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Research article

Some well known inequalities on two dimensional convex mappings by means of Pseudo $\mathcal{L}$-$\mathcal{R}$ interval order relations via fractional integral operators having non-singular kernel

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Abstract: Fractional calculus and convex inequalities combine to form a comprehensive mathematical framework for understanding and analyzing a variety of problems. This note develops Hermite-Hadamard, Fejér, and Pachpatte type integral inequalities within pseudo left-right order relations by applying fractional operators with non-singular kernels. Recently, results have been developed using classical Riemann integral operators in addition to various other partial order relations that have some defects that we explained in literature in order to demonstrate the unique characteristics of pseudo order relations. To verify the developed results, we constructed several interesting examples and provided a number of remarks that demonstrate that this type of fractional operator generalizes several previously published results when different things are set up. This work can lead to improvements in mathematical theory, computational methods, and applications across a wide range of disciplines.

Keywords: Hermite-Hadamard; Fejér; Pachpatte type inequality; pseudo order; interval-valued
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1. Introduction

Fractional calculus (FC) replaces ordinary derivatives with fractional derivatives in mathematical analysis. By doing this, we improve the theory by enabling the dynamics to be represented by a non-integer order derivative, which helps to better characterize natural processes, especially when there are certain degrees of uncertainty involved. FC has been shown to be an effective tool for modeling real-world issues, allowing for more accurate adjustments of theoretical models to actual data. Applications are found, for instance, in the following fields: economy [1], engineering [2] and physics [3].

In FC, fractional integral inequalities are very useful for investigating the behavior of fractional integrals and their link to classical integrals. They establish constraints on fractional differential equation solutions and help to prove their existence and uniqueness. Although FC and convex analysis are two distinct branches of mathematics, they may be combined to provide intriguing new ideas and practical applications in a range of domains, including nonlocal modeling, differential equations, and optimization.

Convex mappings may be used for a wide range of mathematical structures, such as topological spaces, function spaces, and metric spaces. Generalized convexity introduces certain changes of classical convex mappings in order to accommodate a broader class of functions and sets. Some newly presented classes of generalized convex mappings are as follows (see Refs. [4–8]). In [9], the authors examined exponentially convex functions generated by Wulbert's and Stolarsky-type inequalities. In [10], the authors employed (h_1, h_2)-convex stochastic processes to derive three well-known inequalities with interesting applications. In [11], the authors utilized Kulisch-Miranker type relations and created Hermite-Hadamard, Ostrowski, and Jensen type inclusions for Godunova-Levin mappings. Afzal et al. [12] created several novel Hermite-Hadamard inequalities with applications to special means using two distinct concepts of generalized convex mappings. Stojiljkovic [13] applied the twice differentiable tensorial norm inequality of Ostrowski type for self-adjoint operators' continuous functions in Hilbert space. In [14], the authors explored several convex mappings to refine the tensorial inequalities in Hilbert spaces. Saeed et al. [15] employed (h_1, h_2)-convex mappings to create Hermite-Hadamard inequalities utilizing completely interval order relations.

These distinct classes prompted multiple academics to create the following double inequality for convex functions from various perspectives; this double inequality is the most significant feature of optimization. It is defined as follows for some convex function G_1 on some interval of subset of real numbers (see Refs. [16]),

$$\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2}\right) \leq \frac{1}{\partial_2-\partial_1}\int_{\partial_1}^{\partial_2} \mathrm{G}_1(\theta)\,\mathrm{d}\theta \leq \frac{\mathrm{G}_1(\partial_1)+\mathrm{G}_1(\partial_2)}{2}. \tag{1.1}$$

Using a classical integral operator and a basic order relation, renowned mathematician Dragomir [17] extended inequality (1.1) into coordinated form for the first time in 2001 as follows:

$$\begin{aligned}
&\mathrm{G}_1\Big(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\Big) \leq \frac{1}{2}\Big[\frac{1}{\partial_2-\partial_1}\int_{\partial_1}^{\partial_2} \mathrm{G}_1\Big(\mathrm{x}, \frac{\partial_3+\partial_4}{2}\Big)\mathrm{dx} + \frac{1}{\partial_4-\partial_3}\int_{\partial_3}^{\partial_4} \mathrm{G}_1\Big(\frac{\partial_1+\partial_2}{2}, \mathrm{y}\Big)\mathrm{dy}\Big]\\
&\leq \frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} \mathrm{G}_1(\mathrm{x},\mathrm{y})\,\mathrm{dy}\,\mathrm{dx}\\
&\leq \frac{1}{4}\Big[\frac{1}{\partial_2-\partial_1}\int_{\partial_1}^{\partial_2}[\mathrm{G}_1(\mathrm{x},\partial_3)+\mathrm{G}_1(\mathrm{x},\partial_4)]\,\mathrm{dx} + \frac{1}{\partial_4-\partial_3}\int_{\partial_3}^{\partial_4}[\mathrm{G}_1(\partial_1,\mathrm{y})+\mathrm{G}_1(\partial_2,\mathrm{y})]\,\mathrm{dy}\Big]
\end{aligned}$$

$$\leq \frac{G_1(\partial_1, \partial_3) + G_1(\partial_1, \partial_4) + G_1(\partial_2, \partial_3) + G_1(\partial_2, \partial_4)}{4}. \tag{1.2}$$

Owing to the wide range of applications of set-valued analysis across several fields, different authors have used different kinds of operators and order relations to create integral disparities in the context of interval-valued mappings (I.V.F). Inspired by this concept, Zhao et al. [18] recently proved inequality (1.2) in the setting of interval inclusion relations by using the standard integral operator. In [19], authors introduced interval-valued pre-invex functions on plane and developed several double inequalities. Wannalookkhee et al. [20] used quantum integrals to find the double inequality on plane, with several applications. Using the concept of quantum integrals, Kalsoom et al. [21] developed an inequality of the Hermite-Hadamard type (H.H) related to coordinated higher-order generalized pre and quasi-invex mappings. Akkurt et al. [22] developed novel H.H type fractional integral inequalities for double fractional integrals by using two intriguing identities for functions of two variables. Shi et al. [23] used two types of generalized convex functions to create H.H and its symmetric variation, using (I.V.F). Using coordinated up and down convex mappings with fuzzy-number values, Saeed et al. [24] developed H.H and Pachpatte-type integral inequalities. Wu et al. [25] used fractional integrals with exponential kernel to develop three fundamental integral identities based on first- and second-order derivatives of a given function. For convex functions, Ahmad et al. [26] developed H.H, Hermite-Hadamard-Fejér, Dragomir-Agarwal, and Pachpatte type inequalities based on fractional operators with non-singular kernels. Khan et al. [27] used fuzzy fractional integral operators with exponential kernels and established certain H.H and its other different variants of inequalities for exponential trigonometric convex fuzzy-number valued mappings. Alomari and Darus [28, 29] utilized bidimensional generalized convex functions to derive new bounds for H.H inequalities, including s-convex functions in the first sense and log-convex functions. Du and Zhou [30] used convex two-dimensional mappings and established the H.H inequality and its weighted and product forms based on partial order inclusion relations via fractional integral operators with exponential kernels. Budak et al. [31] establish quantum H.H-type inequalities utilizing newly defined quantum integrals for coordinated convex functions according to two-variable functions.

Our primary focus in this article is on pseudo left-right interval order relations. Some recent developed outcomes in this direction is as follows: Saeed et al. [32] created three well-known inequalities in the context of pseudo order relations using coordinated h-convex mappings. Stojiljkovic et al. [33] employed Katugampola integrals to create a new class of p-convex mappings and exploited left-right order relations to create many new generalized bounds of that double inequality. Khan et al. [34] employed convex interval-valued functions via log convex functions based on the pseudo-order relation to produce numerous new intriguing characteristics and inequalities. Srivastava et al. [35] developed new generalized inequalities for pseudo-order relations by applying fractional operators to convex mappings. Motivated by these findings, the authors in [36, 37] first connected H.H inequality using two different kinds of convex functions in the setting of cr-order interval-valued mappings. Afzal et al. [38] used different types of generalized convex functions and developed different types of set-valued H.H inequalities with applications. For more recent findings about related conclusions utilizing different types of convex functions and order relations, we consult these papers: (see Refs. [39–41]).

When it comes to adjusting inequalities in interval mappings, the key notions are "order relations" and "convex functions". Recently, authors utilized an order relation "$\subseteq_p$" that has some defects. As an example of this case, the authors in the following reference [42] showed that some results are not adjusted in the setting of set-valued mappings. To handle this problem, authors introduced a new type of relation called the left-right order relation "$\leq_p$" which allows us to compare intervals with ease and consider as an extension of the classical order "$\leq$". Our work is novel in the case of pseudo left-right order relations, and we have derived H.H, Fejér, and Pachpatte-type inequalities for fractional integral operators with non-singular kernels of exponential type. These results generalize a number of previously reported findings. Furthermore, we developed a few interesting nontrivial examples which demonstrate the accuracy of our results, as well as several remarks that show that we can recover various existing results when different parameters are set. Our motivations stem from the rich literature on developed results, including these articles [18, 33, 34], that have inspired us to create new and improved versions of H.H, Fejér-type, and Pachpatte-type inequalities.

The work is organized in the following manner: In Section 2, we begin by reviewing some known definitions, interval calculus findings, and introductory facts of fractional calculus theory. We summarize the article's key results in Section 3, along with three well-known inequalities in a fresh setting with insightful remarks and illustrations. In Section 4, we discuss the obtained results and conclusions and provide some future recommendations.

2. Preliminaries

This section discusses several fundamental concepts related to fractional calculus and interval calculus, such as definitions and properties. Furthermore, we begin this section by correcting some notations used throughout the text.

- $\mathsf{R_i}^+$: a space of positive intervals in R;
- $\mathsf{R_i}$: a space of positive and negative intervals in R;
- $\underline{\mathrm{G}} = \overline{\mathrm{G}}$: set-valued mapping deformed;
- $\subseteq$: inclusion relation;
- $\leq$: standard relation;
- $\leq_\mathrm{p}$: pseudo left-right relation.

2.1. *Interval analysis*

Let R be the one-dimensional Euclidean space, and consider $\mathsf{R_i}$ the family of all nonempty compact convex subsets of R, that is,

$$\mathsf{R_i} = \{[\partial_1, \partial_2] : \partial_1, \partial_2 \in \mathsf{R} \textit{ and } \partial_1 \leq \partial_2\}.$$

The Hausdorff metric on $\mathsf{R_i}$ is defined by

$$\mathrm{H(P, Q) = \max\{d(P, Q), d(P, Q)\}}, \tag{2.1}$$

where $\mathrm{d(P, Q)} = \max_{\partial_1 \in \mathrm{P}} \mathrm{d}(\partial_1, \mathrm{Q})$, and $\mathrm{d}(\partial_1, \mathrm{Q}) = \min_{\partial_2 \in \mathrm{Q}} \mathrm{d}(\partial_1, \partial_2) = \min_{\partial_2 \in \mathrm{Q}} |\partial_1 - \partial_2|$.

Remark 2.1. *For the Hausdorff metric defined in* (2.1)*, an analogous form is:*

$$\mathrm{H}([\underline{\partial_1}, \overline{\partial_1}], [\underline{\partial_2}, \overline{\partial_2}]) = \max\{|\underline{\partial_1} - \underline{\partial_2}|, |\overline{\partial_1} - \overline{\partial_2}|\},$$

which is also known as the Moore metric on the space of intervals. On $\mathsf{R_i}$, we define the Minkowski sum and scalar multiplication using

$$\mathrm{P} + \mathrm{Q} = \{\partial_1 + \partial_2 \mid \partial_1 \in \mathrm{P}, \partial_2 \in \mathrm{Q}\} \text{ and } \gamma\mathrm{P} = \{\gamma\partial_1 \mid \partial_1 \in \mathrm{P}\}.$$

Also, if $\mathrm{P} = [\underline{\partial_1}, \overline{\partial_1}]$ and $\mathrm{Q} = [\underline{\partial_2}, \overline{\partial_2}]$ are two compact intervals, then we define the difference as follows:

$$\mathrm{P} - \mathrm{Q} = [\underline{\partial_1} - \overline{\partial_2}, \overline{\partial_1} - \underline{\partial_2}],$$

with the product

$$\mathrm{P} \cdot \mathrm{Q} = [\min\{\underline{\partial_1\partial_2}, \underline{\partial_1}\overline{\partial_2}, \overline{\partial_1}\underline{\partial_2}, \overline{\partial_1\partial_2}\}, \sup\{\underline{\partial_1\partial_2}, \underline{\partial_1}\overline{\partial_2}, \overline{\partial_1}\underline{\partial_2}, \overline{\partial_1\partial_2}\}],$$

and the division

$$\frac{\mathrm{P}}{\mathrm{Q}} = \left[\min\left\{\frac{\underline{\partial_1}}{\underline{\partial_2}}, \frac{\underline{\partial_1}}{\overline{\partial_2}}, \frac{\overline{\partial_1}}{\underline{\partial_2}}, \frac{\overline{\partial_1}}{\overline{\partial_2}}\right\}, \sup\left\{\frac{\underline{\partial_1}}{\underline{\partial_2}}, \frac{\underline{\partial_1}}{\overline{\partial_2}}, \frac{\overline{\partial_1}}{\underline{\partial_2}}, \frac{\overline{\partial_1}}{\overline{\partial_2}}\right\}\right],$$

where $0 \notin \mathrm{Q}$. The definition of the pseudo interval order relation "$\leq_{\mathrm{p}}$" is given in [34] and defined as follows:

$$[\underline{\partial_1}, \overline{\partial_1}] \leq_{\mathrm{p}} [\underline{\partial_2}, \overline{\partial_2}] \Leftrightarrow \underline{\partial_1} \leq \underline{\partial_2} \text{ and } \overline{\partial_1} \leq \overline{\partial_2}.$$

