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Research article

Sufficient criteria for oscillation of even-order neutral differential equations with distributed deviating arguments

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Abstract: This paper presents novel criteria for investigating the oscillatory behavior of even-order neutral differential equations. By employing a comparative approach, we established the oscillation properties of the studied equation through comparisons with well-understood first-order equations with known oscillatory behavior. The findings of this study introduce fresh perspectives and enrich the existing body of oscillation criteria found in the literature. To illustrate the practical application of our results, we provide an illustrative example.

Keywords: even-order; oscillatory behavior; neutral differential equations; distributed deviating arguments
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this study, we focus on even-order neutral differential equations (NDE) of the form:

$$\left(\xi(\mathrm{s})\,\mathrm{U}^{(r-1)}(\mathrm{s})\right)' + \int_{\alpha}^{\beta} \mathcal{K}(\mathrm{s},\mathfrak{h})\,\mathrm{y}(\rho(\mathrm{s},\mathfrak{h}))\,d\mathfrak{h} = 0, \quad \mathrm{s} \geq \mathrm{s}_0, \tag{1.1}$$

where $r \geq 4$ is an even integer and $\mathrm{U} := \mathrm{y}(\mathrm{s}) + \upsilon(\mathrm{s})\,\mathrm{y}(\tau(\mathrm{s}))$. Throughout our analysis, we make the following assumptions:

(A_1) $\xi \in C([s_0,\infty))$, $\xi > 0$, $\xi' \geq 0$ and
$$\int_{s_0}^{\infty} \frac{1}{\xi(\mathfrak{h})} d\mathfrak{h} < \infty; \tag{1.2}$$
(A_2) $\mathcal{K} \in C([s_0,\infty) \times (\alpha,\beta), \mathbb{R})$, and $\mathcal{K}(s,\mathfrak{h}) \geq 0$;
(A_3) $\upsilon \in C[s_0,\infty)$, and $0 < \upsilon < \upsilon_0$;
(A_4) $\rho \in C([s_0,\infty) \times (\alpha,\beta), \mathbb{R})$, $\rho(s,\mathfrak{h}) < s$ for $\mathfrak{h} \in [\alpha,\beta]$, and $\lim_{s\to\infty} \rho(s,\mathfrak{h}) = \infty$ for $\mathfrak{h} \in [\alpha,\beta]$;
(A_5) $\tau \in C[s_0,\infty)$, $\tau(s) < s$, $\tau'(s) \geq 0$ and $\lim_{s\to\infty} \tau(s) = \infty$.

A function $y \in C^{r-1}([s_y,\infty))$, $s_y \geqslant s_0$ is said to be a solution of (1.1), which has the property $\xi U^{(r-1)} \in C^1[s_y,\infty)$, and satisfies the Eq (1.1) for all $y \in [s_y,\infty)$. We consider only those solutions y of (1.1), which exist on some half-line $[s_y,\infty)$ and satisfy the condition
$$\sup\{|y(s)| : s \geqslant s_y\} > 0, \text{ for all } s \geq s_y.$$

Definition 1.1. *A solution of (1.1) is considered oscillatory if it alternates between neither positive nor negative values, otherwise, it is classified as nonoscillatory.*

Differential equations are a foundational tool in mathematics and science, serving as a bridge between theory and real-world phenomena. They are essential for modeling and understanding a wide array of dynamic processes in fields as diverse as physics, engineering, biology, economics, and ecology; see [1–6]. Differential equations describe how quantities change with respect to one another, capturing the rate of change and providing a means to predict future behavior. They are classified into various types, such as ordinary differential equations (ODEs) and partial differential equations (PDEs), depending on the nature of the variables involved. Several techniques, including numerical and symbolic techniques, can be used to solve ODEs so they can contribute to the knowledge, see [7–9]. Differential equations have been instrumental in solving complex problems, from predicting the trajectory of celestial bodies to optimizing industrial processes; see [10]. This paper delves into the realm of differential equations, specifically focusing on even-order neutral differential equations, and presents novel criteria to analyze their oscillatory behavior, thereby contributing to the ongoing exploration of these mathematical tools in practical applications.

The highest-order derivative of the unknown function appears in a neutral delay differential equation both with and without delay. In addition to its theoretical significance, the qualitative analysis of these equations holds great practical significance. This is because neutral differential equations are involved in a number of phenomena, such as the study of vibrating masses attached to elastic bars, the solution of variational problems with time delays, and problems involving electric networks with lossless transmission lines (such as those found in high-speed computers, where these lines are used to connect switching circuits); see [1, 2]. Furthermore, it is evident that the ongoing advancements in science and technology give rise to a multitude of phenomena and unresolved challenges; see [11–14].

As a result, numerous theories have surfaced, including oscillation theory, a subset of qualitative theory, aimed at addressing inquiries concerning the oscillatory patterns and affinity characteristics exhibited by solutions of differential equations (DEs); see [15, 16]. Over the past few decades, the oscillation theory pertaining to second-order differential equations has garnered significant attention in the research community. For the latest advancements and comprehensive summaries of established findings in this field, we direct the reader to references [17–19].

Delay differential equations (DDEs) belong to the class of functional differential equations designed to incorporate the temporal memory of dynamic processes. Consequently, their wide-ranging applications are evident across the realms of physics, engineering, biology (see [20, 21]), and various other scientific disciplines, as documented in references [1, 21, 22]. Furthermore, a comprehensive body of research, documented in monographs [23, 24], has been compiled, encompassing a plethora of results, methodologies, and approaches dedicated to the analysis of oscillatory behavior in solutions of DDEs.

Given the significance of neutral differential equations in representing a wide range of phenomena in the natural sciences and engineering [25, 26], researchers have extensively investigated the qualitative properties of solutions to such equations using diverse analytical techniques; see [27–31].

In their work, Baculikova et al. [32] examined the oscillation criteria for the differential equation represented by

$$\left[\xi(\mathrm{s})\left(\mathrm{y}^{(r-1)}(\mathrm{s})\right)^{\alpha}\right]' + \mathcal{K}(\mathrm{s}) f(\mathrm{y}(\rho(\mathrm{s}))) = 0. \tag{1.3}$$

They established that (1.3) exhibits oscillatory behavior under the conditions where the delay differential equation, denoted as:

$$y'(\mathrm{s}) + \mathcal{K}(\mathrm{s}) f\left(\frac{\delta\rho^{r-1}(\mathrm{s})}{(r-1)!\,\xi^{\frac{1}{\alpha}}(\rho(\mathrm{s}))}\right) f\left(y^{\frac{1}{\alpha}}(\rho(\mathrm{s}))\right) = 0$$

is also oscillatory, while concurrently satisfying the assumption expressed by

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\xi^{1/\alpha}(s)} \mathrm{d}s = \infty. \tag{1.4}$$

In [33], Zhang et al. investigated the asymptotic behavior of solutions to the equation:

$$\left(\xi(\mathrm{s})\left(\mathrm{y}^{(r-1)}(\mathrm{s})\right)^{\alpha}\right)' + \mathcal{K}(\mathrm{s})\,\mathrm{y}^{\beta}(\rho(\mathrm{s})) = 0, \tag{1.5}$$

where α and β are ratios of odd positive integers, $\beta \leq \alpha$ and

$$\int_{\mathrm{s}_0}^{\infty} \xi^{-1/\alpha}(s)\,\mathrm{d}s < \infty. \tag{1.6}$$

Meanwhile, Elabbasy et al. in [34] examined a fourth-order delay differential equation:

$$(\xi(\mathrm{s})(\mathrm{y}'''(\mathrm{s}))^{\alpha})' + \upsilon(\mathrm{s})(\mathrm{y}'''(\mathrm{s}))^{\alpha} + \mathcal{K}(\mathrm{s})\,\mathrm{y}^{\beta}(\rho(\mathrm{s})) = 0, \tag{1.7}$$

where $\alpha = \beta = 1$, and they demonstrated that (1.7) is oscillatory if

$$\int_{\mathrm{s}_0}^{\infty} \left(\rho_*(s)\mathcal{K}(s)\frac{\mu}{2}\rho^2(s) - \frac{1}{4\rho_*(s)\xi(s)}\left[\frac{\rho'_{*+}(s)}{\rho_*(s)} - \frac{\upsilon_*(s)}{\xi(s)}\right]^2\right)\mathrm{d}s = \infty,$$

for some $\mu \in (0, 1)$, and

$$\int_{\mathrm{s}_0}^{\infty} \left[\vartheta(s)\int_s^{\infty}\left[\frac{1}{\xi(\upsilon)}\int_{\upsilon}^{\infty}\mathcal{K}(\nu)\left(\frac{\rho^2(\nu)}{\nu^2}\right)d\nu\right]d\upsilon - \frac{(\vartheta'(s))^2}{4\vartheta(s)}\right]\mathrm{d}s = \infty,$$

under the condition

$$\int_{s_0}^{\infty} \left[\frac{1}{\xi(s)} \exp\left(-\int_{s_0}^{s} \frac{\upsilon(u)}{\xi(u)} du \right) \right]^{1/\alpha} ds = \infty. \tag{1.8}$$

