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Research article

Edge-coloring of generalized lexicographic product of graphs

Shuangliang Tian[1,*] **and Ping Chen**[2]

[1] College of Mathematics and Computer Science, Northwest Minzu University, Lanzhou 730030, Gansu, China

[2] College of Management, Northwest Minzu University, Lanzhou 730030, Gansu, China

* **Correspondence:** Email: sl_tian@163.com; Tel: +86-13993177682.

Abstract: An edge-coloring of a graph G is an assignment of colors to its edges so that no two edges incident to the same vertex receive the same color. The chromatic index of G, denoted by $\chi'(G)$, is the least k for which G has a k edge-coloring. Graphs with $\chi'(G) = \Delta(G)$ are said to be Class 1, and graphs with $\chi'(G) = \Delta(G) + 1$ are said to be Class 2. Let G be a graph with $V(G) = \{t_1, t_2, \ldots, t_n\}$, $n \geq 2$, and $h_n = (H_i)_{i\in\{1,2,\ldots,n\}}$ be a sequence of vertex-disjoint graphs with $V(H_i) = \{(t_i, y_1), (t_i, y_2), \ldots, (t_i, y_{m_i})\}$, $m_i \geq 1$. The generalized lexicographic product $G[h_n]$ of G and $h_n = (H_i)_{i\in\{1,2,\ldots,n\}}$ is a simple graph with vertex set $\bigcup_{i=1}^{n} V(H_i)$, in which (t_i, y_p) is adjacent to (t_j, y_q) if and only if either $t_i = t_j$ and $(t_i, y_p)(t_i, y_q) \in E(H_i)$ or $t_it_j \in E(G)$. If G is a complete graph with order 2, then $G[h_2]$ denotes a join $H_1 + H_2$ of vertex-disjoint graphs H_1 and H_2. If $H_i \cong H$ for $i = 1, 2, \ldots, n$, then $G[h_n] = G[H]$, where $G[H]$ denotes the lexicographic product of two graphs G and H. In this paper, we provide sufficient conditions for the generalized lexicographic product $G[h_n]$ of G and $h_n = (H_i)_{i\in\{1,2,\ldots,n\}}$ to be Class 1, where all graphs in h_n have the same number of vertices.

Keywords: edge-coloring; decomposition; generalized lexicographic product
Mathematics Subject Classification: 05C15

1. Introduction

In this paper, we consider only undirected, connected, and simple graphs. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph G, respectively.

An edge-coloring of a graph G is an assignment of colors to its edges so that no two edges incident to the same vertex receive the same color. An edge-coloring σ of G using k colors (k edge-coloring) is then a partition of the edge set $E(G)$ into k disjoint matchings and it can be written as $\sigma = (M_1, M_2, \ldots, M_k)$, where every M_i is a matching of G. The chromatic index of G, denoted by $\chi'(G)$, is the least k for which G has a k edge-coloring. The Vizing Theorem states that $\chi'(G) = \Delta(G)$

or $\chi'(G) = \Delta(G) + 1$. Graphs with $\chi'(G) = \Delta(G)$ are said to be Class 1; graphs with $\chi'(G) = \Delta(G) + 1$ are said to be Class 2. It is NP-complete to determine whether a graph is Class 1 [3]. The classification problem is extremely difficult even for regular graphs. For this problem, Chetwynd and Hilton [1] proposed the following conjecture:

Conjecture 1.1 (1-Factorization Conjecture)**.** *Let G be a k-regular graph with n vertices, n even. If $k \geq \frac{n}{2}$ then $\chi'(G) = k$.*

In this paper, we consider the generalized lexicographic product of graphs, which is defined as follows [10]: Let G be a graph with $V(G) = \{t_1, t_2, \ldots, t_n\}$, $n \geq 2$, and $h_n = (H_i)_{i\in\{1,2,\ldots,n\}}$ be a sequence of vertex-disjoint graphs with $V(H_i) = \{(t_i, y_1), (t_i, y_2), \ldots, (t_i, y_{m_i})\}$, $m_i \geq 1$. The generalized lexicographic product $G[h_n]$ of G and $h_n = (H_i)_{i\in\{1,2,\ldots,n\}}$ is a simple graph with vertex set $\bigcup_{i=1}^{n} V(H_i)$, in which (t_i, y_p) is adjacent to (t_j, y_q) if and only if either $t_i = t_j$ and $(t_i, y_p)(t_i, y_q) \in E(H_i)$ or $t_it_j \in E(G)$. A generalized lexicographic product is also called an expansion or composition (see [11]).

