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Research article

Global bounded solution of a 3D chemotaxis-Stokes system with slow p-Laplacian diffusion and logistic source

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Abstract: In this paper, the chemotaxis-Stokes system with slow p-Laplacian diffusion and logistic source as follows

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (|\nabla n|^{p-2}\nabla n) - \nabla \cdot (n\nabla c) + \mu n(1-n), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - cn, & x \in \Omega, t > 0, \\ u_t + \nabla P = \Delta u + n\nabla\Phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0 \end{cases}$$

was considered in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary under homogeneous Neumann-Neumann-Dirichlet boundary conditions. Subject to the effect of logistic source, we proved the system exists a global bounded weak solution for any $p > 2$.

Keywords: boundedness; p-Laplacian diffusion; logistic source; chemotaxis-Stokes
Mathematics Subject Classification: 35K55, 35B35

1. Introduction

In 1970, Keller and Segel first proposed the classical chemotaxis model in [10], which explained the phenomenon of cellular slime mold aggregation in response to a chemical signal of increased concentration. The mathematical expression of the classical chemotaxis model with consumption is as follows:

$$\begin{cases} n_t = \Delta n - \chi\nabla \cdot (n\nabla c), \\ c_t = \Delta c - cn. \end{cases} \tag{1.1}$$

After extensive research conducted by mathematicians, this model has produced excellent results, and relevant research results on the properties of solutions can be consulted in [1, 12, 14, 20, 22, 26, 31].

Nevertheless, the interaction between chemicals and cells in their surroundings is evident from a variety of research and is unavoidable. Therefore, Tuval et al. [23] constructed the chemotaxis-fluid system with consumption in order to characterize that aerobic bacterial populations are suspended in sessile water droplets as follows:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t + u \cdot \nabla c = \Delta c - nf(c), \\ u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\Phi, \\ \nabla \cdot u = 0. \end{cases} \tag{1.2}$$

The function n indicates the density of bacteria and c represents the concentration of oxygen, respectively. The fluid velocity field is denoted by u, and the rate at which bacteria consumes substrate is expressed by the function $f(c)$. P is the associated pressure, and Φ is a function that represents potential, while the strength of nonlinear fluid convection is measured by $\kappa \in \mathbb{R}$. In 2010, Lorz [17] first obtained the local existence result in a bounded domain. In two-dimensional domains, according to [4], the classical solution exists globally with the small initial datum, and Winkler [29] established that classical solutions exist globally for the chemotaxis-Navier-Stokes model. Even some scholars have researched that the scalar chemotactic sensitivity function $\chi(c)$ is replaced by the matrix function $S(x, n, c) \in C^2(\bar{\Omega} \times [0, \infty)^2; \mathbb{R}^{3\times3})$ in the system (1.2), and we can make reference to [3, 9, 15, 24, 32].

Consulting some chemotaxis literature [2, 11], we know that the random movement of bacteria appears to be enhanced in close proximity to high concentrations, owing to the limited size of the bacteria. Therefore, it is natural to investigate nonlinear diffusion. Some researchers conclude that the solution is bounded when the diffusion parameters meet certain conditions for the 3D chemotaxis-fluid model with nonlinear diffusion in [5, 25, 27, 28]. We discuss one of the forms of nonlinear diffusion known as p-Laplacian diffusion. Regarding the chemotaxis-fluid model with p-Laplacian diffusion, mathematicians have conducted research on various biological populations within three-dimensional space. Liu [16] studied a three-dimensional chemotaxis-Stokes model describing coral fertilization with arbitrarily slow p-Laplacian diffusion, and it is demonstrated that the global boundedness of solutions exists whenever $p > 2$. Han and Liu [6] investigated a 3D chemotaxis-Navier-Stokes system involving two species and p-Laplacian diffusion within smooth bounded domains and proved that if $p > 2$, the model admits a global weak solution. Tao and Li [19] changed the system (1.2) by setting $\chi(c) = \chi$ and replacing the Δn term with p-Laplacian diffusion $\nabla \cdot (|\nabla n|^{p-2}\nabla n)$, investigated the subsequent model

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (|\nabla n|^{p-2}\nabla n) - \chi\nabla \cdot (n\nabla c), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - cn, & x \in \Omega, t > 0, \\ u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\Phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0 \end{cases} \tag{1.3}$$

in a three-dimensional bounded domain Ω, and they proved that there exists the weak solution under the assumption that $p > \frac{32}{15}$. Subsequently, let $\chi = 1$ and $\kappa = 0$ in (1.3), and it becomes a chemotaxis-Stokes system; they improved the result presented in [18]. It indicates that for any $p > \frac{23}{11} (\approx 2.09091)$, the system exists global bounded weak solutions. In a recent literature by Jin [8], the range of values has been extended to $p > p^*(\approx 2.01247)$.

As we all know, the classical chemotaxis model (1.1), which contains the logistic source term that promotes the boundedness of solutions, can successfully inhibit the bacterial aggregation effect. Does the chemotaxis-fluid model retain this property? The answer is yes. In the three-dimensional domain, Lankeit [13] investigated the chemotaxis-Navier-Stokes model with a logistic growth term and demonstrated that the weak solution eventually becomes smooth and converges to a steady state after some waiting time. The 3D chemotaxis-Stokes model, which involves porous diffusion and the logistic source, was recently considered by Yang and Jin [30]. They demonstrated that under the large time limit, the solutions converge to the constant steady state and proved the boundedness of the weak solution with $m > 1, 0 < \alpha < 2m - 1$. Captured by the papers above, a natural question struck us: Does a bounded weak solution exist for any slow p-Laplacian diffusion model with a logistic source? Hence, this work considers the chemotaxis-Stokes model

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (|\nabla n|^{p-2}\nabla n) - \nabla \cdot (n\nabla c) + \mu n(1-n), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - cn, & x \in \Omega, t > 0, \\ u_t + \nabla P = \Delta u + n\nabla\Phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \\ \dfrac{\partial n}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = 0, u = 0, & x \in \partial\Omega, t > 0, \\ n(x,0) = n_0(x), c(x,0) = c_0(x), u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

which includes slow p-Laplacian diffusion and a logistic growth term, inside a bounded domain $\Omega \subset \mathbb{R}^3$. The classical logistic term $\mu n(1-n)$ denotes the rate at which cells proliferate or die and, here, the parameter $\mu > 0$ and the function $\Phi \in W^{1,\infty}(\Omega)$.

Furthermore, it is assumed that every triple (n_0, c_0, u_0) of initial data satisfies

$$\begin{cases} n_0 \in C(\bar{\Omega}) \text{ and } n_0 > 0, \\ c_0 \in W^{2,s}(\Omega), \forall s > 1, \text{ and } c_0 \geq 0, \\ u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \text{ and } \mathrm{div} u_0 = 0. \end{cases} \tag{1.5}$$

Ahead of presenting the main result, it is necessary to give a concise definition for global weak solutions.

Definition 1.1. *If* $(n, c, u) \in X_1 \times X_2 \times X_3$, $\nabla \cdot u = 0$, *such that for arbitrary testing functions* $\phi_1, \phi_2 \in C_0^\infty(\bar{\Omega} \times [0, \infty))$, $\psi \in C_0^\infty(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$, *and* $\nabla \cdot \psi = 0$, *the equalities*

$$\begin{aligned} &\int_0^\infty \int_\Omega n\phi_{1t} dxdt + \int_\Omega n_0\phi_1(\cdot, 0)dx + \int_0^\infty \int_\Omega n\nabla c \cdot \nabla\phi_1 dxdt + \int_0^\infty \int_\Omega nu \cdot \nabla\phi_1 dxdt \\ &\quad + \mu \int_0^\infty \int_\Omega \phi_1 n(1-n)dxdt - \int_0^\infty \int_\Omega |\nabla n|^{p-2}\nabla n \cdot \nabla\phi_1 dxdt = 0, \end{aligned} \tag{1.6}$$

$$\begin{aligned} &\int_0^\infty \int_\Omega c\phi_{2t} dxdt + \int_\Omega c_0\phi_2(\cdot, 0)dx + \int_0^\infty \int_\Omega cu \cdot \nabla\phi_2 dxdt - \int_0^\infty \int_\Omega \nabla c \cdot \nabla\phi_2 dxdt \\ &\quad - \int_0^\infty \int_\Omega nc\phi_2 dxdt = 0, \end{aligned} \tag{1.7}$$

$$\int_0^\infty \int_\Omega u \cdot \psi_t dxdt + \int_\Omega u_0 \psi(\cdot,0)dx - \int_0^\infty \int_\Omega \nabla u \cdot \nabla \psi dxdt + \int_0^\infty \int_\Omega n \nabla \Phi \cdot \psi dxdt = 0 \tag{1.8}$$

hold, then we call (n, c, u) *a global weak solution of the model* (1.4)*, satisfying the initial conditions* (1.5)*. Here,*

$$\begin{aligned} X_1 &= \Big\{ n \in L^\infty(\bar{\Omega} \times [0,\infty)), n \geq 0; \nabla n \in L^\infty(\mathbb{R}^+; L^p(\Omega)), n_t \in L^2_{loc}([0,\infty); L^2(\Omega)), \\ &\qquad n^{-\frac{1}{p}} \nabla n \in L^p_{loc}([0,\infty); L^p(\Omega)) \Big\}, \\ X_2 &= \Big\{ c \in L^\infty(\mathbb{R}^+; W^{1,\infty}(\Omega)), c \geq 0; c_t, \nabla^2 c \in L^s_{loc}([0,\infty); L^s(\Omega)), \forall s > 1 \Big\}, \\ X_3 &= \Big\{ u \in L^\infty(\bar{\Omega} \times [0,\infty)); A^\beta u \in L^\infty([0,\infty); L^2(\Omega)), \forall \beta \in (\frac{3}{4}, 1), u_t, \nabla^2 u \in L^2_{loc}([0,\infty); L^2(\Omega)) \Big\}. \end{aligned}$$

Based on the Definition 1.1 and previous assumptions (1.5), our major result is given below.