Under the canonical case $\int_{s_0}^{\infty} \xi^{-1/\gamma}(s)\, ds = \infty$, Xing et al. [35] studied the oscillatory behavior of higher-order quasi-linear neutral differential equation

$$\left\{ \xi(s) \left(y(s) + p(s)\, y(\tau(s))^{(n-1)} \right)^{\gamma} \right\}' + \mathcal{K}(s)\, y^{\gamma}(\rho(s)) = 0, \tag{1.9}$$

where $n \geq 2$ and $\gamma \leq 1$ is the quotient of odd positive integers. Various theorems and lemmas were presented to establish oscillation conditions for these differential equations, with a particular emphasis on odd-order equations by using a comparison technique. By using the Riccati transformation technique and some inequalities, Dzurina et al. [36] established oscillation theorems for all solutions to even order quasilinear neutral differential equation

$$\left(\left(y(s) + p(s)\, y(\tau(s))^{(n-1)} \right)^{\gamma} \right)' + \mathcal{K}(s)\, y^{\gamma}(\rho(s)) = 0. \tag{1.10}$$

Under the condition (1.2), Baculíková et al. [37] studied the oscillatory behavior of a class of fourth-order neutral differential equations with a $p-$Laplacian-like operator using the Riccati transformation and integral averaging technique. A Kamenev-type oscillation criterion is presented

$$\left(\xi(s) \left| U'''(s) \right|^{p-2} U'''(s) \right)' + \sum_{i=1}^{l} \mathcal{K}_i(s)\, y(\rho_i(s)) = 0, \tag{1.11}$$

where $n \geq 2$ is an even integer, $p > 1$ is constant, and $U := y(s) + \upsilon(s)\, y(\tau(s))$.

under the assumptions that $\xi \in C([s_0, \infty))$, $\xi > 0$, $\xi' \geq 0$. In [38], the asymptotic properties of the solutions of a class of even-order damped differential equations

$$\left(\xi(s) \left| y^{(n-1)}(s) \right|^{p-2} y^{(n-1)}(s) \right)' + r(s) \left| y^{(n-1)}(s) \right|^{p-2} y^{(n-1)}(s) + \mathcal{K}(s) \left| y(\rho(s)) \right|^{p-2} y(\rho(s)) = 0, \tag{1.12}$$

with $p-$Laplacian-like operators, delayed and advanced arguments, was examined by Liu et al, where $n \geq 2$ is an even integer, and $p > 1$ is constant. Moaaz et al. in [39] examined the asymptotic behavior of solutions of a class of higher-order delay differential equations (DDEs) of the form

$$\left(\xi(s)\, v^{(n-1)}(s) \right)' + (h.(f \circ v \circ g))(s) = 0, \tag{1.13}$$

where $n \in \mathbb{Z}^+$ an even number, $n \geq 4$. They obtained a new condition that excludes a class of positive solutions of this type of differential equation (1.13), and constructed a fluctuation criterion that simplifies, improves, and complements previous results in the literature. The simplification lies in obtaining the volatility criterion with two conditions, in contrast to previous results that required at least three conditions. The primary objective of this investigation is to enhance the asymptotic and monotonic properties of solutions to Eq (1.1). Additionally, it aims to ascertain the circumstances that lead to the emergence of oscillations. To illustrate our primary findings, we provide an example.

2. Properties of asymptotes and monotonic behavior

In this section, we will introduce notations designed to enhance the clarity of our main results presentation. Furthermore, we will establish enhanced asymptotic and monotonic properties for the positive solutions of the equation under investigation. Our approach begins with the classification of positive solutions based on the signs of their derivatives. We make the assumption that $y(s), y(\tau(s))$, and $y(\rho(s, \mathfrak{h}))$ all become eventually positive, thereby asserting the eventual positivity of the solution x. Consequently, z(t) approaches positivity over time.

Equation (1.1) provides insight into the behavior of $\xi(s)\, U^{(r-1)}(s)$, indicating that U falls into one of the following categories:

(1) $U' > 0,\ U^{(r-1)} > 0$ are $U^{(r)} \leq 0$;
(2) $U' > 0, U^{(r-2)} > 0,$ and $U^{(r-1)} \leq 0$;
(3) $(-1)^{i} U^{(i)} > 0$, for $i = 1, 2, ..., r-1$.

Notation 2.1. *The set of all solutions that eventually become positive for Eq (1.1) and meet the condition*

$$U^{(j)}(s)\, U^{(j+1)}(s) < 0 \quad \textit{for} \quad j = 0, 1, 2, ..., r-2, \tag{2.1}$$

is denoted as Ω^*. *Additionally, we introduce the functions* μ_i *defined as follows:*

$$\mu_0(s) := \int_s^{\infty} \xi^{-1}(\mathfrak{h})\, d\mathfrak{h},$$

and

$$\mu_i(s) := \int_s^{\infty} \mu_{i-1}(\mathfrak{h})\, d\mathfrak{h},\ i = 1, 2, ..., r-2.$$

Lemma 2.1. *If* y *represents an eventually positive solution to Eq (1.1), then* U *will eventually meet the condition expressed by*

$$y(s) > \sum_{i=0}^{k} \left(\prod_{j=0}^{2i} \upsilon\left(\tau^{[j]}(s)\right) \right) \left[\frac{U\left(\tau^{[2i]}(s)\right)}{\upsilon\left(\tau^{[2i]}(s)\right)} - U\left(\tau^{[2i+1]}(s)\right) \right],\ k \in \mathbb{N}. \tag{2.2}$$

Proof. The following can be deduced from the definition of U

$$\begin{aligned} y(s) &\geq U(s) - \upsilon(s)\, U(\tau(s)) \\ &= U(s) - \upsilon(s) \left[U(\tau(s)) - \upsilon\left(\tau(s)\, y\left(\tau^2(s)\right)\right)\right] \\ &= U(s) - \upsilon(s)\, U(\tau(s)) + \upsilon(s)\, \upsilon(\tau(s))\, y\left(\tau^2(s)\right). \end{aligned} \tag{2.3}$$

By evaluating (2.3) at $\tau^2(s)$, we derive

$$y\left(\tau^2(s)\right) = U\left(\tau^2(s)\right) - \upsilon\left(\tau^2(s)\right) U\left(\tau^3(s)\right) + \upsilon\left(\tau^2(s)\right) \upsilon\left(\tau^3(s)\right) y\left(\tau^4(s)\right). \tag{2.4}$$

Now, employing (2.3) in (2.4), we have

$$\begin{aligned} y(s) \geq\ & U(s) - \upsilon(s)\, U(\tau(s)) \\ & + \upsilon(s)\, U(\tau(s)) \left[U\left(\tau^2(s)\right) - \upsilon\left(\tau^2(s)\right) U\left(\tau^3(s)\right)\right] \end{aligned}$$

$$+\upsilon(s)\,\upsilon(\tau(s))\,\upsilon\left(\tau^2(s)\right)\upsilon\left(\tau^3(s)\right)y\left(\tau^4(s)\right).$$

