\&-\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\Big\}\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\}\\&+\Big\{\lambda_2\int_1^{\mathcal{T}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1\int_1^{\psi_{\mathfrak{u}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1\int_1^{\sigma_{\mathfrak{v}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma\\&-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1\int_1^{\pi_{\mathfrak{w}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\psi_{\mathfrak{u}}}\Big(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\&+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\sigma_{\mathfrak{v}}}\Big(\log\frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\&+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\pi_{\mathfrak{w}}}\Big(\log\frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma-\frac{1}{\Gamma(\alpha_2)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\Big\}\\&\Big\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{i}}(\log\xi_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log\zeta_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})^{\gamma_2-1}\Big\}\Big]\end{aligned}$$

$$-\lambda_1\int_1^{\varpi}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma+\frac{1}{\Gamma(\alpha_1)}\int_1^{\varpi}\left(\log\frac{\varpi}{\varsigma}\right)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma, \tag{3.2}$$

and

$$\begin{aligned}\Upsilon_2(\mathcal{S},\mathcal{Z})(\varpi)=&\frac{1}{\Delta}\Bigg[\lambda_2\int_1^{\mathcal{T}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1\int_1^{\psi_{\mathfrak{u}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1\int_1^{\sigma_{\mathfrak{v}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma\\
&-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1\int_1^{\pi_{\mathfrak{w}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\psi_{\mathfrak{u}}}\left(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\right)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\sigma_{\mathfrak{v}}}\left(\log\frac{\sigma_{\mathfrak{v}}}{\varsigma}\right)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\pi_{\mathfrak{w}}}\left(\log\frac{\pi_{\mathfrak{w}}}{\varsigma}\right)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\alpha_2)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{\varsigma}\right)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\Big\{(\log\mathcal{T})^{\gamma_1-1}\Big\}+\lambda_1\int_1^{\mathcal{T}}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma\\
&-\lambda_2\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\int_1^{\xi_{\mathfrak{j}}}\frac{(\mathcal{Z})\varsigma}{\varsigma}d\varsigma-\lambda_2\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\int_1^{\zeta_{\mathfrak{i}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\lambda_2\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\int_1^{\mu_{\mathfrak{k}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\xi_{\mathfrak{j}}}\left(\log\frac{\xi_{\mathfrak{j}}}{\varsigma}\right)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\zeta_{\mathfrak{i}}}\left(\log\frac{\zeta_{\mathfrak{i}}}{\varsigma}\right)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\mu_{\mathfrak{k}}}\left(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\right)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma-\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{\varsigma}\right)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&\times\left\{\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}(\log\psi_{\mathfrak{u}})^{\gamma_1-1}-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}(\log\sigma_{\mathfrak{v}})^{\gamma_1-1}-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}(\log\pi_{\mathfrak{w}})^{\gamma_1-1}\right\}\Bigg]\\
&-\lambda_2\int_1^{\varpi}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma+\frac{1}{\Gamma(\alpha_2)}\int_1^{\varpi}\left(\log\frac{\varpi}{\varsigma}\right)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma.\end{aligned} \tag{3.3}$$

We need the following hypothesis in the sequel:

$$\begin{aligned}\Omega_1=&\frac{1}{\Delta}\Bigg[\left\{\lambda_1(\log\mathcal{T})+\frac{(\log\mathcal{T})^{\alpha_1}}{\Gamma(\alpha_1+1)}\right\}\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\}\\
&+\Bigg\{\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1(\log\psi_{\mathfrak{u}})-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1(\log\sigma_{\mathfrak{v}})-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1(\log\pi_{\mathfrak{w}})\\
&+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{(\log\psi_{\mathfrak{u}})^{\alpha_1}}{\Gamma(\alpha_1+1)}+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{(\log\sigma_{\mathfrak{v}})^{\alpha_1}}{\Gamma(\alpha_1+1)}+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{(\pi_{\mathfrak{w}})^{\alpha_1}}{\Gamma(\alpha_1+1)}\Bigg\}\\
&\times\left\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{i}}(\log\xi_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log\zeta_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})^{\gamma_2-1}\right\}\Bigg]\\
&-\lambda_1(\log\mathcal{T})+\frac{(\log\mathcal{T})^{\alpha_1}}{\Gamma(\alpha_1+1)},\end{aligned} \tag{3.4}$$

$$\Omega_2 = \frac{1}{\Delta}\Bigg[\Bigg\{-\lambda_2 \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}}(\log \xi_{\mathfrak{j}}) - \lambda_2 \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log \zeta_{\mathfrak{i}}) - \lambda_2 \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log \mu_{\mathfrak{k}})$$

$$+ \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \frac{(\log \xi_{\mathfrak{j}})^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} \frac{(\log \zeta_{\mathfrak{i}})^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} \frac{(\log \mu_{\mathfrak{k}})^{\alpha_2}}{\Gamma(\alpha_2+1)}\Bigg\}\Bigg\{(\log \mathcal{T})^{\gamma_2-1}\Bigg\}$$

$$+ \Bigg\{\lambda_2(\log \mathcal{T}) - \frac{(\log \mathcal{T})^{\alpha_2}}{\Gamma(\alpha_2+1)}\Bigg\}\Bigg\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{i}}(\log \xi_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log \zeta_{\mathfrak{i}})^{\gamma_2-1}$$

$$+ \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log \mu_{\mathfrak{k}})^{\gamma_2-1}\Bigg\}\Bigg], \tag{3.5}$$

$$\bar{\Omega}_1 = \frac{1}{\Delta}\Bigg[\Bigg\{-\sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}}\lambda_1(\log \psi_{\mathfrak{u}}) - \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1(\log \sigma_{\mathfrak{v}}) - \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1(\log \pi_{\mathfrak{w}})$$

$$+ \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \frac{(\log \psi_{\mathfrak{u}})^{\alpha_1}}{\Gamma(\alpha_1+1)} + \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}} \frac{(\log \sigma_{\mathfrak{v}})^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

$$+ \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}} \frac{(\log \pi_{\mathfrak{w}})^{\alpha_1}}{\Gamma(\alpha_1+1)}\Bigg\}\Bigg\{(\log \mathcal{T})^{\gamma_1-1}\Bigg\} + \lambda_1(\log \mathcal{T}) - \frac{(\log \mathcal{T})^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

$$\times \Bigg\{\sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}}(\log \psi_{\mathfrak{u}})^{\gamma_1-1} - \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}(\log \sigma_{\mathfrak{v}})^{\gamma_1-1} - \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}(\log \pi_{\mathfrak{w}})^{\gamma_1-1}\Bigg\}\Bigg], \tag{3.6}$$

$$\bar{\Omega}_2 = \frac{1}{\Delta}\Bigg[\Bigg\{\lambda_2(\log \mathcal{T}) - \frac{(\log \mathcal{T})^{\alpha_2}}{\Gamma(\alpha_2+1)}\Bigg\}\Bigg\{(\log \mathcal{T})^{\gamma_1-1}\Bigg\} + -\lambda_2 \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}}(\log \xi_{\mathfrak{j}})$$

$$- \lambda_2 \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log \zeta_{\mathfrak{i}}) - \lambda_2 \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log \mu_{\mathfrak{k}})$$

$$+ \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \frac{(\log \xi_{\mathfrak{j}})^{\alpha_2}}{\Gamma(\alpha_2)} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} \frac{(\log \zeta_{\mathfrak{i}})^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} \frac{(\log \mu_{\mathfrak{k}})^{\alpha_2}}{\Gamma(\alpha_2+1)}$$

$$\times \Bigg\{\sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}}(\log \psi_{\mathfrak{u}})^{\gamma_1-1} - \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}(\log \sigma_{\mathfrak{v}})^{\gamma_1-1} - \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}(\log \pi_{\mathfrak{w}})^{\gamma_1-1}\Bigg\}\Bigg]$$

$$- \lambda_2(\log \mathcal{T}) + \frac{(\log \mathcal{T})}{\Gamma(\alpha_2)}, \tag{3.7}$$

and

$$\begin{cases} \Psi_0 = \Big(\Omega_1 + \bar{\Omega}_1\Big)\mathfrak{M}_0 + \Big(\Omega_2 + \bar{\Omega}_2\Big)\mathfrak{N}_0, \\ \Psi_1 = \Big(\Omega_1 + \bar{\Omega}_1\Big)\Big(\mathfrak{M}_1 + \frac{\mathfrak{M}_3}{\Gamma(\mathfrak{p}_1+1)}\Big) + \Big(\Omega_2 + \bar{\Omega}_2\Big)\Big(\mathfrak{N}_1 + \frac{\mathfrak{N}_3}{\Gamma(\mathfrak{q}_1+1)}\Big), \\ \Psi_2 = \Big(\Omega_1 + \bar{\Omega}_1\Big)\Big(\mathfrak{M}_2 + \frac{\mathfrak{M}_4}{\Gamma(\mathfrak{p}_2+1)}\Big) + \Big(\Omega_2 + \bar{\Omega}_2\Big)\Big(\mathfrak{N}_2 + \frac{\mathfrak{N}_4}{\Gamma(\mathfrak{q}_2+1)}\Big). \end{cases} \tag{3.8}$$

We give now the assumptions we will use in this section.

($\mathcal{H}_1$) Assume that there exist real constants $\mathfrak{M}_i, \mathfrak{N}_i \geq 0 (i = 1, 2)$ and $\mathfrak{M}_0 > 0, \mathfrak{N}_0$ such that, for all $\varpi \in [1, \mathcal{T}], \mathcal{S}_i \in \mathbb{R}, i = 1, 2, 3, 4,$

$$|\mathcal{F}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4)| \leq \mathfrak{M}_0 + \mathfrak{M}_1|\mathcal{S}_1| + \mathfrak{M}_2|\mathcal{S}_2| + \mathfrak{M}_3|\mathcal{S}_3| + \mathfrak{M}_4|\mathcal{S}_4|,$$
$$|\mathcal{G}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4)| \leq \mathfrak{N}_0 + \mathfrak{N}_1|\mathcal{S}_1| + \mathfrak{N}_2|\mathcal{S}_2| + \mathfrak{N}_3|\mathcal{S}_3| + \mathfrak{N}_4|\mathcal{S}_4|, \forall\ \varpi \in [1, \mathcal{T}].$$

($\mathcal{H}_2$) There exists positive constant $\mathcal{L}, \mathcal{L}_1$, such that, for all $\varpi \in [1, \mathcal{T}], \mathcal{S}_i, \mathcal{Z}_i \in \mathbb{R}, i = 1, 2.$

$$\begin{aligned}
&|\mathcal{F}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4) - \mathcal{F}(\varpi, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4)| \\
\leq &\mathcal{L}\Big(|\mathcal{S}_1 - \mathcal{Z}_1| + |\mathcal{S}_2 - \mathcal{Z}_2| + |\mathcal{S}_3 - \mathcal{Z}_3| + |\mathcal{S}_4 - \mathcal{Z}_4|\Big), \\
&|\mathcal{G}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4) - \mathcal{G}(\varpi, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4)| \\
\leq &\mathcal{L}_1\Big(|\mathcal{S}_1 - \mathcal{Z}_1| + |\mathcal{S}_2 - \mathcal{Z}_2| + |\mathcal{S}_3 - \mathcal{Z}_3| + |\mathcal{S}_4 - \mathcal{Z}_4|\Big).
\end{aligned}$$

4. Existence result via Leray-Schauder alternative

The first theorem uses the Leray-Schauder alternative to establish the existence of a result.

Lemma 4.1. *[1] Let $\theta(\Xi) = \{\mathcal{S} \in \mathcal{E} : \mathcal{S} = \kappa\Xi(\mathcal{S})$ for some $0 < \kappa < 1\}$, where $\Xi : \mathcal{E} \to \mathcal{E}$ is a completely continuous operator. Then, either the set $\theta(\Xi)$ is unbounded or there exists at least one fixed for the operator Ξ.*

Theorem 4.2. *Suppose condition ($\mathcal{H}1$) is satisfied. Additionally, assume that*

$$\max\{\Psi_1, \Psi_2\} < 1. \tag{4.1}$$

Under these conditions, there exists at least one solution to the problems (1.1) and (1.2) on $\mathcal{E}$.