Theorem 1.1. *If* $p > 2$*, then the system* (1.4) *with the initial conditions* (1.5) *exists global weak solutions* (n, c, u) *in the bounded domain* $\Omega \subset \mathbb{R}^3$*, such that for all* $\beta \in (\frac{3}{4}, 1)$, $s > 1$*, satisfying*

$$\begin{aligned} &\sup_{t\in(0,+\infty)} \Big(\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla n(\cdot,t)\|_{L^p(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|A^\beta u(\cdot,t)\|_{L^2(\Omega)} \Big) \leq M_1, \\ &\sup_{t\in(0,+\infty)} \Big(\|\nabla n^{\frac{p-1}{p}}\|_{L^p(Q(t))} + \|n_t\|_{L^2(Q(t))} + \|c\|_{W^{2,1}_s(Q(t))} + \|u\|_{W^{2,1}_s(Q(t))} \Big) \leq M_2, \end{aligned} \tag{1.9}$$

where positive constants M_1, M_2 *only depend on* n_0, c_0, u_0, p, Ω*. Here* $Q(t) = \Omega \times (t, t+1)$*.*

The subsequent sections of this work are structured in the following approach. Section 2 will present a set of fundamental lemmas. Section 3 will provide an analysis of the energy estimate to demonstrate that the solution to the approximation problem exists globally and is bounded. Subsequently, we give the proof of Theorem 1.1 and the boundedness of global weak solutions by utilizing the approximation method.

2. Preliminaries

Within the section, we present fundamental conclusions that will be referenced multiple times in the subsequent sections of the work.

Lemma 2.1. *[7, Lemma 2.4] Suppose* $a > 0, \delta \geq 0, b \geq 0, T > 0$ *and* $\tau \in (0, T)$*. Let* $g : [0, T) \to [0, \infty)$ *be an absolutely continuous function satisfying*

$$g'(t) + a g^{1+\delta}(t) \leq h(t) \quad \text{for } t \in \mathbb{R},$$

where $h(t) \in L^1_{loc}([0, T))$ *is a nonnegative function, and for all* $t \in [\tau, T)$ *fulfills*

$$\int_{t-\tau}^t h(s)ds \leq b.$$

Then we have

$$\sup_{t\in(0,T)} g(t) + a \sup_{t\in(\tau,T)} \int_{t-\tau}^t g^{1+\delta}(s)ds \leq b + 2\max\left\{ g(0) + b + a\tau, \frac{b}{a\tau} + 1 + 2b + 2a\tau \right\}.$$

Lemma 2.2. *[7, Lemma 2.5] Let $\omega_0 \in W^{2,q}(\Omega)$, $h \in L^q_{loc}((0,+\infty); L^q(\Omega))$, $\forall 1 < q < \infty$, and one can find a fixed positive constant τ such that*

$$\sup_{t\in(\tau,+\infty)} \int_{t-\tau}^{t} \|h\|^q_{L^q}\, ds \le A.$$

Then the system

$$\begin{cases} \omega_t - \Delta\omega + \omega = h(x,t), \\ \dfrac{\partial \omega}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \omega(x,0) = \omega_0(x) \end{cases}$$

exists a unique solution $\omega \in L^q_{loc}((0,+\infty); W^{2,q}(\Omega))$, $\omega_t \in L^q_{loc}((0,+\infty); L^q(\Omega))$ such that

$$\sup_{t\in(\tau,+\infty)} \int_{t-\tau}^{t} \left(\|\omega_t\|^q_{L^q} + \|\omega\|^q_{W^{2,q}} \right) ds \le AM \frac{e^{q\tau}}{e^{\frac{q}{2}\tau} - 1} + M e^{\frac{q}{2}\tau} \|\omega_0\|^q_{W^{2,q}},$$

where the constant M is independent of τ.

Lemma 2.3. *[30, Lemma 4.2] If Ω is a bounded domain and $\varphi \in C^2(\bar{\Omega})$ satisfying $\frac{\partial \varphi}{\partial \nu}\big|_{\partial\Omega} = 0$, then there exists an upper bound on the curvatures of Ω given by $\kappa > 0$, such that*

$$\frac{\partial |\nabla \varphi|^2}{\partial \nu} \le 2\kappa |\nabla \varphi|^2 \qquad \text{on } \partial\Omega.$$

Lemma 2.4. *[8, Lemma 2.3] If $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $\omega \in C^2(\bar{\Omega})$ satisfying $\frac{\partial \omega}{\partial \nu}\big|_{\partial\Omega} = 0$, then the following two inequalities hold:*

$$\int_\Omega \frac{|\nabla \omega|^4}{\omega^3} dx \le (2 + \sqrt{N})^2 \int_\Omega \omega |\nabla^2 \ln \omega|^2 dx, \tag{2.1}$$

$$\int_\Omega |\nabla \omega|^{b+2} dx \le (\sqrt{N} + b)^2 \|\omega\|^2_{L^\infty} \int_\Omega |\nabla \omega|^{b-2} |\nabla^2 \omega|^2 dx \qquad \text{for any } b \ge 2. \tag{2.2}$$

3. Global existence and boundedness of the weak solution

With the goal to get the weak solution under the Definition 1.1 in three dimensions, we concerned the following regularized problem with $\varepsilon \in (0,1)$:

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \nabla \cdot \left((|\nabla n_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla n_\varepsilon \right) - \nabla \cdot (n_\varepsilon \nabla c_\varepsilon) + \mu n_\varepsilon (1 - n_\varepsilon), & x \in \Omega, t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon n_\varepsilon, & x \in \Omega, t > 0, \\ u_{\varepsilon t} + \nabla P_\varepsilon = \Delta u_\varepsilon + n_\varepsilon \nabla \Phi_\varepsilon, & x \in \Omega, t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega, t > 0, \\ \dfrac{\partial n_\varepsilon}{\partial \nu} = \dfrac{\partial c_\varepsilon}{\partial \nu} = 0, u_\varepsilon = 0, & x \in \partial\Omega, t > 0, \\ n_\varepsilon(x,0) = n_{\varepsilon 0}(x), c_\varepsilon(x,0) = c_{\varepsilon 0}(x), u_\varepsilon(x,0) = u_{\varepsilon 0}(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where $n_{\varepsilon0}, c_{\varepsilon0}, u_{\varepsilon0} \in C^{2+\alpha}(\bar{\Omega}), \Phi_\varepsilon \in C^{1+\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times[0,+\infty))$ with

$$\left.\frac{\partial n_{\varepsilon0}}{\partial\nu}\right|_{\partial\Omega} = \left.\frac{\partial c_{\varepsilon0}}{\partial\nu}\right|_{\partial\Omega} = 0, \quad u_{\varepsilon0}|_{\partial\Omega} = 0,$$

$$n_{\varepsilon0} \to n_0, c_{\varepsilon0} \to c_0, u_{\varepsilon0} \to u_0, \nabla\Phi_\varepsilon \to \nabla\Phi \quad \text{strongly in } L^r \text{ for any } r \geq 1,$$

$$\|n_{\varepsilon0}\|_{L^\infty} + \|\nabla n_{\varepsilon0}\|_{L^p} + \|c_{\varepsilon0}\|_{W^{2,s}} + \|u_{\varepsilon0}\|_{W^{2,s}} + \|\nabla\Phi_\varepsilon\|_{L^\infty}$$
$$\leq 2(\|n_0\|_{L^\infty} + \|\nabla n_0\|_{L^p} + \|c_0\|_{W^{2,s}} + \|u_0\|_{W^{2,s}} + \|\nabla\Phi\|_{L^\infty}) \quad \text{for any } s > 1.$$

Firstly, to obtain our result, we recall the local existence result of the chemotaxis-Stokes model (1.4) as follows. The proof is similar to [18], thus we omit it.