By iterating this process, it becomes evident through induction that

$$\begin{aligned} y(s) &= U(s) - \upsilon(s)\,U(\tau(s)) \\ &\quad + \sum_{i=0}^{k}\left(\prod_{j=0}^{2i-1}\upsilon\left(\tau^{j}(s)\right)\right)[U\left(\tau^{[2i]}(s)\right) - \upsilon\left(\tau^{[2i+1]}(s)\right)] \\ &\quad + \left(\prod_{j=0}^{2i+1}\upsilon\left(\tau^{[j]}(s)\right)\right)y\left(\tau^{[2k+2]}(s)\right), \end{aligned}$$

and so on. Thus,

$$y(s) > \sum_{i=0}^{k}(-1)^k\left(\prod_{j=0}^{i}\upsilon\left(\tau^{[j]}(s)\right)\right)\frac{U\left(\tau^{[i]}(s)\right)}{\upsilon\left(\tau^{[i]}(s)\right)},$$

for every positive odd integer k, or

$$y(s) > \sum_{i=0}^{k}\left(\prod_{j=0}^{2i}\upsilon\left(\tau^{[j]}(s)\right)\right)\left[\frac{U\left(\tau^{[2i]}(s)\right)}{\upsilon\left(\tau^{[2i]}(s)\right)} - U\left(\tau^{[2i+1]}(s)\right)\right],$$

which implies (2.2). The proof is complete. □

Lemma 2.2. *Assume that* $y \in \Omega^*$. *Then,*

(C_1) $(-1)^{i+1}U^{(r-i-2)}(s) \le \xi(s)\,U^{(r-1)}(s)\,\mu_i(s)$ *for* $i = 0, 1, 2, ..., r-2$;

(C_2) $(U(s)/\mu_{r-2}(s))' > 0$;

(C_3) $y(s) \ge U(s)\,H_1(s,k), \quad k \in \mathbb{N}_0$;

where

$$H_1(s,k) = \sum_{i=0}^{k}\left(\prod_{j=0}^{2i}\upsilon\left(\tau^{[j]}(s)\right)\right)\left[\frac{1}{\upsilon\left(\tau^{[2i]}(s)\right)} - \frac{\mu_{r-2}\left(\tau^{[2i+1]}(s)\right)}{\mu_{r-2}\left(\tau^{[2i]}(s)\right)}\right].$$

Proof. Assume that $y \in \Omega^*$. Thus, for some $s_2 \ge s_1$, we have $y(\rho(s)) > 0$ for all $s \ge s_2$. Hence, from (1.1), we obtain

$$\left(\xi(s)\,U^{(r-1)}(s)\right)' = -\int_{\alpha}^{\beta}\mathcal{K}(s,\mathfrak{h})\,y(\rho(s,\mathfrak{h}))\,d\mathfrak{h} \le 0. \tag{2.5}$$

(C_1) Using (2.5), we have that $\xi.U^{(r-1)}$ is nonincreasing and hence

$$\begin{aligned} \xi(s)\,U^{(r-1)}(s)\,\mu_0(s) &\ge \int_{s}^{\infty}\frac{\xi(\mathfrak{h})\,U^{(r-1)}(\mathfrak{h})}{\xi(\mathfrak{h})}d\mathfrak{h} \\ &= \lim_{s\to\infty}U^{(r-2)}(s) - U^{(r-2)}(s). \end{aligned} \tag{2.6}$$

Given that $U^{(n-2)}$ is a positive decreasing function, it follows that as $s \to \infty$, $U^{(n-2)}(s)$ converges to a nonnegative constant. Consequently, (2.6) transforms into:

$$U^{(r-2)} \ge -\xi U^{(r-1)}\mu_0. \tag{2.7}$$

Using the fact that $(-1)^r \mathrm{U}^{(r)}(\mathrm{s}) > 0$ for $r = 0, 1, ..., n-1$, and integrating the inequality (2.7) along with its subsequent derivations, repeated $r-2$ times over $[\mathrm{s}, \infty)$, we arrive at the following:

$$(-1)^{\mathrm{i}+1} \mathrm{U}^{(r-\mathrm{i}-2)} \leq \xi \mathrm{U}^{(r-1)} \mu_{\mathrm{i}}. \tag{2.8}$$

(C_2) Using (C_1) at i = 0, we get

$$\left(\frac{\mathrm{U}^{r-2}}{\mu_0}\right)' = \frac{\left(\mu_0 \mathrm{U}^{(r-1)} + \xi^{-1} \mathrm{U}^{(r-2)}\right)}{\mu_0^2} \geq 0,$$

which leads to

$$-\mathrm{U}^{(r-3)}(\mathrm{s}) \geq \int_{\mathrm{s}}^{\infty} \mu_0(\varrho) \frac{\mathrm{U}^{(r-2)}(\varrho)}{\mu_0(\varrho)} \mathrm{d}\varrho \geq \frac{\mathrm{U}^{(r-2)}(\mathrm{s})}{\mu_0(\mathrm{s})} \mu_1(\mathrm{s}).$$

This implies

$$\left(\frac{\mathrm{U}^{(r-3)}}{\mu_1}\right)' = \frac{1}{\mu_1^2}\left(\mu_1 \mathrm{U}^{(r-2)} + \mu_0 \mathrm{U}^{(r-3)}\right) \leq 0.$$

By employing a comparable method repeatedly, we derive $(\mathrm{U}/\mu_{r-2})' > 0$. So

$$(-1)^k \frac{\mathrm{d}}{\mathrm{ds}}\left(\frac{\mathrm{U}^{n-k-2}(\mathrm{s})}{\mu_k(\mathrm{s})}\right) \geq 0. \tag{2.9}$$

(C_3) Since $\tau(\mathrm{s}) \leq \mathrm{s}$. From Lemma (2.1), we have (2.2) holds. From (C_2), we conclude that

$$\mathrm{U}(\mathrm{s}) - \upsilon(\mathrm{s}) \mathrm{U}(\tau(\mathrm{s})) \geq \mathrm{U}(\mathrm{s}) - \upsilon(\mathrm{s}) \frac{\mu(\tau(\mathrm{s}))}{\mu(\mathrm{s})} \mathrm{U}(\mathrm{s}). \tag{2.10}$$

Evaluating (2.10) in $\tau^{[2\mathrm{i}+1]}$ and using that U is decreasing, we obtain

$$\mathrm{U}\left(\tau^{[2\mathrm{i}+1]}(\mathrm{s})\right) \leq \frac{\mu_{r-2}\left(\tau^{[2\mathrm{i}+1]}(\mathrm{s})\right)}{\mu_{r-2}\left(\tau^{[2\mathrm{i}+1]}(\mathrm{s})\right)} \mathrm{U}\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right). \tag{2.11}$$

Using (2.10) and (2.11) in (2.2), we get

$$\mathrm{y}(\mathrm{s}) > \sum_{\mathrm{i}=0}^{k} \left(\prod_{\mathrm{j}=0}^{2\mathrm{i}} \upsilon\left(\tau^{[\mathrm{j}]}(\mathrm{s})\right)\right) \left[\frac{1}{\upsilon\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right)} - \frac{\mu_{r-2}\left(\tau^{[2\mathrm{i}+1]}(\mathrm{s})\right)}{\mu_{r-2}\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right)}\right] \mathrm{U}\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right), \quad k \in \mathbb{N}_0. \tag{2.12}$$

Since $\mathrm{U}' < 0$, and $\tau^{[2\mathrm{i}]}(\mathrm{s}) < \mathrm{s}$, then

$$\mathrm{U}\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right) \geq \mathrm{U}(\mathrm{s}),$$

which, with (2.12), leads to

$$\begin{aligned} \mathrm{y}(\mathrm{s}) &> \mathrm{U}(\mathrm{s}) \sum_{\mathrm{i}=0}^{k} \left(\prod_{\mathrm{j}=0}^{2\mathrm{i}} \upsilon\left(\tau[^{\mathrm{j}]}(\mathrm{s})\right)\right) \left[\frac{1}{\upsilon\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right)} - \frac{\mu_{r-2}\left(\tau^{[2\mathrm{i}+1]}(\mathrm{s})\right)}{\mu_{r-2}\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right)}\right] \\ &= H_1(\mathrm{s}, k) \mathrm{U}(\mathrm{s}), \end{aligned}$$

hence, (C_3) holds. □

Remark 2.1. *It is easy to verify that*

$$H_1(\mathrm{s},0)=1-\upsilon(\mathrm{s})\frac{\mu_{r-2}(\tau(\mathrm{s}))}{\mu_{r-2}(\mathrm{s})}.$$