Proof. First, let's establish that the operator $\Upsilon : \mathcal{E} \times \mathcal{E} \to \mathcal{E} \times \mathcal{E}$, as defined in (3.1), is completely continuous. The continuity of the operator Υ (in terms of Υ_1 and Υ_2) is evident from the continuity of $\mathcal{F}$ and $\mathcal{G}$.

Next, we aim to demonstrate that the operator Υ is uniformly bounded. To achieve this, consider a bounded set $\mathcal{B}_{\mathrm{r}} \subset \mathcal{E} \times \mathcal{E}$. Then, we can find positive constants $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfying

$$\begin{cases} |\mathcal{F}(\varpi, \mathcal{S}(\varpi), \mathcal{Z}(\varpi), \mathcal{I}^{\mathrm{p}_1}\mathcal{S}(\varpi), \mathcal{I}^{\mathrm{p}_2}\mathcal{Z}(\varpi))| \leq \mathcal{N}_1, \\ |\mathcal{N}(\varpi, \mathcal{S}(\varpi), \mathcal{Z}(\varpi), \mathcal{I}^{\mathrm{q}_1}\mathcal{S}(\varpi), \mathcal{I}^{\mathrm{q}_2}\mathcal{Z}(\varpi))| \leq \mathcal{N}_2, \forall (\mathcal{S}, \mathcal{Z}) \in \mathcal{B}_{\mathrm{r}}. \end{cases} \tag{4.2}$$

Consequently, we obtain

$$\begin{aligned}
\|\Upsilon_1(\mathcal{S}, \mathcal{Z})(\varpi)\| =& \frac{1}{\Delta}\Bigg[\Bigg\{\lambda_1 \int_1^{\mathcal{T}} \frac{\mathcal{S}(\varsigma)}{\varsigma} d\varsigma - \lambda_2 \sum_{\mathrm{j}=1}^{\mathrm{m}} \eta_{\mathrm{j}} \int_1^{\xi_{\mathrm{j}}} \frac{(\mathcal{Z})\varsigma}{\varsigma} d\varsigma - \lambda_2 \sum_{\mathrm{i}=1}^{\mathrm{n}} \theta_{\mathrm{i}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathrm{i}}} \int_1^{\zeta_{\mathrm{i}}} \frac{\mathcal{Z}(\varsigma)}{\varsigma} d\varsigma \\
&- \lambda_2 \sum_{\mathrm{t}=1}^{\mathrm{r}} \lambda_{\mathrm{t}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathrm{t}}} \int_1^{\mu_{\mathrm{t}}} \frac{\mathcal{Z}(\varsigma)}{\varsigma} d\varsigma + \sum_{\mathrm{j}=1}^{\mathrm{m}} \eta_{\mathrm{j}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\xi_{\mathrm{j}}} \Big(\log \frac{\xi_{\mathrm{j}}}{\varsigma}\Big)^{\alpha_2 - 1} \frac{|\mathcal{N}_2|(\varsigma)}{\varsigma} d\varsigma \\
&+ \sum_{\mathrm{i}=1}^{\mathrm{n}} \theta_{\mathrm{i}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathrm{i}}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\zeta_{\mathrm{i}}} \Big(\log \frac{\zeta_{\mathrm{i}}}{\varsigma}\Big)^{\alpha_2 - 1} \frac{|\mathcal{N}_2|(\varsigma)}{\varsigma} d\varsigma
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\mu_{\mathfrak{k}}}\Big(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\Big)^{\alpha_2-1}\frac{|\mathcal{N}_2|(\varsigma)}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_1-1}\frac{|\mathcal{N}_1|(\varsigma)}{\varsigma}d\varsigma\Big\}\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\}\\
&+\Big\{\lambda_2\int_1^{\mathcal{T}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1\int_1^{\psi_{\mathfrak{u}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1\int_1^{\sigma_{\mathfrak{v}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma\\
&-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1\int_1^{\pi_{\mathfrak{w}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\psi_{\mathfrak{u}}}\Big(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1}\frac{|\mathcal{N}_1|(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\sigma_{\mathfrak{v}}}\Big(\log\frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1}\frac{|\mathcal{N}_1|(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\pi_{\mathfrak{w}}}\Big(\log\frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1}\frac{|\mathcal{N}_1|(\varsigma)}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\alpha_2)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1}\frac{|\mathcal{N}_2|(\varsigma)}{\varsigma}d\varsigma\Big\}\times\Big\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{i}}(\log\xi_{\mathfrak{i}})^{\gamma_2-1}\\
&+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log\zeta_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})^{\gamma_2-1}\Big\}\Big]\\
&-\lambda_1\int_1^{\varpi}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma+\frac{1}{\Gamma(\alpha_1)}\int_1^{\varpi}\left(\log\frac{\varpi}{\varsigma}\right)^{\alpha_1-1}\frac{|\mathcal{N}_1|(\varsigma)}{\varsigma}d\varsigma\\
\le&\mathcal{N}_1\Big\{\frac{1}{\Delta}\Big[\lambda_1(\log\mathcal{T})+\frac{(\log\mathcal{T})^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big]\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\}\\
&+\Big\{\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1(\log\psi_{\mathfrak{u}})-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1(\log\sigma_{\mathfrak{v}})-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1(\log\pi_{\mathfrak{w}})\\
&+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{(\log\psi_{\mathfrak{u}})^{\alpha_1}}{\Gamma(\alpha_1+1)}+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{(\log\sigma_{\mathfrak{v}})^{\alpha_1}}{\Gamma(\alpha_1+1)}+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{(\pi_{\mathfrak{w}})^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big\}\\
&+\Big\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{i}}(\log\xi_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log\zeta_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})^{\gamma_2-1}\Big\}\\
&-\lambda_1(\log\mathcal{T})+\frac{(\log\mathcal{T})^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big\}+\mathcal{N}_2\Big\{\frac{1}{\Delta}\Big[\Big\{-\lambda_2\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}(\log\xi_{\mathfrak{j}})-\lambda_2\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log\zeta_{\mathfrak{i}})\\
&-\lambda_2\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})+\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\frac{(\log\xi_{\mathfrak{j}})^{\alpha_2}}{\Gamma(\alpha_2+1)}+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\frac{(\log\zeta_{\mathfrak{i}})^{\alpha_2}}{\Gamma(\alpha_2+1)}\\
&+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\frac{(\log\mu_{\mathfrak{k}})^{\alpha_2}}{\Gamma(\alpha_2+1)}\Big\}\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\}+\Big\{\lambda_2(\log\mathcal{T})-\frac{(\log\mathcal{T})^{\alpha_2}}{\Gamma(\alpha_2+1)}\Big\}\Big]\Big\}.
\end{aligned}\tag{4.3}
$$

This observation, in light of the notation (3.4) and (3.5), yields

$$\|\Upsilon_1(\mathcal{S},\mathcal{Z})\|\le\Omega_1\mathcal{N}_1+\Omega_2\mathcal{N}_2.\tag{4.4}$$

Similarly, employing the notation (3.6) and (3.7), we obtain

$$\|\Upsilon_2(\mathcal{S},\mathcal{Z})\| \le \bar{\Omega}_1\mathcal{N}_1 + \bar{\Omega}_2\mathcal{N}_2. \tag{4.5}$$

Then, it follows from (4.4) and (4.5) that

$$\|\Upsilon(\mathcal{S},\mathcal{Z})\| \le (\Omega_1 + \bar{\Omega}_1)\mathcal{N}_1 + (\Omega_2 + \bar{\Omega}_2)\mathcal{N}_2. \tag{4.6}$$

This demonstrates that the operator Υ is uniformly bounded.

To establish the equicontinuity of Υ, let $\varpi_1, \varpi_2 \in \mathcal{E}$ with $\varpi_1 < \varpi_2$. Then, we find that

$$\begin{aligned}
&|\Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi_2) - \Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi_1)|\\
=&\Bigg\{\frac{1}{\Delta}\Bigg[\Bigg\{\lambda_1\int_1^{\mathcal{T}}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma - \lambda_2\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\int_1^{\xi_{\mathfrak{j}}}\frac{(\mathcal{Z})\varsigma}{\varsigma}d\varsigma - \lambda_2\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\int_1^{\zeta_{\mathfrak{i}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma\\
&-\lambda_2\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\int_1^{\mu_{\mathfrak{k}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma + \sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\xi_{\mathfrak{j}}}\Big(\log\frac{\xi_{\mathfrak{j}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\zeta_{\mathfrak{i}}}\Big(\log\frac{\zeta_{\mathfrak{i}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma + \sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\mu_{\mathfrak{k}}}\Big(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\Bigg\}\Bigg\{(\log\mathcal{T})^{\gamma_2-1}\Bigg\}\\
&+\Bigg\{\lambda_2\int_1^{\mathcal{T}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma - \sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1\int_1^{\psi_{\mathfrak{u}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma - \sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1\int_1^{\sigma_{\mathfrak{v}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma\\
&-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1\int_1^{\pi_{\mathfrak{w}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma + \sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\psi_{\mathfrak{u}}}\Big(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\sigma_{\mathfrak{v}}}\Big(\log\frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\pi_{\mathfrak{w}}}\Big(\log\frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma - \frac{1}{\Gamma(\alpha_2)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathcal{H}_2(\varsigma)}{\varsigma}d\varsigma\Bigg\}\\
&\Bigg\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{i}}(\log\xi_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log\zeta_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})^{\gamma_2-1}\Bigg\}\Bigg]\\
&+\lambda_1[(\log\varpi_2)^{\gamma_2-1} - (\log\varpi_1)^{\gamma_1-1}]\int_1^{\varpi_1}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma + \int_{\varpi_1}^{\varpi_2}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma\\
&+\frac{1}{\Gamma(\alpha_1)}\int_1^{\varpi_1}\left(\log\frac{\varpi_2}{\varsigma}\right)^{\alpha_1-1} - \left(\log\frac{\varpi_1}{\varsigma}\right)^{\alpha_1-1}\frac{\mathcal{H}_1(\varsigma)}{\varsigma}d\varsigma + \int_{\varpi_1}^{\varpi_2}\Big(\log\frac{\varpi_2}{\varsigma}\Big)^{\alpha_1-1}\frac{d\varsigma}{\varsigma}\Bigg\}, \to 0 \text{ as } \varpi_2 \to \varpi_1,
\end{aligned} \tag{4.7}$$

independent of $(\mathcal{S},\mathcal{Z}) \in \mathcal{B}_{\mathfrak{r}}$. Likewise, it can be shown that $|\Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi_2) - \Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi_1)| \to 0$ as $\varpi_2 \to \varpi_1$ is independent of $(\mathcal{S},\mathcal{Z}) \in \mathcal{B}_{\mathfrak{r}}$. Consequently, the equicontinuity of Υ_1 and Υ_2 implies the equicontinuity of the operator Υ. Therefore, by Arzela-Ascoli's theorem, the operator Υ is compact. Finally, we establish the boundedness of the set $\Theta(\Upsilon) = \{\mathcal{S},\mathcal{Z} \in \mathcal{E}\times\mathcal{E} : \mathcal{S},\mathcal{Z} = \kappa\Upsilon(\mathcal{S},\mathcal{Z}); 0 \le \kappa \le 1\}$.

