AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 15928–15936.
DOI: 10.3934/math.2024770
Received: 01 March 2024
Revised: 01 April 2024
Accepted: 15 April 2024
Published: 06 May 2024

Research article

Graphs with a given conditional diameter that maximize the Wiener index

Junfeng An and Yingzhi Tian*

College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, China

* **Correspondence:** Email: tianyzhxj@163.com.

Abstract: The Wiener index $W(G)$ of a graph G is one of the most well-known topological indices, which is defined as the sum of distances between all pairs of vertices of G. The diameter $D(G)$ of G is the maximum distance between all pairs of vertices of G, and the conditional diameter $D(G; s)$ is the maximum distance between all pairs of vertex subsets with cardinality s of G. When $s = 1$, the conditional diameter $D(G; s)$ is just the diameter $D(G)$. The authors in [18] characterized the graphs with the maximum Wiener index among all graphs with diameter $D(G) = n - c$, where $1 \le c \le 4$. In this paper, we will characterize the graphs with the maximum Wiener index among all graphs with conditional diameter $D(G; s) = n - 2s - c$ $(-1 \le c \le 1)$, which extends partial results above.

Keywords: Wiener index; diameter; conditional diameter
Mathematics Subject Classification: 05C09

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The order and the size of G are $n = |V(G)|$ and $m = |E(G)|$, respectively. The distance between two vertices u and v, denoted by $d(u, v) = d_G(u, v)$, is the length of the shortest path connecting u and v in G. There are plenty of distance-based topological indices, which are widely used in mathematical chemistry in order to describe and predict the properties of chemical compounds. One of the most well-known topological indices is the Wiener index, which was introduced in 1947 by Wiener [20]. The Wiener index $W(G)$ of a graph G is defined as the sum of distances between all (unordered) pairs of vertices of G, that is,

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v).$$

Mathematical properties and applications of the Wiener index have been extensively studied, see [1, 3, 5–7, 9–12, 16, 17, 21, 22] for references.

The diameter $D(G)$ of G is the maximum distance between all pairs of vertices in $V(G)$, that is, $D(G) = max_{u,v\in V(G)} d(u, v)$. For two nonempty vertex subsets V_1 and V_2, the distance between V_1 and

V_2, denoted by $d(V_1, V_2) = d_G(V_1, V_2)$, is the minimum of the distances $d(x, y)$ among all $x \in V_1$ and $y \in V_2$. Given a graphical property $\mathcal{P}$ satisfied by at least one pair (V_1, V_2) of nonempty subsets of $V(G)$, the conditional diameter $D_{\mathcal{P}}(G)$ of G is

$$D_{\mathcal{P}}(G) = max\{d(V_1, V_2) : \emptyset \neq V_1, V_2 \subseteq V(G), (V_1, V_2)\ satisfies\ \mathcal{P}\}.$$

Note that $D_{\mathcal{P}}(G) = 0$ holds if and only if V_1 and V_2 overlap for every $(V_1, V_2) \subseteq V(G) \times V(G)$ that satisfies $\mathcal{P}$. Conditional diameter measures the maximum distance between subgraphs satisfying a given property. So, their consideration could be of some interest if in some applications we need to control the communication delays between the network clusters modeled by such subgraphs.

The first choice of such a graphical property $\mathcal{P}_1$ is defined as follows: (V_1, V_2) satisfies $\mathcal{P}_1$ if and only if $|V_1| = |V_2| = s$, where s is a positive integer. In this case, the conditional diameter is denoted by $D(G; s)$, which is defined as

$$D(G; s) = max\{d(V_1, V_2) : V_1, V_2 \subseteq V(G), |V_1| = |V_2| = s\}.$$

Clearly, $D(G; 1)$ is the standard diameter $D(G)$ of G. Thus $D(G; s)$ can be seen as a generalization of the diameter $D(G)$. When $|V(G)| < 2s$, then $D(G; s) = 0$. Moreover, when $|V(G)| \geq 2s$, it is easy to see that the inequality $D(G; s) \leq n - 2s + 1$ holds.

Although the Wiener index has been extensively studied, there are still some unsolved interesting questions. For example, Plesník [15] posed the open problem "What is the maximum average distance among graphs of order n and diameter d?"; DeLaViña and Waller [8] conjectured that $W(G) \leq W(C_{2d+1})$ for any graph G with diameter $d \geq 3$ and order $2d + 1$, where C_{2d+1} is the cycle of length $2d + 1$. Some results related to the Wiener indices of graphs with given diameter can be seen in [13, 14, 19]. Particularly, Cambie [4] gave an asymptotic solution to the open problem of Plesník, and Sun et al. [18] characterized the graphs with the maximum Wiener index among all graphs with diameter $D(G) = n - c$, where $1 \leq c \leq 4$.