Lemma 3.1. *Let $p > 2$, then there exists a maximal time $T_{max} \in (0,+\infty)$ and a unique nonnegative solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[0,T_{max}))$ of the model* (3.1). *Moreover, if $T_{max} < \infty$, then*

$$\lim_{t\nearrow T_{max}} \sup\Big(\|n_\varepsilon(\cdot,t)\|_{L^\infty} + \|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}} + \|A^\beta u_\varepsilon(\cdot,t)\|_{L^2}\Big) = \infty \quad \textit{for some } \beta \in (\frac{3}{4},1).$$

Subsequently, we derive some estimates by using the maximum principle and straightforward calculations. These estimates are fundamental in the demonstration of our result.

Lemma 3.2. *There exist nonnegative constants $C_1, C_2,$ and C_3 such that the solution of* (3.1) *fulfills*

$$\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t \|n_\varepsilon(\cdot,t)\|_{L^2}^2 ds + \sup_{t\in(0,T_{max})}\|n_\varepsilon(\cdot,t)\|_{L^1} \leq C_1, \tag{3.2}$$

$$\sup_{t\in(0,T_{max})}\|c_\varepsilon(\cdot,t)\|_{L^\infty} \leq \sup_{t\in(0,T_{max})}\|c_{\varepsilon0}(\cdot,t)\|_{L^\infty} \leq C_2, \tag{3.3}$$

$$\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t \Big(\|c_\varepsilon\|_{H^2}^2 + \|u_\varepsilon\|_{H^2}^2 + \|u_{\varepsilon t}\|_{L^2}^2 + \|\nabla P_\varepsilon\|_{L^2}^2\Big)\, ds + \sup_{t\in(0,T_{max})}\|u_\varepsilon(\cdot,t)\|_{H^1}^2 \leq C_3, \tag{3.4}$$

where C_1, C_3 are independent of $\varepsilon, T_{max}, \tau$, while C_2 is independent of ε and T_{max}.

Proof. By integrating the first equation in (3.1), we can derive

$$\frac{d}{dt}\int_\Omega n_\varepsilon dx + \mu\int_\Omega n_\varepsilon^2 dx = \mu\int_\Omega n_\varepsilon dx \leq \frac{\mu}{2}\int_\Omega n_\varepsilon^2 dx + \tilde{C}_1.$$

Shifting the terms gives

$$\frac{d}{dt}\int_\Omega n_\varepsilon dx + \frac{\mu}{2}\int_\Omega n_\varepsilon^2 dx \leq \tilde{C}_1.$$

Then employing Lemma 2.1, we deduce

$$\sup_{t\in(0,T_{max})}\|n_\varepsilon(\cdot,t)\|_{L^1} + \sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t \|n_\varepsilon(\cdot,t)\|_{L^2}^2 ds \leq C_1, \tag{3.5}$$

that is (3.2).

Then one can apply the maximum principle to $(3.1)_2$, and it can yield (3.3), which is the basic estimate of the solution component c_ε.

Regarding (3.4), we first multiply the second equation of (3.1) by c_ε and make use of an integration by parts, which implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c_\varepsilon^2 dx + \int_\Omega |\nabla c_\varepsilon|^2 dx \le 0,$$

collecting with (3.3), then we have

$$\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^{t} \|c_\varepsilon(\cdot,s)\|_{H^2}^2 ds \le \tilde{C}_2. \tag{3.6}$$

Furthermore, we test the third equation in (3.1) by $u_\varepsilon + u_{\varepsilon t}$ and use the integration by parts to obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega (u_\varepsilon^2 + |\nabla u_\varepsilon|^2)dx + \int_\Omega (|\nabla u_\varepsilon|^2 + |u_{\varepsilon t}|^2)dx\\ &\le \int_\Omega (n_\varepsilon \nabla\Phi_\varepsilon)(u_\varepsilon + u_{\varepsilon t})dx.\end{aligned}$$

Applying Poincaré's inequality and Hölder's inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_\varepsilon^2 + |\nabla u_\varepsilon|^2)dx + C\int_\Omega (u_\varepsilon^2 + |\nabla u_\varepsilon|^2)dx + \frac{1}{2}\|u_{\varepsilon t}\|_{L^2}^2 \le \tilde{C}_3\|n_\varepsilon\|_{L^2}^2,$$

substituting (3.5) into the above inequality, then we gain

$$\sup_{t\in(0,T_{max})}\|u_\varepsilon(\cdot,t)\|_{H^1}^2 + \sup_{t\in(\tau,T_{max})}\int_{t-\tau}^{t}\|u_{\varepsilon t}\|_{L^2}^2 ds \le \tilde{C}_4. \tag{3.7}$$

On the other hand, we first move the term of the third equation (3.1), which yields $-\Delta u_\varepsilon + \nabla P_\varepsilon = -u_{\varepsilon t} + n_\varepsilon\nabla\Phi_\varepsilon$, and use the L^2 theory of Stokes operator to obtain

$$\|u_\varepsilon\|_{H^2}^2 + \|\nabla P_\varepsilon\|_{L^2}^2 \le \tilde{C}_5(\|u_{\varepsilon t}\|_{L^2}^2 + \|n_\varepsilon\|_{L^2}^2). \tag{3.8}$$

By a combination of (3.5)–(3.7), we can derive (3.4).

Next, an energy inequality related to the variables n_ε and c_ε in the model (3.1) will be built.

Lemma 3.3. *Let $p > 2$, then there exists a constant $C > 0$, which is independent of ε, T_{max} and τ such that*

$$\begin{aligned}&\sup_{t\in(0,T_{max})}\int_\Omega\left(n_\varepsilon\ln n_\varepsilon + |\nabla c_\varepsilon|^2\right)dx\\ &+\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^{t}\int_\Omega\left(\frac{|\nabla n_\varepsilon|^p}{n_\varepsilon} + n_\varepsilon^2|\ln n_\varepsilon| + |\Delta c_\varepsilon|^2\right)dxds \le C.\end{aligned} \tag{3.9}$$

Proof. The first equation of (3.1) is multiplied by $K(1+\ln n_\varepsilon)$, where $K > 0$ is a constant to be determined, to obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega K n_\varepsilon\ln n_\varepsilon dx + K\int_\Omega \frac{|\nabla n_\varepsilon|^p}{n_\varepsilon}dx + \frac{3K\mu}{4}\int_\Omega n_\varepsilon^2|\ln n_\varepsilon|dx\\ &\le \eta\int_\Omega |\Delta c_\varepsilon|^2 dx + C_\eta\int_\Omega |n_\varepsilon|^2 dx + C_1\\ &\le \eta\int_\Omega |\Delta c_\varepsilon|^2 dx + \frac{K\mu}{4}\int_\Omega n_\varepsilon^2|\ln n_\varepsilon|dx + C_\eta^{'},\end{aligned}$$

which implies

$$\frac{d}{dt}\int_\Omega Kn_\varepsilon \ln n_\varepsilon dx + K\int_\Omega \frac{|\nabla n_\varepsilon|^p}{n_\varepsilon}dx + \frac{K\mu}{2}\int_\Omega n_\varepsilon^2|\ln n_\varepsilon|dx \\ \leq \eta\int_\Omega |\Delta c_\varepsilon|^2 dx + C_\eta^{'}. \tag{3.10}$$

By testing the second equation of (3.1) by $-\Delta c_\varepsilon$ and an application of Young's inequality, then one can find constants $C_2, C_3 > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 dx + \int_\Omega |\Delta c_\varepsilon|^2 dx + \int_\Omega n_\varepsilon |\nabla c_\varepsilon|^2 dx \\
&= \int_\Omega u_\varepsilon \nabla c_\varepsilon \Delta c_\varepsilon dx - \int_\Omega c_\varepsilon \nabla c_\varepsilon \nabla n_\varepsilon dx \\
&\leq \|u_\varepsilon\|_{L^6}\|\nabla c_\varepsilon\|_{L^3}\|\Delta c_\varepsilon\|_{L^2} + \frac{p-1}{p}\int_\Omega n_\varepsilon^{\frac{1}{p-1}}|\nabla c_\varepsilon|^{\frac{p}{p-1}}dx + \frac{1}{p}\|c_\varepsilon\|_{L^\infty}^p\int_\Omega \frac{|\nabla n_\varepsilon|^p}{n_\varepsilon}dx \\
&\leq C_2\|u_\varepsilon\|_{H^1}\|\nabla c_\varepsilon\|_{L^2}^{\frac{1}{2}}\|\Delta c_\varepsilon\|_{L^2}^{\frac{3}{2}} + C_2\int_\Omega \frac{|\nabla n_\varepsilon|^p}{n_\varepsilon}dx + \int_\Omega n_\varepsilon^{\frac{1}{p-1}}|\nabla c_\varepsilon|^{\frac{p}{p-1}}dx \\
&\leq \frac{1}{4}\|\Delta c_\varepsilon\|_{L^2}^2 + C_2\|u_\varepsilon\|_{H^1}^4\|\nabla c_\varepsilon\|_{L^2}^2 + C_2\int_\Omega \frac{|\nabla n_\varepsilon|^p}{n_\varepsilon}dx + \int_\Omega n_\varepsilon|\nabla c_\varepsilon|^2 dx + C_3.
\end{aligned}$$