Let $(\mathcal{S},\mathcal{Z}) \in \Theta(\Upsilon)$. Then, $(\mathcal{S},\mathcal{Z}) = \kappa\Upsilon(\mathcal{S},\mathcal{Z})$, which implies that $\mathcal{S}(\varpi) = \kappa\Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi), \mathcal{Z}(\varpi) = \kappa\Upsilon_2(\mathcal{S},\mathcal{Z})(\varpi)$ for any $\varpi \in \mathcal{E}$, and so $|\mathcal{S}(\varpi)| = |\Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi)|, \mathcal{Z}(\varpi) = |\Upsilon_2(\mathcal{S},\mathcal{Z})(\varpi)|\ \forall\ \varpi \in [0,1]$. Using some inequality proved at the beginning of the proof, we find

$$
\begin{aligned}
\|\Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi)\| = \frac{1}{\Delta}\Bigg[\Bigg\{&\lambda_1\int_1^{\mathcal{T}}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma - \lambda_2\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\int_1^{\xi_{\mathfrak{j}}}\frac{(\mathcal{Z})\varsigma}{\varsigma}d\varsigma - \lambda_2\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\int_1^{\zeta_{\mathfrak{i}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma \\
&- \lambda_2\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\int_1^{\mu_{\mathfrak{k}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma \\
&+ \sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\xi_{\mathfrak{j}}}\Big(\log\frac{\xi_{\mathfrak{j}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathfrak{N}_0+\mathfrak{N}_1|\mathcal{S}_1|+\mathfrak{N}_2|\mathcal{S}_2|+\mathfrak{N}_3|\mathcal{S}_3|+\mathfrak{N}_4|\mathcal{S}_4|}{\varsigma}d\varsigma \\
&+ \sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\zeta_{\mathfrak{i}}}\Big(\log\frac{\zeta_{\mathfrak{i}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathfrak{N}_0+\mathfrak{N}_1|\mathcal{S}_1|+\mathfrak{N}_2|\mathcal{S}_2|+\mathfrak{N}_3|\mathcal{S}_3|+\mathfrak{N}_4|\mathcal{S}_4|}{\varsigma}d\varsigma \\
&+ \sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\mu_{\mathfrak{k}}}\Big(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathfrak{N}_0+\mathfrak{N}_1|\mathcal{S}_1|+\mathfrak{N}_2|\mathcal{S}_2|+\mathfrak{N}_3|\mathcal{S}_3|+\mathfrak{N}_4|\mathcal{S}_4|}{\varsigma}d\varsigma \\
&- \frac{1}{\Gamma(\alpha_1)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathfrak{M}_0+\mathfrak{M}_1|\mathcal{S}_1|+\mathfrak{M}_2|\mathcal{S}_2|+\mathfrak{M}_3|\mathcal{S}_3|+\mathfrak{M}_4|\mathcal{S}_4|}{\varsigma}d\varsigma\Bigg\}\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\} \\
&+ \Bigg\{\lambda_2\int_1^{\mathcal{T}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma - \sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1\int_1^{\psi_{\mathfrak{u}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma - \sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1\int_1^{\sigma_{\mathfrak{v}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma \\
&- \sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1\int_1^{\pi_{\mathfrak{w}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma \\
&+ \sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\psi_{\mathfrak{u}}}\Big(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathfrak{M}_0+\mathfrak{M}_1|\mathcal{S}_1|+\mathfrak{M}_2|\mathcal{S}_2|+\mathfrak{M}_3|\mathcal{S}_3|+\mathfrak{M}_4|\mathcal{S}_4|}{\varsigma}d\varsigma \\
&+ \sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\sigma_{\mathfrak{v}}}\Big(\log\frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathfrak{M}_0+\mathfrak{M}_1|\mathcal{S}_1|+\mathfrak{M}_2|\mathcal{S}_2|+\mathfrak{M}_3|\mathcal{S}_3|+\mathfrak{M}_4|\mathcal{S}_4|}{\varsigma}d\varsigma \\
&+ \sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\pi_{\mathfrak{w}}}\Big(\log\frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1}\frac{\mathfrak{M}_0+\mathfrak{M}_1|\mathcal{S}_1|+\mathfrak{M}_2|\mathcal{S}_2|+\mathfrak{M}_3|\mathcal{S}_3|+\mathfrak{M}_4|\mathcal{S}_4|}{\varsigma}d\varsigma \\
&- \frac{1}{\Gamma(\alpha_2)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1}\frac{\mathfrak{N}_0+\mathfrak{N}_1|\mathcal{S}_1|+\mathfrak{N}_2|\mathcal{S}_2|+\mathfrak{N}_3|\mathcal{S}_3|+\mathfrak{N}_4|\mathcal{S}_4|}{\varsigma}d\varsigma\Bigg\} \\
&\times\Bigg\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{i}}(\log\xi_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log\zeta_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})^{\gamma_2-1}\Bigg\}\Bigg] \\
&- \lambda_1\int_1^{\varpi}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma + \frac{1}{\Gamma(\alpha_1)}\int_1^{\varpi}\left(\log\frac{\varpi}{\varsigma}\right)^{\alpha_1-1}\frac{\mathfrak{M}_0+\mathfrak{M}_1|\mathcal{S}_1|+\mathfrak{M}_2|\mathcal{S}_2|+\mathfrak{M}_3|\mathcal{S}_3|+\mathfrak{M}_4|\mathcal{S}_4|}{\varsigma}d\varsigma \\
\leq\Omega_1\Bigg\{&\mathfrak{M}_0+\mathfrak{M}_1|\mathcal{S}_1|+\mathfrak{M}_2|\mathcal{S}_2|+\frac{\mathfrak{M}_3}{\Gamma(\mathfrak{p}_1+1)|\mathcal{S}_1|}+\frac{\mathfrak{M}_4}{\Gamma(\mathfrak{p}_2+1)|\mathcal{S}_1|}\Bigg\} \\
&+ \Omega_2\Bigg\{\mathfrak{N}_0+\mathfrak{N}_1|\mathcal{Z}_1|+\mathfrak{N}_2|\mathcal{Z}_2|+\frac{\mathfrak{N}_3}{\Gamma(\mathfrak{q}_1+1)|\mathcal{Z}_1|}+\frac{\mathfrak{N}_4}{\Gamma(\mathfrak{q}_2+1)|\mathcal{Z}_1|}\Bigg\},
\end{aligned}
\tag{4.8}
$$

which implies that

$$\begin{aligned}\|\mathcal{S}\| &= \sup_{\varpi\in[1,\mathcal{T}]}|\mathcal{S}(\varpi)|\\ &\leq \Omega_1\left\{\mathfrak{M}_0+\mathfrak{M}_1\|\mathcal{S}_1\|+\mathfrak{M}_2\|\mathcal{S}_2\|+\frac{\mathfrak{M}_3}{\Gamma(\mathfrak{p}_1+1)\|\mathcal{S}_1\|}+\frac{\mathfrak{M}_4}{\Gamma(\mathfrak{p}_2+1)\|\mathcal{S}_1\|}\right\}\\ &\quad+\Omega_2\left\{\mathfrak{N}_0+\mathfrak{N}_1\|\mathcal{S}_1\|+\mathfrak{N}_2\|\mathcal{S}_2\|+\frac{\mathfrak{N}_3}{\Gamma(\mathfrak{q}_1+1)\|\mathcal{S}_1\|}+\frac{\mathfrak{N}_4}{\Gamma(\mathfrak{q}_2+1)\|\mathcal{S}_1\|}\right\}.\end{aligned}\tag{4.9}$$

Similarly, one can find that

$$\begin{aligned}\|\mathcal{Z}\| &= \sup_{\varpi\in[1,\mathcal{T}]}|\mathcal{Z}(\varpi)|\\ &\leq \bar{\Omega}_1\left\{\mathfrak{M}_0+\mathfrak{M}_1\|\mathcal{S}_1\|+\mathfrak{M}_2\|\mathcal{S}_2\|+\frac{\mathfrak{M}_3}{\Gamma(\mathfrak{p}_1+1)\|\mathcal{S}_1\|}+\frac{\mathfrak{M}_4}{\Gamma(\mathfrak{p}_2+1)\|\mathcal{S}_1\|}\right\}\\ &\quad+\bar{\Omega}_2\left\{\mathfrak{N}_0+\mathfrak{N}_1\|\mathcal{S}_1\|+\mathfrak{N}_2\|\mathcal{S}_2\|+\frac{\mathfrak{N}_3}{\Gamma(\mathfrak{q}_1+1)\|\mathcal{S}_1\|}+\frac{\mathfrak{N}_4}{\Gamma(\mathfrak{q}_2+1)\|\mathcal{S}_1\|}\right\}.\end{aligned}\tag{4.10}$$

From (4.9) and (4.10), we obtained

$$\begin{aligned}\|\mathcal{S}\|+\|\mathcal{Z}\| \leq&\left(\Omega_1+\bar{\Omega}_1\right)\mathfrak{M}_0+\left(\Omega_2+\bar{\Omega}_2\right)\mathfrak{N}_0\\ &+\|\mathcal{S}\|\left\{\left(\Omega_1+\bar{\Omega}_1\right)\left(\mathfrak{M}_1+\frac{\mathfrak{M}_3}{\Gamma(\mathfrak{p}_1+1)}\right)+\left(\Omega_2+\bar{\Omega}_2\right)\left(\mathfrak{N}_1+\frac{\mathfrak{N}_3}{\Gamma(\mathfrak{q}_2+1)}\right)\right\}\\ &+\|\mathcal{Z}\|\left\{\left(\Omega_1+\bar{\Omega}_1\right)\left(\mathfrak{M}_2+\frac{\mathfrak{M}_4}{\Gamma(\mathfrak{p}_1+1)}\right)+\left(\Omega_2+\bar{\Omega}_2\right)\left(\mathfrak{N}_2+\frac{\mathfrak{N}_4}{\Gamma(\mathfrak{q}_2+1)}\right)\right\},\\ =&\Psi_0+\Psi_1\|\mathcal{S}\|+\Psi_2\|\mathcal{Z}\|\leq\Psi_0+\max\{\Psi_1,\Psi_2\}\|(\mathcal{S},\mathcal{Z})\|_{\mathcal{E}},\end{aligned}\tag{4.11}$$

where Ψ_i $i=0,1,2$ are given by (3.8). By (4.1), we deduce that

$$\|(\mathcal{S},\mathcal{Z})\|_{\mathcal{E}}=\frac{\Psi_0}{1-\max\{\Psi_1,\Psi_2\}}.\tag{4.12}$$

As a result, $\Theta(\Upsilon)$ is bounded. Consequently, the conclusion of Lemma 2.6 applies, implying that the operator Υ has at least one fixed point, which indeed serves as a solution to the problems (1.1) and (1.2). □

Now, we introduce the constants

$$\mathcal{F}_0=\sup_{\varpi\in[0,1]}|\mathcal{F}(\varpi,0,0,0,0)|,\ \ \mathcal{G}_0=\sup_{\varpi\in[0,1]}|\mathcal{G}(\varpi,0,0,0,0)|,\tag{4.13}$$

$$\begin{cases} \rho_1 = \max\left\{1 + \frac{1}{\Gamma(\mathfrak{p}_1+1)}, 1 + \frac{1}{\Gamma(\mathfrak{p}_2+1)}\right\}, \\ \rho_2 = \max\left\{1 + \frac{1}{\Gamma(\mathfrak{q}_1+1)}, 1 + \frac{1}{\Gamma(\mathfrak{q}_2+1)}\right\}, \\ \mathcal{D}_1 = \mathfrak{z}_0\rho_1\Omega_1 + \mathfrak{k}_0\rho_2\Omega_2, \\ \mathcal{D}_2 = \mathfrak{z}_0\rho_1\bar{\Omega}_1 + \mathfrak{k}_0\rho_2\bar{\Omega}_2, \\ \mathcal{G}_1 = \mathcal{F}_0\Omega_1 + \mathcal{G}_0\Omega_2, \\ \mathcal{G}_2 = \mathcal{F}_0\bar{\Omega}_1 + \mathcal{G}_0\bar{\Omega}_2, \end{cases} \tag{4.14}$$

where $\Omega_1, \Omega_2, \bar{\Omega}_1, \bar{\Omega}_2$ are given by (3.4)–(3.7).

The subsequent result will establish the existence of a unique solution to the problems (1.1) and (1.2) through the application of a fixed point theorem attributed to Banach.

Theorem 4.3. *If the assumption ($\mathcal{H}_2$) is satisfied and that*

$$\mathcal{D}_1 + \mathcal{D}_2 < 1, \tag{4.15}$$

where Ω_i and $\bar{\Omega}_i$, (i=1,2) are given in (3.4)–(3.7), then the problems (1.1) and (1.2) have a unique solution on $\mathcal{E}$.

