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Research article

Common fixed points for (κ_{G_m})-contractions with applications

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Abstract: In this publication, our objective was to introduce and establish the concepts of κ_{G_m}-contraction and generalized (α, κ_{G_m})-contraction in complete G_m-metric spaces, which led to the discovery of novel fixed points, coincidence points, and common fixed points. Additionally, we demonstrated the usefulness of our main results by applying it to the investigation of the integral equation. Also, we presenting a noteworthy example demonstrating the practicality of our primary hypothesis.

Keywords: G_m-metric space; fixed point; (κ_{G_m})-contraction; generalized (α, κ_{G_m})-contraction; integral equation
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1. Introduction

Fixed point theory, a cornerstone of mathematical analysis, investigates the existence and uniqueness of solutions represented by "fixed points" of a function. This theory plays a crucial role in various scientific disciplines [1–3]. In this particular theory, the foundational breakthrough emerges with the Banach contraction principle [4], notable for its application within the realm of complete metric spaces. The concept of the metric space itself was introduced by M. Frechet [5] in 1906. Inspired by the impact of this seminal work on fixed point theory, numerous researchers have undertaken endeavors to extend these concepts in recent years (see. [6–8]). The concept of G_m-metric space was first introduced in 2006 by Mustafa et al. [9]. They established some outcomes in fixed point theory for contractive functions in this space. Thereafter, Mustafa et al. [10] obtained coincidence point theorems for generalized-weakly contractive mappings. Kaewchareon et al. [11] introduced the concept of Housdorff distance function in the setting of G_m-metric spaces and established fixed point theorems for multivalued mappings. Afterward, Tahat et al. [12] utilized the idea of foregoing the Housdorff distance function to establish coincidence point and common fixed point results. Following the pioneer article of Mustafa et al. [9], a number of authors have established various results (see [13–18]). Subsequently, Samet et al. [19,20] observed that several previously published theorems in the context of a quasimetric spaces may be used to deduce some results in the setting of G_m-metric space. According to Samet et al., one may construct an analogous result in the configuration of a quasimetric space if the contractive condition employed in the result constructed in the framework of G_m-metric space can be reduced to two variables from three variables. More specifically, they noted that the G_m-metric produces a quasimetric d, defined by $d(h,\omega) = G_m(h,\omega,\omega)$.

On the other hand, Samet et al. [21] introduced the notions of α-admissble mapping and (α,ψ)-contraction in the framework of complete metric spaces and the generalized Banach contraction principle. Subsequently, Alghamdi et al. [22] extended the concept of α-admissible mapping to G-metric spaces. Later on, Mustafa et al. [23] gave the idea of multivalued α-admissible mapping in the context of G-metric spaces.

Recently, Jleli et al. [24] introduced a new type of contraction named the κ-contraction and established some fixed point results. Li et al. [25] used this new contraction and proved some generalized fixed point theorems. Al-Rawashdeh et al. [26] established common fixed point results for κ-contraction and extended some well-known results of literature.

In this research article, we introduce new concepts such as (κ_{G_m})-contractions and generalized (α,κ_{G_m})-contraction to establish new fixed point, coincidence point and common fixed point theorems. These findings extend and generalize several results found in existing literature.

2. Preliminaries

We present a few needed definitions and outcomes in this part.

Definition 1. *([9]) A nonempty set $\mathcal{M}$ with the $G_m : \mathcal{M}\times\mathcal{M}\times\mathcal{M} \to \mathbb{R}^+$ is a mapping with the following characteristics.*

(G_m1) $0 < G_m(h,h,\omega)$, for all $h,\omega \in \mathcal{M}$ with $h \neq \omega$,
(G_m2) $G_m(h,\omega,\Phi) = 0$ if $h = \omega = \Phi$,
(G_m3) $G_m(h,\omega,\Phi) = G_m(h,\Phi,\omega) = G_m(\omega,\Phi,h) = \cdots$ (symmetry in all three variables),
(G_m4) $G_m(h,h,\omega) \leq G_m(h,\omega,\Phi)$, for all $h,\omega,\Phi \in \mathcal{M}$ with $\omega \neq \Phi$,
(G_m5) $G_m(h,\omega,\Phi) \leq G_m(h,a_1,a_1) + G_m(a_1,\omega,\Phi)$, for all $h,\omega,\Phi,a_1 \in \mathcal{M}$ (rectangle inequality).

The pair $(\mathcal{M},G_m)$ is referred to a generalized metric space, and the mapping is known as a generalized metric or G_m metric on $\mathcal{M}$.

Definition 2. *([9]) Considering $(\mathcal{M},G_m)$ to be a generalized-metric space and $\left(h_g\right)$ to be a sequence of $\mathcal{M}$ points, we may say that (h_n) is G_m-convergent to $h \in M$. if $\lim_{n,p\to\infty} G_m\left(h,h_n,h_p\right) = 0$, that is, considering $\epsilon > 0$, there exists $s \in \mathbb{N}$ such that $G_m\left(h,h_n,h_p\right) < \epsilon$, for all $n,p \geq s$. A point of the series is named h so $h_n \to h$ or $lim_{n\to\infty} h_n = h$.*

Proposition 1. *([9]) A generalized metric space would be $(\mathcal{M},G_m)$. The following claims are equivalent.*

(1) (h_n) is G_m-convergent to h,
(2) $G_m(h_n, h_n, h) \to 0$ as $n \to \infty$,
(3) $G_m(h_n, h, h) \to 0$ as $n \to \infty$,
(4) $G_m\left(h_n, h_p, h\right) \to 0$ as $n, p \to \infty$.

Definition 3. *([9]) In a generalized metric space $(\mathcal{M}, G_m)$, if for each $\epsilon > 0$, there is $s \in \mathbb{N}$ such that $G_m\left(h_n, h_p, h_q\right) < \epsilon$, for all n, p, $q \geq s$, then the sequence (h_t) is said to be G_m-Cauchy sequence that is $G_m\left(h_n, h_p, h_q\right) \to 0$ as n, p, $q \to +\infty$.*

Definition 4. *([9]) Every G_m-Cauchy sequence must be G_m-convergent in a G_m-metric space $(\mathcal{M}, G_m)$ which is G_m-complete.*

The metric d_{G_m} on $\mathcal{M}$ defined by any generalized metric on $\mathcal{M}$ is given below

$$d_{G_m}(h, \omega) = G_m(h, \omega, \omega) + G_m(\omega, h, h), \tag{2.1}$$

for all $h, \omega \in \mathcal{M}$.

Example 1. *([9]) Let $(\mathcal{M}, d)$ be a metric space. The mapping $G_m : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0, +\infty)$, defined by*

$$G_m(h, \omega, \Phi) = \max\{d(h, \omega), d(\omega, \Phi), d(\Phi, h)\},$$

$$G_m(h, \omega, \Phi) = d(h, \omega) + d(\omega, \Phi) + d(\Phi, h),$$

for all $h, \omega, \Phi \in \mathcal{M}$, is a generalized metric on $\mathcal{M}$.