By V_i, $i = 1, 2, \ldots, n$ we will denote the vertex set of graph H_i in $G[h_n]$, and call the sets $V_1, V_2, \ldots, V_n$ the partition sets of $G[h_n]$. If $H_i \cong H$ for $i = 1, 2, \ldots, n$, then $G[h_n] = G[H]$, where $G[H]$ is the *lexicographic product* of two graphs G and H. For example, the *join* $H_1 + H_2$ of vertex-disjoint graphs H_1 and H_2 is $K_2[h_2]$, where $h_2 = (H_i)_{i\in\{1,2\}}$. In addition, *Turán graphs* $T^{r-1}(n)$ (see [2]) are complete $(r-1)$-partite graphs with $n \geq r-1$ vertices whose partition sets differ in size by at most 1, that is, $T^{r-1}(n)$ are $K_{r-1}[h_{r-1}]$, where $h_{r-1} = (H_i)_{i\in\{1,2,\ldots,r-1\}}$ is a sequence of vertex-disjoint empty graphs with $\lfloor \frac{n}{r-1} \rfloor$ vertices or $\lceil \frac{n}{r-1} \rceil$ vertices.

De Simone and Picinin de Mello [9] gave the following sufficient conditions for a join graph to be Class 1:

Theorem 1.1. *Let $G = H_1 + H_2$ be a join graph with $|V(H_1)| \leq |V(H_2)|$. If $\Delta(H_1) > \Delta(H_2)$, then G is Class 1.*

Theorem 1.2. *Let $G = H_1 + H_2$ be a join graph with $\Delta(H_1) = \Delta(H_2)$. If both H_1 and H_2 are Class 1, or if H_1 is a subgraph of H_2, or if both H_1 and H_2 are disjoint unions of cliques, then G is Class 1.*

Theorem 1.3. *Every regular join graph $G = H_1 + H_2$ with $\Delta(H_1) = \Delta(H_2)$ is Class 1.*

De Simone and Galluccio [8] showed that 1-Factorization Conjecture is true for graphs that are join of two graphs:

Theorem 1.4. *Every regular join graph with even order is Class1.*

De Simone and Galluccio [6, 7] extended the above result, and proved the following conclusions:

Theorem 1.5. *Every even graph that is the join of two regular graphs is Class 1.*

Mohar [5] and Jaradat [4] gave the following sufficient conditions for a lexicographic product of graphs to be Class 1:

Theorem 1.6. *Let G and H be two graphs. If G is Class 1, then $G[H]$ is Class 1.*

Theorem 1.7. *Let G and H be two graphs. If $\chi'(H) = \Delta(H)$ and H is of even order, then $\chi'(G[H]) = \Delta(G[H])$.*

By Theorems 1.6 and 1.7, it is easy to see that for any two regular graphs G and H, if 1-Factorization Conjecture is true for the graph G, or 1-Factorization Conjecture is true for the graph H and $\Delta(G) \geq \frac{|V(G)|-1}{2}$, then this conjecture is also true for graphs that are lexicographic product $G[H]$ of G and H.

In this paper, our goal is to find sufficient conditions for a generalized lexicographic product $G[h_n]$ of G and $h_n = (H_i)_{i\in\{1,2,\ldots,n\}}$ to be Class 1, where G is a graph with n vertices and all graphs in h_n have the same number of vertices.

The following lemma will be used later:

Lemma 1. *(Jaradat [8]) Let G and H be two graphs such that $\chi'(H) = \Delta(H)$. Then $\chi'(G \times H) = \Delta(G \times H)$, where $G \times H$ denotes the direct product of graphs G and H.*

In Section 2, we shall present two decompositions of the edge set in the generalized lexicographic product of graphs.

2. The decompositions

Let G be a graph with $V(G) = \{t_1, t_2, \ldots, t_n\}$, $n \geq 2$, and $h_n = (H_i)_{i\in\{1,2,\ldots,n\}}$ be a sequence of vertex-disjoint graphs with $V(H_i) = \{(t_i, y_1), (t_i, y_2), \ldots, (t_i, y_m)\}$, $m \geq 1$. Let $G^* = G[h_n]$, and let $U_j = \{(t_1, y_j), (t_2, y_j), \ldots, (t_n, y_j)\}$, where $j = 1, 2, \ldots, m$. Then $G_j = G^*[U_j] \cong G$ for each $j = 1, 2, \ldots, m$. We will provide two decompositions of the generalized lexicographic product G^*.