Then, putting $k=0$ *in* (C_3), *we get classical relation (2.4).*

Lemma 2.3. *Assume that* $\mathrm{y}\in\Omega^*$. *If*

$$\int_{\mathrm{s}_0}^{\infty}\mu_{r-3}(\varsigma)\left(\int_{\mathrm{s}_0}^{\varsigma}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})H_1(\rho(\varrho,\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho\right)\mathrm{d}\varsigma=\infty, \tag{2.13}$$

and there exists a $\ell_0\in(0,1)$ *such that*

$$\frac{\mu_{r-2}^2(\mathrm{s})}{\mu_{r-3}(\mathrm{s})}\int_{\alpha}^{\beta}\mathcal{K}(\mathrm{s},\mathfrak{h})H_1(\rho(\mathrm{s},\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}\geq\ell_0, \tag{2.14}$$

then,

(C_4) $\lim_{\mathrm{s}\to\infty}\mathrm{U}(\mathrm{s})=0$;
(C_5) $\left(\mathrm{U}(\mathrm{s})/\mu_{r-2}^{\ell_0}(\mathrm{s})\right)'<0$;
(C_6) $\lim_{\mathrm{s}\to\infty}\mathrm{U}(\mathrm{s})/\mu_{r-2}^{\ell_0}(\mathrm{s})=0$;
(C_7) $\mathrm{y}(\mathrm{s})>\mathrm{U}(\mathrm{s})\widetilde{H_1}(\mathrm{s},k)$;

where

$$\widetilde{H_1}(\mathrm{s},k)=\sum_{\mathrm{i}=0}^{k}\left(\prod_{\mathrm{j}=0}^{2\mathrm{i}}\upsilon\left(\tau^{[\mathrm{j}]}(\mathrm{s})\right)\right)\left[\frac{1}{\upsilon(\tau^{[2\mathrm{i}]}(\mathrm{s}))}-\frac{\mu_{r-2}\left(\tau^{[2\mathrm{i}+1]}(\mathrm{s})\right)}{\mu_{r-2}(\tau^{[2\mathrm{i}]}(\mathrm{s}))}\right]\frac{\mu_{r-2}^{\ell_0}\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right)}{\mu_{r-2}^{\ell_0}(\mathrm{s})}.$$

Proof. (C_4) Since U is positive decreasing, we obtain that $\lim_{\mathrm{s}\to\infty}\mathrm{U}(\mathrm{s})=D\geq0$. Assume the contrary that $D>0$. Then, there is a $\mathrm{s}_2\geq\mathrm{s}_1$ with $\mathrm{U}(\mathrm{s})\geq D$ for $\mathrm{s}\geq\mathrm{s}_2$. Then (1.1) becomes

$$\left(\xi(\mathrm{s})\mathrm{U}^{(r-1)}(\mathrm{s})\right)'\leq-\int_{\alpha}^{\beta}\mathcal{K}(\mathrm{s},\mathfrak{h})\,\mathrm{y}(\rho(\mathrm{s},\mathfrak{h}))\,\mathrm{d}\mathfrak{h}.$$

From (C_3) we get

$$\begin{aligned}\left(\xi(\mathrm{s})\mathrm{U}^{(r-1)}(\mathrm{s})\right)' &\leq -\int_{\alpha}^{\beta}\mathcal{K}(\mathrm{s},\mathfrak{h})H_1(\rho(\mathrm{s},\mathfrak{h}),k)\,\mathrm{U}(\rho(\mathrm{s},\mathfrak{h}))\,\mathrm{d}\mathfrak{h}\\ &\leq -D\int_{\alpha}^{\beta}\mathcal{K}(\mathrm{s},\mathfrak{h})H_1(\rho(\mathrm{s},\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}.\end{aligned} \tag{2.15}$$

Integrating (2.15) from s_2 to s, we obtain the following inequality

$$\xi(\mathrm{s})\mathrm{U}^{(r-1)}(\mathrm{s})-\xi(\mathrm{s}_2)\mathrm{U}^{(r-1)}(\mathrm{s}_2)\leq-D\int_{\mathrm{s}_2}^{\mathrm{s}}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})H_1(\rho(\varrho,\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho.$$

From (2.1), we have $\mathrm{U}^{(r-1)}<0$ for $\mathrm{s}\geq\mathrm{s}_1$. Then $\xi(\mathrm{s}_2)\mathrm{U}^{(r-1)}(\mathrm{s}_2)<0$, and so

$$\xi(\mathrm{s})\mathrm{U}^{(r-1)}(\mathrm{s})\leq-D\int_{\mathrm{s}_2}^{\mathrm{s}}\left(\int_{\alpha}^{\beta}\mathcal{K}(\mathrm{s},\mathfrak{h})H_1(\rho(\varrho,\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho. \tag{2.16}$$

From (C_1) at i $= r-3$, we obtain

$$\frac{U'(s)}{\mu_{r-3}(s)} \le \xi U^{(r-1)}(s),$$

which, with (2.16), yields

$$U'(s) \le -D\mu_{r-3}(s)\int_{s_2}^{s}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})H_1(\rho(\varrho,\mathfrak{h}),k)\,d\mathfrak{h}\right)d\varrho.$$

Then,

$$U(s) \le U(s_2) - D\int_{s_2}^{s}\mu_{r-3}(\varsigma)\left(\int_{s_2}^{\varsigma}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})H_1(\rho(\varrho,\mathfrak{h}),k)\,d\mathfrak{h}\right)d\varrho\right)d\varsigma,$$

which, with (2.13), gives $U(s) \to -\infty$ as $s \to \infty$, a contradiction. Then, $U \to 0$ as $s \to \infty$.

(C_5) Given that $\rho(s,s)$ increases as s increases, it follows that $\rho(s,s) \ge \rho(s,\alpha)$ for $s \in (\alpha,\beta)$. Integrating (1.1) over $[s_2, s)$ and using (2.13), we find

$$\xi(s)U^{(r-1)}(s) = \xi(s_2)U^{(r-1)}(s_2) - \int_{s_2}^{s}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,s)\,y(\rho(\varrho,s))\,ds\right)d\varrho.$$

Using (C_3), we have

$$\begin{aligned}\xi(s)U^{(r-1)}(s) &\le \xi(s_2)U^{(r-1)}(s_2) - \int_{s_2}^{s}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})H_1(\rho(\varrho,s),k)\,U(\rho(\varrho,s))\,ds\right)d\varrho \\ &\le \xi(s_2)U^{(r-1)}(s_2) - U(\rho(s,\beta))\int_{s_2}^{s}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,s)H_1(\rho(\varrho,s),k)\,ds\right)d\varrho.\end{aligned}$$

From (2.14), we see that

$$\begin{aligned}\xi(s)U^{(r-1)}(s) &\le \xi(s_2)U^{(r-1)}(s_2) - \ell_0 U(s)\int_{s_2}^{s}\frac{\mu_{r-3}(\varrho)}{\mu_{r-2}^2(\varrho)}d\varrho \\ &= \xi(s_2)U^{(r-1)}(s_2) + \ell_0\frac{U(s)}{\mu_{r-2}(s_2)} - \ell_0\frac{U(s)}{\mu_{r-2}(s)},\end{aligned}$$

which, with (C_4), gives

$$\xi(s)U^{(r-1)}(s) \le -\ell_0\frac{U(s)}{\mu_{r-2}(s)}. \tag{2.17}$$

Therefore, by considering (C_1) at i $= r-3$, we derive the following inequality:

$$\frac{U'(s)}{\mu_{r-3}(s)} \le -\ell_0\frac{U(s)}{\mu_{r-2}(s)}.$$

Consequently,

$$\left(\frac{U(s)}{\mu_{r-2}^{\ell_0}(s)}\right)' = \frac{1}{\mu_{r-2}^{\ell_0+1}(s)}\left(\mu_{r-2}(s)U'(s) + \ell_0\mu_{r-3}(s)U(s)\right) \le 0.$$

(C_6) Now, since $U/\mu_{r-2}^{\ell_0}$ is positive and decreasing, we get that $\lim_{s\to\infty} U(s)/\mu_{r-2}^{\ell_0}(s) = l_0 \ge 0$. Suppose that $l_0 > 0$. Thus, for some $s_2 \ge s_1$, we obtain that $U(s)/\mu_{r-2}^{\ell_0}(s) \ge l_0$ for $s \ge s_2$. Now, let

$$F(s) := \frac{U(s) + \mu_{r-2}(s)\xi(s)U^{(r-1)}(s)}{\mu_{r-2}^{\ell_0}(s)}. \tag{2.18}$$