Motivated by the results above, we will investigate the maximum Wiener index among all graphs with given conditional diameter in this paper. Specifically, we will characterize the graphs with the maximum Wiener index among all graphs with conditional diameter $D(G; s) = n - 2s - c$, where $-1 \leq c \leq 1$. Some lemmas will be given in the next section. Main results will be presented in the last section.

2. Preliminaries

The graphs considered in this paper are simple and undirected. For undefined notation and terminologies, we follow [2]. For a graph G, we denote by $G - u$ and $G - uv$ the graphs obtained from G by deleting the vertex $u \in V(G)$ and the edge $uv \in E(G)$, respectively. Similarly, $G + xy$ is the graph obtained from G by adding an edge $xy \notin E(G)$. The induced subgraph $G[U]$ for a vertex subset $U \subseteq V(G)$ is $G - V(G) \setminus U$. The neighborhood of u in G is $N_G(u) = \{v|uv \in E(G)\}$. The degree $d_G(u)$ of u in G is $|N_G(v)|$. If $d_G(u) = 1$, then u is called a pendent vertex of G. Denote by P_n and C_n the path and the cycle on n vertices, respectively.

The sum of distances between u and all other vertices of G is $D_G(u) = \sum_{v \in V(G)} d(u, v)$.

Lemma 2.1. *([9]) Let G be a graph of order n, v a pendent vertex of G, and u the vertex adjacent to v. Then $W(G) = W(G - v) + D_{G-v}(u) + n - 1$.*

Lemma 2.2. *([13]) Let $P_n = v_1v_2 \ldots v_n$ be a path on n vertices, then $D_{P_n}(v_j) > D_{P_n}(v_k)$ for $1 \leq j < k \leq \lfloor n/2 \rfloor$.*

Lemma 2.3. *([13]) Let G be a non-trivial connected graph on n vertices and let $v \in V(G)$. Suppose that two paths $P = vv_1v_2 \cdots v_k$, $Q = vu_1u_2 \cdots u_l$ of lengths k, l are attached to G by their end vertices at v, respectively, to form $G_{k,l}$, as shown in Figure 1. If $l \geq k \geq 1$, then $W(G_{k,l}) < W(G_{k-1,l+1})$.*

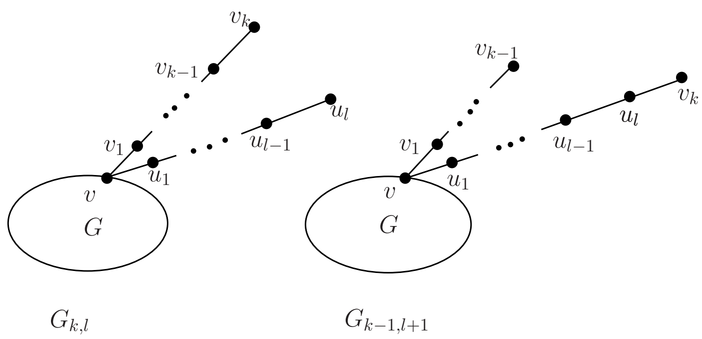

Figure 1. $G_{k,l}$ and $G_{k-1,l+1}$.

3. Main results

It was proved in [11] that $W(P_n)$ is maximum among all trees on n vertices. Since removing of an edge from a connected graph results in an increased Wiener index, it is observed that the Wiener index of a connected graph is less than or equal to the Wiener index of its spanning tree. Thus, $W(P_n)$ is maximum among all connected graphs on n vertices. By $D(P_n; s) = n - 2s + 1$ $(n \geq 2s)$, we have the following theorem.

Theorem 3.1. *Let G be a connected graph on n vertices and $D(G; s) = n - 2s + 1$, where s is a positive integer and $n \geq 2s$. Then $W(G) \leq W(P_n)$, and equality holds if and only if $G \cong P_n$.*

Let T_n^i be the tree on n vertices obtained from $P_{n-1} = x_1x_2 \cdots x_{n-1}$ by attaching a pendent vertex to x_i. See Figure 2 for an illustration.

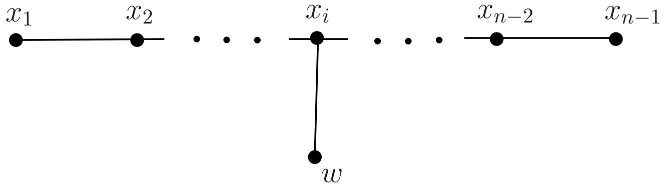

Figure 2. Tree T_n^i.