We first consider the case where G is a Class 1 graph and provide a decomposition of G^* with respect to a matching of G. If there exists a matching M in G such that $\chi'(G - M) = \Delta(G) - 1$ and any distinct vertices with the maximum degree of G are saturated by distinct edges of M, then G is said to be Subclass 1; otherwise, G is said to be Subclass 2. For example, every Class 1 graph in which no two vertices of maximum degree are adjacent is Subclass 1, and every regular Class 1 graph is Subclass 2.

For every matching M of G such that $\chi'(G - M) = \Delta(G) - 1$, let G_M denote the subgraph of G induced by M. Moreover, let

$$G_M^* = G_M[\overline{K}_m] \cup (\bigcup_{t_i \in V_\Delta} H_i),\ G_1^* = G_1[\overline{K}_m] \cup (\bigcup_{t_i \in V(G)-V_\Delta} H_i),$$

where V_Δ denotes the set of vertices with the maximum degree in G, $\overline{K}_m$ denotes an empty graph with m vertices, and $G_1 = G - M$. Then G^* is the union of edge-disjoint graphs G_M^* and G_1^*, that is,

$$G^* = G_M^* \cup G_1^*. \tag{2.1}$$

Figure 1 shows a Subclass 1 graph G and the decomposition of $G^* = G[h_5]$ with respect to a matching M in G, where $h_n = (H_i)_{i\in\{1,2,\ldots,5\}}$ is a sequence of vertex-disjoint graphs, each with m vertices. Figure 2 shows a Subclass 2 graph G and the decomposition of $G^* = G[h_6]$ with respect to a matching M in G, where $h_n = (H_i)_{i\in\{1,2,\ldots,6\}}$ is a sequence of vertex-disjoint graphs, each with m vertices.

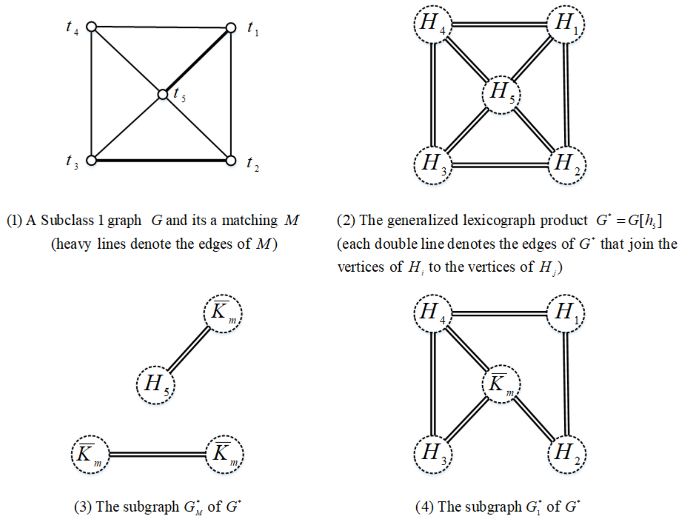

Figure 1. The decomposition of G^* with respect to a matching M in a Subclass 1 graph G.

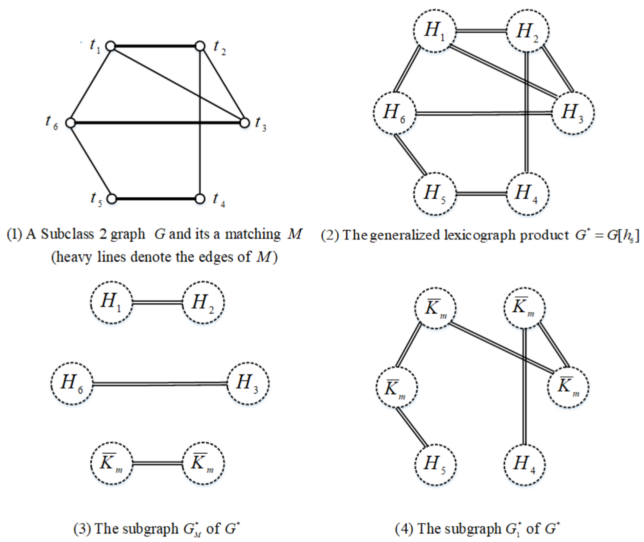

Figure 2. The decomposition of G^* with respect to a matching M in a Subclass 2 graph G.