Proof. By using condition (4.15), we define the positive number

$$\mathcal{R} \geq \frac{\mathcal{G}_1 + \mathcal{G}_2}{1 - (\mathcal{D}_1 + \mathcal{D}_2)}, \tag{4.16}$$

where $\mathcal{G}_1, \mathcal{G}_2, \mathcal{D}_1, \mathcal{D}_2$ are given by (4.14). We will prove that $\mathcal{A}(\mathcal{B}_\mathcal{R}) \subset \mathcal{B}_\mathcal{R}$, where $\mathcal{B}_\mathcal{R} = \{(\mathcal{S}, \mathcal{Z}) \in \mathcal{E} \times \mathcal{E} : \|(\mathcal{S}, \mathcal{Z}) \leq \mathcal{R}\}$. For $(\mathcal{S}, \mathcal{Z}) \in \mathcal{B}_\mathcal{R}$ and $\varpi \in [0, 1]$, we obtain

$$\begin{aligned} &|\mathcal{F}(\varpi, \mathcal{S}(\varpi), \mathcal{Z}(\varpi), \mathcal{I}^{\mathfrak{p}_1}\mathcal{S}(\varpi), \mathcal{I}^{\mathfrak{p}_2}\mathcal{Z}(\varpi))| \\ &\leq |\mathcal{F}(\varpi, \mathcal{S}(\varpi), \mathcal{Z}(\varpi), \mathcal{I}^{\mathfrak{p}_1}\mathcal{S}(\varpi), \mathcal{I}^{\mathfrak{p}_2}\mathcal{Z}(\varpi)) - \mathcal{F}(\varpi, 0, 0, 0, 0)| + |\mathcal{F}(\varpi, 0, 0, 0, 0)| \\ &\leq \mathfrak{z}_0(|\mathcal{S}(\varpi)| + |\mathcal{Z}(\varpi)| + |\mathcal{I}^{\mathfrak{p}_1}\mathcal{S}(\varpi)| + |\mathcal{I}^{\mathfrak{p}_2}\mathcal{Z}(\varpi)|) + \mathcal{F}_0 \\ &\leq \mathfrak{z}_0\Big(\|\mathcal{S}\| + \|\mathcal{Z}\| + \frac{1}{\Gamma(\mathfrak{p}_1 + 1)}\|\mathcal{S}\| + \frac{1}{\Gamma(\mathfrak{p}_2 + 1)}\|\mathcal{Z}\|\Big) + \mathcal{F}_0 \\ &\leq \mathfrak{z}_0 \max\left\{1 + \frac{1}{\Gamma(\mathfrak{p}_1 + 1)}, 1 + \frac{1}{\Gamma(\mathfrak{p}_2 + 1)}\right\}\|\mathcal{S}, \mathcal{Z}\|_\mathcal{E} + \mathcal{F}_0 \\ &\leq \mathfrak{z}_0 \max\left\{1 + \frac{1}{\Gamma(\mathfrak{p}_1 + 1)}, 1 + \frac{1}{\Gamma(\mathfrak{p}_2 + 1)}\right\}\mathcal{R} + \mathcal{F}_0 \\ &\leq \mathfrak{z}_0\rho_1\mathcal{R} + \mathcal{F}_0, \end{aligned}$$

and

$$\begin{aligned} &|\mathcal{G}(\varpi, \mathcal{S}(\varpi), \mathcal{Z}(\varpi), \mathcal{I}^{\mathfrak{q}_1}\mathcal{S}(\varpi), \mathcal{I}^{\mathfrak{q}_2}\mathcal{Z}(\varpi))| \\ &\leq |\mathcal{G}(\varpi, \mathcal{S}(\varpi), \mathcal{Z}(\varpi), \mathcal{I}^{\mathfrak{q}_1}\mathcal{S}(\varpi), \mathcal{I}^{\mathfrak{q}_2}\mathcal{Z}(\varpi)) - \mathcal{G}(\varpi, 0, 0, 0, 0)| + |\mathcal{G}(\varpi, 0, 0, 0, 0)| \\ &\leq \mathfrak{k}_0(|\mathcal{S}(\varpi)| + |\mathcal{Z}(\varpi)| + |\mathcal{I}^{\mathfrak{q}_1}\mathcal{S}(\varpi)| + |\mathcal{I}^{\mathfrak{q}_2}\mathcal{Z}(\varpi)|) + \mathcal{G}_0 \end{aligned}$$

$$\leq \mathfrak{k}_0\Big(\|\mathcal{S}\| + \|\mathcal{Z}\| + \frac{1}{\Gamma(\mathfrak{q}_1+1)}\|\mathcal{S}\| + \frac{1}{\Gamma(\mathfrak{q}_2+1)}\|\mathcal{Z}\|\Big) + \mathcal{G}_0$$
$$\leq \mathfrak{k}_0 \max\left\{1 + \frac{1}{\Gamma(\mathfrak{q}_1+1)}, 1 + \frac{1}{\Gamma(\mathfrak{q}_2+1)}\right\}\|\mathcal{S},\mathcal{Z}\|_{\mathcal{E}} + \mathcal{G}_0$$
$$\leq \mathfrak{k}_0 \max\left\{1 + \frac{1}{\Gamma(\mathfrak{q}_1+1)}, 1 + \frac{1}{\Gamma(\mathfrak{q}_2+1)}\right\}\mathcal{R} + \mathcal{G}_0$$
$$\leq \mathfrak{k}_0\rho_2\mathcal{R} + \mathcal{G}_0.$$

Then we deduce that

$$\begin{aligned}|\Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi)| \leq& (\mathfrak{z}_0\rho_1\mathcal{R} + \mathcal{F}_0)\Omega_1 + (\mathfrak{k}_0\rho_2\mathcal{R} + \mathcal{G}_0)\Omega_2 \\ =& (\mathfrak{z}_0\rho_1\Omega_1 + \mathfrak{k}_1\rho_2\Omega_2)\mathcal{R} + \mathcal{F}_0\Omega_1 + \mathcal{G}_0\Omega_2 = \mathfrak{D}_1\mathcal{R} + \mathfrak{G}_1,\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}|\Upsilon_2(\mathcal{S},\mathcal{Z})(\varpi)| \leq& (\mathfrak{k}_0\rho_1\mathcal{R} + \mathcal{F}_0)\bar{\Omega}_1 + (\mathfrak{k}_0\rho_2\mathcal{R} + \mathcal{G}_0)\bar{\Omega}_2 \\ =& (\mathfrak{z}_0\rho_1\bar{\Omega}_1 + \mathfrak{k}_1\rho_2\bar{\Omega}_2)\mathcal{R} + \mathcal{F}_0\bar{\Omega}_1 + \mathcal{G}_0\bar{\Omega}_2 = \mathfrak{D}_2\mathcal{R} + \mathfrak{G}_2.\end{aligned} \tag{4.18}$$

Therefore, by (4.17), (4.18), and the definition of $\mathcal{R}$, we conclude that

$$\|\Upsilon(\mathcal{S},\mathcal{Z})\|_{\mathcal{E}} = \|\Upsilon_1(\mathcal{S},\mathcal{Z})\| + \|\Upsilon_2(\mathcal{S},\mathcal{Z})\| \leq (\mathfrak{D}_1 + \mathfrak{D}_2)\mathcal{R} + \mathcal{G}_1 + \mathcal{G}_2 = \mathcal{R}, \forall(\mathcal{S},\mathcal{Z}) \in \mathcal{B}_{\mathcal{R}}, \tag{4.19}$$

which gives us $\Upsilon(\mathcal{B}_{\mathcal{R}}) \subset \mathcal{B}_{\mathcal{R}}$.

We will prove next that Υ is a contraction operator. By using $(\mathcal{H}_2)$, for $(\mathcal{S}_i,\mathcal{Z}_i) \in \mathcal{B}_{\mathcal{R}}, i = 1,2$, and for any $\varpi \in [0,1]$, we find:

Letting $\mathcal{K}_1 = \sup_{\varpi\in[1,\mathcal{T}]} |\mathcal{F}(\varpi,0,0)| < \infty$ and $\mathcal{K}_2 = \sup_{\varpi\in[1,\mathcal{T}]} |\mathcal{G}(\varpi,0,0)| < \infty$, it follows by the assumption $(\mathcal{H}_1)$ that

$$|\mathcal{F}(\varpi,\mathcal{S},\mathcal{Z})| \leq \mathcal{L}_1(\|\mathcal{S}\| + \|\mathcal{Z}\|) + \mathcal{K}_1 \leq \mathcal{L}_1(\|\mathcal{S}\| + \|\mathcal{Z}\|) + \mathcal{K}_1,$$

and

$$|\mathcal{G}(\varpi,\mathcal{S},\mathcal{Z})| \leq \mathcal{L}_2(\|\mathcal{S}\| + \|\mathcal{Z}\|) + \mathcal{K}_2.$$

To begin, we show that $\Upsilon\mathcal{B}_\rho \subset \mathcal{B}_\rho$, where $\mathcal{B}_\rho = \{(\mathcal{S},\mathcal{Z}) \in \mathcal{E}\times\mathcal{E} : \|(\mathcal{S},\mathcal{Z}) \leq \rho\}$, with

$$\rho \geq \frac{(\Omega_1 + \bar{\Omega}_1)\mathcal{K}_1 + (\Omega_2 + \bar{\Omega}_2)\mathcal{K}_2}{1 - ((\Omega_1 + \bar{\Omega}_1)\mathcal{L}_1 + (\Omega_2 + \bar{\Omega}_2)\mathcal{L}_2)}. \tag{4.20}$$

For $(\mathcal{S},\mathcal{Z}) \in \mathcal{B}_\rho$, we have

$$\|\Upsilon_1(\mathcal{S},\mathcal{Z})\| = \sup_{\varpi\in[1,\mathcal{T}]} |\Upsilon_1(\mathcal{S},\mathcal{Z})(\varpi)|$$
$$\leq \frac{1}{\Delta}\Bigg[\Bigg\{\lambda_1 \int_1^{\mathcal{T}} \frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma - \lambda_2 \sum_{\mathrm{j}=1}^{\mathfrak{m}} \eta_{\mathrm{j}} \int_1^{\xi_{\mathrm{j}}} \frac{(\mathcal{Z})\varsigma}{\varsigma}d\varsigma - \lambda_2 \sum_{\mathrm{i}=1}^{\mathfrak{n}} \theta_{\mathrm{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathrm{i}}} \int_1^{\zeta_{\mathrm{i}}} \frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma$$