Theorem 1. *([9]) Considering $(\mathcal{M}, d)$ to be a metric space, $(\mathcal{M}, d)$ is a complete metric space if and only if, $(\mathcal{M}, G_m)$ is a complete generalized metric space.*

The following ideas were recently suggested by Kaewchareon et al. [11]. We will refer to the family of all closed, bounded subsets of $\mathcal{M}$ that are not empty as $CB(\mathcal{M})$. The Hausdorff G_m-distance on $CB(\mathcal{M})$ is denoted by $H(A_1, B_2, C_3)$ and defined as:

$$H_{G_m}(A_1, B_2, C_3) = \max\left\{\sup_{h \in A_1} G_m(h, B_2, C_3), \sup_{h \in B_2} G_m(h, C_3, A_1), \sup_{h \in C_3} G_m(h, A_1, B_2)\right\},$$

where

$$G_m(h, B_2, C_3) = d_{G_m}(h, B_2) + d_{G_m}(B_2, C_3) + d_{G_m}(h, C_3),$$

$$d_{G_m}(A_1, B_2) = \inf\{d_{G_m}(a_1, b_2),\ a_1 \in A_1,\ b_2 \in B_2\},$$

$$d_{G_m}(h, B_2) = \inf\{d_{G_m}(h, \omega),\ \omega \in B_2\}.$$

Remember that $G_m(h, \omega, C_3) = \inf\{G_m(h, \omega, \Phi),\ \Phi \in C_3\}$. A function $\hat{w} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$ is named as a multivalued function. If $h \in \hat{w}h$, then the point $h \in \mathcal{M}$ is referred to as a fixed point of $\hat{w}$.

Lemma 1. *If $A_1, B_2 \in CB(\mathcal{M})$ and $a_1 \in A_1$, at the point $\forall\ \varepsilon > 0$, there remains $b_2 \in B_2$ such that*

$$G_m(a_1, b_2, b_2) \leq H_{G_m}(A_1, B_2, B_2) + \epsilon.$$

Definition 5. *([11, 12]) Let $\mathcal{M}$ be a given set containing at least one element. Suppose that $j : \mathcal{M} \longrightarrow \mathcal{M}$ and $\hat{w} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$. If $f = j(h) \in \hat{w}(h)$ for some $h \in \mathcal{M}$, then h is named a coincidence point of mapping $\hat{w}$ and j. Also, f is said to be a point of coincidence of j and $\hat{w}$. If $f = h$, then f is said to be a common fixed point of j and $\hat{w}$. Functions j and $\hat{w}$ are named as weakly compatible if $j(h) \in \hat{w}(h)$ for some $h \in \mathcal{M}$ implies $j\hat{w}(h) \subseteq \hat{w}j(h)$.*

Proposition 2. *([11, 12]) Let $\mathcal{M}$ be a given set containing at least one element. Suppose two weakly compatible functions j and $\hat{w}$, where $j : \mathcal{M} \longrightarrow \mathcal{M}$ and $\hat{w} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$. If the point of coincidence 'f' of j and $\hat{w}$ is unique, then f will be the unique common fixed point of j and $\hat{w}$.*

A new contraction and a related fixed point theorem was established by Jleli et al. [24], which is given below.

Definition 6. *Consider a mapping $\kappa : (0, \infty) \to (1, \infty)$ fulfilling:*

(κ_1) κ is a nondecreasing function,
(κ_2) for every sequence $\{\alpha_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \kappa(\alpha_n) = 1$ if, and only if, $\lim_{n\to\infty}(\alpha_n) = 0$,
(κ_3) there exist $z \in (0, \infty]$ and $0 < r < 1$ such that $\lim_{\alpha\to 0^+} \frac{\kappa(\alpha)-1}{\alpha^r} = z$;

A mapping $\mathcal{L} : \mathcal{M} \to \mathcal{M}$ is said to be a κ-contraction if there exist any constant $\lambda \in (0, 1)$ and a function κ satisfying (κ_1)-(κ_3) and

$$d(\mathcal{L}h, \mathcal{L}\omega) \neq 0 \Longrightarrow \kappa(d(\mathcal{L}h, \mathcal{L}\omega)) \leq [\kappa(d(h, \omega))]^\lambda, \tag{2.2}$$

for all $h, \omega \in \mathcal{M}$.

Theorem 2. *([24]) Let $(\mathcal{M}, d)$ be a complete metric space and $\mathcal{L} : \mathcal{M} \to \mathcal{M}$ be a κ-contraction, then $\mathcal{L}$ has a unique fixed point.*

Subsequently, Hancer et al. [27] added a general condition (κ_4) to the aforementioned Definition 6, which is stated as follows:

(κ_4) If $A_1 \subset (0, \infty)$ with $\inf A_1 > 0$, then $\inf \kappa(A_1) = \kappa(\inf A_1)$.

We represent the set of all continuous functions $\kappa : (0, \infty) \to (1, \infty)$ satisfying the conditions (κ_1)-(κ_4) by Ω, in accordance with Hancer et al. [27].

3. Main result

We introduce the notion of (κ_{G_m})-contraction in this section and present our main result with corollaries and examples.

Definition 7. *Consider the generalized metric space $(\mathcal{M}, G_m)$, the multivalued function $\mathcal{L} : \mathcal{M} \longrightarrow CB(\mathcal{M})$, and the self function $j : \mathcal{M} \longrightarrow \mathcal{M}$. The functions $\mathcal{L}$ and j satisfy (κ_{G_m})-contraction if there exist $\kappa \in \Omega$ and $\lambda \in (0, 1)$ such that*

$$H_{G_m}(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi) > 0 \text{ implies } \kappa(H_{G_m}(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi)) \leq [\kappa(G_m(jh, j\omega, j\Phi))]^\lambda, \tag{3.1}$$

for all $h, \omega, \Phi \in \mathcal{M}$.

Theorem 3. *Let $(\mathcal{M}, G_m)$ be a generalized metric space , $\mathcal{L} : \mathcal{M} \longrightarrow CB(\mathcal{M})$ be a multivalued function, and $j : \mathcal{M} \longrightarrow \mathcal{M}$ is a self-mapping. Suppose that there exist $\kappa \in \Omega$ and $\lambda \in (0,1)$ such that the functions $\mathcal{L}$ and j satisfy (κ_{G_m})-contraction. Then, j and $\mathcal{L}$ have a point of coincidence in $\mathcal{M}$, if for any $h \in \mathcal{M}$, $\mathcal{L}h \subseteq j(\mathcal{M})$ and $j(\mathcal{M})$ is a G_m-complete subspace of $\mathcal{M}$. Moreover, if we suppose that $ju \in \mathcal{L}u$ and $jv \in \mathcal{L}v$ implies $G_m(jv, ju, ju) \leq H_{G_m}(\mathcal{L}v, \mathcal{L}u, \mathcal{L}u)$, then*
(i) j and $\mathcal{L}$ have a unique point of coincidence.
(ii) Furthermore, if j and $\mathcal{L}$ are weakly compatible, then j and $\mathcal{L}$ have a unique common fixed point.