Taking $K = 4C_2$ and $\eta = \frac{1}{2}$ in (3.10), then combining (3.4) and the above inequality, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (Kn_\varepsilon \ln n_\varepsilon + |\nabla c_\varepsilon|^2)dx + \frac{K}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^p}{n_\varepsilon}dx + \frac{K\mu}{2}\int_\Omega n_\varepsilon^2|\ln n_\varepsilon|dx + \int_\Omega |\Delta c_\varepsilon|^2 dx \\
&\leq C_4\|\nabla c_\varepsilon\|_{L^2}^2 + C_4,
\end{aligned}$$

where C_4 is a positive constant. Finally, using (3.4), we can deduce (3.9).

Lemma 3.4. *The triple* $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ *is the solution to the approximation system* (3.1)*. If*

$$\sup_{t\in(0,T_{max})}\int_\Omega n_\varepsilon^{\frac{3}{2}}dx \leq C, \tag{3.11}$$

where $C > 0$ *is a constant, then for all* $r > 2$

$$\sup_{t\in(0,T_{max})}\|u_\varepsilon\|_{L^r} \leq C.$$

Proof. Taking advantage of Duhamel's principle, we can express u_ε by

$$u_\varepsilon = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}(n_\varepsilon\nabla\Phi_\varepsilon)ds.$$

Then we have

$$\begin{aligned}
\|u_\varepsilon\|_{L^p} &\leq e^{-\lambda t}\|u_{\varepsilon 0}\|_{L^p} + \int_0^t \|e^{-(t-s)A}\mathcal{P}(n_\varepsilon\nabla\Phi_\varepsilon)\|_{L^p}ds \\
&\leq e^{-\lambda t}\|u_{\varepsilon 0}\|_{L^p} + \int_0^t e^{-\lambda(t-s)}(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|n_\varepsilon\nabla\Phi_\varepsilon\|_{L^q}ds \\
&\leq e^{-\lambda t}\|u_{\varepsilon 0}\|_{L^p} + C\int_0^t e^{-\lambda(t-s)}(t-s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|n_\varepsilon\|_{L^q}ds.
\end{aligned}$$

Let $q = \frac{3}{2}$ in the above inequality, in view of (3.11), we can derive that

$$\sup_{t\in(0,T_{max})} \|u_\varepsilon\|_{L^r} \le C \qquad \text{for all } t > 0.$$

Lemma 3.5. *If $p > 2$, then we can find some positive constant $C_r = C_r(r)$ such that*

$$\sup_{t\in(0,T_{max})} \int_\Omega n_\varepsilon^{r+1} dx + \sup_{t\in(\tau,T_{max})} \int_{t-\tau}^t \int_\Omega \Big(n_\varepsilon^{r-1} |\nabla n_\varepsilon|^p + n_\varepsilon^{r+2} \Big) dxds \le C_r \qquad \textit{for all } r > 0. \tag{3.12}$$

Proof. The first equation of (3.1) is multiplied by n_ε^r, then we integrate it by Ω to get

$$\begin{aligned}
&\frac{1}{r+1}\frac{d}{dt}\int_\Omega n_\varepsilon^{r+1} dx + r\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + \mu \int_\Omega n_\varepsilon^{r+2} dx \\
&\le r\int_\Omega n_\varepsilon^r \nabla n_\varepsilon \nabla c_\varepsilon dx + \mu \int_\Omega n_\varepsilon^{r+1} dx.
\end{aligned}$$

Combining the Young inequality and the assumption $p > 2$ ensures that $\frac{p}{p-1} < 2$, then we have

$$\begin{aligned}
&\frac{1}{r+1}\frac{d}{dt}\int_\Omega n_\varepsilon^{r+1} dx + r\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + \mu \int_\Omega n_\varepsilon^{r+2} dx \\
&\le \frac{r}{p}\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + \frac{r(p-1)}{p}\int_\Omega n_\varepsilon^{r+\frac{1}{p-1}} |\nabla c_\varepsilon|^{\frac{p}{p-1}} dx + \frac{\mu}{2}\int_\Omega n_\varepsilon^{r+2} dx + 2^{r+1}\mu|\Omega| \\
&\le \frac{r}{2}\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + r\int_\Omega n_\varepsilon^{r+\frac{1}{p-1}} |\nabla c_\varepsilon|^2 dx + C\int_\Omega n_\varepsilon^{r+\frac{1}{p-1}} dx + \frac{\mu}{2}\int_\Omega n_\varepsilon^{r+2} dx + 2^{r+1}\mu|\Omega|.
\end{aligned}$$

Shifting the above inequality, then using the Young inequality once again, we gain

$$\begin{aligned}
&\frac{1}{r+1}\frac{d}{dt}\int_\Omega n_\varepsilon^{r+1} dx + \frac{r}{2}\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + \frac{\mu}{4} \int_\Omega n_\varepsilon^{r+2} dx \\
&\le r\int_\Omega n_\varepsilon^{r+\frac{1}{p-1}} |\nabla c_\varepsilon|^2 dx + C_1 \\
&\le \frac{\mu}{8}\int_\Omega n_\varepsilon^{r+2} dx + \Big(\frac{8}{\mu}\Big)^{\frac{r(p-1)+1}{2(p-1)-1}} r^{\frac{(r+2)(p-1)}{2(p-1)-1}} \int_\Omega |\nabla c_\varepsilon|^{\frac{2(r+2)(p-1)}{2p-3}} dx + C_1.
\end{aligned}$$

Next, applying the Gagliardo-Nirenberg interpolation inequality to the above inequality, it yields

$$\begin{aligned}
&\sup_{t\in(0,T_{max})} \int_\Omega n_\varepsilon^{r+1} dx + \sup_{t\in(\tau,T_{max})} \int_{t-\tau}^t \int_\Omega (n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p + n_\varepsilon^{r+2}) dxds \\
&\le C_2 \sup_{t\in(\tau,T_{max})} \int_{t-\tau}^t \int_\Omega |\nabla c_\varepsilon|^{\frac{2(r+2)(p-1)}{2p-3}} dxds + C_2 \\
&\le C_3 \sup_{t\in(\tau,T_{max})} \int_{t-\tau}^t \int_\Omega |\Delta c_\varepsilon|^{\frac{(r+2)(p-1)}{2p-3}} dxds + C_3.
\end{aligned} \tag{3.13}$$

Meanwhile, combining Lemma 2.2 with (3.4), we arrive at

$$\begin{aligned}
\sup_{t\in(\tau,T_{max})} \int_{t-\tau}^t (\|c_\varepsilon\|_{W^{2,q}}^q + \|c_{\varepsilon t}\|_{L^q}^q) ds &\le \bar{C}_3 \sup_{t\in(\tau,T_{max})} \int_{t-\tau}^t (\|u_\varepsilon \cdot \nabla c_\varepsilon\|_{L^q}^q + \|c_\varepsilon(1-n_\varepsilon)\|_{L^q}^q) ds + \bar{C}_3 \\
&\le C_4 \sup_{t\in(\tau,T_{max})} \int_{t-\tau}^t (\|u_\varepsilon \cdot \nabla c_\varepsilon\|_{L^q}^q + \|n_\varepsilon\|_{L^q}^q) ds + \bar{C}_3.
\end{aligned} \tag{3.14}$$