$$
\begin{aligned}
&-\lambda_2\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\int_1^{\mu_{\mathfrak{k}}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma+\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\xi_{\mathfrak{j}}}\Big(\log\frac{\xi_{\mathfrak{j}}}{\varsigma}\Big)^{\alpha_2-1}\frac{|\mathcal{G}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\zeta_{\mathfrak{i}}}\Big(\log\frac{\zeta_{\mathfrak{i}}}{\varsigma}\Big)^{\alpha_2-1}\frac{|\mathcal{G}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}\frac{1}{\Gamma(\alpha_2)}\int_1^{\mu_{\mathfrak{k}}}\Big(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\Big)^{\alpha_2-1}\frac{|\mathcal{G}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_1-1}\frac{|\mathcal{F}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma\Bigg\}\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\}\\
&+\Bigg\{\lambda_2\int_1^{\mathcal{T}}\frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma-\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1\int_1^{\psi_{\mathfrak{u}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1\int_1^{\sigma_{\mathfrak{v}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma\\
&-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1\int_1^{\pi_{\mathfrak{w}}}\frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\psi_{\mathfrak{u}}}\Big(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1}\frac{|\mathcal{F}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\sigma_{\mathfrak{v}}}\Big(\log\frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1}\frac{|\mathcal{F}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma\\
&+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{1}{\Gamma(\alpha_1)}\int_1^{\pi_{\mathfrak{w}}}\Big(\log\frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1}\frac{|\mathcal{F}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\alpha_2)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1}\frac{|\mathcal{G}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma\Bigg\}\\
&\times\Bigg\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{i}}(\log\xi_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log\zeta_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})^{\gamma_2-1}\Bigg\}\Bigg]\\
&-\lambda_1\int_1^{\varpi}\frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma+\frac{1}{\Gamma(\alpha_1)}\int_1^{\varpi}\left(\log\frac{\varpi}{\varsigma}\right)^{\alpha_1-1}\frac{|\mathcal{F}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|\Upsilon_1(\mathcal{S},\mathcal{Z})\|\leq&(\mathcal{L}_1\rho+\mathcal{K}_1)\Bigg\{\frac{1}{\Delta}\Bigg[\lambda_1(\log\mathcal{T})+\frac{(\log\mathcal{T})^{\alpha_1}}{\Gamma(\alpha_1+1)}\Bigg]\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\}\\
&+\Bigg\{\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\lambda_1(\log\psi_{\mathfrak{u}})-\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1(\log\sigma_{\mathfrak{v}})-\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1(\log\pi_{\mathfrak{w}})\\
&+\sum_{\mathfrak{u}=1}^{\mathfrak{a}}\mathcal{P}_{\mathfrak{u}}\frac{(\log\psi_{\mathfrak{u}})^{\alpha_1}}{\Gamma(\alpha_1+1)}+\sum_{\mathfrak{v}=1}^{\mathfrak{b}}\mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\frac{(\log\sigma_{\mathfrak{v}})^{\alpha_1}}{\Gamma(\alpha_1+1)}+\sum_{\mathfrak{w}=1}^{\mathfrak{c}}\mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\frac{(\pi_{\mathfrak{w}})^{\alpha_1}}{\Gamma(\alpha_1+1)}\Bigg\}\\
&+\Bigg\{\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{i}}(\log\xi_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log\zeta_{\mathfrak{i}})^{\gamma_2-1}+\sum_{\mathfrak{k}=1}^{\mathfrak{r}}\lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log\mu_{\mathfrak{k}})^{\gamma_2-1}\Bigg\}\\
&-\lambda_1(\log\mathcal{T})+\frac{(\log\mathcal{T})^{\alpha_1}}{\Gamma(\alpha_1+1)}\Bigg\}+\Bigg\{\frac{1}{\Delta}\Bigg[\Bigg\{-\lambda_2\sum_{\mathfrak{j}=1}^{\mathfrak{m}}\eta_{\mathfrak{j}}(\log\xi_{\mathfrak{j}})
\end{aligned}
$$

$$- \lambda_2 \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}}(\log \zeta_{\mathfrak{i}}) - \lambda_2 \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}}(\log \mu_{\mathfrak{k}}) + \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \frac{(\log \xi_{\mathfrak{j}})^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} \frac{(\log \zeta_{\mathfrak{i}})^{\alpha_2}}{\Gamma(\alpha_2+1)}$$

$$+ \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} \frac{(\log \mu_{\mathfrak{k}})^{\alpha_2}}{\Gamma(\alpha_2+1)} \Bigg\} \Bigg\{ (\log \mathcal{T})^{\gamma_2 - 1} \Bigg\} + \Bigg\{ \lambda_2 (\log \mathcal{T}) - \frac{(\log \mathcal{T})^{\alpha_2}}{\Gamma(\alpha_2+1)} \Bigg\} \Bigg] \Bigg\}.$$

Using the notation (3.4)-(3.5), we get

$$\|\Upsilon_1(\mathcal{S},\mathcal{Z})\| \leq (\mathcal{L}_1\Omega_1 + \mathcal{L}_2\Omega_2) + \Omega_1\mathcal{K}_1 + \Omega_2\mathcal{K}_2. \tag{4.21}$$

Likewise, we can find that

$$\|\Upsilon_2(\mathcal{S},\mathcal{Z})\| \leq (\mathcal{L}_1\bar{\Omega}_1 + \mathcal{L}_2\bar{\Omega}_2) + \bar{\Omega}_1\mathcal{K}_1 + \bar{\Omega}_2\mathcal{K}_2. \tag{4.22}$$

Then, it follows from (4.21)-(4.22) that

$$\|\Upsilon(\mathcal{S},\mathcal{Z})\| = \|\Upsilon_1(\mathcal{S},\mathcal{Z})\| + \|\Upsilon_2(\mathcal{S},\mathcal{Z})\| \leq \rho.$$

Therefore, $\Upsilon\mathcal{B}_\rho \subset \mathcal{B}_\rho$ as $(\mathcal{S},\mathcal{Z}) \in \mathcal{B}_\rho$ is an arbitrary element.

In order to verify that the operator Υ is a contraction, let $\mathcal{S}_i, \mathcal{Z}_i \in \mathcal{B}_\rho, i = 1, 2$. Then, we get

$$\|\Upsilon_1(\mathcal{S}_1,\mathcal{Z}_1) - \Upsilon_1(\mathcal{S}_1,\mathcal{Z}_1)\|$$

$$\leq \frac{1}{\Delta} \Bigg[\Bigg\{ \lambda_1 \int_1^{\mathcal{T}} \frac{\mathcal{S}(\varsigma)}{\varsigma} d\varsigma - \lambda_2 \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \int_1^{\xi_{\mathfrak{j}}} \frac{(\mathcal{Z})\varsigma}{\varsigma} d\varsigma - \lambda_2 \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} \int_1^{\zeta_{\mathfrak{i}}} \frac{\mathcal{Z}(\varsigma)}{\varsigma} d\varsigma$$

$$- \lambda_2 \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} \int_1^{\mu_{\mathfrak{k}}} \frac{\mathcal{Z}(\varsigma)}{\varsigma} d\varsigma$$

$$+ \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\xi_{\mathfrak{j}}} \Big(\log \frac{\xi_{\mathfrak{j}}}{\varsigma} \Big)^{\alpha_2 - 1} \frac{|\mathcal{G}(\varsigma, \mathcal{S}_1(\varsigma), \mathcal{Z}_1(\varsigma)) - \mathcal{G}(\varsigma, \mathcal{S}_2(\varsigma), \mathcal{Z}_2(\varsigma))|}{\varsigma} d\varsigma$$

$$+ \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\zeta_{\mathfrak{i}}} \Big(\log \frac{\zeta_{\mathfrak{i}}}{\varsigma} \Big)^{\alpha_2 - 1} \frac{|\mathcal{G}(\varsigma, \mathcal{S}_1(\varsigma), \mathcal{Z}_1(\varsigma)) - \mathcal{G}(\varsigma, \mathcal{S}_2(\varsigma), \mathcal{Z}_2(\varsigma))|}{\varsigma} d\varsigma$$

$$+ \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\mu_{\mathfrak{k}}} \Big(\log \frac{\mu_{\mathfrak{k}}}{\varsigma} \Big)^{\alpha_2 - 1} \frac{|\mathcal{G}(\varsigma, \mathcal{S}_1(\varsigma), \mathcal{Z}_1(\varsigma)) - \mathcal{G}(\varsigma, \mathcal{S}_2(\varsigma), \mathcal{Z}_2(\varsigma))|}{\varsigma} d\varsigma$$

$$- \frac{1}{\Gamma(\alpha_1)} \int_1^{\mathcal{T}} \Big(\log \frac{\mathcal{T}}{\varsigma} \Big)^{\alpha_1 - 1} \frac{|\mathcal{F}(\varsigma, \mathcal{S}_1(\varsigma), \mathcal{Z}_1(\varsigma)) - \mathcal{F}(\varsigma, \mathcal{S}_2(\varsigma), \mathcal{Z}_2(\varsigma))|}{\varsigma} d\varsigma \Bigg\} \Bigg\{ (\log \mathcal{T})^{\gamma_2 - 1} \Bigg\}$$

$$+ \Bigg\{ \lambda_2 \int_1^{\mathcal{T}} \frac{\mathcal{Z}(\varsigma)}{\varsigma} d\varsigma - \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \lambda_1 \int_1^{\psi_{\mathfrak{u}}} \frac{(\mathcal{S})\varsigma}{\varsigma} d\varsigma - \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}} \lambda_1 \int_1^{\sigma_{\mathfrak{v}}} \frac{(\mathcal{S})\varsigma}{\varsigma} d\varsigma$$

$$- \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}} \lambda_1 \int_1^{\pi_{\mathfrak{w}}} \frac{(\mathcal{S})\varsigma}{\varsigma} d\varsigma$$

$$+ \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \frac{1}{\Gamma(\alpha_1)} \int_1^{\psi_{\mathfrak{u}}} \Big(\log \frac{\psi_{\mathfrak{u}}}{\varsigma} \Big)^{\alpha_1 - 1} \frac{|\mathcal{F}(\varsigma, \mathcal{S}_1(\varsigma), \mathcal{Z}_1(\varsigma)) - \mathcal{F}(\varsigma, \mathcal{S}_2(\varsigma), \mathcal{Z}_2(\varsigma))|}{\varsigma} d\varsigma$$

$$+\sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}} \frac{1}{\Gamma(\alpha_1)} \int_1^{\sigma_{\mathfrak{v}}} \left(\log \frac{\sigma_{\mathfrak{v}}}{\varsigma}\right)^{\alpha_1-1} \frac{|\mathcal{F}(\varsigma, \mathcal{S}_1(\varsigma), \mathcal{Z}_1(\varsigma)) - \mathcal{F}(\varsigma, \mathcal{S}_2(\varsigma), \mathcal{Z}_2(\varsigma))|}{\varsigma} d\varsigma$$
$$+\sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}} \frac{1}{\Gamma(\alpha_1)} \int_1^{\pi_{\mathfrak{w}}} \left(\log \frac{\pi_{\mathfrak{w}}}{\varsigma}\right)^{\alpha_1-1} \frac{|\mathcal{F}(\varsigma, \mathcal{S}_1(\varsigma), \mathcal{Z}_1(\varsigma)) - \mathcal{F}(\varsigma, \mathcal{S}_2(\varsigma), \mathcal{Z}_2(\varsigma))|}{\varsigma} d\varsigma$$
$$-\frac{1}{\Gamma(\alpha_2)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{\varsigma}\right)^{\alpha_2-1} \frac{|\mathcal{G}(\varsigma, \mathcal{S}_1(\varsigma), \mathcal{Z}_1(\varsigma)) - \mathcal{G}(\varsigma, \mathcal{S}_2(\varsigma), \mathcal{Z}_2(\varsigma))|}{\varsigma} d\varsigma \Bigg\}$$
$$\left\{\sum_{\mathrm{j}=1}^{\mathfrak{m}} \eta_{\mathrm{i}} (\log \xi_{\mathrm{i}})^{\gamma_2-1} + \sum_{\mathrm{i}=1}^{\mathfrak{n}} \theta_{\mathrm{i}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathrm{i}}} (\log \zeta_{\mathrm{i}})^{\gamma_2-1} + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} (\log \mu_{\mathfrak{k}})^{\gamma_2-1}\right\}\Bigg]$$
$$-\lambda_1 \int_1^{\varpi} \frac{\mathcal{S}(\varsigma)}{\varsigma} d\varsigma + \frac{1}{\Gamma(\alpha_1)} \int_1^{\varpi} \left(\log \frac{\varpi}{\varsigma}\right)^{\alpha_1-1} \frac{|\mathcal{F}(\varsigma, \mathcal{S}_1(\varsigma), \mathcal{Z}_1(\varsigma)) - \mathcal{F}(\varsigma, \mathcal{S}_2(\varsigma), \mathcal{Z}_2(\varsigma))|}{\varsigma} d\varsigma,$$

which, by $(\mathcal{H}_2)$, yields

$$\|\Upsilon_1(\mathcal{S}_1, \mathcal{Z}_1) - \Upsilon_1(\mathcal{S}_2, \mathcal{Z}_2)\| \leq (\Omega_1 \mathcal{L}_1 + \Omega_2 \mathcal{L}_2)[\|\mathcal{S}_1 - \mathcal{S}_2\| + \|\mathcal{Z}_1 - \mathcal{Z}_2\|]. \tag{4.23}$$