Therefore, $F(\mathrm{s})>0$ for $\mathrm{s}\geq \mathrm{s}_2$. From (2.14) and (2.18), we obtain

$$\begin{aligned}F'(\mathrm{s}) &= \frac{1}{\mu_{r-2}^{2\ell_0}(\mathrm{s})}\Big[\mu_{r-2}^{\ell_0}(\mathrm{s})\Big(\mathrm{U}'(\mathrm{s})-\mu_{r-3}(\mathrm{s})\Big(\xi\mathrm{U}^{(r-1)}(\mathrm{s})\Big)+\mu_{r-2}(\mathrm{s})\Big(\xi\mathrm{U}^{(r-1)}(\mathrm{s})\Big)'\Big)\\
&+\ell_0\mu_{r-2}^{\ell_0-1}(\mathrm{s})\,\mu_{r-3}(\mathrm{s})\Big(\mathrm{U}(\mathrm{s})+\Big(\xi\mathrm{U}^{(r-1)}(\mathrm{s})\Big)\mu_{r-2}(\mathrm{s})\Big)\Big]\\
&\leq \frac{1}{\mu_{r-2}^{\ell_0+1}(\mathrm{s})}\left[-\mu_{r-2}^2(\mathrm{s})\left(\int_\alpha^\beta \mathcal{K}(\mathrm{s},\mathfrak{h})\,\mathrm{y}(\rho(\mathrm{s},\mathfrak{h}))\,\mathrm{d}\mathfrak{h}\right)+\ell_0\mu_{r-3}(\mathrm{s})\Big(\mathrm{U}(\mathrm{s})+\mu_{r-2}(\mathrm{s})\,\xi(\mathrm{s})\,\mathrm{U}^{(r-1)}(\mathrm{s})\Big)\right]\\
&\leq \frac{1}{\mu_{r-2}^{\ell_0+1}(\mathrm{s})}\left[-\mu_{r-2}^2(\mathrm{s})\left(\int_\alpha^\beta \mathcal{K}(\mathrm{s},\mathfrak{h})\,H_1(\rho(\mathrm{s},\mathfrak{h}),k)\,\mathrm{U}(\rho(\mathrm{s},\mathfrak{h}))\,\mathrm{d}\mathfrak{h}\right)+\ell_0\mu_{r-3}(\mathrm{s})\Big(\mathrm{U}(\mathrm{s})+\mu_{r-2}(\mathrm{s})\,\xi(\mathrm{s})\,\mathrm{U}^{(r-1)}(\mathrm{s})\Big)\right]\\
&\leq \frac{1}{\mu_{r-2}^{\ell_0+1}(\mathrm{s})}\left[-\mu_{r-2}^2(\mathrm{s})\,\mathrm{U}(\mathrm{s})\left(\int_\alpha^\beta \mathcal{K}(\mathrm{s},\mathfrak{h})\,H_1(\rho(\mathrm{s},\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}\right)+\ell_0\mu_{r-3}(\mathrm{s})\Big(\mathrm{U}(\mathrm{s})+\mu_{r-2}(\mathrm{s})\,\xi(\mathrm{s})\,\mathrm{U}^{(r-1)}(\mathrm{s})\Big)\right].\end{aligned}$$

Hence,

$$\begin{aligned}F'(\mathrm{s}) &\leq \frac{1}{\mu_{r-2}^{\ell_0+1}(\mathrm{s})}\left[-\mu_{r-2}^2(\mathrm{s})\,\mathrm{U}(\mathrm{s})\,\frac{\ell_0\mu_{r-3}(\mathrm{s})}{\mu_{r-2}^2(\mathrm{s})}+\ell_0\mu_{r-3}(\mathrm{s})\Big(\mathrm{U}(\mathrm{s})+\mu_{r-2}(\mathrm{s})\,\xi(\mathrm{s})\,\mathrm{U}^{(r-1)}(\mathrm{s})\Big)\right]\\
&= \frac{1}{\mu_{r-2}^{\ell_0+1}(\mathrm{s})}\Big[-\ell_0\mu_{r-3}(\mathrm{s})\,\mathrm{U}(\mathrm{s})+\ell_0\mu_{r-3}(\mathrm{s})\,\mathrm{U}(\mathrm{s})+\ell_0\mu_{r-3}(\mathrm{s})\,\mu_{r-2}(\mathrm{s})\,\xi(\mathrm{s})\,\mathrm{U}^{(r-1)}(\mathrm{s})\Big]\\
&= \frac{\ell_0}{\mu_{r-2}^{\ell_0}(\mathrm{s})}\mu_{r-3}(\mathrm{s})\,\xi(\mathrm{s})\,\mathrm{U}^{(r-1)}(\mathrm{s}).\end{aligned} \tag{2.19}$$

Using the fact that $\mathrm{U}(\mathrm{s})/\mu_{r-2}^{\ell_0}(\mathrm{s})\geq l_0$ with (2.17), we obtain

$$\xi(\mathrm{s})\,\mathrm{U}^{(r-1)}(\mathrm{s})\leq -\ell_0\frac{\mathrm{U}(\mathrm{s})}{\mu_{r-2}(\mathrm{s})}\leq -\ell_0 l_0\mu_{r-2}^{\ell_0-1}(\mathrm{s}). \tag{2.20}$$

Combining (2.19) and (2.20), yields

$$F'(\mathrm{s})\leq -\ell_0^2 l_0\frac{\mu_{r-3}(\mathrm{s})}{\mu_{r-2}(\mathrm{s})}<0.$$

Integrating the above inequality over $[\mathrm{s}_2,\mathrm{s})$, we find

$$-F(\mathrm{s}_2)\leq -\ell_0^2 l_0\log\frac{\mu_{r-2}(\mathrm{s}_2)}{\mu_{r-2}(\mathrm{s})}\to\infty \text{ as } \mathrm{s}\to\infty,$$

a contradiction, and thus, $l_0=0$.

(C_7) As in the proof of Lemma 2.2, we arrive at

$$\mathrm{y}(\mathrm{s})>\sum_{\mathrm{i}=0}^{k}\left(\prod_{\mathrm{j}=0}^{2\mathrm{i}}\upsilon\left(\tau^{[\mathrm{j}]}(\mathrm{s})\right)\right)\left[\frac{1}{\upsilon(\tau^{[2\mathrm{i}]}(\mathrm{s}))}-\frac{\mu_{r-2}\left(\tau^{[2\mathrm{i}+1]}(\mathrm{s})\right)}{\mu_{r-2}(\tau^{[2\mathrm{i}]}(\mathrm{s}))}\right]\mathrm{U}\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right). \tag{2.21}$$

From (C_5), we conclude that

$$\mathrm{U}\left(\tau^{[2\xi]}(\mathrm{s})\right)\geq\frac{\mu_{r-2}^{\ell_0}\left(\tau^{[2\xi]}(\mathrm{s})\right)}{\mu_{r-2}^{\ell_0}}\mathrm{U}(\mathrm{s}),$$

which, with (2.21), gives

$$\begin{aligned} \mathrm{y}(\mathrm{s}) &> \mathrm{U}(\mathrm{s}) \sum_{\mathrm{i}=0}^{k} \left(\prod_{\mathrm{j}=0}^{2\mathrm{i}} \upsilon\left(\tau^{[\mathrm{j}]}(\mathrm{s})\right) \right) \left[\frac{1}{\upsilon\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right)} - \frac{\mu_{r-2}\left(\tau^{[2\mathrm{i}+1]}(\mathrm{s})\right)}{\mu_{r-2}\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right)} \right] \frac{\mu_{r-2}^{\ell_0}\left(\tau^{[2\mathrm{i}]}(\mathrm{s})\right)}{\mu_{r-2}^{\ell_0}} \\ &= \widetilde{H_1}(\mathrm{s}, k)\, \mathrm{U}(\mathrm{s}). \end{aligned}$$

The lemma's proof has been finalized. □

Lemma 2.4. *Let's suppose that* $\mathrm{y} \in \Omega^*$. *If condition (2.14) is satisfied for* $\ell_0 \in (0,1)$*, then condition (2.13) also holds.*