Proof. Let h_0 represent any chosen point in $\mathcal{M}$. Since $\mathcal{L}h_0 \subseteq j(\mathcal{M})$, choose h_1 in the set $\mathcal{M}$ such that $jh_1 \in \mathcal{L}h_0$. If $jh_1 = jh_0$, then j and $\mathcal{L}$ have a point of coincidence. So, we suppose that $jh_0 \neq jh_1$. Now, $\mathcal{L}h_1 \neq \emptyset$, and if $\mathcal{L}h_0 = \mathcal{L}h_1$, then, again, j and $\mathcal{L}$ have a point of coincidence by the fact that $jh_1 \in \mathcal{L}h_0 = \mathcal{L}h_1$. So, we assume that $\mathcal{L}h_0 \neq \mathcal{L}h_1$. Then, $H_{G_m}(\mathcal{L}h_0, \mathcal{L}h_1, \mathcal{L}h_1) > 0$.

Now, by the inequality (3.1), we have

$$\kappa(G_m(jh_1, \mathcal{L}h_1, \mathcal{L}h_1)) \leq \kappa(H_{G_m}(\mathcal{L}h_0, \mathcal{L}h_1, \mathcal{L}h_1)) \leq [\kappa(G_m(jh_0, jh_1, jh_1))]^{\lambda}. \tag{3.2}$$

From (κ_4), we know that

$$\kappa(G_m(jh_1, \mathcal{L}h_1, \mathcal{L}h_1)) = \inf_{\omega \in \mathcal{L}h_1} \kappa(G_m(jh_1, \omega, \omega)).$$

Thus from (3.2), we get

$$\inf_{\omega \in \mathcal{L}h_1} \kappa(G_m(jh_1, \omega, \omega)) \leq [\kappa(G_m(jh_0, jh_1, jh_1))]^{\lambda}. \tag{3.3}$$

Since $\mathcal{L}h_1 \subseteq j(\mathcal{M})$, we deduce that there exists $h_2 \in \mathcal{M}$ and $\omega = jh_2 \in \mathcal{L}h_1$ such that

$$\kappa(G_m(jh_1, jh_2, jh_2)) \leq [\kappa(G_m(jh_0, jh_1, jh_1))]^{\lambda}. \tag{3.4}$$

Similarly, as $jh_2 \in \mathcal{L}h_1$, if $jh_2 = jh_1$, then $w = jh_1$ is a point of coincidence of mapping j and $\mathcal{L}$ and we obtain the required result. Suppose that $jh_1 \neq jh_2$. Now, if $\mathcal{L}h_1 = \mathcal{L}h_2$, then, again, by $jh_2 \in \mathcal{L}h_1 = \mathcal{L}h_2$, j and $\mathcal{L}$ have point of coincidence. So, we assume that $\mathcal{L}h_1 \neq \mathcal{L}h_2$. Then, $H_{G_m}(\mathcal{L}h_1, \mathcal{L}h_2, \mathcal{L}h_2) > 0$. Now, by (3.1), we have

$$\kappa(G_m(jh_2, \mathcal{L}h_2, \mathcal{L}h_2)) \leq \kappa(H_{G_m}(\mathcal{L}h_1, \mathcal{L}h_2, \mathcal{L}h_2)) \leq [\kappa(G_m(jh_1, jh_2, jh_2))]^{\lambda}. \tag{3.5}$$

From the condition (κ_4), we know that

$$\kappa(G_m(jh_2, \mathcal{L}h_2, \mathcal{L}h_2)) = \inf_{\omega \in \mathcal{L}h_2} \kappa(G_m(jh_2, \omega, \omega)).$$

Thus from (3.5), we get

$$\inf_{\omega \in \mathcal{L}h_2} \kappa(G_m(jh_2, \omega, \omega)) \leq [\kappa(G_m(jh_1, jh_2, jh_2))]^{\lambda}. \tag{3.6}$$

Since $\mathcal{L}h_2 \subseteq j(\mathcal{M})$, we deduce that there exists $h_3 \in \mathcal{M}$ and $\omega = jh_3 \in \mathcal{L}h_2$ such that

$$\kappa(G_m(jh_2, jh_3, jh_3)) \leq [\kappa(G_m(jh_1, jh_2, jh_2))]^{\lambda}. \tag{3.7}$$

In the same way, we will define a sequence $\{jh_n\} \subset \mathcal{M}$ such that $jh_n \notin \mathcal{L}h_n$, $jh_{n+1} \in \mathcal{L}h_n$ and

$$\kappa(G_m(jh_n, jh_{n+1}, jh_{n+1})) \leq [\kappa(G_m(jh_{n-1}, jh_n, jh_n))]^{\lambda}, \tag{3.8}$$

for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} 1 &< \kappa(G_m(jh_n, jh_{n+1}, jh_{n+1})) \leq [\kappa(G_m(jh_{n-1}, jh_n, jh_n))]^{\lambda} \\ &\leq [\kappa(G_m(jh_{n-2}, jh_{n-1}, jh_{n-1}))]^{\lambda^2} \\ &\leq \cdots \\ &\leq [\kappa(G_m(jh_0, jh_1, jh_1))]^{\lambda^n}, \end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$. Since $\kappa \in \Omega$, by taking the limit as $n \longrightarrow \infty$ in (3.9), we have

$$\lim_{n \longrightarrow \infty} \kappa(G_m(jh_n, jh_{n+1}, jh_{n+1})) = 1. \tag{3.10}$$

From the condition (κ_2), we have

$$\lim_{n \longrightarrow \infty} G_m(jh_n, jh_{n+1}, jh_{n+1}) = 0.$$

From the condition (κ_3), there exist $z \in (0, \infty]$ and $0 < r < 1$ such that

$$\lim_{n \to \infty} \frac{\kappa(G_m(jh_n, jh_{n+1}, jh_{n+1})) - 1}{G_m(jh_n, jh_{n+1}, jh_{n+1})^r} = z. \tag{3.11}$$

Let us consider $z < \infty$. For the above condition, take $B_2 = \frac{z}{2} > 0$. Using the condition of the limit of a sequence, there exists $n_0 \in \mathbb{N}$, and we have

$$\left|\frac{\kappa(G_m(jh_n, jh_{n+1}, jh_{n+1})) - 1}{G_m(jh_n, jh_{n+1}, jh_{n+1})^r} - z\right| \leq B_2$$

for all $n > n_0$. This implies that

$$\frac{\kappa(G_m(jh_n, jh_{n+1}, jh_{n+1})) - 1}{G_m(jh_n, jh_{n+1}, jh_{n+1})^r} \geq z - B_2 = \frac{z}{2} = B_2$$

for all $n > n_0$. We get

$$nG_m(jh_n, jh_{n+1}, jh_{n+1})^r \leq A_1 n[\kappa(G_m(jh_n, jh_{n+1}, jh_{n+1})) - 1] \tag{3.12}$$

for all $n > n_0$, where $A_1 = \frac{1}{B_2}$. Let us take $z = \infty$. We take $B_2 > 0$ any random positively number. Using condition of limit,

$$B_2 \leq \frac{\kappa(G_m(jh_n, jh_{n+1}, jh_{n+1})) - 1}{G_m(jh_n, jh_{n+1}, jh_{n+1})^r},$$

for all $n > n_0$. This implies that

$$nG_m(jh_n, jh_{n+1}, jh_{n+1})^r \leq A_1 n[\kappa(G_m(jh_n, jh_{n+1}, jh_{n+1})^r - 1],$$

for all $n > n_0$, where $A_1 = \frac{1}{B_2}$. For every case, there exist $A_1 > 0$ and $n_0 \in \mathbb{N}$,

$$nG_m(jh_n, jh_{n+1}, jh_{n+1})^r \leq A_1 n[\kappa(G_m(jh_n, jh_{n+1}, jh_{n+1})^r) - 1], \tag{3.13}$$

for all $n > n_0$. Thus, by (3.9) and (3.13), we get

$$nG_m(jh_n, jh_{n+1}, jh_{n+1})^r \leq A_1 n([\kappa(G_m(jh_0, jh_1, jh_1))]^{r^n} - 1). \tag{3.14}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to+\infty} nG_m(jh_n, jh_{n+1}, jh_{n+1})^r = 0.$$