Theorem 3.2. *Let G be a connected graph on n vertices and $D(G; s) = n - 2s$, where s is a positive integer and $n \geq 2s + 3$. Then $W(G) \leq W(T_n^{s+1})$, and equality holds if and only if $G \cong T_n^{s+1}$.*

Proof. Let $d(L, R) = n - 2s$, where $L = \{x_1, \ldots x_s\}$ and $R = \{x_{n-s}, \ldots, x_{n-1}\}$. Assume $P = x_sx_{s+1} \cdots x_{n-s}$ is a path of length $n - 2s$ connecting L and R in G. Denote $M = \{x_{s+1}, \ldots, x_{n-s-1}\}$ and $W = V(G) \setminus (L \cup M \cup R) = \{w\}$.

Claim. We can choose L and R such that w is adjacent to vertices in M.

By $n \geq 2s + 3$, w can not be adjacent to both vertices in L and R. If w is only adjacent to vertices in L, then x_{s+1} must be adjacent to another vertex $w' \in L$ other than x_s. Otherwise, the distance between $L - x_s + w$ and R would be $n - 2s + 1$, a contradiction. Thus, we replace L by L' and W by W', where $L' = L - w' + w$ and $W' = \{w'\}$. The case w is only adjacent to vertices in R and can be analyzed similarly. So, the claim holds.

Case 1. w is only adjacent to the vertices in M.

If w is adjacent to more than one vertex in M, then delete all but one of the edges incident with w. Note that this operation does not change the conditional diameter and increases the Wiener index. So, we assume that w is a pendent vertex. Without loss of generality, assume w is adjacent to x_i, where $s + 1 \leq i \leq n - s - 1$.

Consider the induced subgraph $G[L \cup \{x_{s+1}\}]$. First, we transform it to a tree by removing edges. Removing edges in this way does not change the conditional diameter and increases the Wiener index (by removing an edge from a graph, the distance of the two ends of this edge must increase, and the distance of other pairs of vertices does not decrease). Then, we transform it to a path as follows: We take one of the longest paths from $\{x_{s+1}\}$ and gradually enlarge it to an even longer path by appending the rest of the vertices in $L \cup \{x_{s+1}\}$ to the current end-vertex on the other side of this path, one after another. By Lemma 2.3, each such transformation increases the Wiener index and retains the conditional diameter. Similarly, we can transform $G[R \cup \{x_{n-s-1}\}]$ to a path with one endvertex $\{x_{n-s-1}\}$.

Now the graph G is changed to the graph isomorphic to T_n^i, where $s + 1 \leq i \leq n - s - 1$. Let $T_n^{s+1} = T_n^i - x_i w + x_{s+1} w$. Since $T_n^{s+1} - w \cong T_n^i - w \cong P_{n-1}$, by Lemmas 2.1 and 2.2, we have $W(T_n^i) \leq W(T_n^{s+1})$, and equality holds if and only if $i = s + 1$. Thus, $W(G) \leq W(T_n^{s+1})$, and equality holds if and only if $G \cong W(T_n^{s+1})$.

Case 2. w is adjacent to vertices in both L (or R) and M.

We only need to consider the case in which w is adjacent to vertices in both L and M. Since $P = x_s x_{s+1} \cdots x_{n-s}$ is a shortest path connecting L and R, we obtain that $N_G(w) \cap \{x_{s+1}, \ldots, x_{n-s}\} \subseteq \{x_{s+1}, x_{s+2}\}$. Let $x' = x_{s+2}$ if $x_{s+2} \in N_G(w)$ and let $x' = x_{s+1}$ otherwise.

If $x' = x_{s+2}$, then consider the induced subgraph $G[L \cup \{x_{s+1}, x_{s+2}, w\}]$. First, we change it to a tree by removing some edges in $E(G[L \cup \{x_{s+1}, x_{s+2}, w\}]) \setminus \{x_{s+1}x_{s+2}, x_{s+2}w\}$. Then, we transform it to a path such that x_{s+2} is adjacent to one endvertex of this path as follows: We take one of the longest paths from $\{x_{s+2}\}$ and gradually enlarge it to an even longer path by appending the rest of the vertices in L to the current endvertex on the other side of this path, one after another. Note that x_{s+2} is still adjacent to vertices x_{s+1} and w, and one of x_{s+1} and w must be an endvertex of this path. Now we change G to a graph isomorphic to T_n^{s+2}. By Case 1, we get $W(G) \leq W(T_n^{s+1})$.