We now provide another decomposition of G^*. Let

$$G_2^* = (\bigcup_{j=1}^{m} G_j) \cup (\bigcup_{i=1}^{n} H_i),\ G_3^* = G \times K_m.$$

Then the graph G^* is the union of edge-disjoint graphs G_2^* and G_3^*, that is,

$$G^* = G_2^* \cup G_3^*. \tag{2.2}$$

In Section 3, we shall study sufficient conditions for G^* to be Class 1 using the above two decompositions of G^*.

3. Sufficient conditions for $G[h_n]$ to be Class 1

Through the decomposition of G^* in formula 2.1, we can make the following observation:

Observation 3.1. *Let G be a Class 1 graph. If there exists a matching M in G such that $\chi'(G-M)=\Delta(G)-1$ and the corresponding G^*_M is Class 1, then G^* is also Class 1.*

Proof. Let M be a matching of G such that $\chi'(G-M)=\Delta(G)-1$ and the corresponding G^*_M is Class 1. It is easy to see that $\Delta(G^*_M)=\max\{\Delta(H_i)|t_i\in V_\Delta\}+m$ and $\Delta(G^*_1)=(\Delta(G)-1)m$. Note that $\Delta(G^*)=\max\{\Delta(H_i)|t_i\in V_\Delta\}+\Delta(G)m$. Hence, $\Delta(G^*)=\Delta(G^*_M)+\Delta(G^*_1)$. It follows that if both G^*_M and G^*_1 are Class 1, then G^* is Class 1. Thus, we only need to verify that $\chi'(G^*_1)=(\Delta(G)-1)m$.

Since $\chi'(G_1)=\Delta(G)-1$, it follows that we can color the edges of $G_1[\overline{K}_m]$ with $(\Delta(G)-1)m$ colors such that for each positive integer i, $t_i\in V(G)-V_\Delta$, there are at least m colors are missing at all vertices in V_i. Note that $G^*_1=G_1[\overline{K}_m]\cup(\bigcup_{t_i\in V(G)-V_\Delta}H_i)$. Hence, we can extend the $(\Delta(G)-1)m$ edge-coloring of $G_1[\overline{K}_m]$ to all the edges of $\bigcup_{t_i\in V(G)-V_\Delta}H_i$, so that $\chi'(G^*_1)=(\Delta(G)-1)m$. □

Theorem 3.1. *If G is Subclass 1, then there exists a matching M in G such that $\chi'(G-M)=\Delta(G)-1$ and the corresponding G^*_M is Class 1.*

Proof. Let M be a matching of G such that $\chi'(G-M)=\Delta(G)-1$ and any distinct vertices with the maximum degree in G are saturated by distinct edges of M. Note that each connected component of G^*_M is either a join of a graph on m vertices and an empty graph on m vertices, or a balanced complete bipartite graph on $2m$ vertices. It is easy to see that these connected components are all Class 1. Thus G^*_M is Class 1. □

An instant corollary of Theorem 3.1 and Observation 3.1 is:

Corollary 3.1. *If G is Subclass 1, then G^* is Class 1.*

Theorem 3.2. *Let G be a Subclass 2 graph, and let M be a matching of G such that $\chi'(G-M)=\Delta(G)-1$. For every pair of vertices t_p and t_q of maximum degree in G which are saturated by the same edge of M, if one of the following five conditions holds:*

(i) both H_p and H_q are Class 1;

(ii) H_p is a subgraph of H_q;

(iii) both H_p and H_q are disjoint unions of cliques;

(iv) $\Delta(H_p)\neq\Delta(H_q)$;

(v) join graph H_p+H_q is regular;

then G^*_M is Class 1.

Proof. Since G is Subclass 2 , every connected component of G^*_M can be denoted by H_p+H_q, or $H_p+\overline{K}_m$, or $H_q+\overline{K}_m$, or $\overline{K}_m+\overline{K}_m$, where $\overline{K}_m$ denotes an empty graph on m vertices, and $\overline{K}_m+\overline{K}_m$ denotes the join of two vertex-disjoint empty graphs on m vertices. Note that $H_p+\overline{K}_m$, $H_q+\overline{K}_m$ and $\overline{K}_m+\overline{K}_m$ are all Class 1. Hence, it is only necessary to prove that: if one of the conditions (i)–(v) holds, then H_p+H_q is Class 1.

Assume that one of the conditions (i)–(iv) holds. Since $|V(H_p)|=|V(H_q)|=m$, it follows from Theorems 1.1 and 1.2 that H_p+H_q is Class 1.