Hence, there is $n_1 \in \mathbb{N}$ such that

$$G_m(jh_n, jh_{n+1}, jh_{n+1}) \leq \frac{1}{n^{1/r}}, \tag{3.15}$$

for all $n > n_1$. We are now going to prove that $\{jh_n\}$ is a G_m-Cauchy sequence.
For $p > n > n_1$, we have

$$\begin{aligned} G_m\left(jh_n, jh_p, jh_p\right) &\leq \sum_{i=n}^{p-1} G_m(jh_i, jh_{i+1}, jh_{i+1}) \\ &\leq \sum_{i=n}^{p-1} \frac{1}{i^{\frac{1}{r}}} \leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{r}}}. \end{aligned} \tag{3.16}$$

Since $r \in (0,1)$, the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ converges. As a result, $G_m\left(jh_n, jh_p, jh_p\right) \longrightarrow 0$ as $p, n \longrightarrow \infty$. Hence, $\{jh_n\}$ is a G_m-Cauchy sequence in complete subspace $j(\mathcal{M})$, and this confirms the existence of $v \in j(\mathcal{M})$ such that

$$\lim_{n\longrightarrow\infty} G_m(jh_n, jh_n, v) = \lim_{n\longrightarrow\infty} G_m(jh_n, v, v) = 0. \tag{3.17}$$

Since $v \in j(\mathcal{M})$, there exists $u \in \mathcal{M}$ such that $v = ju$. Thus from (3.17), we have

$$\lim_{n\longrightarrow\infty} G_m(jh_n, jh_n, ju) = \lim_{n\longrightarrow\infty} G_m(jh_n, ju, ju) = 0.$$

We are going to prove that $ju \in \mathcal{L}u$. If there exists a sequence $\{n_\mu\}$ such that $jh_{n_\mu} \in \mathcal{L}u$, for all $\mu \in \mathbb{N}$, as $jh_{n_\mu} \to ju$, the proof is successfully finished, since we have obtained $ju \in \mathcal{L}u$ because $\mathcal{L}u$ is closed. Suppose that there is $n_0 \in \mathbb{N}$ such that $jh_{n+1} \notin \mathcal{L}u$, for all $n \in \mathbb{N}$ and $n \geq n_0$, then $\mathcal{L}h_n \neq \mathcal{L}u$, therefore,

$$G_m(jh_{n+1}, \mathcal{L}u, \mathcal{L}u) \leq H_{G_m}(\mathcal{L}h_n, \mathcal{L}u, \mathcal{L}u). \tag{3.18}$$

So, by (3.1), we get

$$\begin{aligned} \kappa(G_m(jh_{n+1}, \mathcal{L}u, \mathcal{L}u)) &\leq \kappa(H_{G_m}(\mathcal{L}h_n, \mathcal{L}u, \mathcal{L}u)) \\ &\leq [\kappa(G_m(jh_n, ju, ju))]^\lambda \leq \kappa(G_m(jh_n, ju, ju)). \end{aligned}$$

From the condition (κ_1), we have

$$G_m(jh_{n+1}, \mathcal{L}u, \mathcal{L}u) \leq G_m(jh_n, ju, ju). \tag{3.19}$$

Using the assumption that the function G_m is continuous on its three variables and allowing $n \to \infty$ in the preceding inequality, we obtain $G_m(ju, \mathcal{L}u, \mathcal{L}u) = 0$. As $\mathcal{L}u$ is closed, we obtained $ju \in \mathcal{L}u$. It follows that there exists a point of coincidence v of $\mathcal{L}$ and j. We shall demonstrate the uniqueness of the point of coincidence of $\mathcal{L}$ and j. Assume that there exists another point of coincidence σ of $\mathcal{L}$ and j such that $\sigma = j\varpi \in \mathcal{L}\varpi$ and $ju \neq j\varpi$. Thus, we have

$$G_m(j\varpi, ju, ju) \leq H_{G_m}(\mathcal{L}\varpi, \mathcal{L}u, \mathcal{L}u).$$

We get by (3.1):

$$\kappa(G_m(j\varpi, ju, ju) \leq \kappa(H_{G_m}(\mathcal{L}\varpi, \mathcal{L}u, \mathcal{L}u) \leq [\kappa(G_m(j\varpi, ju, ju)]^{\lambda}.$$

Additionally, we get

$$1 < \kappa(G_m(j\varpi, ju, ju)) \leq [\kappa(G_m(j\varpi, ju, ju)]^{\lambda}. \tag{3.20}$$

Letting $n \to \infty$ in (3.20), we have

$$\lim_{n\to+\infty} \kappa(G_m(j\varpi, ju, ju)) = 1.$$

By the condition (κ_2), we get

$$G_m(j\varpi, ju, ju) = \lim_{n\to+\infty} G_m(j\varpi, ju, ju) = 0.$$

That is, $j\varpi = ju$. Hence, the point of coincidence for j and $\mathcal{L}$ is unique. Assume that j and $\mathcal{L}$ are weakly compatible. By using the proposition 2, we can easily obtain the common fixed point of j and $\mathcal{L}$ which will be unique. □

Example 2. *Let $\mathcal{M} = [0, 1]$. Define function $\mathcal{L} : \mathcal{M} \longrightarrow CB(\mathcal{M})$ by $\mathcal{L}h = \left[0, \frac{h}{25}\right]$ and define $j : \mathcal{M} \longrightarrow \mathcal{M}$ by $j(h) = \frac{3h}{4}$. Define a generalized metric on $\mathcal{M}$ by $G_m(h, \omega, \Phi) = |h - \omega| + |\omega - \Phi| + |h - \Phi|$. We get*

(1) the mappings $\mathcal{L}$ and j are weakly compatible;
(2) $j(\mathcal{M})$ is G_m-complete;
(3) $\mathcal{L}h \subseteq j(\mathcal{M})$;
(4) the functions $\mathcal{L}$ and j satisfy (κ_{G_m})-contraction, where $\kappa(\alpha) = \exp\sqrt{\alpha}$ and $\lambda = \sqrt{\frac{32}{75}} \in (0, 1)$.