Proof. Suppose we have $\mathrm{y} \in \Omega^*$. By applying (2.14), we obtain the following inequality:

$$\begin{aligned} \int_{\mathrm{s}_0}^{\varsigma} \left(\int_{\alpha}^{\beta} \mathcal{K}(\varrho, \mathfrak{h})\, H_1(\rho(\varrho, \mathfrak{h}), k)\, d\mathfrak{h} \right) d\varrho &\geq \int_{\mathrm{s}_0}^{\varsigma} \ell_0 \frac{\mu_{r-3}(\varrho)}{\mu_{r-2}^2(\varrho)} d\varrho \\ &= \ell_0 \left(\frac{1}{\mu_{r-2}(\varsigma)} - \frac{1}{\mu_{r-2}(\mathrm{s}_0)} \right). \end{aligned}$$

By leveraging the fact that as $\mu_{n-2} \to 0$ as $\mathrm{s} \to \infty$, we can eventually derive the following inequality:

$$\frac{1}{\mu_{r-2}(\mathrm{s})} - \frac{1}{\mu_{r-2}(\mathrm{s}_0)} \geq \frac{\mu}{\mu_{r-2}(\mathrm{s})},$$

for $\mu \in (0,1)$. Therefore,

$$\mu_{r-3}(\varsigma) \int_{\mathrm{s}_0}^{\varsigma} \left(\int_{\alpha}^{\beta} \mathcal{K}(\varrho, \mathfrak{h})\, H_1(\rho(\varrho, \mathfrak{h}), k)\, d\mathfrak{h} \right) d\varrho \geq \ell_0 \mu \frac{\mu_{r-3}(\varsigma)}{\mu_{r-2}(\varsigma)}.$$

Thus,

$$\int_{\mathrm{s}_0}^{\mathrm{s}} \mu_{r-3}(\varsigma) \left(\int_{\mathrm{s}_2}^{\varsigma} \left(\int_{\alpha}^{\beta} \mathcal{K}(\varrho, \mathfrak{h})\, d\mathfrak{h} H_1(\rho(\varrho, \mathfrak{h}), k) \right) d\varrho \right) d\varsigma \geq \ell_0 \mu \ln \frac{\mu_{r-3}(\mathrm{s}_0)}{\mu_{r-2}(\mathrm{s})} \to \infty \text{ as } \mathrm{s} \to \infty.$$

The proof is complete. □

Theorem 2.1. *Assume that* $\mathrm{y} \in \Omega^*$, *(2.14) holds for some* $\ell_0 \in (0,1)$. *If there exists a natural number* n *such that* $\ell_{\mathrm{i}} \leq \ell_{\mathrm{i}+1} < 1$ *for all* $\mathrm{i} = 0, 1, 2, ..., n-1$, *the following conditions holds:*

$(\mathrm{C}_{1,n})$ $\left(\mathrm{U}(\mathrm{s}) / \mu_{r-2}^{\ell_n}(\mathrm{s})\right)' < 0$*;*

$(\mathrm{C}_{2,n})$ $\lim_{\mathrm{s}\to\infty} \mathrm{U}(\mathrm{s}) / \mu_{r-2}^{\ell_n}(\mathrm{s}) = 0$*;*

where ℓ_{i} *is defined as:*

$$\ell_{\mathrm{j}} = \ell_0 \frac{\lambda^{\ell_{\mathrm{j}-1}}}{1 - \ell_{\mathrm{j}-1}}, \quad \mathrm{j} = 1, 2, ..., n,$$

and for some $\lambda \geq 1$, *the inequality:*

$$\frac{\mu_{r-2}(\rho(\mathrm{s}, \beta))}{\mu_{r-2}(\mathrm{s})} \geq \lambda, \tag{2.22}$$

is satisfied.

Proof. Assume that $y \in \Omega^*$. Then, from Lemma 2.1, we have that (C_1)–(C_3) hold. Using induction, we have from Lemmas 2.1 and 2.2 that $(C_{1,0})$ and $(C_{2,0})$ hold. Now, we assume that $(C_{1,s-1})$ and $(C_{2,s-1})$ hold. Over $[s_1, s)$, integration (1.1) yields

$$\begin{aligned} \xi(s)\,U^{(r-1)}(s) &= \xi(s_2)\,U^{(r-1)}(s_2) - \int_{s_2}^{s}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})\,y(\rho(\varrho,\mathfrak{h}))\,d\mathfrak{h}\right)d\varrho \\ &\le \xi(s_2)\,U^{(r-1)}(s_2) - \int_{s_2}^{s} U(\rho(\varrho,\beta))\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})\,H_1(\rho(\varrho,\mathfrak{h}),k)\,d\mathfrak{h}\right)d\varrho. \quad (2.23) \end{aligned}$$

Using $(C_{1,n-1})$, we have that

$$U(\rho(s,\mathfrak{h})) \ge \mu_{r-2}^{\ell_{n-1}}(\rho(s,\mathfrak{h}))\,\frac{U(s)}{\mu_{r-2}^{\ell_{n-1}}(s)},$$

then (2.23) becomes

$$\xi(s)\,U^{(r-1)}(s) \le \xi(s_2)\,U^{(r-1)}(s_2) - \int_{s_2}^{s}\mu_{r-2}^{\ell_{n-1}}(\rho(\varrho,\beta))\,\frac{U(\varrho)}{\mu_{r-2}^{\ell_{n-1}}(\varrho)}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})\,H_1(\rho(\varrho,\mathfrak{h}),k)\,d\mathfrak{h}\right)d\varrho.$$

Since $\left(U(s)/\mu_{r-2}^{\ell_{s-1}}(s)\right)' \le 0$, we can conclude that

$$\xi(s)\,U^{(r-1)}(s) \le \xi(s_2)\,U^{(r-1)}(s_2) - \frac{U(s)}{\mu_{r-2}^{\ell n-1}(s)}\int_{s_2}^{s}\mu_{r-2}^{\ell_{n-1}}(\varrho)\,\frac{\mu_{r-2}^{\ell_{n-1}}(\rho(\varrho,\beta))}{\mu_{r-2}^{\ell_{n-1}}(\varrho)}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})\,H_1(\rho(\varrho,\mathfrak{h}),r)\,d\mathfrak{h}\right)d\varrho.$$

Hence, from (2.14) and (2.22), we obtain

$$\begin{aligned} \xi(s)\,U^{(r-1)}(s) &\le \xi(s_2)\,U^{(r-1)}(s_2) - \ell_0\lambda^{\ell_{n-1}}\frac{U(s)}{\mu_{r-2}^{\ell n-1}(s)}\int_{s_2}^{s}\frac{\mu_{r-3}(\varrho)}{\mu_{r-2}^{2-\ell_{n-1}}(\varrho)}d\varrho \\ &= \xi(s_2)\,U^{(r-1)}(s_2) - \ell_0\frac{\lambda^{\ell_{n-1}}}{1-\ell_{n-1}}\frac{U(s)}{\mu_{r-2}^{\ell_{n-1}}(s)}\left(\frac{1}{\mu_{r-2}^{1-\ell_{n-1}}(s)} - \frac{1}{\mu_{r-2}^{1-\ell_{n-1}}(s_2)}\right), \end{aligned}$$

hence,

$$\xi(s)\,U^{(r-1)}(s) \le \xi(s_2)\,U^{(r-1)}(s_2) - \ell_s\frac{U(s)}{\mu_{r-2}^{\ell_{n-1}}(s)}\frac{1}{\mu_{r-2}^{1-\ell_{n-1}}(s_2)} - \ell_n\frac{U(s)}{\mu_{r-2}(s)}.$$

Using the property $\lim_{s\to\infty} U(s)/\mu_{r-2}^{\ell_{n-1}}(s) = 0$, we get

$$\xi U^{(r-1)} \le -\ell_s\frac{U}{\mu_{r-2}}. \quad (2.24)$$

Thus, from (C_1) at $k = r-3$, we obtain

$$\frac{U'}{\mu_{r-3}} \le -\ell_n\frac{U}{\mu_{r-2}}.$$

Consequently,

$$\left(\frac{U}{\mu_{r-2}^{\ell_n}}\right)' = \frac{1}{\mu_{r-2}^{\ell_{n+1}}}\left(\mu_{r-2}U' + \ell_n\mu_{r-3}U\right) \le 0.$$

The proof's remaining steps align precisely with those found in the proof of (C_6), as demonstrated in Lemma 2.2. Consequently, we can conclude that the proof is now finished. □