If $x' = x_{s+1}$, then, by a similar argument as above, we can change G to a graph isomorphic to T_n^{s+1}. Thus, $W(G) \leq W(T_n^{s+1})$.

From the arguments above, we obtain that $W(G) \leq W(T_n^{s+1})$, and the equality holds if and only if $G \cong T_n^{s+1}$. □

Let $T_n^{i,j}$ be a tree on n vertices obtained from $P_{n-2} = x_1 x_2 \cdots x_{n-2}$ by attaching two pendent vertices to x_i and x_j, respectively. See Figure 3 for an illustration.

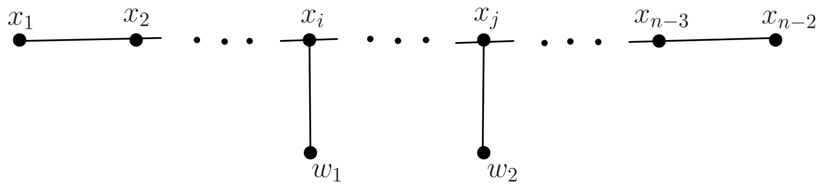

Figure 3. Tree $T_n^{i,j}$.

Let $T_n^{i(2)}$ be a tree on n vertices obtained from $P_{n-2} = x_1x_2\cdots x_{n-2}$ by attaching the endvertex of a path of order 2 to x_i. See Figure 4 for an illustration.

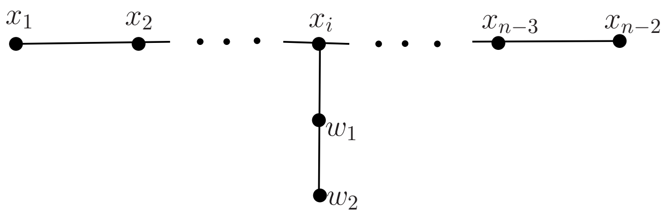

Figure 4. Tree $T_n^{i(2)}$.

Theorem 3.3. *Let G be a connected graph on n vertices and $D(G;s) = n-2s-1$, where s is a positive integer and $n \geq 2s+5$. Then $W(G) \leq W(T_n^{s+1,n-s-2})$, and equality holds if and only if $G \cong T_n^{s+1,n-s-2}$.*

Proof. Let $d(L,R) = n-2s-1$, where $L = \{x_1,\ldots x_s\}$ and $R = \{x_{n-s-1},\ldots,x_{n-2}\}$. Assume $P = x_sx_{s+1}\cdots x_{n-s-1}$ is a path of length $n-2s-1$ connecting L and R. Denote $M = \{x_{s+1},\ldots,x_{n-s-2}\}$ and $W = V(G)\setminus(L\cup M\cup R) = \{w_1,w_2\}$.

Case 1. Neither w_1 nor w_2 is adjacent to vertices in $L\cup R$.

Subcase 1.1. $w_1w_2 \notin E(G)$.

Note that $N_G(w_i) \subseteq \{x_{s+1},\cdots,x_{n-s-2}\}$ for $i = 1,2$. If w_i is adjacent to more than one vertex in M, then delete all but one of the edges incident with w_i, where $i \in \{1,2\}$. Note that this operation does not change the conditional diameter and increases the Wiener index. So, we assume that w_i is pendent vertex for $i = 1,2$. Without loss of generality, assume that w_1 is attached to x_a and w_2 is attached to x_b, where $s+1 \leq a \leq b \leq n-s-2$.

By a similar argument as in the proof of Case 1 in Theorem 3.2, we transform $G[L\cup\{x_{s+1}\}]$ to a path with one endvertex x_{s+1}, and transform $G[R\cup\{x_{n-s-2}\}]$ to a path with one endvertex x_{n-s-2}. That is, we change G to a graph isomorphic to $T_n^{a,b}$, where $s+1 \leq a \leq b \leq n-s-2$.

Let $T_n^{a,n-s-2} = T_n^{a,b} - x_bw_2 + x_{n-s-2}w_2$, $k_{w_1} = d(x_{s+1},x_a)$, and $k_{w_2} = d(x_{n-s-2},x_b)$. Since $T_n^{a,n-s-2} - w_2 \cong T_n^{a,b} - w_2$, we have

$$
\begin{aligned}
&W(T_n^{a,n-s-2}) - W(T_n^{a,b})\\
&= 1+2+\cdots+(n-s-2)+2+\cdots+(s+1)+(n-s-a)\\
&-1-2-\cdots-(n-s-2-k_{w_2})-2-\cdots-(s+1+k_{w_2})-(n-s-a-k_{w_2})\\
&= \sum_{1\leq i\leq k_{w_2}}(n-s-2-k_{w_2}+i) - \sum_{1\leq i\leq k_{w_2}}(s+1+i)+k_{w_2}.
\end{aligned}
$$

Since $n-2s-3-k_{w_2} \geq 0$, we have $W(T_n^{i,n-m-2}) - W(T_n^{i,j}) \geq 0$, and equality holds if and only if $k_{w_2} = 0$.