Assume that (v) holds. Since H_p+H_q is regular and $|V(H_p)|=|V(H_q)|$, H_p+H_q is Class 1 by Theorem 1.3 or Theorem 1.4. □

An instant corollary of Theorem 3.2 and Observation 3.1 is:

Corollary 3.2. *Let G be a Subclass 2 graph, and let M be a matching of G such that $\chi'(G-M) = \Delta(G)-1$. For every pair of vertices t_p and t_q of maximum degree in G which are saturated by the same edge of M, if one of five conditions of Theorem 3.2 holds, then G^* is Class 1.*

Note that the join graph $H_1 + H_2$ corresponds to $P_2[h_2]$, where P_2 is a Class 1 graph. By Corollaries 3.1 and 3.2, we can effortlessly generalize the results of Theorem 1.5 in the case where $|V(H_1)| = |V(H_2)|$ to the generalized lexicographic product, yielding the following theorem:

Theorem 3.3. *If G is Class 1 and all graphs in h_n are regular, then G^* is Class 1.*

In addition, by Corollaries 3.1 and 3.2, we can directly obtain Theorem 1.6. Furthermore, one easy consequence of Corollary 3.2 is the following result, which is similar to Theorem 1.2.

Theorem 3.4. *Let $G = H_1 + H_2$ be a join graph with $|V(H_1)| = |V(H_2)|$. If G is regular, or if both H_1 and H_2 are Class 1, or if H_1 is a subgraph of H_2, or if both H_1 and H_2 are disjoint unions of cliques, then G is Class 1.*

Through the decomposition of G^* in formula 2.2, we can make the following observation:

Observation 3.2. *Suppose that all graphs in h_n have the same maximum degree. If both G_2^* and G_3^* are Class 1, then G^* is Class 1.*

Proof. Let $\Delta(H_i) = \Delta_H$, where $i = 1, 2, \ldots, n$. It is easy to see that $\Delta(G_2^*) = \Delta(G) + \Delta_H$, $\Delta(G_3^*) = \Delta(G)(m-1)$ and $\Delta(G^*) = \Delta(G)m + \Delta_H$. Hence, $\Delta(G^*) = \Delta(G_2^*) + \Delta(G_3^*)$. Since $\chi'(G_2^*) = \Delta(G_2^*)$, and since $\chi'(G_3^*) = \Delta(G_3^*)$, it follows that $\chi'(G^*) = \Delta(G^*)$, that is, G^* is Class 1. □

By Observation 3.2, we can obtain the following theorem:

Theorem 3.5. *Suppose that all graphs in h_n are Class 1 graphs with the same maximum degree. If m is even, then G^* is Class 1.*

Proof. Let $\Delta(H_i) = \Delta_H$, where $i = 1, 2, \ldots, n$. We can first color the edges of each subgraph G_j of G_2^* with $\Delta(G) + 1$ colors $1, 2, \ldots, \Delta(G) + 1$ such that edges which are corresponding to the same edge of G receive the same color. Since we use $\Delta(G) + 1$ colors, it follows that each vertex (t_i, y_j) of H_i misses at least one color c_i in $\{1, 2, \ldots, \Delta(G) + 1\}$, where $i = 1, 2, \ldots, n$. Hence, we can color the edges of each subgraph H_i of G_2^* with the color c_i and an additional $\Delta_H - 1$ new colors. Thus, $\chi'(G_2^*) \le \Delta(G) + \Delta_H = \Delta(G_2^*)$, that is, G_2^* is Class 1. On the other hand, since $G_3^* = G \times K_m$, and since m is even, G_3^* is Class 1 according to Lemma 1.1. Therefore, G^* is also Class 1. □

An instant corollary of Theorem 3.5 is:

Corollary 3.3. *If all graphs in h_n are regular Class 1 graphs with the same maximum degree, then G^* is Class 1.*

By applying Theorem 3.5, we can derive Theorem 1.7. Furthermore, by utilizing Theorem 3.3 and Corollary 3.3, we can formulate the following theorem:

Theorem 3.6. *Suppose that $G^* = G[h_n]$ is regular. If G is Class 1, or each graph of h_n is Class 1, then G^* is Class 1.*

Proof. Since G^* is regular, and since all graphs in h_n have the same number of vertices, it follows that all graphs in h_n are regular graphs with the same maximum degree. If G is Class 1, then G^* is Class 1 by Theorem 3.3. If each graph of h_n is Class 1, then G^* is Class 1 according to Corollary 3.3. □