Definition 2.1. [36] *Let* $\mathrm{G}_1 : [\partial_1, \partial_2] \to \mathsf{R_i}$ *be an interval-valued mapping defined by* $\mathrm{G}_1(\eta) = [\underline{\mathrm{F}_1}(\eta), \overline{\mathrm{F}_1}(\eta)]$. $\mathrm{G}_1 \in \mathsf{IR}_{([\partial_1,\partial_2])}$ *if* $\underline{\mathrm{F}_1}(\eta), \overline{\mathrm{F}_1}(\eta) \in \mathrm{R}_{([\partial_1,\partial_2])}$ *and*

$$(\mathsf{IR}) \int_{\partial_1}^{\partial_2} \mathrm{G}_1(\eta)\, \mathsf{d}\eta = \left[(\mathsf{R}) \int_{\partial_1}^{\partial_2} \underline{\mathrm{F}_1}(\eta)\, \mathsf{d}\eta, (\mathsf{R}) \int_{\partial_1}^{\partial_2} \overline{\mathrm{F}_1}(\eta)\, \mathsf{d}\eta\right].$$

2.2. *Interval-valued double integral*

A collection of numeral $\{\mathrm{a_{i-1}}, \xi_\mathrm{i}, \mathrm{a_i}\}_{i=1}^{m}$ is called to be nonoverlapping partition P' of $[\partial_1, \partial_2]$ if $\mathrm{P}' : \partial_1 = \mathrm{t}_0 < \mathrm{t}_1 < \cdots < \mathrm{t_n} = \partial_2$ with $\mathrm{a_{i-1}} \leq \xi_\mathrm{i} \leq \mathrm{a_i}$ for all $\mathrm{i} = 1, 2, 3, \ldots, \mathrm{m}$. Further, if we consider $\Delta\mathrm{a_i} = \mathrm{a_i} - \mathrm{a_{i-1}}$, then P' is called to be δ-fine if $\Delta\mathrm{a_i} < \delta$ for all i. Let $\mathrm{P}(\delta, [\partial_1, \partial_2])$ be taken to be the pack of all δ-fine partitions of $[\partial_1, \partial_2]$. If $\{\mathrm{a_{i-1}}, \xi_\mathrm{i}, \mathrm{a_i}\}_{i=1}^{m}$ and $\{\mathrm{b}_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}, \mathrm{b}_{\mathfrak{J}}\}_{\mathrm{j}=1}^{\mathrm{n}}$ are partitions of P' and P'', respectively, then one has $\Delta_{i,j} = [\mathrm{a_{i-1}}, \mathrm{a_i}] \times [\mathrm{b}_{\mathfrak{J}-1}, \mathrm{b}_{\mathfrak{J}}]$ called the partition rectangle of $\Delta = [\partial_1, \partial_2] \times [\partial_3, \partial_4]$ with the points $(\xi_\mathrm{i}, \eta_{\mathfrak{J}})$ inside the rectangles $[\mathrm{a_{i-1}}, \mathrm{a_i}] \times [\mathrm{b}_{\mathfrak{J}-1}, \mathrm{b}_{\mathfrak{J}}]$. Furthermore, if $\mathrm{P}(\delta, \Delta)$, we denote the set of all δ-fine partitions of Δ where $\mathrm{P}' \in \mathrm{P}(\delta, [\partial_1, \partial_2])$ and $\mathrm{P}'' \in \mathrm{P}(\delta, [\partial_3, \partial_4])$, then one has

$$\mathrm{S}(\mathrm{G}_1, \mathrm{P}, \delta, \Delta) = \sum_{\mathrm{i}=1}^{\mathrm{m}} \sum_{\mathfrak{J}=1}^{\mathrm{n}} \mathrm{G}_1(\xi_\mathrm{i}, \eta_{\mathfrak{J}}) \Delta \mathrm{A}_{\mathrm{i,j}}.$$

We call $\mathrm{S}(\mathrm{G}_1, \mathrm{P}, \delta, \Delta)$ an integral sum of G_1 related with $\mathrm{P} \in \mathrm{P}(\delta, \Delta)$. We refer to the reference [18] for a more detailed description of the principles and notations of interval-valued double integrals.

Taking motivation from the typical double integrals defined in the article [43], we propose the following double fractional integrals.

Definition 2.2. *Let* $G_1 : [\partial_1, \partial_2] \times [\partial_3, \partial_4] \subset \mathsf{R}^2 \to \mathsf{R}_i$ *be an interval-valued function defined as* $G_1(\eta_1, \eta_2) = [\underline{G_1}(\eta_1, \eta_2), \overline{G_1}(\eta_1, \eta_2)]$. *The double fractional operators are represented as* $\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^+}, \Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-}, \Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^+}$, *and* $\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^-}$ *of order* $\theta_1 \in (0,1), \theta_2 \in (0,1)$ *along with* $\partial_1, \partial_4 \geq 0$ *defined by*

$$\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^+} G_1(x,y) = \frac{1}{\theta_1\theta_2}\int_{\partial_1}^{x}\int_{\partial_4}^{y} e^{-\frac{1-\theta_1}{\theta_1}(x-t)} e^{-\frac{1-\theta_2}{\theta_2}(y-s)} G_1(t,s)dsdt, x > \partial_1, y > \partial_4,$$
$$\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-} G_1(x,y) = \frac{1}{\theta_1\theta_2}\int_{\partial_1}^{x}\int_{y}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(x-t)} e^{-\frac{1-\theta_2}{\theta_2}(s-y)} G_1(t,s)dsdt, x > \partial_1, y < \partial_4,$$
$$\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^+} G_1(x,y) = \frac{1}{\theta_1\theta_2}\int_{x}^{\partial_2}\int_{\partial_4}^{y} e^{-\frac{1-\theta_1}{\theta_1}(t-x)} e^{-\frac{1-\theta_2}{\theta_2}(y-s)} G_1(t,s)dsdt, x < \partial_2, y > \partial_4,$$

as well as

$$\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^-} G_1(x,y) = \frac{1}{\theta_1\theta_2}\int_{x}^{\partial_2}\int_{y}^{v} e^{-\frac{1-\theta_1}{\theta_1}(t-x)} e^{-\frac{1-\theta_2}{\theta_2}(s-y)} G_1(t,s)dsdt, x < \partial_2, y < \partial_4,$$

respectively, then one has

$$\lim_{\substack{\theta_1\to 1\\ \theta_2\to 1}} \Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^+} G_1(x,y) = \frac{1}{\theta_1\theta_2}\int_{\partial_1}^{x}\int_{\partial_4}^{y} G_1(t,s)dsdt.$$

It is simple to give sequential interval-valued fractional integrals in line with Definition 2.2.

$$\Im^{\theta_1}_{\partial_1{}^+} G_1\left(x, \frac{\partial_4+\partial_3}{2}\right) = \frac{1}{\theta_1}\int_{\partial_1}^{x} e^{-\frac{1-\theta_1}{\theta_1}(x-t)} G_1\left(t, \frac{\partial_4+\partial_3}{2}\right)dt, x > \partial_1,$$
$$\Im^{\theta_1}_{\partial_2{}^-} G_1\left(x, \frac{\partial_4+\partial_3}{2}\right) = \frac{1}{\theta_1}\int_{x}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(t-x)} G_1\left(t, \frac{\partial_4+\partial_3}{2}\right)dt, x < \partial_2,$$
$$\Im^{\theta_2}_{\partial_4{}^+} G_1\left(\frac{\partial_1+\partial_2}{2}, y\right) = \frac{1}{\theta_2}\int_{\partial_4}^{y} e^{-\frac{1-\theta_2}{\theta_2}(y-s)} G_1\left(\frac{\partial_1+\partial_2}{2}, s\right)ds, y > \partial_4,$$

along with

$$\Im^{\theta_2}_{\partial_4{}^-} G_1\left(\frac{\partial_1+\partial_2}{2}, y\right) = \frac{1}{\theta_2}\int_{y}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(s-y)} G_1\left(\frac{\partial_1+\partial_2}{2}, s\right)ds, y < \partial_4.$$

Definition 2.3. [17] *Let Nonnegative real-valued function* $G_1 : \Omega = [\partial_1, \partial_2] \times [\partial_3, \partial_4] \subseteq \mathsf{R}^2 \to \mathsf{K}^+$ *is said to be a coordinated convex function if*

$$\begin{aligned} G_1(\Im_1\partial_1 + (1-\Im_1)\partial_2, \Im_2\partial_3 + (1-\Im_2)\partial_4) &\leq \partial_1\partial_2 G_1(\Im_1, \partial_3) + \partial_2(1-\partial_1)G_1(\Im_1, \partial_4) \\ &+ \partial_1(1-\partial_2)G_1(\Im_2.\partial_3) + (1-\partial_1)(1-\partial_2)G_1(\Im_2, \partial_4) \end{aligned}$$

holds true for every $(\partial_1, \partial_2), (\partial_3, \partial_4) \in \Omega$ *along with* $\Im_1, \Im_2 \in [0,1]$.

Definition 2.4. [18] *Let* $G_1 : \Omega = [\partial_1, \partial_2] \times [\partial_3, \partial_4] \subseteq \mathsf{R}^2 \to \mathsf{R_i}^+$ *is said to be an interval-valued coordinated convex function if*

$$\begin{aligned} G_1(\Im_1\partial_1 + (1-\Im_1)\partial_2, \Im_2\partial_3 + (1-\Im_2)\partial_4) &\supseteq \partial_1\partial_2 G_1(\Im_1, \partial_3) + \partial_2(1-\partial_1)G_1(\Im_1, \partial_4) \\ &+ \partial_1(1-\partial_2)G_1(\Im_2.\partial_3) + (1-\partial_1)(1-\partial_2)G_1(\Im_2, \partial_4) \end{aligned}$$

holds true for every $(\partial_1, \partial_2), (\partial_3, \partial_4) \in \Omega$ *along with* $\Im_1, \Im_2 \in [0,1]$.

Definition 2.5. [44] $G_1 : \Omega = [\partial_1, \partial_2] \times [\partial_3, \partial_4] \subseteq \mathsf{R}^2 \rightarrow \mathsf{R_i}^+$ *is said to be an interval-valued coordinated convex function if*

$$G_1(\mathfrak{I}_1\partial_1 + (1-\mathfrak{I}_1)\partial_2, \mathfrak{I}_2\partial_3 + (1-\mathfrak{I}_2)\partial_4) \leq_p \partial_1\partial_2 G_1(\mathfrak{I}_1, \partial_3) + \partial_2(1-\partial_1)G_1(\mathfrak{I}_1, \partial_4) + \partial_1(1-\partial_2)G_1(\mathfrak{I}_2.\partial_3) + (1-\partial_1)(1-\partial_2)G_1(\mathfrak{I}_2, \partial_4)$$

holds true for every $(\partial_1, \partial_2), (\partial_3, \partial_4) \in \Omega$ *along with* $\mathfrak{I}_1, \mathfrak{I}_2 \in [0, 1]$.

Example 2.1. $G_1 : [\partial_1, \partial_2] \times [\partial_3, \partial_4] \subseteq \mathsf{R}^2 \rightarrow \mathsf{R_i}^+$ *is a coordinated interval-valued function defined as*

$$G_1 = [4e^{5x+1} + 6e^{9y+3} + 2, 11e^{3x+5} + 2e^{7y+1} + 7], \quad (x, y) \in [0, 1] \times [0, 1].$$

The terminanl point functions $\underline{G_1}(a, b), \overline{G_1}(a, b)$ *are convex functions on* Ω. *Hence,* G_1 *is a convex interval-valued function on* Ω.

3. H.H-type double fractional Pseudo order relations

Theorem 3.1. *Let* $G_1 : [\partial_1, \partial_2] \times [\partial_3, \partial_4] \subseteq \mathsf{R}^2 \rightarrow \mathsf{R_i}^+$ *be an interval-valued coordinated convex mapping defined as* $G_1 = [\underline{G_1}(a, b), \overline{G_1}(a, b)]$ *with* $0 \leq \partial_1 < \partial_2, 0 \leq \partial_3 < \partial_4$*, , then the following relations hold:*

$$\begin{aligned} &G_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right) \\ &\leq_p \frac{(1-\theta_1)(1-\theta_2)}{4(1-e^{-\eth_1})(1-e^{-\eth_2})}\Big[\mathfrak{I}^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_3{}^+}G_1(\partial_2, \partial_4) + \mathfrak{I}^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-}G_1(\partial_2, \partial_3) \\ &\quad + \mathfrak{I}^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_3{}^+}G_1(\partial_1, \partial_4) + \mathfrak{I}^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4-}G_1(\partial_1, \partial_3)\Big] \\ &\leq_p \frac{G_1(\partial_1, \partial_3) + G_1(\partial_2, \partial_3) + G_1(\partial_1, \partial_4) + G_1(\partial_2, \partial_4)}{4}, \end{aligned}$$

where $\eth_1 = \frac{1-\theta_1}{\theta_1}(\partial_2 - \partial_1)$ *and* $\eth_2 = \frac{1-\theta_2}{\theta_2}(\partial_4 - \partial_3)$.