Taking $q = \frac{(r+2)(p-1)}{2p-3}$ in the above inequality, then substituting it into (3.13), we get

$$\begin{aligned}
&\sup_{t\in(0,T_{max})}\int_\Omega n_\varepsilon^{r+1}dx + \sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\int_\Omega (n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p + n_\varepsilon^{r+2})dxds \\
&\le \bar{C}_3 + C_4\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\Big(\|u_\varepsilon\cdot\nabla c_\varepsilon\|_{L^{\frac{(r+2)(p-1)}{2p-3}}}^{\frac{(r+2)(p-1)}{2p-3}} + \|n_\varepsilon\|_{L^{\frac{(r+2)(p-1)}{2p-3}}}^{\frac{(r+2)(p-1)}{2p-3}}\Big)ds \\
&\le C_4\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\|u_\varepsilon\cdot\nabla c_\varepsilon\|_{L^{\frac{(r+2)(p-1)}{2p-3}}}^{\frac{(r+2)(p-1)}{2p-3}}ds + \frac{1}{2}\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\int_\Omega n_\varepsilon^{r+2}dxds + C_5,
\end{aligned} \tag{3.15}$$

thanks to $\frac{p-1}{2p-3} < 1$. Next, let $\frac{(r+2)(p-1)}{2p-3} = \frac{5}{2}$, which implies $r = 3 - \frac{5}{2(p-1)}$. Substituting it into (3.15) and shifting the terms, we arrive at

$$\begin{aligned}
&\sup_{t\in(0,T_{max})}\int_\Omega n_\varepsilon^{4-\frac{5}{2(p-1)}}dx + \frac{1}{2}\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\int_\Omega (n_\varepsilon^{5-\frac{5}{2(p-1)}} + n_\varepsilon^{2-\frac{5}{2(p-1)}}|\nabla n_\varepsilon|^p)dxds \\
&\le C_4\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\|u_\varepsilon\cdot\nabla c_\varepsilon\|_{L^{\frac{5}{2}}}^{\frac{5}{2}}ds + C_5.
\end{aligned} \tag{3.16}$$

By employing Hölder's inequality and the Gagliardo-Nirenberg inequality, it becomes evident that

$$\begin{aligned}
\|u_\varepsilon\cdot\nabla c_\varepsilon\|_{L^{\frac{5}{2}}} &\le \left(\int_\Omega |u_\varepsilon|^{\frac{5}{2}}|\nabla c_\varepsilon|^{\frac{5}{2}}dx\right)^{\frac{2}{5}} \\
&\le \left(\left(\int_\Omega |u_\varepsilon|^6 dx\right)^{\frac{5}{12}}\left(\int_\Omega |\nabla c_\varepsilon|^{\frac{30}{7}}dx\right)^{\frac{7}{12}}\right)^{\frac{2}{5}} \\
&= \|u_\varepsilon\|_{L^6}\|\nabla c_\varepsilon\|_{L^{\frac{30}{7}}} \\
&\le C_6\|\nabla u_\varepsilon\|_{L^2}\Big(\|\nabla c_\varepsilon\|_{L^2}^{\frac{1}{5}}\|\Delta c_\varepsilon\|_{L^2}^{\frac{4}{5}} + \|\nabla c_\varepsilon\|_{L^2}\Big).
\end{aligned} \tag{3.17}$$

Combining Lemma 3.2, (3.16), and (3.17), we can deduce that

$$\sup_{t\in(0,T_{max})}\int_\Omega n_\varepsilon^{4-\frac{5}{2(p-1)}}dx + \sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\int_\Omega (n_\varepsilon^{5-\frac{5}{2(p-1)}} + n_\varepsilon^{2-\frac{5}{2(p-1)}}|\nabla n_\varepsilon|^p)dxds \le C_7.$$

We notice that $4 - \frac{5}{2(p-1)} > \frac{3}{2}$. Combining Lemma 3.4, for any $r > 0$, we have

$$\sup_{t\in(0,T_{max})}\|u_\varepsilon\|_{L^r} \le C. \tag{3.18}$$

Then collecting (3.4) and (3.18), we notice that

$$\begin{aligned}
\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\|u_\varepsilon\cdot\nabla c_\varepsilon\|_{L^3}^3ds &\le \sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\|u_\varepsilon\|_{L^{12}}^3\|\nabla c_\varepsilon\|_{L^4}^3ds \\
&\le C_8\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\|u_\varepsilon\|_{L^{12}}^3\|c_\varepsilon\|_{L^\infty}^{\frac{3}{2}}\|c_\varepsilon\|_{H^2}^{\frac{3}{2}}ds \\
&\le C_9.
\end{aligned}$$

Taking $r = 4 - \frac{3}{p-1}$ in (3.15), and combining the above inequality, we can deduce

$$\sup_{t\in(0,T_{max})}\int_\Omega n_\varepsilon^{5-\frac{3}{p-1}}dx + \sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\int_\Omega\Big(n_\varepsilon^{6-\frac{3}{p-1}} + n_\varepsilon^{3-\frac{3}{p-1}}|\nabla n_\varepsilon|^p\Big)dxds \le M. \tag{3.19}$$

We employ the gradient operator ∇ on the first equation of (3.1) and test the resulting identity by $|\nabla c_\varepsilon|^{r-2}\nabla c_\varepsilon$, $\forall r > 2$, then a combination of Lemma 2.3 entails

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int_\Omega|\nabla c_\varepsilon|^r dx + (r-2)\int_\Omega|\nabla c_\varepsilon|^{r-2}(\nabla|\nabla c_\varepsilon|)^2dx + \int_\Omega|\nabla c_\varepsilon|^{r-2}|\nabla^2 c_\varepsilon|^2dx\\
&= \frac{1}{2}\int_{\partial\Omega}\frac{\partial(|\nabla c_\varepsilon|^2)}{\partial\nu}|\nabla c_\varepsilon|^{r-2}dS + \int_\Omega n_\varepsilon c_\varepsilon \mathrm{div}(|\nabla c_\varepsilon|^{r-2}\nabla c_\varepsilon)dx + \int_\Omega u_\varepsilon\nabla c_\varepsilon\mathrm{div}(|\nabla c_\varepsilon|^{r-2}\nabla c_\varepsilon)dx\\
&\le \kappa\int_{\partial\Omega}|\nabla c_\varepsilon|^r dS + (2r-2)\int_\Omega|\nabla c_\varepsilon|^{r-2}|n_\varepsilon c_\varepsilon|^2dx + (2r-2)\int_\Omega|u_\varepsilon|^2|\nabla c_\varepsilon|^r dx\\
&\quad + \frac{r-2}{4}\int_\Omega|\nabla c_\varepsilon|^{r-2}(\nabla|\nabla c_\varepsilon|)^2dx + \frac{1}{4}\int_\Omega|\nabla c_\varepsilon|^{r-2}|\nabla^2c_\varepsilon|^2dx\\
&\le \kappa\int_{\partial\Omega}|\nabla c_\varepsilon|^r dS + \eta_1\int_\Omega|\nabla c_\varepsilon|^{r+2}dx + C_{\eta_1}\int_\Omega|n_\varepsilon|^{\frac{r+2}{2}}dx + C'_{\eta_1}\int_\Omega|u_\varepsilon|^{r+2}dx\\
&\quad + \frac{r-2}{4}\int_\Omega|\nabla c_\varepsilon|^{r-2}(\nabla|\nabla c_\varepsilon|)^2dx + \frac{1}{4}\int_\Omega|\nabla c_\varepsilon|^{r-2}|\nabla^2c_\varepsilon|^2dx.
\end{aligned}$$

Utilizing the boundary trace embedding inequality and (3.9), for any small $\eta_2 > 0$, we get

$$\begin{aligned}
\kappa\int_{\partial\Omega}|\nabla c_\varepsilon|^r dS &\le \eta_2\|\nabla(|\nabla c_\varepsilon|^{\frac{r}{2}})\|^2_{L^2} + C_{\eta_2}\||\nabla c_\varepsilon|^{\frac{r}{2}}\|^2_{L^{\frac{4}{r}}}\\
&\le \eta_2\|\nabla(|\nabla c_\varepsilon|^{\frac{r}{2}})\|^2_{L^2} + C'_{\eta_2}.
\end{aligned}$$

Thus, by employing inequality (2.2) and a combination of the above two inequalities, we can find a fixed positive constant σ such that

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int_\Omega|\nabla c_\varepsilon|^r dx + \frac{r-2}{2}\int_\Omega|\nabla c_\varepsilon|^{r-2}(\nabla|\nabla c_\varepsilon|)^2dx + \frac{1}{4}\int_\Omega|\nabla c_\varepsilon|^{r-2}|\nabla^2c_\varepsilon|^2dx + \sigma\int_\Omega|\nabla c_\varepsilon|^{r+2}dx\\
&\le C_{10}\int_\Omega\Big(|n_\varepsilon|^{\frac{r+2}{2}} + |u_\varepsilon|^{r+2}\Big)dx + C_{11}.
\end{aligned} \tag{3.20}$$