Similarly, we can observe that

$$\|\Upsilon_2(\mathcal{S}_1, \mathcal{Z}_1) - \Upsilon_2(\mathcal{S}_2, \mathcal{Z}_2)\| \leq (\bar{\Omega}_1 \mathcal{L}_1 + \bar{\Omega}_2 \mathcal{L}_2)[\|\mathcal{S}_1 - \mathcal{S}_2\| + \|\mathcal{Z}_1 - \mathcal{Z}_2\|]. \tag{4.24}$$

Consequently, it follows from (4.23) and (4.24) that

$$\begin{aligned} \|\Upsilon(\mathcal{S}_1, \mathcal{Z}_1) - \Upsilon(\mathcal{S}_2, \mathcal{Z}_2)\| &= \|\Upsilon_1(\mathcal{S}_1, \mathcal{Z}_2) - \Upsilon_1(\mathcal{S}_1, \mathcal{Z}_2)\| + \|\Upsilon_2(\mathcal{S}_1, \mathcal{Z}_2) - \Upsilon_2(\mathcal{S}_1, \mathcal{Z}_2)\| \\ &\leq [(\Omega_1 + \bar{\Omega}_1)\mathcal{L}_1 + (\Omega_2 + \bar{\Omega}_2)\mathcal{L}_2][\|\mathcal{S}_1 - \mathcal{S}_2\| + \|\mathcal{Z}_1 - \mathcal{Z}_2\|], \end{aligned} \tag{4.25}$$

and by condition (4.15), it follows that Υ is a contraction. Consequently, the operator Υ possesses a unique fixed point as a direct application of the Banach fixed point theorem. Thus, there exists a unique solution for the problems (1.1) and (1.2) on $\mathcal{E}$. □

5. Hyers-Ulam stability of system

This section is devoted to the investigation of Hyers-Ulam stability for our proposed system. Consider the following inequality:

$$\begin{cases} ({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1,\beta_1} + \lambda_1 {}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1-1,\beta_1})\mathcal{S}(\varpi) - \mathcal{F}(\varpi, \mathcal{S}(\varpi), \mathcal{Z}(\varpi), \mathcal{I}^{\mathfrak{p}_1}\mathcal{S}(\varpi), \mathcal{I}^{\mathfrak{p}_2}\mathcal{Z}(\varpi)) \leq \varepsilon_1, & \varpi \in [1, \mathcal{T}], \\ ({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2,\beta_2} + \lambda_2 {}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2-1,\beta_2})\mathcal{Z}(\varpi) - \mathcal{G}(\varpi, \mathcal{S}(\varpi), \mathcal{Z}(\varpi), \mathcal{I}^{\mathfrak{q}_1}\mathcal{S}(\varpi), \mathcal{I}^{\mathfrak{q}_2}\mathcal{Z}(\varpi)) \leq \varepsilon_2, & \varpi \in [1, \mathcal{T}], \end{cases} \tag{5.1}$$

where $\varepsilon_1, \varepsilon_2$ are given two positive real numbers.

Definition 5.1. *Problem (1.1) is Hyers-Ulam stable if there exist $\Omega_i > 0, i = 1, 2, 3, 4$ such that for a given $\varepsilon_1, \varepsilon_2 > 0$ and for each solution $(\mathcal{S}, \mathcal{Z}) \in C([1, \mathcal{T}], \times\mathbb{R}^2)$ of inequality (5.1), there exists a solution $(\mathcal{S}^*, \mathcal{Z}^*) \in C([1, \mathcal{T}], \times\mathbb{R}^2)$ of problem (1.1) with*

$$\begin{cases} |\mathcal{S}(\varpi) - \mathcal{S}^*(\varpi)| \leq \Omega_1\varepsilon_1 + \Omega_2\varepsilon_2, & \varpi \in [1, \mathcal{T}], \\ |\mathcal{Z}(\varpi) - \mathcal{Z}^*(\varpi)| \leq \bar{\Omega}_1\varepsilon_1 + \bar{\Omega}_2\varepsilon_2, & \varpi \in [1, \mathcal{T}]. \end{cases} \tag{5.2}$$

Remark 5.1. *$(\mathcal{S},\mathcal{Z})$ is a solution of inequality (5.1) if there exist functions $\mathcal{Q}_i \in C([1,\mathcal{T}],\mathbb{R}), i = 1,2$, which depend upon $\mathcal{S},\mathcal{Z}$, respectively, such that*

$$i)|\mathcal{Q}_1(\varpi)| \leq \varepsilon_1, \quad ii)|\mathcal{Q}_2(\varpi)| \leq \varepsilon_2, \quad \varpi \in [1,\mathcal{T}]. \tag{5.3}$$

$$\begin{cases} ({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1,\beta_1} + \lambda_1 {}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1-1,\beta_1})\mathcal{S}(\varpi) = \mathcal{F}(\varpi,\mathcal{S}(\varpi),\mathcal{Z}(\varpi),\mathcal{I}^{\mathfrak{p}_1}\mathcal{S}(\varpi),\mathcal{I}^{\mathfrak{p}_2}\mathcal{Z}(\varpi)) + \mathcal{Q}_1(\varpi), & \varpi \in [1,\mathcal{T}], \\ ({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2,\beta_2} + \lambda_2 {}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2-1,\beta_2})\mathcal{Z}(\varpi) = \mathcal{G}(\varpi,\mathcal{S}(\varpi),\mathcal{Z}(\varpi),\mathcal{I}^{\mathfrak{q}_1}\mathcal{S}(\varpi),\mathcal{I}^{\mathfrak{q}_2}\mathcal{Z}(\varpi)) + \mathcal{Q}_2(\varpi), & \varpi \in [1,\mathcal{T}], \end{cases} \tag{5.4}$$

Remark 5.2. *If $(\mathcal{S},\mathcal{Z})$, respectively, is a solution of inequality (5.1), then $(\mathcal{S},\mathcal{Z})$ is a solution of the following inequality:*

$$\begin{cases} |\mathcal{S}(\varpi) - \mathcal{S}^*(\varpi)| \leq \Omega_1\varepsilon_1 + \Omega_2\varepsilon_2, & \varpi \in [1,\mathcal{T}], \\ |\mathcal{Z}(\varpi) - \mathcal{Z}^*(\varpi)| \leq \bar{\Omega}_1\varepsilon_1 + \bar{\Omega}_2\varepsilon_2, & \varpi \in [1,\mathcal{T}]. \end{cases} \tag{5.5}$$

As from Remark 5.1, we have

$$\begin{cases} ({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1,\beta_1} + \lambda_1 {}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1-1,\beta_1})\mathcal{S}(\varpi) \\ = \mathcal{F}(\varpi,\mathcal{S}(\varpi),\mathcal{Z}(\varpi),\mathcal{I}^{\mathfrak{p}_1}\mathcal{S}(\varpi),\mathcal{I}^{\mathfrak{p}_2}\mathcal{Z}(\varpi)) + \mathcal{Q}_1(\varpi), \quad \varpi \in [1,\mathcal{T}], \\ ({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2,\beta_2} + \lambda_2 {}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2-1,\beta_2})\mathcal{Z}(\varpi) \\ = \mathcal{G}(\varpi,\mathcal{S}(\varpi),\mathcal{Z}(\varpi),\mathcal{I}^{\mathfrak{q}_1}\mathcal{S}(\varpi),\mathcal{I}^{\mathfrak{q}_2}\mathcal{Z}(\varpi)) + \mathcal{Q}_2(\varpi), \quad \varpi \in [1,\mathcal{T}]. \end{cases} \tag{5.6}$$

With the help of Definition 5.1 and Remark 5.1, we verified Remark 5.2 in the following lines:

$$\begin{aligned} |\mathcal{S}(\varpi) - \mathcal{S}^*(\varpi)| \leq & \Bigg|\frac{1}{\Delta}\Bigg[\Bigg\{\lambda_1 \int_1^{\mathcal{T}} \frac{\mathcal{S}(\varsigma)}{\varsigma}d\varsigma - \lambda_2 \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \int_1^{\xi_{\mathfrak{j}}} \frac{(\mathcal{Z})\varsigma}{\varsigma}d\varsigma - \lambda_2 \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} \int_1^{\zeta_{\mathfrak{i}}} \frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma \\ & - \lambda_2 \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} \int_1^{\mu_{\mathfrak{k}}} \frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma + \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\xi_{\mathfrak{j}}} \Big(\log\frac{\xi_{\mathfrak{j}}}{\varsigma}\Big)^{\alpha_2-1} \frac{|\mathcal{G}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma \\ & + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\zeta_{\mathfrak{i}}} \Big(\log\frac{\zeta_{\mathfrak{i}}}{\varsigma}\Big)^{\alpha_2-1} \frac{|\mathcal{G}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma \\ & + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\mu_{\mathfrak{k}}} \Big(\log\frac{\mu_{\mathfrak{k}}}{\varsigma}\Big)^{\alpha_2-1} \frac{|\mathcal{G}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma \\ & - \frac{1}{\Gamma(\alpha_1)} \int_1^{\mathcal{T}} \Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_1-1} \frac{|\mathcal{F}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma\Bigg\}\Big\{(\log\mathcal{T})^{\gamma_2-1}\Big\} \\ & + \Bigg\{\lambda_2 \int_1^{\mathcal{T}} \frac{\mathcal{Z}(\varsigma)}{\varsigma}d\varsigma - \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}}\lambda_1 \int_1^{\psi_{\mathfrak{u}}} \frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma - \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}}\lambda_1 \int_1^{\sigma_{\mathfrak{v}}} \frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma \\ & - \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}} {}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}}\lambda_1 \int_1^{\pi_{\mathfrak{w}}} \frac{(\mathcal{S})\varsigma}{\varsigma}d\varsigma \\ & + \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \frac{1}{\Gamma(\alpha_1)} \int_1^{\psi_{\mathfrak{u}}} \Big(\log\frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1} \frac{|\mathcal{F}(\varsigma,\mathcal{S}(\varsigma),\mathcal{Z}(\varsigma))|}{\varsigma}d\varsigma \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}} \frac{1}{\Gamma(\alpha_1)} \int_1^{\sigma_{\mathfrak{v}}} \Big(\log \frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1} \frac{|\mathcal{F}(\varsigma, \mathcal{S}(\varsigma), \mathcal{Z}(\varsigma))|}{\varsigma} d\varsigma \\
&+\sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}} \frac{1}{\Gamma(\alpha_1)} \int_1^{\pi_{\mathfrak{w}}} \Big(\log \frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1} \frac{|\mathcal{F}(\varsigma, \mathcal{S}(\varsigma), \mathcal{Z}(\varsigma))|}{\varsigma} d\varsigma \\
&-\frac{1}{\Gamma(\alpha_2)} \int_1^{\mathcal{T}} \Big(\log \frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1} \frac{|\mathcal{G}(\varsigma, \mathcal{S}(\varsigma), \mathcal{Z}(\varsigma))|}{\varsigma} d\varsigma \Big\} \\
&\times \Big\{ \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{i}} (\log \xi_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} (\log \zeta_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} (\log \mu_{\mathfrak{k}})^{\gamma_2-1} \Big\} \Big] \\
&-\lambda_1 \int_1^{\varpi} \frac{\mathcal{S}(\varsigma)}{\varsigma} d\varsigma + \frac{1}{\Gamma(\alpha_1)} \int_1^{\varpi} \Big(\log \frac{\varpi}{\varsigma}\Big)^{\alpha_1-1} \frac{|\mathcal{F}(\varsigma, \mathcal{S}(\varsigma), \mathcal{Z}(\varsigma))|}{\varsigma} d\varsigma \Big| \\
\le& \Big|\frac{1}{\Delta}\Big[\Big\{ \lambda_1 \int_1^{\mathcal{T}} \frac{\mathcal{S}(\varsigma)}{\varsigma} d\varsigma - \lambda_2 \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \int_1^{\xi_{\mathfrak{j}}} \frac{(\mathcal{Z})\varsigma}{\varsigma} d\varsigma - \lambda_2 \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} \int_1^{\zeta_{\mathfrak{i}}} \frac{\mathcal{Z}(\varsigma)}{\varsigma} d\varsigma \\
&-\lambda_2 \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} \int_1^{\mu_{\mathfrak{k}}} \frac{\mathcal{Z}(\varsigma)}{\varsigma} d\varsigma + \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\xi_{\mathfrak{j}}} \Big(\log \frac{\xi_{\mathfrak{j}}}{\varsigma}\Big)^{\alpha_2-1} \frac{|\mathcal{Q}_2(\varsigma)|}{\varsigma} d\varsigma \\
&+\sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\zeta_{\mathfrak{i}}} \Big(\log \frac{\zeta_{\mathfrak{i}}}{\varsigma}\Big)^{\alpha_2-1} \frac{|\mathcal{Q}_2(\varsigma)|}{\varsigma} d\varsigma \\
&+\sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} \frac{1}{\Gamma(\alpha_2)} \int_1^{\mu_{\mathfrak{k}}} \Big(\log \frac{\mu_{\mathfrak{k}}}{\varsigma}\Big)^{\alpha_2-1} \frac{|\mathcal{Q}_2(\varsigma)|}{\varsigma} d\varsigma \\
&-\frac{1}{\Gamma(\alpha_1)} \int_1^{\mathcal{T}} \Big(\log \frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_1-1} \frac{|\mathcal{Q}_1(\varsigma)|}{\varsigma} d\varsigma \Big\} \Big\{ (\log \mathcal{T})^{\gamma_2-1} \Big\} \\
&+\Big\{ \lambda_2 \int_1^{\mathcal{T}} \frac{\mathcal{Z}(\varsigma)}{\varsigma} d\varsigma - \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \lambda_1 \int_1^{\psi_{\mathfrak{u}}} \frac{(\mathcal{S})\varsigma}{\varsigma} d\varsigma - \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}} \lambda_1 \int_1^{\sigma_{\mathfrak{v}}} \frac{(\mathcal{S})\varsigma}{\varsigma} d\varsigma \\
&-\sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}} \lambda_1 \int_1^{\pi_{\mathfrak{w}}} \frac{(\mathcal{S})\varsigma}{\varsigma} d\varsigma + \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \frac{1}{\Gamma(\alpha_1)} \int_1^{\psi_{\mathfrak{u}}} \Big(\log \frac{\psi_{\mathfrak{u}}}{\varsigma}\Big)^{\alpha_1-1} \frac{|\mathcal{Q}_1(\varsigma)|}{\varsigma} d\varsigma \\
&+\sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}} \frac{1}{\Gamma(\alpha_1)} \int_1^{\sigma_{\mathfrak{v}}} \Big(\log \frac{\sigma_{\mathfrak{v}}}{\varsigma}\Big)^{\alpha_1-1} \frac{|\mathcal{Q}_1(\varsigma)|}{\varsigma} d\varsigma \\
&+\sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}} \frac{1}{\Gamma(\alpha_1)} \int_1^{\pi_{\mathfrak{w}}} \Big(\log \frac{\pi_{\mathfrak{w}}}{\varsigma}\Big)^{\alpha_1-1} \frac{|\mathcal{Q}_1(\varsigma)|}{\varsigma} d\varsigma \\
&-\frac{1}{\Gamma(\alpha_2)} \int_1^{\mathcal{T}} \Big(\log \frac{\mathcal{T}}{\varsigma}\Big)^{\alpha_2-1} \frac{|\mathcal{Q}_2(\varsigma)|}{\varsigma} d\varsigma \Big\} \\
&\times \Big\{ \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{i}} (\log \xi_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} (\log \zeta_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} (\log \mu_{\mathfrak{k}})^{\gamma_2-1} \Big\} \Big] \\
&-\lambda_1 \int_1^{\varpi} \frac{\mathcal{S}(\varsigma)}{\varsigma} d\varsigma + \frac{1}{\Gamma(\alpha_1)} \int_1^{\varpi} \Big(\log \frac{\varpi}{\varsigma}\Big)^{\alpha_1-1} \frac{|\mathcal{Q}_1(\varsigma)|}{\varsigma} d\varsigma \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \varepsilon_1 \Bigg\{ \frac{1}{\Delta} \Bigg[\Bigg\{ \lambda_1 (\log \mathcal{T}) + \frac{(\log \mathcal{T})^{\alpha_1}}{\Gamma(\alpha_1+1)} \Bigg\} \Bigg\{ (\log \mathcal{T})^{\gamma_2 - 1} \Bigg\} \\
&+ \Bigg\{ \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \lambda_1 (\log \psi_{\mathfrak{u}}) - \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}} \lambda_1 (\log \sigma_{\mathfrak{v}}) - \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}} \lambda_1 (\log \pi_{\mathfrak{w}}) \\
&+ \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \frac{(\log \psi_{\mathfrak{u}})^{\alpha_1}}{\Gamma(\alpha_1+1)} + \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_{\mathfrak{v}}} \frac{(\log \sigma_{\mathfrak{v}})^{\alpha_1}}{\Gamma(\alpha_1+1)} + \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\vartheta_{\mathfrak{w}}} \frac{(\pi_{\mathfrak{w}})^{\alpha_1}}{\Gamma(\alpha_1+1)} \Bigg\} \\
&\times \Bigg\{ \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{i}} (\log \xi_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} (\log \zeta_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} (\log \mu_{\mathfrak{k}})^{\gamma_2-1} \Bigg\} \Bigg] \\
&- \lambda_1 (\log \mathcal{T}) + \frac{(\log \mathcal{T})^{\alpha_1}}{\Gamma(\alpha_1+1)} \Bigg\} + \varepsilon_2 \Bigg\{ \frac{1}{\Delta} \Bigg[\Bigg\{ -\lambda_2 \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} (\log \xi_{\mathfrak{j}}) - \lambda_2 \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} (\log \zeta_{\mathfrak{i}}) \\
&- \lambda_2 \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} (\log \mu_{\mathfrak{k}}) + \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \frac{(\log \xi_{\mathfrak{j}})^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} \frac{(\log \zeta_{\mathfrak{i}})^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&+ \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} \frac{(\log \mu_{\mathfrak{k}})^{\alpha_2}}{\Gamma(\alpha_2+1)} \Bigg\} \Bigg\{ (\log \mathcal{T})^{\gamma_2-1} \Bigg\} + \Bigg\{ \lambda_2 (\log \mathcal{T}) - \frac{(\log \mathcal{T})^{\alpha_2}}{\Gamma(\alpha_2+1)} \Bigg\} \Bigg\{ \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{i}} (\log \xi_{\mathfrak{i}})^{\gamma_2-1} \\
&+ \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\phi_{\mathfrak{i}}} (\log \zeta_{\mathfrak{i}})^{\gamma_2-1} + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}{}^{\mathcal{H}}\mathcal{D}_{1^+}^{\omega_{\mathfrak{k}}} (\log \mu_{\mathfrak{k}})^{\gamma_2-1} \Bigg\} \Bigg] \Bigg\},
\end{aligned}
$$

$$
|\mathcal{S}(\varpi) - \mathcal{S}^*(\varpi)| \leq \Omega_1 \varepsilon_1 + \Omega_2 \varepsilon_2. \tag{5.7}
$$

By the same method, we can obtain that

$$
|\mathcal{Z}(\varpi) - \mathcal{Z}^*(\varpi)| \leq \bar{\Omega}_1 \varepsilon_1 + \bar{\Omega}_2 \varepsilon_2, \tag{5.8}
$$

where $\Omega_1, \Omega_2, \bar{\Omega}_1, \bar{\Omega}_2$ are given by (3.4)–(3.7). Hence, Remark 5.2 is verified, with the help of (5.7) and (5.8). Thus, the nonlinear sequential coupled system of HHFDEs is Hyers-Ulam stable and, consequently, the system (1.1) is Hyers-Ulam stable.

6. Examples

Consider the following Hilfer-Hadamard fractional BVP:

$$
\begin{cases}
({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1,\beta_1} + \lambda_1 {}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_1-1,\beta_1}) \mathcal{S}(\varpi) = \mathcal{F}(\varpi, \mathcal{S}(\varpi), \mathcal{Z}(\varpi), \mathcal{I}^{\mathfrak{p}_1}\mathcal{S}(\varpi), \mathcal{I}^{\mathfrak{p}_2}\mathcal{Z}(\varpi)), \\
({}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2,\beta_2} + \lambda_2 {}^{\mathcal{HH}}\mathcal{D}_{1_+}^{\alpha_2-1,\beta_2}) \mathcal{Z}(\varpi) = \mathcal{G}(\varpi, \mathcal{S}(\varpi), \mathcal{Z}(\varpi), \mathcal{I}^{\mathfrak{q}_1}\mathcal{S}(\varpi), \mathcal{I}^{\mathfrak{q}_2}\mathcal{Z}(\varpi)), \\
\mathcal{S}(1) = 0, \quad \mathcal{S}(\mathcal{T}) = \displaystyle\sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \mathcal{Z}(\xi_{\mathfrak{j}}) + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}}{}^{\mathcal{H}}\mathcal{I}_1^{\phi_{\mathfrak{i}}} \mathcal{Z}(\zeta_{\mathfrak{i}}) + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}^{\mathcal{H}}\mathcal{D}_1^{\omega_{\mathfrak{k}}} \mathcal{Z}(\mu_{\mathfrak{k}}), \\
\mathcal{Z}(1) = 0, \quad \mathcal{Z}(\mathcal{T}) = \displaystyle\sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \mathcal{S}(\psi_{\mathfrak{u}}) + \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}}{}^{\mathcal{H}}\mathcal{I}_{1_+}^{\delta_{\mathfrak{v}}} \mathcal{S}(\sigma_{\mathfrak{v}}) + \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}^{\mathcal{H}}\mathcal{D}_{1_+}^{\vartheta_{\mathfrak{w}}} \mathcal{S}(\pi_{\mathfrak{w}}),
\end{cases} \tag{6.1}
$$

with $\alpha_1 = 5/4, \alpha_2 = 3/2, \beta_1 = 1/2, \beta_2 = 1/4, \mathfrak{m} = 2, \mathfrak{n} = 2, \mathfrak{r} = 2, \mathfrak{a} = 2, \mathfrak{b} = 2, \mathfrak{c} = 2, \eta_1 = 1/5, \eta_2 = 1/10, \xi_1 = 4/3, \xi_2 = 3/2, \phi_1 = 5/3, \phi_2 = 7/3, \theta_1 = 1/2, \theta_2 = 1/2, \zeta_1 = 7/3, \zeta_2 =$

$5/2, \lambda_1 = 1, \lambda_2 = 1, \omega_1 = 1/4, \omega_2 = 2/3, \mu_1 = 4, \mu_2 = 4/3, \mathcal{P}_1 = 1/18, \mathcal{P}_2 = 1/9, \psi_1 = 4/3, \psi_2 = 5/2, Q_1 = 1/4, Q_2 = 1/7, \delta_1 = 1/4, \delta_2 = 3/5, \sigma_1 = 5/3, \sigma_2 = 3/2, \mathcal{M}_1 = 2/3, \mathcal{M}_2 = 2/5, \vartheta_1 = 2/3, \vartheta_2 = 3/5, \pi_1 = 5/2, \pi_2 = 3/2$. Using the given data, it is found that $\gamma_1 = 13/8, \gamma_2 = 13/8$, $\Delta = 0.639100490745, \mathcal{A}_1 = 0.799441, \mathcal{A}_2 = 0.799441, \mathcal{B}_1 = 0.1184655, \mathcal{B}_2 = 0.1251315, \Omega_1 = 1.3283929, \Omega_2 = 0.718823345, \bar{\Omega}_1 = 1.028734, \bar{\Omega}_2 = 0.97432874, \mathcal{T} = 2, \mathfrak{p}_1 = 11/5, \mathfrak{p}_2 = 25/6, \mathfrak{q}_1 = 11/5, \mathfrak{q}_2 = 22/7$.