Proof. Since G_1 is an interval-valued coordinated convex function, for instance we consider if we take $x = \mathfrak{I}_1\partial_1 + (1-\mathfrak{I}_1)\partial_2, y = (1-\mathfrak{I}_1)\partial_1 + \mathfrak{I}_1\partial_2, u = s_1\partial_3 + (1-s_1)\partial_4, w = (1-s_1)\partial_3 + s_1\partial_4$, then one has

$$\begin{aligned} &G_1\left(\frac{x+y}{2}, \frac{u+w}{2}\right) = G_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right) \\ &\leq_p \frac{1}{4}\left[G_1\left(\mathfrak{I}_1\partial_1 + (1-\mathfrak{I}_1)\partial_2, s_1\partial_3 + (1-s_1)\partial_4\right) + G_1\left(\mathfrak{I}_1\partial_1 + (1-\mathfrak{I}_1)\partial_2, (1-s_1)\partial_3 + s_1\partial_4\right)\right. \\ &\quad \left.+ G_1\left((1-\mathfrak{I}_1)\partial_1 + \mathfrak{I}_1\partial_2, s_1\partial_3 + (1-s_1)\partial_4\right) + G_1\left((1-\mathfrak{I}_1)\partial_1 + \mathfrak{I}_1\partial_2, (1-s_1)\partial_3 + s_1\partial_4\right)\right]. \end{aligned} \tag{3.1}$$

Multiplying above pseudo order relation (3.1) with $e^{-\eth_1\mathfrak{I}_1}e^{-\eth_2 s_1}$ and integrating, one has

$$\begin{aligned} &G_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right)\int_0^1\int_0^1 e^{-\eth_1\mathfrak{I}_1}e^{-\eth_2 s_1}\, ds_1\, d\mathfrak{I}_1 \\ &\leq_p \frac{1}{4}\left\{\int_0^1\int_0^1 e^{-\eth_1\mathfrak{I}_1}e^{-\eth_2 s_1}\left[G_1\left(\mathfrak{I}_1\partial_1 + (1-\mathfrak{I}_1)\partial_2, s_1\partial_3 + (1-s_1)\partial_4\right)\right.\right. \end{aligned}$$

$$+G_1(\Im_1\partial_1+(1-\Im_1)\partial_2,(1-s_1)\partial_3+s_1\partial_4)]\,ds_1\,d\Im_1$$
$$+\int_0^1\int_0^1 e^{-\eth_1\Im_1}e^{-\eth_2 s_1}[G_1((1-\Im_1)\partial_1+\Im_1\partial_2,s_1\partial_3+(1-s_1)\partial_4)$$
$$+G_1((1-\Im_1)\partial_1+\Im_1\partial_2,(1-s_1)\partial_3+s_1\partial_4)]\,ds_1\,d\Im_1\}.$$

In order to determine our results, we can adjust the variable and perform different computations:

$$\frac{\left(1-e^{-\eth_1}\right)\left(1-e^{-\eth_2}\right)}{\eth_1\eth_2}G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)$$
$$\leq_p\frac{1}{4(\partial_2-\partial_1)(\partial_4-\partial_3)}\left\{\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4}e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)dxdy\right.$$
$$+\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4}e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)dxdy$$
$$+\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4}e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)dxdy$$
$$\left.+\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4}e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)dxdy\right\}$$
$$=\frac{\theta_1\theta_2}{4(\partial_2-\partial_1)(\partial_4-\partial_3)}\left[\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_3{}^+}G_1(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-}G_1(\partial_2,\partial_3)\right.$$
$$\left.+\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_3{}^+}G_1(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^-}G_1(\partial_1,\partial_3)\right].$$

□

This proves the first part of the main theorem. For the second part, again considering Definition 2.5, we have

$$G_1(\Im_1\partial_1+(1-\Im_1)\partial_2,s_1\partial_3+(1-s_1)\partial_4)\leq_p\Im_1 s_1G_1(\partial_1,\partial_3)+s_1(1-\Im_1)G_1(\partial_2,\partial_3)$$
$$+\Im_1(1-s_1)G_1(\partial_1,\partial_4)+(1-s_1))(1-\Im_1)G_1(\partial_2,\partial_4),$$
$$G_1(\Im_1\partial_1+(1-\Im_1)\partial_2,(1-s_1)\partial_3+s_1\partial_4)\leq_p\Im_1(1-s_1)G_1(\partial_1,\partial_3)+(1-s_1)(1-\Im_1)G_1(\partial_2,\partial_3)$$
$$+\Im_1 s_1G_1(\partial_1,\partial_4)+(1-\Im_1)s_1G_1(\partial_2,\partial_4),$$
$$G_1((1-\Im_1)\partial_1+\Im_1\partial_2,s_1\partial_3+(1-s_1)\partial_4)\leq_p(1-\Im_1)s_1G_1(\partial_1,\partial_3)+\Im_1 s_1G_1(\partial_2,\partial_3)$$
$$+(1-\Im_1)(1-s_1)G_1(\partial_1,\partial_4)+\Im_1(1-s_1)G_1(\partial_2,\partial_4),$$

as well as

$$G_1((1-\Im_1)\partial_1+\Im_1\partial_2,(1-s_1)\partial_3+s_1\partial_4)\leq_p(1-\Im_1)(1-s_1)G_1(\partial_1,\partial_3)+\Im_1(1-s_1)G_1(\partial_2,\partial_3)$$
$$+s_1(1-\Im_1)G_1(\partial_1,\partial_4)+\Im_1 s_1G_1(\partial_2,\partial_4).$$

Adding the aforementioned relationships, it deduces that

$$G_1(\Im_1\partial_1+(1-\Im_1)\partial_2,s_1\partial_3+(1-s_1)\partial_4)+G_1(\Im_1\partial_1+(1-\Im_1)\partial_2,(1-s_1)\partial_3+s_1\partial_4)$$
$$+G_1((1-\Im_1)\partial_1+\Im_1\partial_2,s_1\partial_3+(1-s_1)\partial_4)+G_1((1-\Im_1)\partial_1+\Im_1\partial_2,(1-s_1)\partial_3+s_1\partial_4)$$
$$\leq_pG_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4). \tag{3.2}$$

Multiply aforementioned relation with $e^{-\delta_1 \Im_1} e^{-\delta_2 s_1}$, then integrate the resultant output about $(\Im_1, s_1)$, and we get

$$\begin{aligned}
&\int_0^1 \int_0^1 e^{-\delta_1 \Im_1} e^{-\delta_2 s_1} \left[G_1\left(\Im_1 \partial_1 + (1-\Im_1)\partial_2, s_1 \partial_3 + (1-s_1)\partial_4\right)\right.\\
&\left.+G_1\left(\Im_1 \partial_1 + (1-\Im_1)\partial_2, (1-s_1)\partial_3 + s_1 \partial_4\right)\right] ds_1\, d\Im_1\\
&+\int_0^1 \int_0^1 e^{-\delta_1 \Im_1} e^{-\delta_2 s_1} \left[G_1\left((1-\Im_1)\partial_1 + \Im_1 \partial_2, s_1 \partial_3\right.\right.\\
&\left.\left.+(1-s_1)\partial_4\right) + G_1\left((1-\Im_1)\partial_1 + \Im_1 \partial_2, (1-s_1)\partial_3 + s_1 \partial_4\right)\right] ds_1\, d\Im_1\\
\leq_p &\int_0^1 \int_0^1 e^{-\delta_1 \Im_1} e^{-\delta_2 s_1} [G_1(\partial_1, \partial_3) + G_1(\partial_2, \partial_3) + G_1(\partial_1, \partial_4) + G_1(\partial_2, \partial_4)] ds_1\, d\Im_1.
\end{aligned}$$

Changing the variables results in

$$\begin{aligned}
&\frac{(1-\theta_1)(1-\theta_2)}{4\left(1-e^{-\delta_1}\right)\left(1-e^{-\delta_2}\right)} \left[\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+} G_1(\partial_2, \partial_4) + \Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-} G_1(\partial_2, \partial_3)\right.\\
&\left.+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+} G_1(\partial_1, \partial_4) + \Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-} G_1(\partial_1, \partial_3)\right]\\
\leq_p &\frac{G_1(\partial_1, \partial_3) + G_1(\partial_2, \partial_3) + G_1(\partial_1, \partial_4) + G_1(\partial_2, \partial_4)}{4}.
\end{aligned}$$

Consequently, Theorem 3.1 is proved.

Remark 3.1. • *If one has $\theta_1 \to 1, \theta_2 \to 1$ with $\underline{G_1} \neq \overline{G_1}$, we get the following result by authors in [18].*

$$\begin{aligned}
G_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right) \supseteq &\frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)} \int_{\partial_1}^{\partial_2} \int_{\partial_3}^{\partial_4} G_1(x,y)dxdy\\
\supseteq &\frac{G_1(\partial_1, \partial_3) + G_1(\partial_2, \partial_3) + G_1(\partial_1, \partial_4) + G_1(\partial_2, \partial_4)}{4}.
\end{aligned}$$

• *If one has $\theta_1 \to 1, \theta_2 \to 1$ with $\underline{G_1} = \overline{G_1}$, we get the following result by the author in [17].*

$$\begin{aligned}
G_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right) \leq &\frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)} \int_{\partial_1}^{\partial_2} \int_{\partial_3}^{\partial_4} G_1(x,y)dxdy\\
\leq &\frac{G_1(\partial_1, \partial_3) + G_1(\partial_2, \partial_3) + G_1(\partial_1, \partial_4) + G_1(\partial_2, \partial_4)}{4}.
\end{aligned}$$

• *If one has $\underline{G_1} = \overline{G_1}$, we get the following result which is fresh as well.*

$$\begin{aligned}
&G_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right)\\
\leq &\frac{(1-\theta_1)(1-\theta_2)}{4(1-e^{-\delta_1})(1-e^{-\delta_2})} \left[\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+} G_1(\partial_2, \partial_4) + \Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-} G_1(\partial_2, \partial_3)\right.\\
&\left.+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+} G_1(\partial_1, \partial_4) + \Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-} G_1(\partial_1, \partial_3)\right]\\
\leq &\frac{G_1(\partial_1, \partial_3) + G_1(\partial_2, \partial_3) + G_1(\partial_1, \partial_4) + G_1(\partial_2, \partial_4)}{4}.
\end{aligned}$$

Example 3.1. *Let* $G_1(x,y) = \left[4e^{4x}e^{4y}, (8+e^x)(7+e^y)\right], [\partial_1,\partial_2] = [0,1], [\partial_3,\partial_4] = [0,1], \theta_1 = 1$*, and* $\theta_2 = 1$*, then one has*

$$\begin{aligned}
&G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right) \approx [52.17, 45.77],\\
&\frac{(1-\theta_1)(1-\theta_2)}{4\left(1-e^{-\delta_1}\right)\left(1-e^{-\delta_2}\right)}\left[\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_3{}^+}G_1(\partial_2,\partial_4) + \Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-}G_1(\partial_2,\partial_3)\right.\\
&\left.+\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_3{}^+}G_1(\partial_1,\partial_4) + \Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^-}G_1(\partial_1,\partial_3)\right] \approx [126.27, 77.55],\\
&\frac{G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_4)+G_1(\partial_1,\partial_4)}{4} \approx [250.66, 110.19].
\end{aligned}$$

Thus,

$$[52.17, 45.77] \leq_p [126.27, 77.55] \leq_p [250.66, 110.19].$$

Fejér type double fractional Pseudo order relations

Theorem 3.2. *Let* $G_1 : [\partial_1,\partial_2]\times[\partial_3,\partial_4] \subseteq \mathrm{R}^2 \to \mathrm{R_i}^+$ *be an interval-valued coordinated convex function defined as* $G_1 = [\underline{G_1}(x,y), \overline{G_1}(x,y)]$ *with* $0 \leq \partial_1 < \partial_2, 0 \leq \partial_3 < \partial_4$*. If the mapping* $\varphi : [\partial_1,\partial_2]\times[\partial_3,\partial_4] \subset \mathrm{R}^2 \to \mathrm{R}$ *is symmetric, then one has*

$$\varphi(x,y) = \begin{cases} \varphi(\partial_1+\partial_2-x, y),\\ \varphi(x, \partial_3+\partial_4-y),\\ \varphi(\partial_1+\partial_2-x, \partial_3+\partial_4-y),\end{cases}$$

then we have

$$\begin{aligned}
&G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\left[\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_3{}^+}\varphi(\partial_2,\partial_4) + \Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-}\varphi(\partial_2,\partial_3)\right.\\
&\quad\left.+\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_3{}^+}\varphi(\partial_1,\partial_4) + \Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^-}\varphi(\partial_1,\partial_3)\right]\\
&\leq_p \left[\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_3{}^+}G_1(\partial_2,\partial_4)\varphi(\partial_2,\partial_4) + \Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-}G_1(\partial_2,\partial_3)\varphi(\partial_2,\partial_3)\right.\\
&\quad\left.+\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_3{}^+}G_1(\partial_1,\partial_4)\varphi(\partial_1,\partial_4) + \Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^-}G_1(\partial_1,\partial_3)\varphi(\partial_1,\partial_3)\right]\\
&\leq_p \frac{G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4)}{4}\left[\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_3{}^+}\varphi(\partial_2,\partial_4) + \Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-}\varphi(\partial_2,\partial_3)\right.\\
&\quad\left.+\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_3{}^+}\varphi(\partial_1,\partial_4) + \Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^-}\varphi(\partial_1,\partial_3)\right].
\end{aligned}$$

Proof. By virtue of the following result (3.1) in Theorem 3.1, and on both sides multiplying with

$4e^{-\eth_1\Im_1}e^{-\eth_2 s_1}\varphi(\Im_1\partial_1 + (1-\Im_1)\partial_2, s_1\partial_3 + (1-s_1)\partial_4)$ and integrating, one has

$$
\begin{aligned}
&4G_1\left(\frac{\partial_1+\partial_2}{2}\ \frac{\partial_3+\partial_4}{2}\right)\int_0^1\int_0^1 e^{-\eth_1\Im_1}e^{-\eth_2 s_1}\varphi(\Im_1\partial_1+(1-\Im_1)\partial_2, s_1\partial_3+(1-s_1)\partial_4)\,ds_1\,d\Im_1\\
\leq_p &\int_0^1\int_0^1 e^{-\eth_1\Im_1}e^{-\eth_2 s_1}\varphi(\Im_1\partial_1+(1-\Im_1)\partial_2, s_1\partial_3+(1-s_1)\partial_4)\\
&\times[G_1(\Im_1\partial_1+(1-\Im_1)\partial_2, s_1\partial_3+(1-s_1)\partial_4)+G_1(\Im_1\partial_1+(1-\Im_1)\partial_2,(1-s_1)\partial_3+s_1\partial_4)]\,ds_1\,d\Im_1\\
&+\int_0^1\int_0^1 e^{-\eth_1\Im_1}e^{-\eth_2 s_1}\varphi(\Im_1\partial_1+(1-\Im_1)\partial_2, s_1\partial_3+(1-s_1)\partial_4)\\
&\times[G_1((1-\Im_1)\partial_1+\Im_1\partial_2, s_1\partial_3+(1-s_1)\partial_4)+G_1((1-\Im_1)\partial_1+\Im_1\partial_2,(1-s_1)\partial_3+s_1\partial_4)]\,ds_1\,d\Im_1.
\end{aligned}
$$