Solution: First three conditions are satisfied easily. We need to prove the condition (4).

We have $d_{G_m}(h, \omega) = G_m(h, \omega, \omega) + G_m(\omega, h, h) = 4|h - \omega|$, for all $h, \omega \in \mathcal{M}$. To prove the condition (4), let $h, \omega, \Phi \in \mathcal{M}$. If at least one of h, ω, and Φ being 0, then $\mathcal{L}h = \mathcal{L}\omega = \mathcal{L}\Phi = 0$, and $H_{G_m}(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi) = 0$, thus we may suppose that h, ω, and Φ are nonzero. Without changing in conception, let us suppose $h < \omega < \Phi$. We get

$$H_{G_m}(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi) = H_{G_m}\left(\left[0, \frac{h}{25}\right], \left[0, \frac{\omega}{25}\right], \left[0, \frac{\Phi}{25}\right]\right)$$

$$= \max\left\{\begin{array}{c}\sup_{0\le a_1\le \frac{h}{25}} G_m\left(a_1,\left[0,\frac{\omega}{25}\right],\left[0,\frac{\Phi}{25}\right]\right),\\ \sup_{0\le b_2\le \frac{\omega}{25}} G_m\left(b_2,\left[0,\frac{h}{25}\right],\left[0,\frac{\Phi}{25}\right]\right),\\ \sup_{0\le c_3\le \frac{\Phi}{25}} G_m\left(c_3,\left[0,\frac{h}{25}\right],\left[0,\frac{\omega}{25}\right]\right)\end{array}\right\}. \tag{3.21}$$

Since $h<\omega<\Phi$, then $\left[0,\frac{h}{25}\right]\subseteq\left[0,\frac{\omega}{25}\right]\subseteq\left[0,\frac{\Phi}{25}\right]$, which implies that

$$d_{G_m}\left(\left[0,\frac{h}{25}\right],\left[0,\frac{\omega}{25}\right]\right)=d_{G_m}\left(\left[0,\frac{\omega}{25}\right],\left[0,\frac{\Phi}{25}\right]\right)=d_{G_m}\left(\left[0,\frac{h}{25}\right],\left[0,\frac{\Phi}{25}\right]\right)=0.$$

Now, for each $0\le a_1\le\frac{h}{25}$, we have

$$G_m\left(a_1,\left[0,\frac{\omega}{25}\right],\left[0,\frac{\Phi}{25}\right]\right)=d_{G_m}\left(a_1,\left[0,\frac{\omega}{25}\right]\right)+d_{G_m}\left(\left[0,\frac{\omega}{25}\right],\left[0,\frac{\Phi}{25}\right]\right)+d_{G_m}\left(a_1,\left[0,\frac{\Phi}{25}\right]\right)=0.$$

Also, for each $0\le b_2\le\frac{\omega}{25}$, we have

$$\begin{aligned}G_m\left(b_2,\left[0,\frac{h}{25}\right],\left[0,\frac{\Phi}{25}\right]\right) &= d_{G_m}\left(b_2,\left[0,\frac{h}{25}\right]\right)+d_{G_m}\left(\left[0,\frac{h}{25}\right],\left[0,\frac{\Phi}{25}\right]\right)+d_{G_m}\left(b_2,\left[0,\frac{\Phi}{25}\right]\right)\\ &= \begin{cases}0, \text{ if } 0\le b_2\le\frac{h}{25};\\ 4b_2-\frac{4h}{25}, \text{ if } b_2\ge\frac{h}{25}\end{cases}\end{aligned}$$

which implies that

$$\sup_{0\le b_2\le\frac{\omega}{25}} G_m\left(b_2,\left[0,\frac{h}{25}\right],\left[0,\frac{\Phi}{25}\right]\right)=\frac{4\omega-4h}{25}.$$

Furthermore, for every $0\le c_3\le\frac{\Phi}{25}$,

$$\begin{aligned}G_m\left(c_3,\left[0,\frac{h}{25}\right],\left[0,\frac{\omega}{25}\right]\right) &= d_{G_m}\left(c_3,\left[0,\frac{h}{25}\right]\right)+d_{G_m}\left(\left[0,\frac{h}{25}\right],\left[0,\frac{\omega}{25}\right]\right)+d_{G_m}\left(c_3,\left[0,\frac{\omega}{25}\right]\right)\\ &= \begin{cases}0, \text{ if } 0\le c_3\le\frac{h}{25};\\ 4c_3-\frac{4h}{25}, \text{ if } \frac{h}{25}\le c_3\le\frac{\omega}{25};\\ 8c_3-\frac{4\omega}{25}-\frac{4h}{25}, \text{ if } \frac{\omega}{25}\le c_3\le\frac{\Phi}{25}\end{cases}\end{aligned}$$

which implies that

$$\sup_{0\le c_3\le\frac{\omega}{25}} G_m\left(c_3,\left[0,\frac{h}{25}\right],\left[0,\frac{\omega}{25}\right]\right)=\frac{8\Phi-4\omega-4h}{25}.$$

Thus, we deduce that

$$\begin{aligned}e^{\sqrt{H_{G_m}(\mathcal{L}h,\mathcal{L}\omega,\mathcal{L}\Phi)}} &= e^{\sqrt{\max\left\{0,\frac{4\omega-4h}{25},\frac{8\Phi-4\omega-4h}{25}\right\}}}\\ &= e^{\sqrt{\frac{8\Phi-4\omega-4h}{25}}}\\ &\le e^{\sqrt{\frac{8\Phi-8h}{25}}}\\ &= e^{\sqrt{\frac{8}{25}|\Phi-h|}}\\ &= e^{\sqrt{\frac{32}{75}\left|\frac{3\Phi}{4}-\frac{3h}{4}\right|}}\end{aligned}$$

$$
\begin{aligned}
&= e^{\sqrt{\frac{32}{75}|j\Phi - jh|}} \\
&\leq e^{\sqrt{\frac{32}{75}(|jh - j\omega| + |j\omega - j\Phi| + |jh - j\Phi|)}} \\
&= e^{\sqrt{\frac{32}{75}G_m(jh, j\omega, j\Phi)}} \\
&= e^{\sqrt{\frac{32}{75}}\sqrt{G_m(jh, j\omega, j\Phi)}}
\end{aligned}
$$

By using $\kappa(\alpha) = e^{\sqrt{\alpha}}$, we get

$$\kappa\left(H_{G_m}(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi)\right) \leq \left[\kappa\left(G_m(jh, j\omega, j\Phi)\right)\right]^{\lambda}$$

where $\lambda = \sqrt{\frac{32}{75}} \in (0, 1)$.

Hence, the functions $\mathcal{L}$ and j satisfy the (κ_{G_m})-contraction. Now, all conditions of 3 are satisfied. Hence the functions $\mathcal{L}$ and j have a unique coincidence point and common fixed point, which is 0.