Additionally, by applying the Gagliardo-Nirenberg inequality, we get

$$\|n_\varepsilon\|^{\frac{5p+\beta p+3\beta}{3}}_{L^{\frac{5p+\beta p+3\beta}{3}}} = \|n_\varepsilon^{\frac{p+\beta}{p}}\|^{p+\frac{(2+\beta)p^2}{3(p+\beta)}}_{L^{p+\frac{(2+\beta)p^2}{3(p+\beta)}}} \le C_{12}\|n_\varepsilon^{\frac{p+\beta}{p}}\|^{\frac{(2+\beta)p^2}{3(p+\beta)}}_{L^{\frac{p(2+\beta)}{p+\beta}}}\|\nabla n_\varepsilon^{\frac{p+\beta}{p}}\|^p_{L^p} + C_{13}\|n_\varepsilon\|^{\frac{5p+\beta p+3\beta}{3}}_{L^{2+\beta}},$$

here $\beta = \frac{3(p-2)}{p-1}$, that is $2+\beta = 5 - \frac{3}{p-1}$. Combining (3.9) with (3.19), we infer that

$$\sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\int_\Omega|n_\varepsilon|^{\frac{5p+\beta p+3\beta}{3}}dxds \le C_{14}. \tag{3.21}$$

Next, multiplying the third equation of the system (3.1) by $u_{\varepsilon t}$, and integration by parts, we derive

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla u_\varepsilon|^2dx + \int_\Omega|u_{\varepsilon t}|^2dx = \int_\Omega n_\varepsilon\nabla\Phi_\varepsilon\cdot u_{\varepsilon t}dx \le \frac{1}{2}\int_\Omega|u_{\varepsilon t}|^2dx + C_{15}\int_\Omega|n_\varepsilon|^2dx.$$

Through a simple calculation, one can get

$$\frac{d}{dt}\int_\Omega |\nabla u_\varepsilon|^2 dx + \int_\Omega |u_{\varepsilon t}|^2 dx + \int_\Omega |\nabla u_\varepsilon|^2 dx \le C_{16}\int_\Omega |n_\varepsilon|^2 dx + \int_\Omega |\nabla u_\varepsilon|^2 dx.$$

In accordance with (3.19) and (3.21), using the fact $5-\frac{3}{p-1}>2$, we can arrive that

$$\sup_{t\in(0,T_{max})}\int_\Omega |\nabla u_\varepsilon|^2 dx + \sup_{t\in(\tau,T_{max})}\int_{t-\tau}^t\int_\Omega |u_{\varepsilon t}|^2 dxds \le C_{17}.$$

Moreover, we notice that

$$\sup_{t\in(0,T_{max})}\|u_\varepsilon\|_{L^6} \le \sup_{t\in(0,T_{max})}\|\nabla u_\varepsilon\|_{L^2} \le C_{18}. \tag{3.22}$$

Taking $r=4$ in (3.20) and collecting (3.21), the fact that $\frac{5p+\beta p+3\beta}{3}=\frac{8p^2-2p-18}{3p-3}>3$ for any $p>2$ and (3.22), we have

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^4 dx + \int_\Omega |\nabla c_\varepsilon|^2(\nabla|\nabla c_\varepsilon|)^2 dx + \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^2|\nabla^2 c_\varepsilon|^2 dx + \sigma\int_\Omega |\nabla c_\varepsilon|^6 dx\\
&\le C_{19}\int_\Omega (|n_\varepsilon|^3 + |u_\varepsilon|^6) dx + C_{20}\\
&\le C_{21},
\end{aligned}$$

which implies

$$\sup_{t\in(0,T_{max})}\int_\Omega |\nabla c_\varepsilon|^4 dx \le C_{22}. \tag{3.23}$$

For any $r>0, p>2$, due to $\frac{2p}{p-1}<4, r+\frac{1}{p-1}\le r-1+p, 2r+\frac{2}{p-1}\le\frac{3p}{(3-p)_+}\frac{p+r-1}{p}$, multiplying the first equation in the system (3.1) by n_ε^r, and employing the Sobolev embedding inequality and (3.23), we conduce

$$\begin{aligned}
&\frac{1}{r+1}\frac{d}{dt}\int_\Omega |n_\varepsilon|^{r+1} dx + r\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + \mu\int_\Omega |n_\varepsilon|^{r+2} dx\\
&\le r\int_\Omega n_\varepsilon^r \nabla n_\varepsilon \nabla c_\varepsilon dx + \mu\int_\Omega |n_\varepsilon|^{r+1} dx\\
&\le \frac{r}{4}\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + C_{23}\int_\Omega n_\varepsilon^{r+\frac{1}{p-1}}|\nabla c_\varepsilon|^{\frac{p}{p-1}} dx + \frac{\mu}{2}\int_\Omega n_\varepsilon^{r+2} dx + 2^{r+1}\mu|\Omega|\\
&\le \frac{r}{4}\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + C_{23}\Big(\int_\Omega n_\varepsilon^{2r+\frac{2}{p-1}} dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla c_\varepsilon|^{\frac{2p}{p-1}} dx\Big)^{\frac{1}{2}} + \frac{\mu}{2}\int_\Omega n_\varepsilon^{r+2} dx + C_{24}\\
&\le \frac{r}{4}\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + C_{23}\Big(\int_\Omega n_\varepsilon^{2r+\frac{2}{p-1}} dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla c_\varepsilon|^4 dx\Big)^{\frac{1}{2}} + \frac{\mu}{2}\int_\Omega n_\varepsilon^{r+2} dx + C_{24}\\
&\le \frac{r}{4}\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + C_{25}\|n_\varepsilon\|_{L^{2(r+\frac{1}{p-1})}}^{r+\frac{1}{p-1}} + \frac{\mu}{2}\int_\Omega n_\varepsilon^{r+2} dx + C_{24}\\
&\le \frac{r}{2}\int_\Omega n_\varepsilon^{r-1}|\nabla n_\varepsilon|^p dx + \frac{\mu}{2}\int_\Omega n_\varepsilon^{r+2} dx + C_{26},
\end{aligned}$$

then by concise calculation, it derives (3.12) straightly.

Lemma 3.6. *If $p > 2$, then one can find positive constants $M_1 = M_1(\beta), M_2, M_3 = M_3(r)$, such that*

$$\sup_{t\in(0,T_{max})} \{\|A^{\beta}u_\varepsilon\|_{L^2} + \|u_\varepsilon\|_{L^\infty}\} \le M_1, \tag{3.24}$$

$$\sup_{t\in(0,T_{max})} \|c_\varepsilon\|_{W^{1,\infty}} \le M_2, \tag{3.25}$$

$$\sup_{t\in(\tau,T_{max})} \int_{t-\tau}^{t} (\|c_\varepsilon\|_{W^{2,r}}^r + \|c_{\varepsilon t}\|_{L^r}^r)ds \le M_3 \qquad \textit{for all } r > 0. \tag{3.26}$$

Proof. Noticing $5 - \frac{3}{p-1} > 2$ for any $p > 2$ and (3.19), we can get

$$\begin{aligned}
\|A^{\beta}u_\varepsilon\|_{L^2} &\le e^{-t}\|A^{\beta}u_{\varepsilon 0}\|_{L^2} + \int_0^t \|A^{\beta}e^{-(t-s)A}P(n_\varepsilon(s)\nabla\Phi_\varepsilon(s))\|_{L^2}ds \\
&\le e^{-t}\|A^{\beta}u_{\varepsilon 0}\|_{L^2} + \int_0^t (t-s)^{-\beta}e^{-\lambda(t-s)}\|n_\varepsilon(s)\nabla\Phi_\varepsilon(s)\|_{L^2}ds \\
&\le e^{-t}\|A^{\beta}u_{\varepsilon 0}\|_{L^2} + \int_0^t (t-s)^{-\beta}e^{-\lambda(t-s)}\|n_\varepsilon(s)\|_{L^2}\|\nabla\Phi_\varepsilon(s)\|_{L^\infty}ds \\
&\le C.
\end{aligned}$$