In order to determine our results, we can adjust the variable and perform different computations

$$
\begin{aligned}
&G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\frac{4}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-\xi)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-\delta)}G_1(\xi,\delta)\varphi(\xi,\delta)d\delta d\xi\\
\leq_p&\frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\left\{\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-\xi)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-\delta)}G_1(\xi,\delta)\varphi(\xi,\delta)d\delta d\xi\right.\\
&+\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-\xi)}e^{-\frac{1-\theta_2}{\theta_2}(\delta-\partial_3)}G_1(\xi,\delta)\varphi(\xi,\partial_3+\partial_4-\delta)d\delta d\xi\\
&+\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\xi-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-\delta)}G_1(\xi,\delta)\varphi(\partial_1+\partial_2-\xi,\delta)d\delta d\xi\\
&\left.+\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\xi-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(\delta-\partial_3)}G_1(\xi,\delta)\varphi(\partial_1+\partial_2-\xi,\partial_3+\partial_4-\delta)d\delta d\xi\right\}\\
=&\frac{\theta_1\theta_2}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\left[\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_3}G_1(\partial_2,\partial_4)\varphi(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}G_1(\partial_2,\partial_3)\varphi(\partial_2,\partial_3)\right.\\
&\left.+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}G_1(\partial_1,\partial_4)\varphi(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-}G_1(\partial_1,\partial_3)\varphi(\partial_1,\partial_3)\right].
\end{aligned}
$$

As $\varphi(x,y)$ is symmetry, one has

$$
\begin{aligned}
&G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\frac{4}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-\xi)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-\delta)}G_1(\xi,\delta)\varphi(\xi,\delta)d\delta d\xi\\
=&\frac{\theta_1\theta_2}{(\partial_2-\partial_1)(\partial_4-\partial_3)}G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\left[\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+}\varphi(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}\varphi(\partial_2,\partial_3)\right.\\
&\left.+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}\varphi(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-}\varphi(\partial_1,\partial_3)\right].
\end{aligned}
$$

The initial relation is therefore concluded. For the second part, consider result (3.2) present in Theorem 3.1, and on both sides multiply with $e^{-\eth_1\Im_1}e^{-\eth_2 s_1}\varphi(\Im_1\partial_1+(1-\Im_1)\partial_2, s_1\partial_3+(1-s_1)\partial_4)$ and integrate, one has

$$\int_0^1\int_0^1 e^{-\eth_1\Im_1}e^{-\eth_2 s_1}\varphi(\Im_1\partial_1+(1-\Im_1)\partial_2, s_1\partial_3+(1-s_1)\partial_4)$$
$$\times[G_1(\Im_1\partial_1+(1-\Im_1)\partial_2, s_1\partial_3+(1-s_1)\partial_4)+G_1(\Im_1\partial_1+(1-\Im_1)\partial_2,(1-s_1)\partial_3+s_1\partial_4)]\,ds_1\,d\Im_1$$
$$+\int_0^1\int_0^1 e^{-\eth_1\Im_1}e^{-\eth_2 s_1}\varphi(\Im_1\partial_1+(1-\Im_1)\partial_2, s_1\partial_3+(1-s_1)\partial_4)$$
$$\times[G_1((1-\Im_1)\partial_1+\Im_1\partial_2, s_1\partial_3+(1-s_1)\partial_4)+G_1((1-\Im_1)\partial_1+\Im_1\partial_2,(1-s_1)\partial_3+s_1\partial_4)]\,ds_1\,d\Im_1$$
$$\leq_p\int_0^1\int_0^1 e^{-\eth_1\Im_1}e^{-\eth_2 s_1}\varphi(\Im_1\partial_1+(1-\Im_1)\partial_2, s_1\partial_3+(1-s_1)\partial_4)$$
$$\times[G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4)]ds_1\,d\Im_1.$$

Changing the variables results in

$$\frac{\theta_1\theta_2}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\Big[\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+}G_1(\partial_2,\partial_4)\varphi(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}G_1(\partial_2,\partial_3)\varphi(\partial_2,\partial_3)$$
$$+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}G_1(\partial_1,\partial_4)\varphi(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2^-,-}G_1(\partial_1,\partial_3)\varphi(\partial_1,\partial_3)\Big]$$
$$\leq_p\frac{G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4)}{4}$$
$$\times\frac{\theta_1\theta_2}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\Big[\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+}\varphi(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}\varphi(\partial_2,\partial_3)+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}\varphi(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-}\varphi(\partial_1,\partial_3)\Big].$$

Consequently, Theorem 3.2 is verfied. □

Remark 3.2.
- *Setting* $\underline{G_1}=\overline{G_1}$, *we get the following result which is new as well, that is,*

$$G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\Big[\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+}\varphi(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}\varphi(\partial_2,\partial_3)$$
$$+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}\varphi(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-}\varphi(\partial_1,\partial_3)\Big]$$
$$\leq\Big[\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+}G_1(\partial_2,\partial_4)\varphi(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}G_1(\partial_2,\partial_3)\varphi(\partial_2,\partial_3)$$
$$+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}G_1(\partial_1,\partial_4)\varphi(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-}G_1(\partial_1,\partial_3)\varphi(\partial_1,\partial_3)\Big]$$
$$\leq\frac{G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4)}{4}\Big[\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+}\varphi(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}\varphi(\partial_2,\partial_3)$$
$$+\Im^{\theta_1,\theta_2}_{\partial_2^{-,},\partial_3+}\varphi(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2^-,v}\varphi(\partial_1,\partial_3)\Big].$$

- *Setting symmetric weight function* $\varphi(x,y)=1$ *with* $\theta_1\to1,\theta_2\to1$ *and* $\underline{G_1}=\overline{G_1}$, *we get the following result obtained in [17].*
- *Setting* $\theta_1\to1,\theta_2\to1$, *we get the following result obtained in [44].*
- *Setting* $\theta_1\to1,\theta_2\to1$ *with* $\underline{G_1}\neq\overline{G_1}$, *we get the following result obtained in [18].*

We have now established a H.H-type inequality with regards to the midpoint type intervals.

Theorem 3.3. *Considering the same hypothesis of Theorem 3.1, we get the following double relations:*

$$\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\leq_{\mathrm{p}}\frac{(1-\theta_1)(1-\theta_2)}{4\left(1-\mathrm{e}^{-\frac{\delta_1}{2}}\right)\left(1-\mathrm{e}^{-\frac{\delta_2}{2}}\right)}\Big[\mathfrak{I}_{\left(\frac{\partial_1+\partial_2}{2}\right)^+,\left(\frac{\partial_3+\partial_4}{2}\right)^+}\mathrm{G}_1(\partial_2,\partial_4)$$
$$+\mathfrak{I}^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^+,\left(\frac{\partial_3+\partial_4}{2}\right)^-}\mathrm{G}_1(\partial_2,\partial_3)+\mathfrak{I}^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^-,\left(\frac{\partial_3+\partial_4}{2}\right)^+}\mathrm{G}_1(\partial_1,\partial_4)+\mathfrak{I}^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^-,\left(\frac{\partial_3+\partial_4}{2}\right)^-}\mathrm{G}_1(\partial_1,\partial_3)\Big]$$
$$\leq_{\mathrm{p}}\frac{\mathrm{G}_1(\partial_1,\partial_3)^2+\mathrm{G}_1\left(\partial_2,\partial_3{}^2\right)+\mathrm{G}_1(\partial_1,\partial_4)+\mathrm{G}_1(\partial_2,\partial_4)}{4}.$$

Proof. Since $\mathrm{G}_1$is an interval-valued coordinated convex function, for instance, if one takes $\mathrm{x}=\frac{\mathfrak{I}_1}{2}\partial_1+\frac{2-\mathfrak{I}_1}{2}\partial_2$, $\mathrm{y}=\frac{2-\mathfrak{I}_1}{2}\partial_1+\frac{\mathfrak{I}_1}{2}\partial_2$, $\mathrm{u}=\frac{\mathsf{s}_1}{2}\partial_3+\frac{2-\mathsf{s}_1}{2}\partial_4$, $\mathrm{w}=\frac{2-\mathsf{s}_1}{2}\partial_3+\frac{\mathsf{s}_1}{2}\partial_4$, then one has

$$\mathrm{G}_1\left(\frac{\mathrm{x}+\mathrm{y}}{2},\frac{\mathrm{u}+\mathrm{w}}{2}\right)=\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)$$
$$\leq_{\mathrm{p}}\frac{1}{4}\Bigg[\mathrm{G}_1\left(\frac{\mathfrak{I}_1}{2}\partial_1+\frac{2-\mathfrak{I}_1}{2}\partial_2,\frac{\mathsf{s}_1}{2}\partial_3+\frac{2-\mathsf{s}_1}{2}\partial_4\right)+\mathrm{G}_1\left(\frac{\mathfrak{I}_1}{2}\partial_1+\frac{2-\mathfrak{I}_1}{2}\partial_2,\frac{2-\mathsf{s}_1}{2}\partial_3+\frac{\mathsf{s}_1}{2}\partial_4\right)$$
$$+\mathrm{G}_1\left(\frac{2-\mathfrak{I}_1}{2}\partial_1+\frac{\mathfrak{I}_1}{2}\partial_2,\frac{\mathsf{s}_1}{2}\partial_3+\frac{2-\mathsf{s}_1}{2}\partial_4\right)+\mathrm{G}_1\left(\frac{2-\mathfrak{I}_1}{2}\partial_1+\frac{\mathfrak{I}_1}{2}\partial_2,\frac{2-\mathsf{s}_1}{2}\partial_3+\frac{\mathsf{s}_1}{2}\partial_4\right)\Bigg].$$

Multiplying the above relation with $\mathrm{e}^{-\frac{\delta_1}{2}\mathfrak{I}_1}\mathrm{e}^{-\frac{\delta_2}{2}\mathsf{s}_1}$ and integrating, one has

$$\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\int_0^1\int_0^1\mathrm{e}^{-\frac{\delta_1}{2}\mathfrak{I}_1}\mathrm{e}^{-\frac{\delta_2}{2}\mathsf{s}_1}\,\mathrm{d}\mathsf{s}_1\,\mathrm{d}\mathfrak{I}_1$$
$$\leq_{\mathrm{p}}\frac{1}{4}\Bigg[\int_0^1\int_0^1\mathrm{e}^{-\frac{\delta_1}{2}\mathfrak{I}_1}\mathrm{e}^{-\frac{\delta_2}{2}\mathsf{s}_1}\Bigg[\mathrm{G}_1\left(\frac{\mathfrak{I}_1}{2}\partial_1+\frac{2-\mathfrak{I}_1}{2}\partial_2,\frac{\mathsf{s}_1}{2}\partial_3+\frac{2-\mathsf{s}_1}{2}\partial_4\right)$$
$$+\mathrm{G}_1\left(\frac{\mathfrak{I}_1}{2}\partial_1+\frac{2-\mathfrak{I}_1}{2}\partial_2,\frac{2-\mathsf{s}_1}{2}\partial_3+\frac{\mathsf{s}_1}{2}\partial_4\right)\Bigg]\mathrm{d}\mathsf{s}_1\,\mathrm{d}\mathfrak{I}_1$$
$$+\int_0^1\int_0^1\mathrm{e}^{-\frac{\delta_1}{2}\mathfrak{I}_1}\mathrm{e}^{-\frac{\delta_2}{2}\mathsf{s}_1}\Bigg[\mathrm{G}_1\left(\frac{2-\mathfrak{I}_1}{2}\partial_1+\frac{\mathfrak{I}_1}{2}\partial_2,\frac{\mathsf{s}_1}{2}\partial_3+\frac{2-\mathsf{s}_1}{2}\partial_4\right)$$
$$+\mathrm{G}_1\left(\frac{2-\mathfrak{I}_1}{2}\partial_1+\frac{\mathfrak{I}_1}{2}\partial_2,\frac{2-\mathsf{s}_1}{2}\partial_3+\frac{\mathsf{s}_1}{2}\partial_4\right)\Bigg]\mathrm{d}\mathsf{s}_1\,\mathrm{d}\mathfrak{I}_1\Bigg].$$

In order to determine our results, we can adjust the variable and perform different computations

$$
\begin{aligned}
&\frac{4\left(1-e^{-\frac{\eth_1}{2}}\right)\left(1-e^{-\frac{\eth_2}{2}}\right)}{\eth_1\eth_2}G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\\
\leq_p&\frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\left\{\int_{\frac{\partial_1+\partial_2}{2}}^{\partial_2}\int_{\frac{\partial_3+\partial_4}{2}}^{\partial_4}e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)dxdy\right.\\
&+\int_{\frac{\partial_1+\partial_2}{2}}^{\partial_2}\int_{\partial_3}^{\frac{\partial_3+\partial_4}{2}}e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)dxdy\\
&+\int_{\partial_1}^{\frac{\partial_1+\partial_2}{2}}\int_{\frac{\partial_3+\partial_4}{2}}^{\partial_4}e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)dxdy\\
&\left.+\int_{\partial_1}^{\frac{\partial_1+\partial_2}{2}}\int_{\partial_3}^{\frac{\partial_3+\partial_4}{2}}e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)dxdy\right\}\\
=&\frac{\theta_1\theta_2}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\left[\Im^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^+,\left(\frac{\partial_3+\partial_4}{2}\right)^+}G_1(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^+,\left(\frac{\partial_3+\partial_4}{2}\right)^-}G_1(\partial_2,\partial_3)\right.\\
&\left.+\Im^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^-,\left(\frac{\partial_3+\partial_4}{2}\right)^+}G_1(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^-,\left(\frac{\partial_3+\partial_4}{2}\right)^-}G_1(\partial_1,\partial_3)\right].
\end{aligned}
$$