Corollary 1. *Let $(\mathcal{M}, G_m)$ be a complete generalized metric space and $\mathcal{L} : \mathcal{M} \longrightarrow CB(\mathcal{M})$ be a multivalued mapping. Suppose that there exist $\kappa \in \Omega$ and $\lambda \in (0, 1)$ such that*

$$H_{G_m}(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi) > 0 \Longrightarrow \kappa\left(H_{G_m}(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi)\right) \leq \left[\kappa\left(G_m(h, \omega, \Phi)\right)\right]^{\lambda},$$

for all $h, \omega, \Phi \in \mathcal{M}$, then $\mathcal{L}$ has a fixed point.

Proof. By assuming that j is the identity function in 3, we can obtain the desired outcome. □

Corollary 2. *Let $(\mathcal{M}, G_m)$ be a complete generalized metric space and $\mathcal{L} : \mathcal{M} \longrightarrow \mathcal{M}$ be a self mapping. If there exist $\kappa \in \Omega$ and $\lambda \in (0, 1)$ such that*

$$G_m(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi) > 0 \Longrightarrow \kappa\left(G_m(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi)\right) \leq \left[\kappa\left(G_m(h, \omega, \Phi)\right)\right]^{\lambda},$$

for all $h, \omega, \Phi \in \mathcal{M}$, then $\mathcal{L}$ has a fixed point.

Proof. By assuming that j is the identity function and $\mathcal{L}$ is a single-valued function in 3, we can obtain the desired outcome. □

Alghamdi et al. [22] defined the concept of α-admissible mapping within the framework of G-metric space, providing the following definition:

Definition 8. *([22]) Let $\alpha : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0, +\infty)$. A mapping $\mathcal{L} : \mathcal{M} \longrightarrow \mathcal{M}$ is designated as α-admissible if for all $h, \omega, \Phi \in \mathcal{M}$, we have*

$$\alpha(h, \omega, \Phi) \geq 1 \textit{ implies } \alpha(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi) \geq 1.$$

Mustafa et al. [23] extended the above notion to multivalued mapping as follows:

Definition 9. *Let $\alpha : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0, +\infty)$. A mapping $\mathcal{L} : \mathcal{M} \longrightarrow Cl(\mathcal{M})$ is designated as multivalued α-admissible if for all $h, \omega, \Phi \in \mathcal{M}$, we have*

$$\alpha(h, \omega, \Phi) \geq 1 \textit{ implies } \alpha(\varrho, \varkappa, \rho) \geq 1$$

for $\varrho \in \mathcal{L}h, \varkappa \in \mathcal{L}\omega$ and $\rho \in \mathcal{L}\Phi$.

Definition 10. *Let* $(\mathcal{M}, G_m)$ *be a generalized metric space and* Ξ *be a closed subset of* $\mathcal{M}$. *A multivalued mapping* $\mathcal{L} : \Xi \longrightarrow CB(\mathcal{M})$ *is said to be a generalized* (α, κ_{G_m})*-contraction if there exist* $\kappa \in \Omega$, $\alpha : \Xi \times \Xi \times \Xi \rightarrow [0, +\infty)$, *and* $\lambda \in (0, 1)$ *satisfying the following conditions (i)* $\mathcal{L}h \cap \Xi \neq \emptyset$, *for all* $h \in \Xi$,
(ii) for all $h, \omega, \Phi \in \Xi$, *we have* $H_{G_m}(\mathcal{L}h \cap \Xi, \mathcal{L}\omega \cap \Xi, \mathcal{L}\Phi \cap \Xi) > 0$ *implying*

$$\alpha(h, \omega, \Phi)\, \kappa\left(H_{G_m}(\mathcal{L}h \cap \Xi, \mathcal{L}\omega \cap \Xi, \mathcal{L}\Phi \cap \Xi)\right) \leq \left[\kappa\left(G_m(h, \omega, \Phi)\right)\right]^{\lambda}. \tag{3.22}$$

Theorem 4. *Let* $(\mathcal{M}, G_m)$ *be a complete generalized metric space,* Ξ *be a closed subset of* $\mathcal{M}$, *and* $\mathcal{L} : \Xi \longrightarrow CB(\mathcal{M})$ *is a generalized* (α, κ_{G_m})*-contraction. Let us consider the fulfillment of the following conditions:*
(i) $\mathcal{L}$ *is a multivalued* α*-admissible mapping,*
(ii) there exist $h_0 \in \Xi$ *and* $h_1 \in \mathcal{L}h_0 \cap \Xi$ *such that* $\alpha(h_0, h_1, h_1) \geq 1$,
(iii) $\mathcal{L}$ *is continuous,*
then $\mathcal{L}$ *has a fixed point.*

Proof. By the supposition (ii), $\exists\, h_0 \in \Xi$ and $h_1 \in \mathcal{L}h_0 \cap \Xi$ such that $\alpha(h_0, h_1, h_1) \geq 1$. If $h_0 = h_1$, then h_0 is the required fixed point and we have nothing to prove. So, we suppose that $h_0 \neq h_1$. If $h_1 \in \mathcal{L}h_1 \cap \Xi$, then h_1 is a fixed point. Let $h_1 \notin \mathcal{L}h_1 \cap \Xi$. Then, $H_{G_m}(\mathcal{L}h_0 \cap \Xi, \mathcal{L}h_1 \cap \Xi, \mathcal{L}h_1 \cap \Xi) > 0$. Now, by the inequality (3.22), we have

$$\begin{aligned} \kappa\left(G_m(h_1, \mathcal{L}h_1 \cap \Xi, \mathcal{L}h_1 \cap \Xi)\right) &\leq \kappa\left(H_{G_m}(\mathcal{L}h_0 \cap \Xi, \mathcal{L}h_1 \cap \Xi, \mathcal{L}h_1 \cap \Xi)\right) \\ &\leq \alpha(h_0, h_1, h_1)\, \kappa\left(H_{G_m}(\mathcal{L}h_0 \cap \Xi, \mathcal{L}h_1 \cap \Xi, \mathcal{L}h_1 \cap \Xi)\right) \\ &\leq \left[\kappa\left(G_m(h_0, h_1, h_1)\right)\right]^{\lambda}. \end{aligned} \tag{3.23}$$

From (κ_4), we know that

$$\kappa\left(G_m(h_1, \mathcal{L}h_1 \cap \Xi, \mathcal{L}h_1 \cap \Xi)\right) = \inf_{\omega \in \mathcal{L}h_1 \cap \Xi} \kappa(G_m(h_1, \omega, \omega)).$$

Thus from (3.23), we get

$$\inf_{\omega \in \mathcal{L}h_1 \cap \Xi} \kappa(G_m(h_1, \omega, \omega)) \leq \left[\kappa\left(G_m(h_0, h_1, h_1)\right)\right]^{\lambda}. \tag{3.24}$$

Since $\mathcal{L}h_1 \neq \emptyset$, we deduce that there exists $h_2 \in \Xi$ such that $h_2 \in \mathcal{L}h_1$. Now since $\omega = h_2 \in \mathcal{L}h_1 \cap \Xi$, so by the inequality (3.24), we have

$$\kappa(G_m(h_1, h_2, h_2)) \leq \left[\kappa\left(G_m(h_0, h_1, h_1)\right)\right]^{\lambda}. \tag{3.25}$$