Example 6.1. *For illustrating Theorem 4.2, we take*

$$\begin{cases} |\mathcal{F}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4)| \leq \frac{\varpi}{\varpi^2+1}\Big(\cos\varpi + \frac{1}{8}\sin(\mathcal{S}_1 + \mathcal{S}_2)\Big) - \frac{1}{9(\varpi+1)}\mathcal{S}_3 + \frac{1}{9}\arctan\mathcal{S}_4, \\ |\mathcal{G}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4)| \leq \frac{1}{(\varpi+2)^2}\Big[7e^{-\varpi} + \frac{1}{3}\mathcal{S}_1 + 4\mathcal{S}_2\Big] - \frac{\varpi+3}{5}\sin(\mathcal{S}_3 + \mathcal{S}_4), \end{cases} \tag{6.2}$$

for all $\varpi \in [0, 1], \mathcal{S}_i \in \mathbb{R}, i = 1, 2, 3, 4$.

We obtained the inequalities

$$\begin{cases} |\mathcal{F}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4)| = \frac{1}{2} + \frac{1}{16}|\mathcal{S}_1| + \frac{1}{16}|\mathcal{S}_2| + \frac{1}{18}|\mathcal{S}_3| + \frac{1}{9}|\mathcal{S}_4|, \\ |\mathcal{G}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4)| = \frac{7}{9} + \frac{1}{27}|\mathcal{S}_1| + \frac{4}{27}|\mathcal{S}_2| + \frac{4}{5}|\mathcal{S}_3| + \frac{4}{5}|\mathcal{S}_4|, \end{cases} \tag{6.3}$$

for all $\varpi \in [0, 1]$ *and* $\mathcal{S}_i \in \mathbb{R}, i = 1, 2, 3, 4$. *We also have* $\mathfrak{M}_0 = \frac{1}{2}, \mathfrak{M}_1 = \frac{1}{16}, \mathfrak{M}_2 = \frac{1}{16}, \mathfrak{M}_3 = \frac{1}{18}, \mathfrak{M}_4 = \frac{1}{9}, \mathfrak{N}_0 = \frac{7}{9}, \mathfrak{N}_1 = \frac{1}{27}, \mathfrak{N}_2 = \frac{4}{27}, \mathfrak{N}_3 = \frac{4}{5}, \mathfrak{N}_4 = \frac{4}{5}$,

We find here $\Psi_1 \approx 0.8228588$ *and* $\Psi_2 \approx 0.5948088559$. *We deduce that the condition* $\max\{\Psi_1, \Psi_2\} = \Psi_1 < 1$ *is satisfied. Then, by Theorem 4.2, we conclude that the problem (6.1) with the nonlinearities (6.2) has at least one solution* $\varpi \in [0, 1]$.

Example 6.2. *For illustrating Theorem 4.3, we take*

$$\begin{cases} |\mathcal{F}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4)| \leq \frac{\varpi+1}{3} + \frac{1}{9(\varpi+2)}\Big(\mathcal{S}_1 + \frac{|\mathcal{S}_2|}{1+|\mathcal{S}_2|}\Big) - \frac{1}{(1+\varpi)^2}\cos\mathcal{S}_3 + \frac{\varpi}{4}\arctan\mathcal{S}_4, \\ |\mathcal{G}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4)| \leq \frac{\varpi^2+2}{\varpi^3+2} - \frac{1}{7}\mathcal{S}_1 + \frac{1}{8}\sin\mathcal{S}_2 + \frac{1}{5(\varpi+3)}\sin\mathcal{S}_3 - e^{-2\varpi}\frac{|\mathcal{S}_4|}{8(1+|\mathcal{S}_4|)}, \end{cases} \tag{6.4}$$

for all $\varpi \in [0, 1], \mathcal{S}_i \in \mathbb{R}, i = 1, 2, 3, 4$.

We obtain here the following inequalities

$$\begin{cases} & |\mathcal{F}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4) - \mathcal{F}(\varpi, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4)| \\ \leq & \Big(1/27|\mathcal{S}_1 - \mathcal{Z}_1| + 1/27|\mathcal{S}_2 - \mathcal{Z}_2| + 1/4|\mathcal{S}_3 - \mathcal{Z}_3| + 1/4|\mathcal{S}_4 - \mathcal{Z}_4|\Big), \\ & |\mathcal{G}(\varpi, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4) - \mathcal{G}(\varpi, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4)| \\ \leq & \Big(1/7|\mathcal{S}_1 - \mathcal{Z}_1| + 1/8|\mathcal{S}_2 - \mathcal{Z}_2| + 1/20|\mathcal{S}_3 - \mathcal{Z}_3| + 1/8|\mathcal{S}_4 - \mathcal{Z}_4|\Big), \end{cases} \tag{6.5}$$

for all $\varpi \in [0, 1]$. *So, we have* $\mathfrak{c}_0 = 1/4$ *and* $\mathfrak{d}_0 = 1/20$. *In addition, we find* $\rho_1 \approx 1.4125480, \rho_2 \approx 1.13889158, \mathcal{D}_1 \approx 0.510067622, \mathcal{D}_2 \approx 0.43933889$. *Then,* $\mathcal{D}_1 + \mathcal{D}_2 \approx 0.9410020109 < 1$*, that is, the condition (4.15) is satisfied. Therefore, by Theorem 4.3, we conclude that problem (6.1) with the nonlinearities (6.4) has a unique solution* $\varpi \in [0, 1]$.

7. Discussion & conclusions

We have presented criteria for the existence, uniqueness, and Ulam-Hyers stability of solutions to a coupled system of nonlinear sequential HHFIEs and nonlocal coupled Hadamard fractional integrodifferential and multipoint boundary conditions. We derive the expected results using a methodology that uses modern analytical tools. It is imperative to emphasize that the results offered in this specific context are novel and contribute to the corpus of existing literature on the topic. Furthermore, our results encompass cases where the system reduces to the boundary conditions of the form:

When $\eta_{\mathfrak{j}} = \mathcal{P}_{\mathfrak{u}} = 0$, then we get

$$\begin{cases} \mathcal{S}(1) = 0, \ \ \mathcal{S}(\mathcal{T}) = \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}} {}^{\mathcal{H}}\mathcal{I}^{\phi_{\mathfrak{i}}} \mathcal{Z}(\zeta_{\mathfrak{i}}) + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}^{\mathcal{H}} \mathcal{D}_{1}^{\omega_{\mathfrak{k}}} \mathcal{Z}(\mu_{\mathfrak{k}}), \\ \mathcal{Z}(1) = 0, \ \ \mathcal{Z}(\mathcal{T}) = \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}} {}^{\mathcal{H}}\mathcal{I}^{\delta_{\mathfrak{v}}} \mathcal{S}(\sigma_{\mathfrak{v}}) + \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}^{\mathcal{H}} \mathcal{D}_{1}^{\vartheta_{\mathfrak{w}}} \mathcal{S}(\pi_{\mathfrak{w}}). \end{cases}$$

If $\theta_{\mathfrak{i}} = \mathcal{Q}_{\mathfrak{v}} = 0$, we get:

$$\begin{cases} \mathcal{S}(1) = 0, \ \ \mathcal{S}(\mathcal{T}) = \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \mathcal{Z}(\xi_{\mathfrak{j}}) + \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}^{\mathcal{H}} \mathcal{D}_{1}^{\omega_{\mathfrak{k}}} \mathcal{Z}(\mu_{\mathfrak{k}}), \\ \mathcal{Z}(1) = 0, \ \ \mathcal{Z}(\mathcal{T}) = \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \mathcal{S}(\psi_{\mathfrak{u}}) + \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}^{\mathcal{H}} \mathcal{D}_{1}^{\vartheta_{\mathfrak{w}}} \mathcal{S}(\pi_{\mathfrak{w}}). \end{cases}$$

When $\lambda_{\mathfrak{k}} = \mathcal{M}_{\mathfrak{w}} = 0$, the outcome is:

$$\begin{cases} \mathcal{S}(1) = 0, \ \ \mathcal{S}(\mathcal{T}) = \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \mathcal{Z}(\xi_{\mathfrak{j}}) + \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}} {}^{\mathcal{H}}\mathcal{I}^{\phi_{\mathfrak{i}}} \mathcal{Z}(\zeta_{\mathfrak{i}}), \\ \mathcal{Z}(1) = 0, \ \ \mathcal{Z}(\mathcal{T}) = \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \mathcal{S}(\psi_{\mathfrak{u}}) + \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}} {}^{\mathcal{H}}\mathcal{I}^{\delta_{\mathfrak{v}}} \mathcal{S}(\sigma_{\mathfrak{v}}). \end{cases}$$

In addition, if $\eta_{\mathfrak{j}} = \mathcal{P}_{\mathfrak{u}} = \lambda_{\mathfrak{k}} = \mathcal{M}_{\mathfrak{w}} = 0$, we obtain:

$$\begin{cases} \mathcal{S}(1) = 0, \ \ \mathcal{S}(\mathcal{T}) = \sum_{\mathfrak{i}=1}^{\mathfrak{n}} \theta_{\mathfrak{i}} {}^{\mathcal{H}}\mathcal{I}^{\phi_{\mathfrak{i}}} \mathcal{Z}(\zeta_{\mathfrak{i}}), \\ \mathcal{Z}(1) = 0, \ \ \mathcal{Z}(\mathcal{T}) = \sum_{\mathfrak{v}=1}^{\mathfrak{b}} \mathcal{Q}_{\mathfrak{v}} {}^{\mathcal{H}}\mathcal{I}^{\delta_{\mathfrak{v}}} \mathcal{S}(\sigma_{\mathfrak{v}}). \end{cases}$$

When $\eta_{\mathfrak{j}} = \mathcal{P}_{\mathfrak{u}} = \theta_{\mathfrak{i}} = \mathcal{Q}_{\mathfrak{v}} = 0$, the boundary condition is:

$$\begin{cases} \mathcal{S}(1) = 0, \ \ \mathcal{S}(\mathcal{T}) = \sum_{\mathfrak{k}=1}^{\mathfrak{r}} \lambda_{\mathfrak{k}}^{\mathcal{H}} \mathcal{D}_{1}^{\omega_{\mathfrak{k}}} \mathcal{Z}(\mu_{\mathfrak{k}}), \\ \mathcal{Z}(1) = 0, \ \ \mathcal{Z}(\mathcal{T}) = \sum_{\mathfrak{w}=1}^{\mathfrak{c}} \mathcal{M}_{\mathfrak{w}}^{\mathcal{H}} \mathcal{D}_{1}^{\vartheta_{\mathfrak{w}}} \mathcal{S}(\pi_{\mathfrak{w}}). \end{cases}$$

If $\lambda_{\mathfrak{k}} = \mathcal{M}_{\mathfrak{w}} = \theta_{\mathfrak{i}} = \mathcal{Q}_{\mathfrak{v}} = 0$, we obtain:

$$\begin{cases} \mathcal{S}(1) = 0, \ \ \mathcal{S}(\mathcal{T}) = \sum_{\mathfrak{j}=1}^{\mathfrak{m}} \eta_{\mathfrak{j}} \mathcal{Z}(\xi_{\mathfrak{j}}), \\ \mathcal{Z}(1) = 0, \ \ \mathcal{Z}(\mathcal{T}) = \sum_{\mathfrak{u}=1}^{\mathfrak{a}} \mathcal{P}_{\mathfrak{u}} \mathcal{S}(\psi_{\mathfrak{u}}). \end{cases}$$

These cases represent new findings. Looking ahead, our future plans include extending this work to a coupled system of nonlinear sequential HHFIEs enhanced by the nonlocal coupled mixed integro-differential and discrete type boundary conditions. We also intend to investigate the multivalued analogue of the problem studied in this paper.

Author contributions

Subramanian Muthaiah: Developed the conceptualization and proposed the method, wrote–original draft, reviewed and edited the paper; Manigandan Murugesan: Developed the conceptualization and proposed the method, wrote–original draft, investigated, processed and provided examples; Muath Awadalla: Investigated, processed and provided examples, reviewed and edited the paper; Bundit Unyong: Developed the conceptualization and proposed the method, reviewed and edited the paper; Ria H Egami: Investigated, processed and provided examples, reviewed and edited the paper. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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