This proves the first part of the main theorem. For the second part, again consider Definition 2.5, and we have

$$
\begin{aligned}
G_1\left(\frac{\Im_1}{2}\partial_1+\frac{2-\Im_1}{2}\partial_2,\frac{s_1}{2}\partial_3+\frac{2-s_1}{2}\partial_4\right)\leq_p\frac{1}{4}&\left[\Im_1 s_1 G_1(\partial_1,\partial_3)+s_1(2-\Im_1)G_1(\partial_2,\partial_3)\right.\\
&\left.+\Im_1(2-s_1)G_1(\partial_1,\partial_4)+(2-s_1))(2-\Im_1)G_1(\partial_2,\partial_4)\right]\\
G_1\left(\frac{\Im_1}{2}\partial_1+\frac{2-\Im_1}{2}\partial_2,\frac{2-s_1}{2}\partial_3+\frac{s_1}{2}\partial_4\right)\leq_p\frac{1}{4}&\left[\Im_1(2-s_1)G_1(\partial_1,\partial_3)+(2-s_1)(2-\Im_1)G_1(\partial_2,\partial_3)\right.\\
&\left.+\Im_1 s_1 G_1(\partial_1,\partial_4)+(2-\Im_1)s_1 G_1(\partial_2,\partial_4)\right]\\
G_1\left(\frac{2-\Im_1}{2}\partial_1+\frac{\Im_1}{2}\partial_2,\frac{s_1}{2}\partial_3+\frac{2-s_1}{2}\partial_4\right)\leq_p\frac{1}{4}&\left[(2-\Im_1)s_1 G_1(\partial_1,\partial_3)+\Im_1 s_1 G_1(\partial_2,\partial_3)\right.\\
&\left.+(2-\Im_1)(2-s_1)G_1(\partial_1,\partial_4)+\Im_1(2-s_1)G_1(\partial_2,\partial_4)\right]
\end{aligned}
$$

$$
\begin{aligned}
G_1\left(\frac{2-\Im_1}{2}\partial_1+\frac{\Im_1}{2}\partial_2,\frac{s_1}{2}\partial_3+\frac{2-s_1}{2}\partial_4\right)\leq_p\frac{1}{4}&\left[(2-\Im_1)s_1 G_1(\partial_1,\partial_3)+\Im_1 s_1 G_1(\partial_2,\partial_3)\right.\\
&\left.+(2-\Im_1)(2-s_1)G_1(\partial_1,\partial_4)+\Im_1(2-s_1)G_1(\partial_2,\partial_4)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
G_1\left(\frac{2-\Im_1}{2}\partial_1+\frac{\Im_1}{2}\partial_2,\frac{2-s_1}{2}\partial_3+\frac{s_1}{2}\partial_4\right)\leq_p\frac{1}{4}&\left[(2-\Im_1)(2-s_1)G_1(\partial_1,\partial_3)+\Im_1(2-s_1)G_1(\partial_2,\partial_3)\right.\\
&\left.+s_1(2-\Im_1)G_1(\partial_1,\partial_4)+\Im_1 s_1 G_1(\partial_2,\partial_4)\right].
\end{aligned}
$$

Adding the aforementioned relations, we get

$$\begin{aligned}
&G_1\left(\frac{\Im_1}{2}\partial_1+\frac{2-\Im_1}{2}\partial_2,\frac{s_1}{2}\partial_3+\frac{2-s_1}{2}\partial_4\right)+G_1\left(\frac{\Im_1}{2}\partial_1+\frac{2-\Im_1}{2}\partial_2,\frac{2-s_1}{2}\partial_3+\frac{s_1}{2}\partial_4\right)\\
&+G_1\left(\frac{2-\Im_1}{2}\partial_1+\frac{\Im_1}{2}\partial_2,\frac{s_1}{2}\partial_3+\frac{2-s_1}{2}\partial_4\right)+G_1\left(\frac{2-\Im_1}{2}\partial_1+\frac{\Im_1}{2}\partial_2,\frac{2-s_1}{2}\partial_3+\frac{s_1}{2}\partial_4\right)\\
&\leq_p G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4).
\end{aligned}$$

Multiplying aforementioned relation with $e^{-\frac{\delta_1}{2}\Im_1}e^{-\frac{\delta_2}{2}s_1}$ then integrating, one has

$$\begin{aligned}
&\int_0^1\int_0^1 e^{-\frac{\delta_1}{2}\Im_1}e^{-\frac{\delta_2}{2}s_1}\Bigg[G_1\left(\frac{\Im_1}{2}\partial_1+\frac{2-\Im_1}{2}\partial_2,\frac{s_1}{2}\partial_3+\frac{2-s_1}{2}\partial_4\right)\\
&+G_1\left(\frac{\Im_1}{2}\partial_1+\frac{2-\Im_1}{2}\partial_2,\frac{2-s_1}{2}\partial_3+\frac{s_1}{2}\partial_4\right)\Bigg]ds_1\,d\Im_1\\
&+\int_0^1\int_0^1 e^{-\frac{\delta_1}{2}\Im_1}e^{-\frac{\delta_2}{2}s_1}\Bigg[G_1\left(\frac{2-\Im_1}{2}\partial_1+\frac{\Im_1}{2}\partial_2,\frac{s_1}{2}\partial_3+\frac{2-s_1}{2}\partial_4\right)\\
&+G_1\left(\frac{2-\Im_1}{2}\partial_1+\frac{\Im_1}{2}\partial_2,\frac{2-s_1}{2}\partial_3+\frac{s_1}{2}\partial_4\right)\Bigg]ds_1\,d\Im_1\\
&\leq_p\int_0^1\int_0^1 e^{-\frac{\delta_1}{2}\Im_1}e^{-\frac{\delta_2}{2}s_1}[G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4)]ds_1\,d\Im_1.
\end{aligned}$$

Changing the variables results in

$$\begin{aligned}
&\frac{\theta_1\theta_2}{4(\partial_2-\partial_1)(\partial_4-\partial_3)}\Bigg[\Im^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^+,\left(\frac{\partial_3+\partial_4}{2}\right)^+}G_1(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^+,\left(\frac{\partial_3+\partial_4}{2}\right)^-}G_1(\partial_2,\partial_3)\\
&+\Im_{\left(\frac{\partial_1+\partial_2}{2}\right)^-,\left(\frac{\partial_3+\partial_4}{2}\right)^+}G_1(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^-,\left(\frac{\partial_3+\partial_4}{2}\right)^-}G_1(\partial_1,\partial_3)\Bigg]\\
&\leq_p\frac{G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4)}{4}.
\end{aligned}$$

Consequently, Theorem 3.3 is proved. □

Remark 3.3.
- *If one has $\theta_1\to 1,\theta_2\to 1$, we get the following result by the authors in [18].*
- *If one has $\theta_1\to 1,\theta_2\to 1$ with $\underline{G_1}=\overline{G_1}$, we get the following result by author in [17].*

Example 3.2. *Let* $G_1(x,y)=\left[(2x+2)(2y+2)4e^{5x}e^{3y},(2+3e^x)(2+3e^y)\right],[\partial_1,\partial_2]=[0,1],[\partial_3,\partial_4]=$

$[0,1], \theta_1 = 1$, *then one has*

$$G_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right) \approx [270.12, 90.12],$$

$$\frac{(1-\theta_1)(1-\theta_2)}{4\left(1-e^{-\frac{\rho_1}{2}}\right)\left(1-e^{-\frac{\rho_2}{2}}\right)}\left[\mathfrak{I}^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^+,\left(\frac{\partial_3+\partial_4}{2}\right)^+} G_1(\partial_1,\partial_3) + \mathfrak{I}^{\theta_1}_{\left(\frac{\partial_1+\partial_2}{2}\right)^{\theta_1},\left(\frac{\partial_3+\partial_4}{2}\right)^-} G_1(\partial_1,\partial_4)\right.$$
$$\left. + \mathfrak{I}^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^-,\left(\frac{\partial_3+\partial_4}{2}\right)^+} G_1(\partial_1,\partial_3) + \mathfrak{I}^{\theta_1,\theta_2}_{\left(\frac{\partial_1+\partial_2}{2}\right)^-,\left(\frac{\partial_3+\partial_4}{2}\right)^-} G_1(\partial_1,\partial_3)\right] \approx [310.13, 153.12],$$

$$\frac{G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_4)+G_1(\partial_1,\partial_4)}{4} \approx [350.33, 179.45].$$

Thus,

$$[270.12, 90.12] \leq_p [310.13, 153.12] \leq_p [350.33, 179.45].$$

Theorem 3.4. *Let* $G_1, F_1 : [\partial_1,\partial_2] \times [\partial_3,\partial_4] \subseteq \mathbb{R}^2 \to \mathbb{R}_i^+$ *be an interval-valued coordinated convex function defined as* $G_1 = [\underline{G_1}(x,y), \overline{G_1}(x,y)]$ *and* $F_1 = [\underline{\eta}(x,y), \overline{\eta}(x,y)]$ *where* $0 \leq \partial_1 < \partial_2, 0 \leq \partial_3 < \partial_4$. *Then one has the following relations:*

$$\frac{\theta_1\theta_2}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\left[\mathfrak{I}^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+} G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_4) + \mathfrak{I}^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-} G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_3)\right.$$
$$\left. + \mathfrak{I}^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+} G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_4) + \mathfrak{I}^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-} G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_3)\right]$$
$$\leq_p A_1A_2C(\partial_1,\partial_2,\partial_3,\partial_4) + A_1B_2D(\partial_1,\partial_2,\partial_3,\partial_4) + A_2B_1E(\partial_1,\partial_2,\partial_3,\partial_4) + B_1B_2\Psi(\partial_1,\partial_2,\partial_3,\partial_4),$$

where

$$A_i = \frac{-2\theta_i + \theta_i^2 + 4 - \left(2\theta_i + \theta_i^2 + 4\right)e^{-\theta_i}}{\theta_i^3}, \quad B_i = \frac{-4 + 2\theta_i + (2\theta_i + 4)e^{-\theta_i}}{\theta_i^3},$$

$$C(\partial_1,\partial_2,\partial_3,\partial_4) = G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_3) + G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_4) + G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_3) + G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_4),$$
$$D(\partial_1,\partial_2,\partial_3,\partial_4) = G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_4) + G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_3) + G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_4) + G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_3),$$
$$E(\partial_1,\partial_2,\partial_3,\partial_4) = G_1(\partial_1,\partial_3)F_1(\partial_2,\partial_3) + G_1(\partial_1,\partial_4)F_1(\partial_2,\partial_4) + G_1(\partial_2,\partial_3)F_1(\partial_1,\partial_3) + G_1(\partial_2,\partial_4)F_1(\partial_1,\partial_4),$$

and

$$\Psi(\partial_1,\partial_2,\partial_3,\partial_4) = G_1(\partial_1,\partial_3)F_1(\partial_2,\partial_4) + G_1(\partial_1,\partial_4)F_1(\partial_2,\partial_3) + G_1(\partial_2,\partial_3)F_1(\partial_1,\partial_4) + G_1(\partial_2,\partial_4)F_1(\partial_1,\partial_3).$$

Proof. As G_1, F_1 are interval-valued coordinated convex functions, then one has

$G_{1x}(y) : [\partial_3,\partial_4] \to \mathbb{R}_i^+, G_{1x}(y) = G_1(x,y), F_{1x}(y) : [\partial_3,\partial_4] \to \mathbb{R}_i^+, F_{1x}(y) = F_1(x,y)$, also

$G_{1y}(x) : [\partial_1,\partial_2] \to \mathbb{R}_i^+, G_{1y}(x) = G_1(x,y), F_{1y}(x) : [\partial_1,\partial_2] \to \mathbb{R}_i^+, F_{1y}(x) = F_1(x,y)$

for each $x \in [\partial_1,\partial_2]$ accompanying $y \in [\partial_3,\partial_4]$. Now, by virtue of the Theorem 2.4 within this reference [45], we may have

$$\frac{1}{\partial_4-\partial_3}\left[\int_{\partial_3}^{\partial_4} G_{1x}(y)F_{1x}(y)e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}dy+\int_{\partial_3}^{\partial_4} G_{1y}(y)F_{1x}(y)e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}dy\right]$$
$$\leq_p \frac{\eth_2{}^2-2\eth_2+4-\left(\eth_2{}^2+2\eth_2+4\right)e^{-\eth_2}}{\eth_2{}^3}\left[G_{1x}(\partial_3)F_{1x}(\partial_3)+G_{1x}(\partial_4)F_{1x}(\partial_4)\right]$$
$$+\frac{2\eth_2-4+(2\eth_2+4)e^{-\eth_2}}{\eth_2{}^3}\left[G_{1x}(\partial_3)F_{1x}(\partial_4)+G_{1x}(\partial_4)F_{1x}(\partial_3)\right].$$

This can be written as

$$\frac{1}{\partial_4-\partial_3}\left[\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)F_1(x,y)dy+\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)F_1(x,y)dy\right]$$
$$\leq_p \frac{\eth_2{}^2-2\eth_2+4-\left(\eth_2{}^2+2\eth_2+4\right)e^{-\eth_2}}{\eth_2{}^3}\left[G_1(x,\partial_3)F_1(x,\partial_3)+G_1(x,\partial_4)F_1(x,\partial_4)\right]$$
$$+\frac{2\eth_2-4+(2\eth_2+4)e^{-\eth_2}}{\eth_2{}^3}\left[G_1(x,\partial_3)F_1(x,\partial_4)+G_1(x,\partial_4)F_1(x,\partial_3)\right].$$