Now since $\alpha(h_0, h_1, h_1) \geq 1$ and $\mathcal{L}$ is a multivalued α-admissible mapping, so $\alpha(h_1, h_2, h_2) \geq 1$ for $h_1 \in \mathcal{L}h_0 \cap \Xi$ and $h_2 \in \mathcal{L}h_1 \cap \Xi$. If $h_1 = h_2$, then h_1 is the required fixed point and we have nothing to prove. So, we suppose that $h_1 \neq h_2$. Also, if $h_2 \in \mathcal{L}h_2 \cap \Xi$, then h_2 is a fixed point. Let $h_2 \notin \mathcal{L}h_2 \cap \Xi$. Then, $H_{G_m}(\mathcal{L}h_1 \cap \Xi, \mathcal{L}h_2 \cap \Xi, \mathcal{L}h_2 \cap \Xi) > 0$. Now, by the inequality (3.22), we have

$$\begin{aligned} \kappa\left(G_m(h_2, \mathcal{L}h_2 \cap \Xi, \mathcal{L}h_2 \cap \Xi)\right) &\leq \kappa\left(H_{G_m}(\mathcal{L}h_1 \cap \Xi, \mathcal{L}h_2 \cap \Xi, \mathcal{L}h_2 \cap \Xi)\right) \\ &\leq \alpha(h_1, h_2, h_2)\, \kappa\left(H_{G_m}(\mathcal{L}h_1 \cap \Xi, \mathcal{L}h_2 \cap \Xi, \mathcal{L}h_2 \cap \Xi)\right) \end{aligned}$$

$$\leq \quad [\kappa (G_m (h_1, h_2, h_2))]^{\lambda}. \tag{3.26}$$

From (κ_4), we know that

$$\kappa (G_m (h_2, \mathcal{L}h_2 \cap \Xi, \mathcal{L}h_2 \cap \Xi)) = \inf_{\omega \in \mathcal{L}h_2 \cap \Xi} \kappa(G_m (h_2, \omega, \omega)). \tag{3.27}$$

Thus from (3.26), we get

$$\inf_{\omega \in \mathcal{L}h_2 \cap \Xi} \kappa(G_m (h_1, \omega, \omega)) \leq [\kappa (G_m (h_1, h_2, h_2))]^{\lambda} \tag{3.28}$$

Since $\mathcal{L}h_2 \neq \emptyset$, we deduce that there exists $h_3 \in \Xi$ such that $h_3 \in \mathcal{L}h_2$. Now, since $\omega = h_3 \in \mathcal{L}h_2 \cap \Xi$, by the inequality (3.26), we have

$$\kappa(G_m (h_2, h_3, h_3)) \leq [\kappa (G_m (h_1, h_2, h_2))]^{\lambda}.$$

Continuing in this way, we can find a sequence of points $\{h_n\} \subset \Xi$ such that $h_{n+1} \in \mathcal{L}h_n \cap \Xi$ and

$$\kappa(G_m (h_n, h_{n+1}, h_{n+1})) \leq [\kappa(G_m (h_{n-1}, h_n, h_n))]^{\lambda}, \tag{3.29}$$

for all $n \in \mathbb{N}$.

Therefore

$$\begin{aligned} 1 &< \kappa(G_m (h_n, h_{n+1}, h_{n+1})) \leq [\kappa(G_m (h_{n-1}, h_n, h_n))]^{\lambda} \\ &\leq [\kappa(G_m (h_{n-2}, h_{n-1}, h_{n-1}))]^{\lambda^2} \\ &\leq \cdots \\ &\leq [\kappa(G_m (h_0, h_1, h_1))]^{\lambda^n} \end{aligned} \tag{3.30}$$

for all $n \in \mathbb{N}$. Since $\kappa \in \Omega$, by taking the limit as $n \longrightarrow \infty$ in (3.30), we have

$$\lim_{n \longrightarrow \infty} \kappa(G_m (h_n, h_{n+1}, h_{n+1})) = 1. \tag{3.31}$$

From the condition (κ_2), we have

$$\lim_{n \longrightarrow \infty} G_m (h_n, h_{n+1}, h_{n+1}) = 0.$$

By replicating the methodology employed in establishing the validity of Theorem 3, it can be demonstrated that $\{h_n\}$ conforms to the criteria of being a G_m-Cauchy sequence in Ξ. Since Ξ is a closed subset of complete generalized metric space $(\mathcal{M}, G_m)$, (Ξ, G_m) is also complete. Thus, there exists a point $h^* \in \Xi$ such that $\lim_{n\to\infty} h_n = h^*$. Now, since $h_{n+1} \in \mathcal{L}h_n \cap \Xi$ and the mapping is continuous, taking the limit as $n \to \infty$, we have

$$h^* = \lim_{n\to\infty} h_{n+1} \in \mathcal{L}(\lim_{n\to\infty} h_n) \cap \Xi = \mathcal{L}(h^*) \cap \Xi.$$

Hence, h^* is a fixed point of $\mathcal{L}$.

□

Theorem 5. *Let $(\mathcal{M}, G_m)$ be a complete generalized metric space, Ξ be a closed subset of $\mathcal{M}$, and $\mathcal{L} : \Xi \longrightarrow CB(\mathcal{M})$ is a generalized (α, κ_{G_m})-contraction. Let us consider the fulfillment of the following conditions:*

(i) $\mathcal{L}$ is a multivalued α-admissible mapping,
(ii) there exist $h_0 \in \Xi$ and $h_1 \in \mathcal{L}h_0 \cap \Xi$ such that $\alpha(h_0, h_1, h_1) \geq 1$,
(iii) for any sequence $\{h_n\}$ in Ξ such that $h_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(h_n, h_{n+1}, h_{n+1}) \geq 1$, implying $\alpha(h_n, h, h) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$,
then $\mathcal{L}$ has a fixed point.

Proof. Following the proof of Theorem 4, there exists a G_m-Cauchy sequence $\{h_n\}$ in Ξ with $h_{n+1} \in \mathcal{L}h_n \cap \Xi$ and $h_n \rightarrow h^*$ as $n \rightarrow \infty$ and $\alpha(h_n, h_{n+1}, h_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Then by the assumption (iii), we have $\alpha(h_n, h^*, h^*) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Now by (3.22), we have

$$\begin{aligned} \kappa(G_m(h_{n+1}, \mathcal{L}h^* \cap \Xi, \mathcal{L}h^* \cap \Xi)) &\leq \kappa(H_{G_m}(\mathcal{L}h_n \cap \Xi, \mathcal{L}h^* \cap \Xi, \mathcal{L}h^* \cap \Xi)) \\ &\leq \alpha(h_n, h^*, h^*)\, \kappa(H_{G_m}(\mathcal{L}h_n \cap \Xi, \mathcal{L}h^* \cap \Xi, \mathcal{L}h^* \cap \Xi)) \\ &\leq [\kappa(G_m(h_n, h^*, h^*))]^\lambda < \kappa(G_m(h_n, h^*, h^*)). \end{aligned} \tag{3.32}$$

By (κ_1), we have

$$G_m(h_{n+1}, \mathcal{L}h^* \cap \Xi, \mathcal{L}h^* \cap \Xi) < G_m(h_n, h^*, h^*)$$

for all $n \in \mathbb{N} \cup \{0\}$. Taking the limit as $n \rightarrow \infty$, we get $G_m(h^*, \mathcal{L}h^* \cap \Xi, \mathcal{L}h^* \cap \Xi) \leq 0$. Since $\mathcal{L}h^* \cap \Xi$ is closed, $h^* \in \mathcal{L}h^* \cap \Xi$. Hence, $\mathcal{L}$ has a fixed point.