The Sobolev embedding theorem gives us the following inequality

$$\|u_\varepsilon\|_{L^\infty} \le C(\|u_\varepsilon\|_{L^2} + \|A^{\beta}u_\varepsilon\|_{L^2}),$$

where $\beta > \frac{3}{4}$. Combining the above two inequalities, (3.24) can be deduced. In addition to collecting with Lemma 3.5, we have

$$\begin{aligned}
\|\nabla c_\varepsilon\|_{L^\infty} &\le e^{-t}\|\nabla c_{\varepsilon 0}\|_{L^\infty} + \int_0^t e^{-(t-s)}\|\nabla(e^{-(t-s)\Delta}(u_\varepsilon\cdot\nabla c_\varepsilon - c_\varepsilon n_\varepsilon + c_\varepsilon))\|_{L^\infty}ds \\
&\le e^{-t}\|\nabla c_{\varepsilon 0}\|_{L^\infty} + \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{4}}e^{-(t-s)}\|u_\varepsilon\cdot\nabla c_\varepsilon - c_\varepsilon n_\varepsilon + c_\varepsilon\|_{L^6}ds \\
&\le e^{-t}\|\nabla c_{\varepsilon 0}\|_{L^\infty} + \int_0^t (t-s)^{-\frac{3}{4}}e^{-(t-s)}(\|u_\varepsilon\|_{L^\infty}\|\nabla c_\varepsilon\|_{L^6} + \|c_\varepsilon\|_{L^\infty}\|n_\varepsilon\|_{L^6} + \|c_\varepsilon\|_{L^\infty})ds \\
&\le e^{-t}\|\nabla c_{\varepsilon 0}\|_{L^\infty} + \int_0^t (t-s)^{-\frac{3}{4}}e^{-(t-s)}(\|u_\varepsilon\|_{L^\infty}\|\nabla c_\varepsilon\|_{L^\infty}^{\frac{2}{3}}\|\nabla c_\varepsilon\|_{L^2}^{\frac{1}{3}} + \|c_\varepsilon\|_{L^\infty}\|n_\varepsilon\|_{L^6} + \|c_\varepsilon\|_{L^\infty})ds \\
&\le e^{-t}\|\nabla c_{\varepsilon 0}\|_{L^\infty} + C\Big(1 + \sup_{t\in(0,T_{max})}\|\nabla c_\varepsilon\|_{L^\infty}^{\frac{2}{3}}\Big).
\end{aligned}$$

This along with (3.3) implies (3.25), then we utilize (3.14) and the above inequality to gain

$$\begin{aligned}
\sup_{t\in(\tau,T_{max})} \int_{t-\tau}^{t} (\|c_\varepsilon\|_{W^{2,q}}^q + \|c_{\varepsilon t}\|_{L^q}^q)ds &\le C + C\sup_{t\in(\tau,T_{max})} \int_{t-\tau}^{t} (\|\nabla c_\varepsilon\|_{L^q}^q + \|n_\varepsilon\|_{L^q}^q)ds \\
&\le C + C\sup_{t\in(\tau,T_{max})} \int_{t-\tau}^{t} \|n_\varepsilon\|_{L^q}^q ds.
\end{aligned}$$

According to (3.12), we finally infer (3.26).

Lemma 3.7. *If $p > 2$, then there exist positive constants C and $C_r = C_r(r, \varepsilon)$ such that*

$$\sup_{t\in(0,T_{max})} \|n_\varepsilon(\cdot, t)\|_{L^\infty} \le C, \tag{3.27}$$

$$\sup_{t\in(\tau,T_{max})} \int_{t-\tau}^{t} (\|u_\varepsilon(\cdot, s)\|^r_{W^{2,r}} + \|u_{\varepsilon t}(\cdot, s)\|^r_{L^r})ds \le C_r. \tag{3.28}$$

Proof. The inequality (3.27) can be derived by using a Moser-Alikakos-type method presented in [21]. Since the process of proof is standard, we omit it. In light of Lemma 2.2, it is straightforward to arrive at (3.28).

Lemma 3.8. *If $p > 2$, then we have*

$$\sup_{t\in(0,+\infty)} \int_\Omega \left(|\nabla n_\varepsilon|^2 + \varepsilon\right)^{\frac{p}{2}} dx + \sup_{t\in(0,+\infty)} \int_t^{t+1} \int_\Omega \left|\frac{\partial n_\varepsilon}{\partial t}\right|^2 dxds \le C, \tag{3.29}$$

where $C > 0$ is a constant.

Proof. By multiplying the first equation of (3.1) by $\frac{\partial n_\varepsilon}{\partial t}$ and integrating by parts over Ω, we can utilize the Young inequality to derive

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega \left(|\nabla n_\varepsilon|^2 + \varepsilon\right)^{\frac{p}{2}} dx + \int_\Omega \left|\frac{\partial n_\varepsilon}{\partial t}\right|^2 dx + \int_\Omega \left(|\nabla n_\varepsilon|^2 + \varepsilon\right)^{\frac{p}{2}} dx \\
&\le -\int_\Omega \nabla\cdot(n_\varepsilon\nabla c_\varepsilon)\frac{\partial n_\varepsilon}{\partial t}dx - \int_\Omega u_\varepsilon\cdot\nabla n_\varepsilon\frac{\partial n_\varepsilon}{\partial t}dx + \mu\int_\Omega n_\varepsilon(1-n_\varepsilon)\frac{\partial n_\varepsilon}{\partial t}dx + \int_\Omega \left(|\nabla n_\varepsilon|^2 + \varepsilon\right)^{\frac{p}{2}} dx \\
&\le C_1\int_\Omega (|\Delta c_\varepsilon|^2 + |\nabla n_\varepsilon|^2 + |\nabla n_\varepsilon|^p + 1)dx + \frac{1}{2}\int_\Omega \left|\frac{\partial n_\varepsilon}{\partial t}\right|^2 dx \\
&\le C_2\int_\Omega (|\Delta c_\varepsilon|^2 + |\nabla n_\varepsilon|^p + 1)dx + \frac{1}{2}\int_\Omega \left|\frac{\partial n_\varepsilon}{\partial t}\right|^2 dx.
\end{aligned}$$

A combination of Lemmas 3.5 and 3.6 and the above inequality can yield (3.29). Then, for all $t \in (0, T_{max})$, Lemmas 3.6 and 3.7 entail

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty} + \|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}} + \|A^\beta u_\varepsilon(\cdot, t)\|_{L^2} \le C, \tag{3.30}$$

which paired with the criterion of extensibility in Lemma 3.1 yields $T_{max} = \infty$.

Using the estimates that were gathered in the preceding section, we will show the existence of global weak solutions for system (1.4) in this section.

Proof of Theorem 1.1. If $p > 2$ and $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (3.1), then we employ the Sobolev compact embedding theorem, Aubin-Lions compactness theorem and Lemmas 3.3, 3.5–3.8, there exists some subsequence of $(\varepsilon_j)_{j\in N} \subset (0, 1)$ with $\varepsilon_j \searrow 0$ as $j \to \infty$ such that

$$\begin{aligned}
n_\varepsilon &\to n, && \text{in } L^s(\bar\Omega \times [0, \infty)), \\
n_\varepsilon &\overset{*}{\rightharpoonup} n, && \text{in } L^\infty(\bar\Omega \times [0, \infty)), \\
n_{\varepsilon t} &\rightharpoonup n_t, && \text{in } L^2(\bar\Omega \times [0, \infty)), \\
\nabla n_\varepsilon &\rightharpoonup \nabla n, && \text{in } L^p(\bar\Omega \times [0, \infty)),
\end{aligned}$$

$$
\begin{aligned}
(|\nabla n_\varepsilon|^2+\varepsilon)|^{\frac{p-2}{2}}\nabla n_\varepsilon \rightharpoonup \omega, &\qquad \text{in } L^{\frac{p}{p-1}}(\bar{\Omega}\times[0,\infty)),\\
c_\varepsilon \to c, &\qquad \text{uniformly,}
\end{aligned}
\tag{3.31}
$$

and for any $s \in (1,+\infty)$

$$
\begin{aligned}
c_\varepsilon \rightharpoonup c, &\qquad \text{in } W_s^{2,1}(\bar{\Omega}\times[0,\infty)),\\
\nabla c_\varepsilon \to \nabla c, &\qquad \text{in } L^s(\bar{\Omega}\times[0,\infty)),\\
u_\varepsilon \to u, &\qquad \text{in } L^s(\bar{\Omega}\times[0,\infty)),\\
\nabla u_\varepsilon \to \nabla u, &\qquad \text{in } L^s(\bar{\Omega}\times[0,\infty)),\\
\nabla P_\varepsilon \rightharpoonup \nabla P, &\qquad \text{in } L^2(\bar{\Omega}\times[0,\infty)).
\end{aligned}
$$

To claim that (3.31) holds with $\omega = |\nabla n|^{p-2}\nabla n$, we need to prove that for any $\phi_1 \in C^\infty(\bar{\Omega}\times[0,\infty))$,

$$
\int_0^\infty\int_\Omega |\nabla n|^{p-2}\nabla n\nabla\phi_1 dxdt = \int_0^\infty\int_\Omega \omega\nabla\phi_1 dxdt. \tag{3.32}
$$

We replace ϕ_1 with $n_\varepsilon\phi_1$ in (1.6), which implies

$$
\begin{aligned}
&\int_0^\infty\int_\Omega (n_{\varepsilon t}+u_\varepsilon\cdot\nabla n_\varepsilon)n_\varepsilon\phi_1 dxdt + \int_0^\infty\int_\Omega (|\nabla n_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}(|\nabla n_\varepsilon|^2\phi_1 + n_\varepsilon\nabla n_\varepsilon\cdot\nabla\phi_1)dxdt\\
&= \int_0^\infty\int_\Omega (n_\varepsilon^2\nabla c_\varepsilon\cdot\nabla\phi_1 + n_\varepsilon\phi_1\nabla n_\varepsilon\cdot\nabla c_\varepsilon)dxdt - \mu\int_0^\infty\int_\Omega n_\varepsilon^2(1-n_\varepsilon)\phi_1 dxdt.
\end{aligned}
\tag{3.33}
$$