Multiplying the above relation with $\frac{1}{\partial_2-\partial_1}e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}$ and $\frac{1}{\partial_2-\partial_1}e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}$ and integrating, one has

$$\frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\left[\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{\frac{\theta_1-1}{\theta_1}(\partial_2-y)}e^{\frac{\theta_2-1}{\theta_2}(\partial_4-y)}F_1(x,y)G_1(x,y)dxdy\right.$$
$$\left.+\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)F_1(x,y)dxdy\right]$$
$$\leq_p \frac{\eth_2{}^2-2\eth_2+4-\left(\eth_2{}^2+2\eth_2+4\right)e^{-\eth_2}}{\eth_2{}^3(\partial_2-\partial_1)}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}\left[G_1(x,\partial_3)F_1(x,\partial_3)+G_1(x,\partial_4)F_1(x,\partial_4)\right]dx$$
$$+\frac{2\eth_2-4+(2\eth_2+4)e^{-\eth_2}}{\eth_2{}^3(\partial_2-\partial_1)}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}\left[G_1(x,\partial_3)F_1(x,\partial_4)+G_1(x,\partial_4)F_1(x,\partial_3)\right]dx,$$

and

$$\frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\left[\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)F_1(x,y)dxdy\right.$$
$$\left.+\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)F_1(x,y)dxdy\right]$$
$$\leq_p \frac{\eth_2{}^2-2\eth_2+4-\left(\eth_2{}^2+2\eth_2+4\right)e^{-\eth_2}}{\eth_2{}^3(\partial_2-\partial_1)}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}\left[G_1(x,\partial_3)F_1(x,\partial_3)+G_1(x,\partial_4)F_1(x,\partial_4)\right]dx$$
$$+\frac{2\eth_2-4+(2\eth_2+4)e^{-\eth_2}}{\eth_2{}^3(\partial_2-\partial_1)}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}\left[G_1(x,\partial_3)F_1(x,\partial_4)+G_1(x,\partial_4)F_1(x,\partial_3)\right]dx.$$

Summing the above two relations, one has

$$\frac{\theta_1\theta_2}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\left[\mathfrak{J}^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_3{}^+}G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_4)+\mathfrak{J}^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-}G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_3)\right.$$

$$+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-}G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_3)\Big]$$
$$\leq_p \frac{\eth_2{}^2-2\eth_2+4-\left(\eth_2{}^2+2\eth_2+4\right)e^{-\eth_2}}{\eth_2{}^3(\partial_2-\partial_1)}\theta_1\Big\{\Big[\Im^{\theta_1}_{\partial_1^+}G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_3)$$
$$+\Im^{\theta_1}_{\partial_2^-}G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_3)\Big]+\Big[\Im^{\theta_1}_{\partial_1^+}G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_4)+\Im^{\theta_1}_{\partial_2^-}G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_4)\Big]\Big\}$$
$$+\frac{2\eth_2-4+(2\eth_2+4)e^{-\eth_2}}{\eth_2{}^3(\partial_2-\partial_1)}\theta_1\Big\{\Big[\Im^{\theta_1}_{\partial_1^+}G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_4)$$
$$+\Im^{\theta_1}_{\partial_2^-}G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_4)\Big]+\Big[\Im^{\theta_1}_{\partial_1^+}G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_3)+\Im^{\theta_1}_{\partial_2^-}G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_3)\Big]\Big\}. \tag{3.3}$$

This also indicates that

$$\frac{\theta_1}{\partial_2-\partial_1}\Big[\Im^{\theta_1}_{\partial_1^+}G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_3)+\Im^{\theta_1}_{\partial_2^-}G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_3)\Big]$$
$$\leq_p \frac{\eth_1{}^2-2\eth_1+4-\left(\eth_1{}^2+2\eth_1+4\right)e^{-\eth_1}}{\eth_1{}^3}[G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_3)]$$
$$+\frac{2\eth_1-4+(2\eth_1+4)e^{-\eth_1}}{\eth_1{}^3}[G_1(\partial_1,\partial_3)F_1(\partial_2,\partial_3)+G_1(\partial_2,\partial_3)F_1(\partial_1,\partial_3)], \tag{3.4}$$

$$\frac{\theta_1}{\partial_2-\partial_1}\Big[\Im^{\theta_1}_{\partial_1^+}G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_4)+\Im^{\theta_1}_{\partial_2^-}G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_4)\Big]$$
$$\leq_p \frac{\eth_1{}^2-2\eth_1+4-\left(\eth_1{}^2+2\eth_1+4\right)e^{-\eth_1}}{\eth_1{}^3}[G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_4)]$$
$$+\frac{2\eth_1-4+(2\eth_1+4)e^{-\eth_1}}{\eth_1{}^3}[G_1(\partial_1,\partial_4)F_1(\partial_2,\partial_4)+G_1(\partial_2,\partial_4)F_1(\partial_1,\partial_4)], \tag{3.5}$$

$$\frac{\theta_1}{\partial_2-\partial_1}\Big[\Im^{\theta_1}_{\partial_1^+}G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_4)+\Im^{\theta_1}_{\partial_2^-}G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_4)\Big]$$
$$\leq_p \frac{\eth_1{}^2-2\eth_1+4-\left(\eth_1{}^2+2\eth_1+4\right)e^{-\eth_1}}{\eth_1{}^3}[G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_4)]$$
$$+\frac{2\eth_1-4+(2\eth_1+4)e^{-\eth_1}}{\eth_1{}^3}[G_1(\partial_1,\partial_3)F_1(\partial_2,\partial_4)+G_1(\partial_2,\partial_3)F_1(\partial_1,\partial_4)], \tag{3.6}$$

and

$$\frac{\theta_1}{\partial_2-\partial_1}\Big[\Im^{\theta_1}_{\partial_1+}G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_3)+\Im^{\theta_1}_{\partial_2^-}G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_3)\Big]$$
$$\leq_p \frac{\eth_1{}^2-2\eth_1+4-\left(\eth_1{}^2+2\eth_1+4\right)e^{-\eth_1}}{\eth_1{}^3}[G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_4)F_1(\partial_1,\partial_3)]$$
$$+\frac{2\eth_1-4+(2\eth_1+4)e^{-\eth_1}}{\eth_1{}^3}[G_1(\partial_1,\partial_4)F_1(\partial_2,\partial_3)+G_1(\partial_2,\partial_4)F_1(\partial_1,\partial_3)]. \tag{3.7}$$

Substituting the relations (3.4)–(3.7) into (3.3), we get final output.

□

Remark 3.4. • *If one has $\theta_1 \to 1, \theta_2 \to 1$ with $\underline{G_1} \neq \overline{G_1}$ and $\underline{F_1} \neq \overline{F_1}$, we get the following result by the authors in [18].*

$$\frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} G_1(x,y)F_1(x,y)dxdy$$
$$\supseteq\frac{1}{9}C(\partial_1,\partial_2,\partial_3,\partial_4)+\frac{1}{18}[D(\partial_1,\partial_2,\partial_3,\partial_4)+E(\partial_1,\partial_2,\partial_3,\partial_4)]+\frac{1}{36}\Psi(\partial_1,\partial_2,\partial_3,\partial_4).$$

• *If one has $\theta_1 \to 1, \theta_2 \to 1$ with $\underline{G_1} = \overline{G_1}$ and $\underline{F_1} = \overline{F_1}$, we get the following result by the author in [46].*

$$\frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} G_1(x,y)F_1(x,y)dxdy$$
$$\leq\frac{1}{9}C(\partial_1,\partial_2,\partial_3,\partial_4)+\frac{1}{18}[D(\partial_1,\partial_2,\partial_3,\partial_4)+E(\partial_1,\partial_2,\partial_3,\partial_4)]+\frac{1}{36}\Psi(\partial_1,\partial_2,\partial_3,\partial_4).$$

• *If one has $\theta_1 \to 1, \theta_2 \to 1$, we get the following result by the authors in [44].*

Theorem 3.5. *Considering the same hypothesis of Theorem 3.4, we get the following double relations:*

$$4G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)F_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)$$
$$\leq_p\frac{(1-\theta_2)(1-\theta_1)}{4\left(1-e^{-\delta_1}\right)\left(1-e^{-\delta_2}\right)}\Big[\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+}G_1(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}G_1(\partial_2,\partial_3)$$
$$+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}G_1(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-}G_1(\partial_1,\partial_3)\Big]$$
$$+\left[\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1\right]C(\partial_1,\partial_2,\partial_3,\partial_4)+\left[\alpha_1\alpha_2+\alpha_1\beta_2+\beta_1\beta_2\right]D(\partial_1,\partial_2,\partial_3,\partial_4)$$
$$+\left[\alpha_1\alpha_2+\alpha_2\beta_1+\beta_1\beta_2\right]E(\partial_1,\partial_2,\partial_3,\partial_4)+\left[\alpha_1\beta_2+\alpha_2\beta_1+\beta_1\beta_2\right]\Psi(\partial_1,\partial_2,\partial_3,\partial_4),$$

where

$$\alpha_i=\frac{-2+\theta_i+(\theta_i+2)\,e^{-\theta_i}}{\theta_i^2\left(1-e^{-\theta_i}\right)},\quad \beta_i=\frac{-2\theta_i+\theta_i^2+4-\left(\theta_i^2+4+2\theta_i\right)e^{-\theta_i}}{2\theta_i^2\left(1-e^{-\theta_i}\right)}.$$

Proof. Since G_1 and F_1 are interval-valued coordinated convex functions and considering Theorem 2.5 from the reference [45], one has

$$2G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)F_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)$$
$$\leq_p\frac{1-\theta_1}{2\left(1-e^{-\delta_1}\right)}\left[\Im^{\theta_1}_{\partial_1}G_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)+\Im^{\theta_1}_{\partial_2}G_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\right]$$
$$+\alpha_1\left[G_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)+G_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)\right]$$
$$+\beta_1\left[G_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)+G_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)\right] \quad (3.8)$$

and

$$\begin{aligned}
&2G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)F_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\\
&\leq_p\frac{1-\theta_2}{2\left(1-e^{-\eth_2}\right)}\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}G_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)+\mathfrak{I}^{\theta_2}_{\partial_4^-}G_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\right]\\
&\quad+\alpha_2\left[G_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)+G_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)\right]\\
&\quad+\beta_2\left[G_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)+G_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\right]. \qquad (3.9)
\end{aligned}$$

Summing the relations (3.8) and (3.9), then multiplying the result by constant 2, we get that

$$\begin{aligned}
&8G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)F_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\\
&\leq_p\frac{1-\theta_1}{2\left(1-e^{-\eth_1}\right)}\left[2\mathfrak{I}^{\theta_1}_{\partial_1^+}G_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)+2\mathfrak{I}^{\theta_1}_{\partial_2^-}G_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\right]\\
&\quad+\frac{1-\theta_2}{2\left(1-e^{-\eth_2}\right)}\left[2\mathfrak{I}^{\theta_2}_{\partial_3^+}G_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)+2\mathfrak{I}^{\theta_2}_{\partial_4^-}G_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\right]\\
&\quad+\alpha_1\left[2G_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)+2G_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)\right]\\
&\quad+\alpha_2\left[2G_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)+2G_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)\right]\\
&\quad+\beta_1\left[2G_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)+2G_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\right]\\
&\quad+\beta_2\left[2G_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)+2G_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)F_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\right].
\end{aligned}$$

This further implies that

$$\begin{aligned}
&2G_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\\
&\leq_p\frac{1-\theta_2}{2\left(1-e^{-\eth_2}\right)}\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_4^-}G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_3)\right]\\
&\quad+\alpha_2[G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_3)+G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_4)]\\
&\quad+\beta_2[G_1(\partial_1,\partial_3)F_1(\partial_1,\partial_4)+G_1(\partial_1,\partial_4)F_1(\partial_1,\partial_3)], \qquad (3.10)
\end{aligned}$$

$$\begin{aligned}
&2G_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)F_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)\\
&\leq_p\frac{1-\theta_2}{2\left(1-e^{-\eth_2}\right)}\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_4^-}G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_3)\right]\\
&\quad+\alpha_2[G_1(\partial_2,\partial_3)F_1(\partial_2,\partial_3)+G_1(\partial_2,\partial_4)F_1(\partial_2,\partial_4)]
\end{aligned}$$

$$+\beta_2[\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_4)+\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_3)], \tag{3.11}$$

$$\begin{aligned}
&2\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\mathrm{F}_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\\
&\leq_{\mathrm{p}}\frac{1-\theta_1}{2\left(1-\mathrm{e}^{-\eth_1}\right)}\left[\mathfrak{I}^{\theta_1}_{\partial_1}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)+\mathfrak{I}^{\theta_1}_{\partial_2^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)\right]\\
&\quad+\alpha_1[\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)+\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)]\\
&\quad+\beta_1[\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)+\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)],
\end{aligned}$$

$$\begin{aligned}
&2\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)\mathrm{F}_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)\\
&\leq_{\mathrm{p}}\frac{1-\theta_1}{2\left(1-\mathrm{e}^{-\eth_1}\right)}\left[\mathfrak{I}^{\theta_1}_{\partial_1^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_1}_{\partial_2-1}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)\right]\\
&\quad+\alpha_1[\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)+\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)]\\
&\quad+\beta_1[\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)+\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)],
\end{aligned}$$

$$\begin{aligned}
&2\mathrm{G}_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\mathrm{F}_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)\\
&\leq_{\mathrm{p}}\frac{1-\theta_2}{2\left(1-\mathrm{e}^{-\eth_2}\right)}\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_4^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)\right]\\
&\quad+\alpha_2[\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)+\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)]\\
&\quad+\beta_2[\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_2,\partial_4)+\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_2,\partial_3)],
\end{aligned}$$

$$\begin{aligned}
&2\mathrm{G}_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)\mathrm{F}_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\\
&\leq_{\mathrm{p}}\frac{1-\theta_2}{2\left(1-\mathrm{e}^{-\eth_2}\right)}\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_4^-}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)\right]\\
&\quad+\alpha_2[\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)+\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)]\\
&\quad+\beta_2[\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_1,\partial_4)+\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_1,\partial_3)],
\end{aligned}$$

$$\begin{aligned}
&2\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\mathrm{F}_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)\\
&\leq_{\mathrm{p}}\frac{1-\theta_1}{2\left(1-\mathrm{e}^{-\eth_1}\right)}\left[\mathfrak{I}^{\theta_1}_{\partial_1^+}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_1}_{\partial_2^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_4)\right]\\
&\quad+\alpha_1[\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_4)+\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_4)]\\
&\quad+\beta_1[\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_1,\partial_4)+\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_2,\partial_4)],
\end{aligned}$$

$$\begin{aligned}
&2\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)\mathrm{F}_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\\
&\leq_{\mathrm{p}}\frac{1-\theta_1}{2\left(1-\mathrm{e}^{-\eth_1}\right)}\left[\mathfrak{I}^{\theta_1}_{\partial_1^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_3)+\mathfrak{I}^{\theta_1}_{\partial_2^-}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_3)\right]\\
&\quad+\alpha_1[\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_3)+\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_3)]\\
&\quad+\beta_1[\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_1,\partial_3)+\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_2,\partial_3)].
\end{aligned}$$