3. Criteria for oscillation

Theorem 3.1. *Suppose that* $\mathrm{y} \in \Omega^{*}$*, and, for a certain value* $\ell_0 \in (0,1)$*, (2.14) is satisfied. If there exists a natural number n such that* $\ell_{\mathrm{i}} \leq \ell_{\mathrm{i}+1} < 1$ *for all* i *from* 0 *to* $n-1$*, then*

$$\varphi'(\mathrm{s}) + \frac{1}{1-\ell_n}\left(\int_{\alpha}^{\beta} H_1(\rho(\mathrm{s},\mathfrak{h}),k)\,\mathcal{K}(\mathrm{s},\mathfrak{h})\,\mathrm{d}\mathfrak{h}\right)\mu_{r-2}(\mathrm{s})\,\varphi(\rho(\mathrm{s},\beta)) = 0 \tag{3.1}$$

has a positive solution. Here, ℓ_{j} *and* λ *are defined as per the description in Lemma 2.4.*

Proof. Suppose we have $\mathrm{y} \in \Omega^{*}$. According to Lemma 2.4, it follows that both $(\mathrm{C}_{1,n})$ and $(\mathrm{C}_{2,n})$ are satisfied. Now, define φ as

$$\varphi = \xi\mu_{r-2}\mathrm{U}^{(r-1)} + \mathrm{U}. \tag{3.2}$$

Consequently, based on (C_1) at $\mathrm{i} = r-2$, we can deduce that $\varphi(\mathrm{s}) > 0$ for $\mathrm{s} \geq \mathrm{s}_2$. Additionally,

$$\varphi' = \mu_{r-2}\left(\xi\mathrm{U}^{(r-1)}\right)' - \xi\mu_{r-3}\mathrm{U}^{(r-1)} + \mathrm{U}'.$$

By utilizing (C_1) at $\mathrm{i} = r-3$, we can establish the following inequality

$$\varphi' \leq \mu_{r-2}\left(\xi\mathrm{U}^{(r-1)}\right)' \leq -\mathrm{U}(\rho(\mathrm{s},\beta))\,\mu_{r-2}\int_{\alpha}^{\beta} H_1(\rho(\varrho,\mathfrak{h}),k)\,\mathcal{K}(\mathrm{s},\mathfrak{h})\,\mathrm{d}\mathfrak{h}. \tag{3.3}$$

Based on the proof provided in Lemma 2.4, it is apparent that (2.24) is satisfied. When we merge the Eqs (3.2) and (2.24), we can deduce

$$\varphi(\mathrm{s}) \leq (1-\ell_n)\,\mathrm{U}(\mathrm{s}).$$

Consequently, Eq (3.3) can be rewritten as

$$\varphi'(\mathrm{s}) + \frac{1}{1-\ell_n}\left(\int_{\alpha}^{\beta} H_1(\rho(\mathrm{s},\mathfrak{h}),k)\,\mathcal{K}(\mathrm{s},\mathfrak{h})\,\mathrm{d}\mathfrak{h}\right)\mu_{r-2}(\mathrm{s})\,\varphi(\rho(\mathrm{s},\beta)) \leq 0. \tag{3.4}$$

Thus, we have established that φ is a positive solution to the differential inequality (3.4). Furthermore, according to [22, Theorem 1], Eq (3.1) also possesses a positive solution, thereby concluding our proof. □

Theorem 3.2. *Suppose there exists a value* ℓ_0 *within the interval* $(0,1)$ *such that condition (2.14) is satisfied. Additionally, assume there exists a natural number n such that* $\ell_{\mathrm{i}} \leq \ell_{\mathrm{i}+1} < 1$ *for all* i *from* 0 *to* $n-1$. *Furthermore, consider the delay differential equations (3.1):*

$$\varpi'(\mathrm{s}) + \frac{\epsilon_1\rho^{r-1}(\mathrm{s},\beta)}{(r-1)!\,(\xi(\rho(\mathrm{s},\beta)))}\left(\int_{\alpha}^{\beta}\mathcal{K}(\mathrm{s},\mathfrak{h})\,H_1(\rho(\mathrm{s},\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}\right)\varpi(\rho(\mathrm{s},\beta)) = 0 \tag{3.5}$$

and

$$\varpi'(\mathrm{s}) + \frac{\epsilon_2}{(r-2)!\,\xi(\mathrm{s})}\left(\int_{\mathrm{s}_0}^{\mathrm{s}}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})\,H_1(\rho(\varrho,\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}\right)\rho^{r-2}(\varrho,\beta)\,\mathrm{d}\varrho\right)\varpi(\rho(\mathrm{s},\beta)) = 0, \tag{3.6}$$

which are oscillatory for certain ϵ_1, ϵ_2, $\ell_n \in (0,1)$*, where* ℓ_{j}, λ *are defined according to Theorem 3.1. Under these conditions, it follows that every solution of Eq (1.1) exhibits oscillatory behavior.*

Proof. Let's assume the opposite scenario, where y represents solutions that eventually become positive. In this case, as per [1, Lemma 2.2.1], we encounter three distinct cases denoted as (1)–(3).

By adopting an approach quite akin to the one employed in [32, Theorem 3], we can establish that cases (1) and (2) cannot occur, based on our initial assumption that Eqs (3.4) and (3.6) exhibit oscillatory behavior.

Consequently, we are left with the situation where (3) is true. Utilizing Theorem 3.1, we deduce that (3.1) possesses a positive solution, which contradicts our earlier assumption. Hence, we can conclude that the proof is now fully substantiated. □

Corollary 3.1. *Suppose there exists a value ℓ_0 within the interval $(0,1)$ such that condition (2.14) is satisfied. Additionally, assume there exists a natural number n such that $\ell_{\mathrm{i}} \leq \ell_{\mathrm{i}+1} < 1$ for all i from 0 to $n-1$, and the following inequalities hold*

$$\liminf_{\mathrm{s}\to\infty} \int_{\rho(\mathrm{s},\beta)}^{\mathrm{s}} \mu_{r-2}(\varrho)\left(\int_{\alpha}^{\beta} \mathcal{K}(\varrho,\mathfrak{h}) H_1(\rho(\varrho,\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho > \frac{1-\ell_n}{\mathrm{e}}, \tag{3.7}$$

$$\liminf_{\mathrm{s}\to\infty} \int_{\rho(\mathrm{s},\beta)}^{\mathrm{s}} \frac{1}{\xi(\rho(\varrho,\beta))}\rho^{r-1}(\varrho,\beta)\left(\int_{\alpha}^{\beta} \mathcal{K}(\varrho,\mathfrak{h}) H_1(\rho(\varrho,\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho > \frac{(r-1)!}{\mathrm{e}}, \tag{3.8}$$

and

$$\liminf_{\mathrm{s}\to\infty} \int_{\rho(\mathrm{s},\beta)}^{\mathrm{s}} \frac{1}{\xi(u)}\left(\int_{\mathrm{s}_0}^{u}\left(\int_{\alpha}^{\beta} \mathcal{K}(\varrho,\mathfrak{h}) H_1(\rho(\varrho,\mathfrak{h}),k)\,\mathrm{d}\mathfrak{h}\right)\rho^{r-2}(\varrho,\beta)\,\mathrm{d}\varrho\right)\mathrm{d}u > \frac{(r-2)!}{\mathrm{e}}, \tag{3.9}$$

where $\epsilon, \ell_n \in (0,1)$, then it follows that every solution of (1.1) exhibits oscillatory behavior.