Similarly, replacing ϕ_1 with $n\phi_1$ in (1.6), we can get

$$
\begin{aligned}
&\int_0^\infty\int_\Omega (n_t+u\cdot\nabla n)n\phi_1 dxdt + \int_0^\infty\int_\Omega \omega(\phi_1\nabla n + n\nabla\phi_1)dxdt\\
&= \int_0^\infty\int_\Omega (n^2\nabla c\cdot\nabla\phi_1 + n\phi_1\nabla n\cdot\nabla c)dxdt - \mu\int_0^\infty\int_\Omega n^2(1-n)\phi_1 dxdt.
\end{aligned}
\tag{3.34}
$$

For any $\phi_1 \geq 0, \zeta \in L^p_{loc}(\mathbb{R}^+; W^{1,p}(\Omega))$, we have

$$
\int_0^\infty\int_\Omega \Big((|\nabla n_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla n_\varepsilon - (|\nabla\zeta_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla\zeta_\varepsilon\Big)(\nabla n_\varepsilon - \nabla v)\phi_1 dxdt \geq 0,
$$

and by moving the item, then we can obtain

$$
\begin{aligned}
&\int_0^\infty\int_\Omega (|\nabla n_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla n_\varepsilon|^2\phi_1 dxdt\\
&\geq \int_0^\infty\int_\Omega (|\nabla\zeta_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla\zeta_\varepsilon(\nabla n_\varepsilon - \nabla\zeta)\phi_1 dxdt + \int_0^\infty\int_\Omega \phi_1(|\nabla n_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla n_\varepsilon\nabla\zeta dxdt.
\end{aligned}
\tag{3.35}
$$

By a combination of (3.33) and (3.35), for any $\phi_1 \in C^\infty(\bar{\Omega}\times[0,\infty))$ with $\phi_1 \geq 0$, we can see that

$$
\begin{aligned}
&\int_0^\infty\int_\Omega (n_{\varepsilon t}+u_\varepsilon\cdot\nabla n_\varepsilon)n_\varepsilon\phi_1 dxdt + \int_0^\infty\int_\Omega (|\nabla\zeta_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla\zeta_\varepsilon(\nabla n_\varepsilon - \nabla\zeta)\phi_1 dxdt\\
&+ \int_0^\infty\int_\Omega \phi_1(|\nabla n_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla n_\varepsilon\nabla\varphi dxdt + \int_0^\infty\int_\Omega n_\varepsilon(|\nabla n_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla n_\varepsilon\cdot\nabla\phi_1 dxdt\\
&\leq \int_0^\infty\int_\Omega (n_\varepsilon^2\nabla c_\varepsilon\cdot\nabla\phi_1 + n_\varepsilon\phi_1\nabla n_\varepsilon\cdot\nabla c_\varepsilon)dxdt - \mu\int_0^\infty\int_\Omega n_\varepsilon^2(1-n_\varepsilon)\phi_1 dxdt,
\end{aligned}
$$

then we obtain that

$$\int_0^\infty\int_\Omega (n_t+u\cdot\nabla n)n\phi_1dxdt+\int_0^\infty\int_\Omega|\nabla\zeta|^{p-2}\nabla\zeta(\nabla n-\nabla\zeta)\phi_1dxdt+\int_0^\infty\int_\Omega(\phi_1\omega\nabla\zeta+n\omega\nabla\phi_1)dxdt$$
$$\leq\int_0^\infty\int_\Omega(n^2\nabla c\cdot\nabla\phi_1+n\phi_1\nabla n\cdot\nabla c)dxdt-\mu\int_0^\infty\int_\Omega n^2(1-n)\phi_1dxdt\qquad\text{as }\varepsilon\to0.$$

Inserting (3.34) into the above inequality, for any $\phi_1\in C^\infty(\bar{\Omega}\times[0,\infty))$ with $\phi_1\geq0$, we deduce

$$\int_0^\infty\int_\Omega(|\nabla\zeta|^{p-2}\nabla\zeta-\omega)(\nabla n-\nabla\zeta)\phi_1dxdt\leq0.$$

With $\lambda>0$, $\psi\in C^\infty(\bar{\Omega}\times[0,\infty))$, we select $\zeta=n-\lambda\psi$ to arrive at

$$\int_0^\infty\int_\Omega\nabla\psi\Big(|\nabla(n-\lambda\psi)|^{p-2}\nabla(n-\lambda\psi)-\omega\Big)\phi_1dxdt\leq0.$$

In the above inequality, letting $\lambda\to0$, we attain

$$\int_0^\infty\int_\Omega\nabla\psi\Big(|\nabla n|^{p-2}\nabla n-\omega\Big)\phi_1dxdt\leq0.$$

Similarly, we take $\zeta=n+\lambda\psi$ to get

$$\int_0^\infty\int_\Omega\nabla\psi\Big(|\nabla n|^{p-2}\nabla n-\omega\Big)\phi_1dxdt\geq0.$$

So, we derive

$$\int_0^\infty\int_\Omega\nabla\psi\Big(|\nabla n|^{p-2}\nabla n-\omega\Big)\phi_1dxdt=0.$$

Thus, (3.31) is achieved. Letting $\varepsilon\to0$ in (1.6)–(1.8), we can deduce

$$\int_0^\infty\int_\Omega n\phi_{1t}dxdt+\int_\Omega n_0\phi_1(\cdot,0)dx+\int_0^\infty\int_\Omega n\nabla c\cdot\nabla\phi_1dxdt+\int_0^\infty\int_\Omega nu\cdot\nabla\phi_1dxdt$$
$$+\mu\int_0^\infty\int_\Omega n(1-n)\phi_1dxdt-\int_0^\infty\int_\Omega|\nabla n|^{p-2}\nabla n\cdot\nabla\phi_1dxdt=0,$$
$$\int_0^\infty\int_\Omega c\phi_{2t}dxdt+\int_\Omega c_0\phi_2(\cdot,0)dx+\int_0^\infty\int_\Omega cu\cdot\nabla\phi_2dxdt-\int_0^\infty\int_\Omega\nabla c\cdot\nabla\phi_2dxdt$$
$$-\int_0^\infty\int_\Omega nc\phi_2dxdt=0,$$
$$\int_0^\infty\int_\Omega u\cdot\psi_tdxdt+\int_\Omega u_0\psi(\cdot,0)dx+\int_0^\infty\int_\Omega n\nabla\Phi\cdot\psi dxdt-\int_0^\infty\int_\Omega\nabla u\cdot\nabla\psi dxdt=0,$$

where $\forall\phi_1\in C_0^\infty(\bar{\Omega}\times[0,\infty))$, $\phi_2\in C_0^\infty(\bar{\Omega}\times[0,\infty))$, $\psi\in C_0^\infty(\bar{\Omega}\times[0,\infty);\mathbb{R}^3)$ and $\nabla\cdot\psi=0$. Therefore, the system (1.4) exists a global weak solution (n,c,u). With a combination of (3.6)–(3.8), the proof of Theorem 1.1 has been concluded.

4. Conclusions

In this paper, we considered the chemotaxis-Stokes system (1.4) with slow p-Laplacian diffusion and logistic source in a bounded domain $\Omega \subset \mathbb{R}^3$ with zero-flux boundary conditions and no-slip boundary condition and proved the existence of global bounded weak solutions for any slow p-Laplacian diffusion ($p > 2$) under the action of logistic source. The main result is as follows:

Theorem 1.1. *If $p > 2$, then the system* (1.4) *with the initial conditions* (1.5) *exists global weak solutions (n, c, u) in the bounded domain $\Omega \subset \mathbb{R}^3$, such that for all $\beta \in (\frac{3}{4}, 1)$, $s > 1$, satisfying*

$$\sup_{t\in(0,+\infty)}\Big(\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla n(\cdot,t)\|_{L^p(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|A^\beta u(\cdot,t)\|_{L^2(\Omega)}\Big) \le M_1,$$
$$\sup_{t\in(0,+\infty)}\Big(\|\nabla n^{\frac{p-1}{p}}\|_{L^p(Q(t))} + \|n_t\|_{L^2(Q(t))} + \|c\|_{W_s^{2,1}(Q(t))} + \|u\|_{W_s^{2,1}(Q(t))}\Big) \le M_2,$$

where positive constants M_1, M_2 only depend on n_0, c_0, u_0, p, Ω. Here $Q(t) = \Omega \times (t, t+1)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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