Substituting the above relations into the relation (3), it follows that

$$
\begin{aligned}
&8\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\mathrm{F}_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\\
\leq_{\mathrm{p}}&\frac{1-\theta_1}{2\left(1-\mathrm{e}^{-\delta_1}\right)}\left[2\Im^{\theta_1}_{\partial_1{}^+}\mathrm{G}_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)\mathrm{F}_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)+2\Im^{\theta_1}_{\partial_2{}^-}\mathrm{G}_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\mathrm{F}_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\right]\\
&+\frac{1-\theta_2}{2\left(1-\mathrm{e}^{-\delta_2}\right)}\left[2\Im^{\theta_2}_{\partial_3{}^+}\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)\mathrm{F}_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)+2\Im^{\theta_2}_{\partial_4{}^-}\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\mathrm{F}_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\right]\\
&+\frac{(1-\theta_2)\alpha_1}{2\left(1-\mathrm{e}^{-\delta_2}\right)}\left[\Im^{\theta_2}_{\partial_3{}^+}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)+\Im^{\theta_2}_{\partial_4{}^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)\right.\\
&\left.+\Im^{\theta_2}_{\partial_3{}^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)+\Im^{\theta_2}_{\partial_4{}^-}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)\right]\\
&+\frac{(1-\theta_2)\beta_1}{2\left(1-\mathrm{e}^{-\delta_2}\right)}\left[\Im^{\theta_2}_{\partial_3{}^+}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)+\Im^{\theta_2}_{\partial_4{}^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)\right.\\
&\left.+\Im^{\theta_2}_{\partial_3{}^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)+\Im^{\theta_2}_{\partial_4{}^-}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)\right]\\
&+\frac{(1-\theta_1)\alpha_2}{2\left(1-\mathrm{e}^{-\delta_1}\right)}\left[\Im^{\theta_1}_{\partial_1{}^+}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)+\Im^{\theta_1}_{\partial_2{}^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)\right.\\
&\left.+\Im^{\theta_1}_{\partial_1{}^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)+\Im^{\theta_2}_{\partial_2{}^-}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)\right]\\
&+\frac{(1-\theta_1)\beta_2}{2\left(1-\mathrm{e}^{-\delta_1}\right)}\left[\Im^{\theta_1}_{\partial_1{}^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_3)+\Im^{\theta_1}_{\partial_2{}^-}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_3)\right.\\
&\left.+\Im^{\theta_1}_{\partial_1{}^+}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_4)+\Im^{\theta_1}_{\partial_2{}^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_4)\right]\\
&+2\alpha_1\alpha_2C(\partial_1,\partial_2,\partial_3,\partial_4)+2\alpha_1\beta_2D(\partial_1,\partial_2,\partial_3,\partial_4)+2\alpha_2\beta_1E(\partial_1,\partial_2,\partial_3,\partial_4)+2\beta_1\beta_2\Psi(\partial_1,\partial_2,\partial_3,\partial_4).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&2\Im^{\theta_1}_{\partial_1{}^+}\mathrm{G}_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)\mathrm{F}_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)\\
\leq_{\mathrm{p}}&\frac{1-\theta_2}{2\left(1-\mathrm{e}^{-\delta_2}\right)}\left[\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_3{}^+}\mathrm{G}_1(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-}\mathrm{G}_1(\partial_2,\partial_3)\right]\\
&+\alpha_2\left[\Im^{\theta_1}_{\partial_1{}^+}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)+\Im^{\theta_1}_{\partial_1{}^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)\right]\\
&+\beta_2\left[\Im^{\theta_1}_{\partial_1{}^+}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_4)+\Im^{\theta_1}_{\partial_1{}^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_3)\right],
\end{aligned}
$$

$$
\begin{aligned}
&2\Im^{\theta_1}_{\partial_2{}^-}F\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\mathrm{F}_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\\
\leq_{\mathrm{p}}&\frac{1-\theta_2}{2\left(1-\mathrm{e}^{-\delta_2}\right)}\left[\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_3{}^+}\mathrm{G}_1(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^-}\mathrm{G}_1(\partial_1,\partial_3)\right]\\
&+\alpha_2\left[\Im^{\theta_1}_{\partial_2{}^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)+\Im^{\theta_1}_{\partial_2{}^-}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)\right]\\
&+\beta_2\left[\Im^{\theta_1}_{\partial_2{}^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_4)+\Im^{\theta_1}_{\partial_2{}^-}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_3)\right],
\end{aligned}
$$

$$
\begin{aligned}
&2\mathfrak{I}^{\theta_2}_{\partial_3^+}F\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)\mathrm{F}_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)\\
\leq_{\mathrm{p}}&\frac{1-\theta_1}{2\left(1-\mathrm{e}^{-\delta_1}\right)}\left[\mathfrak{I}^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+}\mathrm{G}_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}\mathrm{G}_1(\partial_1,\partial_4)\right]\\
&+\alpha_1\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_3^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)\right]\\
&+\beta_1\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_3^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)\right],
\end{aligned}
$$

$$
\begin{aligned}
&2\mathfrak{I}^{\theta_2}_{\partial_4^-}F\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\mathrm{F}_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\\
\leq_{\mathrm{p}}&\frac{1-\theta_1}{2\left(1-\mathrm{e}^{-\delta_1}\right)}\left[\mathfrak{I}^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}\mathrm{G}_1(\partial_2,\partial_3)+\mathfrak{I}^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-}\mathrm{G}_1(\partial_1,\partial_3)\right]\\
&+\alpha_1\left[\mathfrak{I}^{\theta_2}_{\partial_4^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)+\mathfrak{I}^{\theta_2}_{\partial_4^-}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)\right]\\
&+\beta_1\left[\mathfrak{I}^{\theta_2}_{\partial_4^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)+\mathfrak{I}^{\theta_2}_{\partial_4^-}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)\right].
\end{aligned}
$$

Summing the above relations, it follows that

$$
\begin{aligned}
&8\mathrm{G}_1\left(\frac{\partial_1+\partial_2}{2}\ ,\frac{\partial_3+\partial_4}{2}\right)\mathrm{F}_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\\
\leq_{\mathrm{p}}&\frac{(1-\theta_2)(1-\theta_1)}{2\left(1-\mathrm{e}^{-\delta_1}\right)\left(1-\mathrm{e}^{-\delta_2}\right)}\left[\mathfrak{I}^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+}\mathrm{G}_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}\mathrm{G}_1(\partial_2,\partial_3)\right.\\
&\left.+\mathfrak{I}^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}\mathrm{G}_1(\partial_1,\partial_4)+\mathfrak{I}^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-}\mathrm{G}_1(\partial_1,\partial_3)\right]\\
&+\frac{(1-\theta_2)\alpha_1}{1-\mathrm{e}^{-\delta_2}}\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_4^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)\right.\\
&\left.+\mathfrak{I}^{\theta_2}_{\partial_3^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_4^-}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)\right]\\
&+\frac{(1-\theta_2)\beta_1}{1-\mathrm{e}^{-\delta_2}}\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_4^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)\right.\\
&\left.+\mathfrak{I}^{\theta_2}_{\partial_3^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_4^-}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)\right]\\
&+\frac{(1-\theta_1)\alpha_2}{1-e_1^{-\delta_1}}\left[\mathfrak{I}^{\theta_1}_{\partial_1^+}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_3)+\mathfrak{I}^{\theta_1}_{\partial_2^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_3)\right.\\
&\left.+\mathfrak{I}^{\theta_1}_{\partial_1+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_1}_{\partial_2^-}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_4)\right]\\
&+\frac{(1-\theta_1)\beta_2}{1-\mathrm{e}^{-\delta_1}}\left[\mathfrak{I}^{\theta_1}_{\partial_1^+}\mathrm{G}_1(\partial_2,\partial_4)\mathrm{F}_1(\partial_2,\partial_3)+\mathfrak{I}^{\theta_1}_{\partial_2^-}\mathrm{G}_1(\partial_1,\partial_4)\mathrm{F}_1(\partial_1,\partial_3)\right.\\
&\left.+\mathfrak{I}^{\theta_1}_{\partial_1^+}\mathrm{G}_1(\partial_2,\partial_3)\mathrm{F}_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_1}_{\partial_2^-}\mathrm{G}_1(\partial_1,\partial_3)\mathrm{F}_1(\partial_1,\partial_4)\right]\\
&+2\alpha_1\alpha_2\mathrm{C}(\partial_1,\partial_2,\partial_3)+2\alpha_1\beta_2\mathrm{D}(\partial_1,\partial_2,\partial_3,\partial_4)\\
&+2\alpha_2\beta_1\mathrm{E}(\partial_1,\partial_2,\partial_3)+2\beta_1\beta_2\Psi(\partial_1,\partial_2,\partial_3).
\end{aligned}
$$

□

As a result, Theorem 3.5 is accomplished.

Remark 3.5. • *If one has $\theta_1 \to 1, \theta_2 \to 1$ with $\underline{G_1} \neq \overline{G_1}$ and $\underline{F_1} \neq \overline{F_1}$, we get the following result by the authors in [18].*

$$\begin{aligned}
&4G_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right)F_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right)\\
\supseteq&\frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} G_1(x,y)F_1(x,y)dxdy\\
&+\frac{5}{36}C(\partial_1,\partial_2,\partial_3,\partial_4)+\frac{7}{36}[D(\partial_1,\partial_2,\partial_3,\partial_4)+E(\partial_1,\partial_2,\partial_3,\partial_4)]+\frac{2}{9}\Psi(\partial_1,\partial_2,\partial_3,\partial_4).
\end{aligned}$$

• *If one has $\theta_1 \to 1, \theta_2 \to 1$ with $\underline{G_1} = \overline{G_1}$ and $\underline{F_1} = \overline{F_1}$, we get the following result by the author in [46].*

$$\begin{aligned}
&4G_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right)F_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right)\\
\leq&\frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} G_1(x,y)F_1(x,y)dxdy\\
&+\frac{5}{36}C(\partial_1,\partial_2,\partial_3,\partial_4)+\frac{7}{36}[D(\partial_1,\partial_2,\partial_3,\partial_4)+E(\partial_1,\partial_2,\partial_3,\partial_4)]+\frac{2}{9}\Psi(\partial_1,\partial_2,\partial_3,\partial_4).
\end{aligned}$$

• *If one has $\theta_1 \to 1, \theta_2 \to 1$, we get the following result by the authors in [44].*

Theorem 3.6. *Considering the same hypothesis of Theorem 3.1, we get the following double relations:*

$$\begin{aligned}
&G_1\left(\frac{\partial_1+\partial_2}{2}, \frac{\partial_3+\partial_4}{2}\right)\leq_p\frac{1-\theta_1}{4\left(1-e^{-\delta_1}\right)}\left[\mathfrak{I}^{\theta_1}_{\partial_1^+}G_1\left(\partial_2, \frac{\partial_3+\partial_4}{2}\right)+\mathfrak{I}^{\theta_1}_{\partial_2^-}G_1\left(\partial_1, \frac{\partial_3+\partial_4}{2}\right)\right]\\
&+\frac{1-\theta_2}{4\left(1-e^{-\delta_2}\right)}\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}G_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)+\mathfrak{I}^{\theta_2}_{\partial_4^-}G_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\right]\\
\leq_p&\frac{(1-\theta_2)(1-\theta_1)}{4\left(1-e^{-\delta_1}\right)\left(1-e^{-\delta_2}\right)}\left[\mathfrak{I}^{\theta_1,\theta_2}_{\partial_1^+,\partial_3^+}G_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_1,\theta_2}_{\partial_1^+,\partial_4^-}G_1(\partial_2,\partial_3)+\mathfrak{I}^{\theta_1,\theta_2}_{\partial_2^-,\partial_3^+}G_1(\partial_1,\partial_4)+\mathfrak{I}^{\theta_1,\theta_2}_{\partial_2^-,\partial_4^-}G_1(\partial_1,\partial_3)\right]\\
\leq_p&\frac{1-\theta_1}{8\left(1-e^{-\delta_1}\right)}\left[\mathfrak{I}^{\theta_1}_{\partial_1^+}G_1(\partial_2,\partial_3)+\mathfrak{I}^{\theta_1}_{\partial_1^+}G_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_1}_{\partial_2^-}G_1(\partial_1,\partial_3)+\mathfrak{I}^{\theta_1}_{\partial_2^-}G_1(\partial_1,\partial_4)\right]\\
&+\frac{1-\theta_2}{8\left(1-e^{-\delta_2}\right)}\left[\mathfrak{I}^{\theta_2}_{\partial_3^+}G_1(\partial_1,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_3^+}G_1(\partial_2,\partial_4)+\mathfrak{I}^{\theta_2}_{\partial_4^-}G_1(\partial_1,\partial_3)+\mathfrak{I}^{\theta_2}_{\partial_4^-}G_1(\partial_2,\partial_3)\right]\\
\leq_p&\frac{G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4)}{4}.
\end{aligned}\tag{3.12}$$