□

Corollary 3. *Let $(\mathcal{M}, G_m)$ be a complete generalized metric space, Ξ be a closed subset of $\mathcal{M}$ and $\mathcal{L} : \Xi \longrightarrow CB(\mathcal{M})$ is continuous. If there exist $\kappa \in \Omega$ and $\lambda \in (0, 1)$ such that*
(i) $\mathcal{L}h \cap \Xi \neq \emptyset$, for all $h \in \Xi$,
(ii) for all $h, \omega, \Phi \in \Xi$, we have $H_{G_m}(\mathcal{L}h \cap \Xi, \mathcal{L}\omega \cap \Xi, \mathcal{L}\Phi \cap \Xi) > 0$ implies

$$\kappa(H_{G_m}(\mathcal{L}h \cap \Xi, \mathcal{L}\omega \cap \Xi, \mathcal{L}\Phi \cap \Xi)) \leq [\kappa(G_m(h, \omega, \Phi))]^\lambda,$$

then $\mathcal{L}$ has a fixed point.

Proof. Define $\alpha : \Xi \times \Xi \times \Xi \rightarrow [0, +\infty)$ by $\alpha(h, \omega, \Phi) = 1$, for all $h, \omega, \Phi \in \Xi$ in Theorem 4.

□

4. Application

We utilize Corollary 2 to demonstrate that the following integral equation has a solution:

$$h(t) = \int_a^b W(t, s)T(s, h(s))ds. \tag{4.1}$$

Here, $h(t)$ belongs to the set $\mathcal{M}$ of all continuous functions from $[a, b]$ to $\mathbb{R}$. The mappings $W : [a, b] \times [a, b] \rightarrow [0, \infty)$ and $T : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous.

Establish a function $\mathcal{L} : \mathcal{M} \rightarrow \mathcal{M}$ by

$$\mathcal{L}h(t) = \int_a^b W(t, s)T(s, h(s))ds \tag{4.2}$$

for all $t \in [a, b]$.

Theorem 6. *Analyze calculation 4.1 to assume the following:*

1. $\max_{t\in[a,b]} \int_a^b W(t, s)ds < \lambda^2$, *for some* $\lambda \in (0, 1)$,
2. *for all* $h(s), \omega(s) \in \mathcal{M}$; $s \in [a, b]$, *we have*

$$|T(s, h(s)) - T(s, \omega(s))| \leq |h(s) - \omega(s)|. \tag{4.3}$$

Then equation (4.1) has a solution.

Proof. For $h, \omega, \Phi \in \mathcal{M}$, define the generalized metric on $\mathcal{M}$ by

$$G_m(h, \omega, \Phi) = d(\omega, \Phi) + d(h, \omega) + d(h, \Phi) \tag{4.4}$$

where

$$d(h, \omega) = \sup_{t\in[a,b]} |h(t) - \omega(t)|.$$

Now, let $h(t), \omega(t) \in \mathcal{M}$, then we have

$$\begin{aligned}
|\mathcal{L}h(t) - \mathcal{L}\omega(t)| &= \Big| \int_a^b W(t, s)[T(s, h(s)) - T(s, \omega(s))]ds \Big| \\
&\leq \int_a^b W(t, s)|T(s, h(s)) - T(s, \omega(s)|ds \\
&\leq \int_a^b W(t, s)|h(s) - \omega(s)|ds \\
&\leq \int_a^b W(t, s) \sup_{s\in[a,b]} |h(s) - \omega(s)|ds \\
&= \sup_{t\in[a,b]} |h(t) - \omega(t)| \int_a^b W(t, s)ds \\
&\leq \lambda^2 \sup_{t\in[a,b]} |h(t) - \omega(t)|.
\end{aligned}$$

Hence,

$$\sup_{t\in[a,b]} |\mathcal{L}h(t) - \mathcal{L}\omega(t)| \leq \lambda^2 \sup_{t\in[a,b]} |h(t) - \omega(t)|. \tag{4.5}$$

Similarly, we have

$$\sup_{t\in[a,b]} |\mathcal{L}\omega(t) - \mathcal{L}w(t)| \leq \lambda^2 \sup_{t\in[a,b]} |\omega(t) - w(t)| \tag{4.6}$$

and

$$\sup_{t\in[a,b]} |\mathcal{L}h(t) - \mathcal{L}\Phi(t)| \leq \lambda^2 \sup_{t\in[a,b]} |h(t) - \Phi(t)|. \tag{4.7}$$

Therefore, from 4.5, 4.6, and 4.7, we have

$$\sup_{t\in[a,b]} |\mathcal{L}h(t) - \mathcal{L}\omega(t)| + \sup_{t\in[a,b]} |\mathcal{L}\omega(t) - \mathcal{L}\Phi(t)| + \sup_{t\in[a,b]} |\mathcal{L}h(t) - \mathcal{L}\Phi(t)|$$

$$\leq \lambda^2 \left[\sup_{t\in[a,b]} |h(t)-\omega(t)| + \sup_{t\in[a,b]} |\omega(t)-\Phi(t)| + \sup_{t\in[a,b]} |h(t)-\Phi(t)| \right]$$

which implies

$$G_m(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi) \leq \lambda^2 G_m(h, \omega, \Phi). \tag{4.8}$$

Taking exponential, we have

$$e^{(G_m(\mathcal{L}h,\mathcal{L}\omega,\mathcal{L}\Phi))} \leq e^{\lambda^2(G_m(h,\omega,\Phi))}.$$

Now, we consider the mapping $\kappa : (0, \infty) \to (1, \infty)$ defined by $\kappa(\alpha) = e^{\sqrt{\alpha}}$. Thus we have

$$\kappa(G_m(\mathcal{L}h, \mathcal{L}\omega, \mathcal{L}\Phi)) \leq [\kappa(G_m(h, \omega, \Phi))]^{\lambda}.$$

Hence, all requirements of Corollary 2 are obtained. As an outcome of 2, $\mathcal{M}$ will contain a fixed point of the function $\mathcal{L}$, which will be the solution of 4.1. □

5. Conclusions

In this research article, we introduced the notion of κ_{G_m}-contraction and generalized (α, κ_{G_m})-contraction in complete G_m-metric spaces and worked to prove some fixed point, coincidence point, and common fixed point theorems. Additionally, we demonstrated the usefulness of our obtained result by applying it to the investigation of the integral equation. Also, we presented a nontrivial example demonstrating the practicality of our primary hypothesis.

Author contributions

J.A., A.S., I.A. and N.M. wrote the main manuscript text. All authors of this manuscript contributed equally.

Conflict of interest

The authors declare no conflicts of interest.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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