By applying Theorem 3.6, take $G = P_2$, we can derive the result of theorem 1.4 in the case where $|V(H_1)| = |V(H_2)|$. In addition, the graph G^* in Theorem 3.6 does not necessarily satisfy the condition "$\Delta(G^*) \geq \frac{|V(G^*)|}{2}$" of 1-Factorization Conjecture. For instance, suppose that G is a k-regular bipartite graph with n vertices, and $h_n = (H_i)_{i\in\{1,2,\ldots,n\}}$ represents a sequence of vertex-disjoint cubic graphs, each with m vertices, where $m \geq 4$. Clearly, G^* is Class 1 according to Theorem 3.6, however, we have $\Delta(G^*) = km + 3 < \frac{nm}{2} = \frac{|V(G^*)|}{2}$ when $k < \frac{n}{2} - 1$.

It is easy to see that if G^* is regular and all graphs in h_n have the same number of vertices, then the inequality $\Delta(G^*) \geq \frac{|V(G^*)|}{2}$ can be derived from the inequality $\Delta(G) \geq \frac{|V(G)|}{2}$. By Theorem 3.6, if the 1-factorial conjecture holds for G, then the conjecture holds for G^*.

Finally, in a generalized lexicographic product $K_p[h_p]$, we consider the case where $|V(K_p[h_p])|$ is even and every graph in the sequence $h_p = (H_i)_{i\in\{1,2,\ldots,p\}}$ has $\lfloor\frac{|V(K_p[h_p])|}{p}\rfloor$ vertices or $\lceil\frac{|V(K_p[h_p])|}{p}\rceil$ vertices. Using Theorem 1.5, we can derive the following theorem:

Theorem 3.7. *Let $G = K_p[h_p]$ and let $h_p = (H_i)_{i\in\{1,2,\ldots,p\}}$ be a sequence of vertex-disjoint k-regular graphs such that every graph in h_p has $\lfloor\frac{|V(G)|}{p}\rfloor$ vertices or $\lceil\frac{|V(G)|}{p}\rceil$ vertices, where $p \geq 2$ and $k \geq 0$. If $|V(G)|$ is even, then G is Class 1.*

Proof. Let $n = |V(G)|$, and let $V_1, V_2, \ldots, V_p$ be the partition sets of G. If $\lfloor\frac{n}{p}\rfloor = \lceil\frac{n}{p}\rceil$, it follows that G is an even graph that is the join of two regular graphs, then G is Class 1 according to Theorem 1.5. If $\lfloor\frac{n}{p}\rfloor \neq \lceil\frac{n}{p}\rceil$, then we may assume that $|V_i| = \lfloor\frac{n}{p}\rfloor$ for $i = 1, 2, \ldots, r$ and $|V_i| = \lceil\frac{n}{p}\rceil$ for $i = r+1, r+2, \ldots, p$, where $1 \leq r \leq p-1$. Let

$$G_1 = G[\bigcup_{i=1}^{r} V_i],\ G_2 = G[\bigcup_{i=r+1}^{p} V_i].$$

Clearly, both G_1 and G_2 are regular graphs, and $G = G_1 + G_2$. By Theorem 1.5, G is Class 1. □

In Theorem 3,7, by letting $k = 0$ and $p = r - 1$, then we can directly derive the following corollary:

Corollary 3.4. *All Turán graphs on an even number of vertices are Class 1.*

4. Conclusions

In this paper, for a generalized lexicographic product $G[h_n]$ of a graph G with n vertices and a sequence $h_n = (H_i)_{i\in\{1,2,\ldots,n\}}$ of vertex-disjoint graphs with m vertices, we obtain the following sufficient conditions for $G[h_n]$ to be Class 1: (i) G is Subclass 1; (ii) G is Class 1 and all graphs in h_n are regular; (iii) all graphs in h_n are Class 1 graphs with the same maximum degree, and m is even; (iv) $G[h_n]$ is regular and either G or each graph in h_n is Class 1.

In addition, for a generalized lexicographic product $G = K_p[h_p]$ of a complete graph K_p on p vertices and a sequence $h_p = (H_i)_{i\in\{1,2,\ldots,p\}}$ of vertex-disjoint k-regular graphs whose partition sets differ in size by at most 1, we prove that G is Class 1 if G has an even number of vertices.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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