Proof. As G_1 is an interval-valued coordinated convex function, it follows that $G_{1x} : [\partial_3, \partial_4] \to \mathbb{R}, G_{1x}(y) = G_1(x, y)$ are convex over $[\partial_3, \partial_4]$ for each $x \in [\partial_1, \partial_2]$, and we have

$$G_{1x}\left(\frac{\partial_3+\partial_4}{2}\right)\leq_p\frac{1-\theta_2}{2\left(1-e^{-\delta_2}\right)}\left[\mathfrak{I}^{\theta_2}_{\partial_3+}G_{1x}(\partial_2)+\mathfrak{I}^{\theta_2}_{\partial_4-}G_{1x}(\partial_3)\right]\leq_p\frac{G_{1x}(\partial_3)+G_{1x}(\partial_4)}{2}, x\in[\partial_1,\partial_2].$$

This indicates that

$$\begin{aligned}
G_1\left(x, \frac{\partial_3+\partial_4}{2}\right)\leq_p&\frac{1-\theta_2}{2\left(1-e^{-\delta_2}\right)}\left[\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)dy+\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)dy\right]\\
\leq_p&\frac{G_1(x,\partial_3)+G_1(x,\partial_4)}{2}.
\end{aligned}$$

Multiplying the above relation with $\frac{1-\theta_1}{2\left(1-e^{-\eth_1}\right)}\frac{1}{\theta_1}e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}$ and $\frac{1-\theta_1}{2\left(1-e^{-\eth_1}\right)}\frac{1}{\theta_1}e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}$ and integrating, one has

$$\begin{aligned}
&\frac{1-\theta_1}{2\left(1-e^{-\eth_1}\right)}\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}G_1\left(x,\frac{\partial_3+\partial_4}{2}\right)dx\\
\leq_p&\frac{(1-\theta_1)(1-\theta_2)}{4\left(1-e^{-\eth_1}\right)\left(1-e^{-\eth_2}\right)}\left[\frac{2}{\theta_1\theta_2}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-y)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)dxdy\right.\\
&\left.+\frac{1}{\theta_1\theta_2}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)dxdy\right]\\
\leq_p&\frac{1-\theta_1}{4\left(1-e^{-\eth_1}\right)}\left[\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}G_1(x,\partial_3)dx+\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}G_1(x,\partial_4)dx\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1-\theta_1}{2\left(1-e^{-\eth_1}\right)}\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}G_1\left(x,\frac{\partial_3+\partial_4}{2}\right)dx\\
\leq_p&\frac{(1-\theta_1)(1-\theta_2)}{4\left(1-e^{-\eth_1}\right)\left(1-e^{-\eth_2}\right)}\left[\frac{1}{\theta_1\theta_2}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)dxdy\right.\\
&\left.+\frac{1}{\theta_1\theta_2}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)dxdy\right]\\
\leq_p&\frac{1-\theta_1}{4\left(1-e^{-\eth_1}\right)}\left[\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}G_1(x,\partial_3)dx+\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}G_1(x,\partial_4)dx\right].
\end{aligned}$$

By a similar argument applied on the mapping $G_{1_y}:[\partial_1,\partial_2]\to\mathsf{R}, G_{1_y}=G_1(x,y)$, it yields that

$$\begin{aligned}
&\frac{1-\theta_2}{2\left(1-e^{-\eth_2}\right)}\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1\left(\frac{\partial_1+\partial_2}{2},y\right)dy\\
\leq_p&\frac{(1-\theta_1)(1-\theta_2)}{4\left(1-e^{-\eth_1}\right)\left(1-e^{-\eth_2}\right)}\left[\frac{1}{\theta_1\theta_2}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)dxdy\right.\\
&\left.+\frac{1}{\theta_1\theta_2}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(x,y)dxdy\right]\\
\leq_p&\frac{1-\theta_2}{4\left(1-e^{-\eth_2}\right)}\left[\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(\partial_1,y)dy+\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(\partial_2,y)dy\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1-\theta_2}{2\left(1-e^{-\eth_2}\right)}\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1\left(\frac{\partial_1+\partial_2}{2},y\right)dy\\
\leq_p&\frac{(1-\theta_1)(1-\theta_2)}{4\left(1-e^{-\eth_1}\right)\left(1-e^{-\eth_2}\right)}\left[\frac{1}{\theta_1\theta_2}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)dxdy\right.\\
&\left.+\frac{1}{\theta_1\theta_2}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(x,y)dxdy\right]\\
\leq_p&\frac{1-\theta_2}{4\left(1-e^{-\eth_2}\right)}\left[\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(\partial_1,y)dy+\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(\partial_2,y)dy\right].
\end{aligned}$$

Adding the above relations, we get

$$
\begin{aligned}
&\frac{1-\theta_1}{4\left(1-e^{-\eth_1}\right)}\left[\Im^{\theta_1}_{\partial_1{}^+}G_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)+\Im^{\theta_1}_{\partial_2{}^-}G_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\right]\\
&\quad+\frac{1-\theta_2}{4\left(1-e^{-\eth_2}\right)}\left[\Im^{\theta_2}_{\partial_3{}^+}G_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)+\Im^{\theta_2}_{\partial_4{}^-}G_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\right]\\
&\leq_p\frac{(1-\theta_2)(1-\theta_1)}{4\left(1-e^{-\eth_1}\right)\left(1-e^{-\eth_2}\right)}\left[\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_3{}^+}G_1(\partial_2,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_1{}^+,\partial_4{}^-}G_1(\partial_2,\partial_3)\right.\\
&\quad\left.+\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_3{}^+}G_1(\partial_1,\partial_4)+\Im^{\theta_1,\theta_2}_{\partial_2{}^-,\partial_4{}^-}G_1(\partial_1,\partial_3)\right]\\
&\leq_p\frac{1-\theta_1}{8\left(1-e^{-\eth_1}\right)}\left[\Im^{\theta_1}_{\partial_1{}^+}G_1(\partial_2,\partial_3)+\Im^{\theta_1}_{\partial_1{}^+}G_1(\partial_2,\partial_4)+\Im^{\theta_1}_{\partial_2{}^-}G_1(\partial_1,\partial_3)+\Im^{\theta_1}_{\partial_2{}^-}G_1(\partial_1,\partial_4)\right]\\
&\quad+\frac{1-\theta_2}{8\left(1-e^{-\eth_2}\right)}\left[\Im^{\theta_2}_{\partial_3{}^+}G_1(\partial_1,\partial_4)+\Im^{\theta_2}_{\partial_3{}^+}G_1(\partial_2,\partial_4)+\Im^{\theta_2}_{\partial_4{}^-}G_1(\partial_1,\partial_3)+\Im^{\theta_2}_{\partial_4{}^-}G_1(\partial_2,\partial_3)\right].
\end{aligned}
$$

The second and third relations in Theorem 3.6 are thus obtained. We may ascertain that by applying the first relation from Theorem 3.3,

$$
\begin{aligned}
G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\leq_p\frac{1-\theta_1}{2\left(1-e^{-\eth_1}\right)}&\left[\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2}e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}G_1\left(x,\frac{\partial_3+\partial_4}{2}\right)dx\right.\\
&\left.+\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2}e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}G_1\left(x,\frac{\partial_3+\partial_4}{2}\right)dx\right],
\end{aligned}
$$

and

$$
\begin{aligned}
G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\leq_p\frac{1-\theta_2}{2\left(1-e^{-\eth_2}\right)}&\left[\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4}e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1\left(\frac{\partial_1+\partial_2}{2},y\right)dy\right.\\
&\left.+\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4}e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1\left(\frac{\partial_1+\partial_2}{2},y\right)dy\right].
\end{aligned}
$$

By addition,

$$
\begin{aligned}
G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\leq_p\frac{1-\theta_1}{4\left(1-e^{\eth_1}\right)}&\left[\Im^{\theta_1}_{\partial_1{}^+}G_1\left(\partial_2,\frac{\partial_3+\partial_4}{2}\right)+\Im^{\theta_1}_{\partial_2{}^-}G_1\left(\partial_1,\frac{\partial_3+\partial_4}{2}\right)\right]\\
+\frac{1-\theta_2}{4\left(1-e^{-\eth_2}\right)}&\left[\Im^{\theta_2}_{\partial_3{}^+}G_1\left(\frac{\partial_1+\partial_2}{2},\partial_4\right)+\Im^{\theta_2}_{\partial_4{}^-}G_1\left(\frac{\partial_1+\partial_2}{2},\partial_3\right)\right].
\end{aligned}
$$

The first relation in Theorem 3.6 is therefore inferred. Lastly, once more, we have

$$\frac{1-\theta_1}{2\left(1-e^{-\eth_1}\right)}\left[\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}G_1(x,\partial_3)dx+\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}G_1(x,\partial_3)dx\right]$$
$$\leq_p \frac{G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)}{2},$$
$$\frac{1-\theta_1}{2\left(1-e^{-\eth_1}\right)}\left[\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(\partial_2-x)}G_1(x,\partial_4)dx+\frac{1}{\theta_1}\int_{\partial_1}^{\partial_2} e^{-\frac{1-\theta_1}{\theta_1}(x-\partial_1)}G_1(x,\partial_4)dx\right]$$
$$\leq_p \frac{G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4)}{2},$$
$$\frac{1-\theta_2}{2\left(1-e^{-\eth_2}\right)}\left[\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(\partial_1,y)dy+\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(\partial_1,y)dy\right]$$
$$\leq_p \frac{G_1(\partial_1,\partial_3)+G_1(\partial_1,\partial_4)}{2},$$
$$\frac{1-\theta_2}{2\left(1-e^{-\eth_2}\right)}\left[\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(\partial_4-y)}G_1(\partial_2,y)dy+\frac{1}{\theta_2}\int_{\partial_3}^{\partial_4} e^{-\frac{1-\theta_2}{\theta_2}(y-\partial_3)}G_1(\partial_2,y)dy\right]$$
$$\leq_p \frac{G_1(\partial_2,\partial_3)+G_1(\partial_2,\partial_4)}{2}.$$

Therefore, the proof of Theorem 3.6 is verfied. □

Remark 3.6.

- *If we take $\theta_1 \to 1, \theta_2 \to 1$, then one has*

$$\lim_{\theta_1\to 1}\frac{1-\theta_1}{4\left(1-e^{-\eth_1}\right)}=\frac{1}{4(\partial_2-\partial_1)},$$
$$\lim_{\theta_2\to 1}\frac{1-\theta_2}{4\left(1-e^{-\eth_2}\right)}=\frac{1}{4(\partial_4-\partial_3)},$$

$$\lim_{\substack{\theta_1\to 1\\ \theta_2\to 1}}\frac{(1-\theta_2)(1-\theta_1)}{4\left(1-e^{\eth_1}\right)\left(1-e^{\eth_2}\right)}=\frac{1}{4(\partial_2-\partial_1)(\partial_4-\partial_3)},$$

  and we get the following result by the authors in [18].
- *If one has $\theta_1 \to 1, \theta_2 \to 1$ with $\underline{G_1}=\overline{G_1}$, we get the following result by the authors in [17].*

$$G_1\left(\frac{\partial_1+\partial_2}{2},\frac{\partial_3+\partial_4}{2}\right)\leq \frac{1}{2}\left[\frac{1}{\partial_2-\partial_1}\int_{\partial_1}^{\partial_2}G_1\left(x,\frac{\partial_3+\partial_4}{2}\right)dx+\frac{1}{\partial_4-\partial_3}\int_{\partial_3}^{\partial_4}G_1\left(\frac{\partial_1+\partial_2}{2},y\right)dy\right]$$
$$\leq \frac{1}{(\partial_2-\partial_1)(\partial_4-\partial_3)}\int_{\partial_1}^{\partial_2}\int_{\partial_3}^{\partial_4}G_1(x,y)dxdy$$
$$\leq \frac{1}{4}\left[\frac{1}{\partial_2-\partial_1}\int_{\partial_1}^{\partial_2}G_1(x,\partial_3)dx+\frac{1}{\partial_2-\partial_1}\int_{\partial_1}^{\partial_2}G_1(x,\partial_4)dx\right.$$
$$\left.+\frac{1}{\partial_4-\partial_3}\int_{\partial_3}^{\partial_4}G_1(\partial_1,y)dy+\frac{1}{\partial_4-\partial_3}\int_{\partial_3}^{\partial_4}G_1(\nu,y)dy\right]$$
$$\leq \frac{G_1(\partial_1,\partial_3)+G_1(\partial_2,\partial_3)+G_1(\partial_1,\partial_4)+G_1(\partial_2,\partial_4)}{4}.$$

4. Discussion and conclusions

Fractional calculus and convex optimization are two strong mathematical techniques with numerous applications in a variety of domains. We are mainly focusing on generalizing the results of these articles [17, 18, 44, 46] that the authors have recently developed by using classical integral operators and order relations. A number of recently developed results have been achieved using partial order, standard order relations, and classical integral operators, which are all special cases of pseudo order relations along with fractional operators having non-singular kernels.

As a part of our key results, we constructed H.H, Fejér, and Pachpatte type integral inequalities. Further, we first fixed some notions related to standard, partial interval, and pseudo order relations in order to demonstrate the differences between all these relations, as well as remarks on all main findings. This motivation inspired us to suggest a unique and fresh idea for readers to apply this novel approach regarding stochastic integration that involves Brownian motion to develop these results that is,

$$\int_0^t \mathrm{F}d\mathrm{B} = \lim_{n\to\infty} \sum_{[t_{i-1},t_i]\in\pi_n} \mathrm{F}_{t_{i-1}} \left(\mathrm{B}_{t_i} - \mathrm{B}_{t_{i-1}}\right),$$

where dB is Brownian motion.

This study's findings are anticipated to have a substantial influence on inequality and the development of optimization theory.

Author contributions

Zareen A. Khan: Methodology, visualization, writing—original draft; Waqar Afzal: Conceptualization, methodology, validation, writing—original draft; Mujahid Abbas: Conceptualization, investigation, visualization, writing—original draft, writing—review and editing; Jong-Suk Ro: Investigation, validation, writing—review and editing; Abdullah A. Zaagan: Methodology, visualization. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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