Proof. According to [40, Corollary 2.1], when conditions (3.7)–(3.9) are met, it indicates the oscillatory nature of the solutions for (3.1), (3.5), and (3.6), respectively. Consequently, based on Theorem 3.2, we can conclude that every solution of (1.1) exhibits oscillatory behavior. □

Example 3.1. *Consider the NDE*

$$\left(\mathrm{s}^4(\mathrm{y}(\mathrm{s}) + \upsilon_0 \mathrm{y}(\tau_0 \mathrm{s}))\right)^{(4)} + \int_{\alpha}^{\beta} \mathcal{K}_0 \mathrm{y}(\rho_0 s)\,\mathrm{d}s = 0, \ \mathrm{s} \geq 1, \tag{3.10}$$

where $\upsilon_0, \rho_0 \in (0,1)$ and $s \in (0.4,1)$. By comparing (1.1) and (3.10), we see that $r=4$, $\xi(\mathrm{s}) = \mathrm{s}^4$, $\mu_i(\mathrm{s}) = \mathrm{e}^{-s}$, $\mathrm{i}=0,1,2$, $\mathcal{K}(\mathrm{s},\mathfrak{h}) = \mathcal{K}_0$, $\rho(\mathrm{s},\mathfrak{h}) = \rho_0 s$. It is easy to verify that

$$\mu_0(\mathrm{s}) = \frac{1}{3\mathrm{s}^3},\ \mu_1(\mathrm{s}) = \frac{1}{6\mathrm{s}^2},\ \mu_2(\mathrm{s}) = \frac{1}{6\mathrm{s}},$$

$$H_1(\mathrm{s},k) = H_1 = \left[1 - \frac{\upsilon_0}{\tau_0}\right]\sum_{\mathrm{i}=0}^{k} \upsilon_0^{2\mathrm{i}},$$

and

$$\widetilde{H_1}(\mathrm{s},k) = \left[1 - \frac{\upsilon_0}{\tau_0}\right]\sum_{\mathrm{i}=0}^{k} \upsilon_0^{2\mathrm{i}} \frac{1}{\tau_0^{2\mathrm{i}\ell_0}}.$$

Condition (3.7) becomes

$$\begin{aligned}
&\liminf_{s\to\infty}\int_{\rho(s,\beta)}^{s}\mu_{r-2}(\varrho)\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})\,H_1(\rho(\varrho,\mathfrak{h}),k)\,d\mathfrak{h}\right)d\varrho\\
&=\liminf_{s\to\infty}\int_{\rho_0 s}^{s}\mu_2(\varrho)\left(\int_{\alpha}^{\beta}\mathcal{K}_0 H_1 d\mathfrak{h}\right)d\varrho\\
&=\liminf_{s\to\infty}\int_{\rho_0 s}^{s}\frac{1}{6\varrho}\left((\beta-\alpha)\mathcal{K}_0 H_1\right)d\varrho\\
&=\frac{1}{6}(\beta-\alpha)\mathcal{K}_0\left[1-\frac{\upsilon_0}{\tau_0}\right]\ln\frac{1}{\rho_0},
\end{aligned}$$

which leads to

$$\mathcal{K}_0>\frac{6(1-\ell_n)}{(\beta-\alpha)H_1\ln\frac{1}{\rho_0}}>\frac{1}{e},\tag{3.11}$$

condition (3.8) becomes

$$\begin{aligned}
&\liminf_{s\to\infty}\int_{\rho(s,\beta)}^{s}\frac{1}{\xi(\rho(\varrho,\beta))}\rho^{r-1}(\varrho,\beta)^3\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})\,H_1(\rho(\varrho,\mathfrak{h}),k)\,d\mathfrak{h}\right)d\varrho\\
=\;&\liminf_{s\to\infty}\int_{\rho_0 s}^{s}\frac{1}{\rho_0^4\varrho^4}\rho_0^3\varrho^3(\beta-\alpha)\mathcal{K}_0\left[1-\frac{\upsilon_0}{\tau_0}\right]\sum_{i=0}^{k}\upsilon_0^{2i}d\varrho\\
=\;&\frac{1}{\rho_0}(\beta-\alpha)\mathcal{K}_0\left[1-\frac{\upsilon_0}{\tau_0}\right]\ln\frac{1}{\rho_0}\sum_{i=0}^{k}\upsilon_0^{2i},
\end{aligned}$$

which leads to

$$\mathcal{K}_0>\frac{6\rho_0}{(\beta-\alpha)H_1\ln\frac{1}{\rho_0}}\frac{1}{e},\tag{3.12}$$

and condition (3.9) becomes

$$\begin{aligned}
&\liminf_{s\to\infty}\int_{\rho(s,\beta)}^{s}\frac{1}{\xi(u)}\left(\int_{s_0}^{u}\left(\int_{\alpha}^{\beta}\mathcal{K}(\varrho,\mathfrak{h})\,H_1(\rho(\varrho,\mathfrak{h}),k)\,d\mathfrak{h}\right)\rho^{r-2}(\varrho,\beta)\,d\varrho\right)du\\
=\;&\liminf_{s\to\infty}\int_{\rho_0 s}^{s}\frac{1}{u^4}\left(\int_{s_0}^{u}\left(\int_{\alpha}^{\beta}\mathcal{K}_0 H_1 d\mathfrak{h}\right)\rho_0^2\varrho^2 d\varrho\right)du\\
=\;&\liminf_{s\to\infty}\int_{\rho_0 s}^{s}\frac{1}{u^4}\left(\left((\beta-\alpha)\mathcal{K}_0\left[1-\frac{\upsilon_0}{\tau_0}\right]\sum_{i=0}^{k}\upsilon_0^{2i}\right)\int_{s_0}^{u}\rho_0^2\varrho^2 d\varrho\right)du\\
=\;&\liminf_{s\to\infty}\int_{\rho_0 s}^{s}\left((\beta-\alpha)\frac{\rho_0^2}{3}\mathcal{K}_0\left[1-\frac{\upsilon_0}{\tau_0}\right]\sum_{i=0}^{k}\upsilon_0^{2i}\right)\frac{1}{u^4}\frac{\rho_0^2}{3}u^3 du\\
=\;&\frac{1}{3}(\beta-\alpha)\rho_0^2\ln\frac{1}{\rho_0}\mathcal{K}_0\left[1-\frac{\upsilon_0}{\tau_0}\right]\sum_{i=0}^{k}\upsilon_0^{2i},
\end{aligned}$$

which is achieved when

$$\mathcal{K}_0>\frac{6}{(\beta-\alpha)H_1\rho_0^2\ln\frac{1}{\rho_0}}\frac{1}{e}.\tag{3.13}$$

From Corollary 3.1, we see that every solution of (3.10) is oscillatory if (3.11)–(3.13) hold.

Example 3.2. *Consider the Eq (3.10) where* $\upsilon_0 = 0.5$, $\tau_0 = 0.9$, $\rho_0 = 0.7$, $\alpha = 0.5$, *and* $\beta = 1$, *we see that*

$$H_1(\mathrm{s}, 20) = H_1 = \left(1 - \frac{0.5}{0.9}\right) \sum_{i=0}^{20} (0.5)^{2i} = 0.59259.$$

Using Corollary 3.1, the conditions

$$\mathcal{K}_0 > \frac{12(1-\ell_n)}{0.59259 * \ln\frac{1}{0.7}} \frac{1}{\mathrm{e}} = 20.886(1-\ell_n),$$

$$\mathcal{K}_0 > \frac{12 * 0.7}{0.59259 * \ln\frac{1}{0.7}} \frac{1}{\mathrm{e}} = 14.62,$$

and

$$\mathcal{K}_0 > \frac{12}{0.59259 * (0.7)^2 * \ln\frac{1}{0.7}} \frac{1}{\mathrm{e}} = 42.625.$$

confirm the oscillation of all solutions of (3.10).

4. Conclusions

This paper has contributed significantly to the field of even-order neutral differential equations by introducing novel sufficient criteria for guaranteeing oscillatory solutions. By drawing comparisons with the oscillatory behavior of first-order delay equations, we have expanded upon and enriched the existing body of knowledge in this area. The findings presented here not only advance our understanding of even-order neutral differential equations in form (1.1) but also hold the potential for further extensions to address half-linear and super-linear cases. Our results can be extended to the following case:

$$\left(\xi(\mathrm{s})\left[\mathrm{U}^{(r-1)}(\mathrm{s})\right]^{\kappa}\right)' + \int_{\alpha}^{\beta} \mathcal{K}(\mathrm{s}, \mathfrak{h})\left[\mathrm{y}(\rho(\mathrm{s}, \mathfrak{h}))\right]^{\delta} \mathrm{d}\mathfrak{h} = 0,$$

where κ and δ are quotients of odd numbers. Moreover, it would be interesting to obtain new oscillation criteria that do not place monotonic constraints on delay functions.

Author contributions

S.E., O.M. and M.Z. developed the conceptualization and proposed the method. S.E. and K.S.N. wrote the original draft. O.M., M.Z. and E.M.E. investigated, processed and provided examples. K.S.N. and E.M.E. reviewed and edited the paper. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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