Let $T_n^{s+1,n-s-2} = T_n^{a,n-s-2} - x_a w_1 + x_{s+1} w_1$. Since $T_n^{s+1,n-s-2} - w_1 \cong T_n^{a,n-s-2} - w_1$, we have

$$\begin{aligned}
&W(T_n^{s+1,n-s-2}) - W(T_n^{a,n-s-2})\\
&= 1+2+\cdots+(s+1)+2+\cdots+(n-s-2)+(n-2s-1)\\
&\quad -1-2-\cdots-(s+1+k_{w_1})-2-\cdots-(n-s-2-k_{w_1})-(n-2s-1-k_{w_1})\\
&= -\sum_{1\leq i\leq k_{w_1}}(s+1+i)+\sum_{1\leq i\leq k_{w_1}}(n-s-2-k_{w_1}+i)+k_{w_1}.
\end{aligned}$$

By $n-2s-3-k_{w_2} \geq 0$, we have $W(T_n^{s+1,n-s-2}) - W(T_n^{a,n-s-2}) \geq 0$, and equality holds if and only if $k_{w_1} = 0$.

In this subcase, we conclude that $W(G) \leq W(T_n^{s+1,n-s-2})$, and equality holds if and only if $G \cong T_n^{s+1,n-s-2}$.

Subcase 1.2. $w_1 w_2 \in E(G)$.

If both w_1 and w_2 have neighbors in M, then by deleting edge $w_1 w_2$, we reduce this situation to Subcase 1.1. So, we assume that only one of the vertices in W, say w_1, has neighbors in M and the other vertex w_2 is a pendent vertex adjacent to w_1. If w_1 has more than one neighbor in W, then delete all but one of the edges incident with w_1. Here, the remaining edge satisfies the property that the end other than w_1 is farthest to the vertex set $\{x_{s+1}, x_{n-s-2}\}$. Assume, without loss of generality, that w_1 is attached to x_i, where $s+2 \leq i \leq n-s-3$.

By a similar argument as in the proof of Case 1 in Theorem 3.2, we transform $G[L \cup \{x_{s+1}\}]$ to a path with one endvertex x_{s+1}, and transform $G[R \cup \{x_{n-s-2}\}]$ to a path with one endvertex x_{n-s-2}. That is, we change G to a graph isomorphic to $T_n^{i(2)}$, where $s+2 \leq i \leq n-s-3$.

Let $T_n^{(s+2)(2)} = T_n^{i(2)} - x_i w_1 + x_{s+2} w_1$ and $k'_{w_1} = d(x_{s+2}, x_i)$. Since $T_n^{(s+2)(2)} - w_1 - w_2 \cong T_n^{i(2)} - w_1 - w_2$, we have

$$\begin{aligned}
W(T_n^{(s+1)(2)}) - W(T_n^{i(2)}) &= 1+2+\cdots+(n-s-1)+3+\cdots+(s+2)\\
&\quad +1+2+\cdots+(n-s-2)+2+\cdots+(s+1)\\
&\quad -1-2-\cdots-(n-s-1-k'_{w_1})-3-\cdots-(s+2+k'_{w_1})\\
&\quad -1-2-\cdots-(n-s-2-k'_{w_1})-2-\cdots-(s+1+k'_{w_1})\\
&= \sum_{1\leq j\leq k'_{w_1}}(n-s-1-k'_{w_1}+j)-\sum_{1\leq j\leq k'_{w_1}}(s+2+j)\\
&\quad +\sum_{1\leq j\leq k'_{w_1}}(n-s-2-k'_{w_1}+j)-\sum_{1\leq j\leq k'_{w_1}}(s+1+j).
\end{aligned}$$

Since $n-2s-3-k'_{w_1} \geq 0$, we have $W(T_n^{(s+2)(2)}) - W(T_n^{i(2)}) \geq 0$, and equality holds if and only if $k'_{w_1} = 0$.

In this subcase, we conclude that $W(G) \leq W(T_n^{(s+2)(2)})$, and equality holds if and only if $G \cong T_n^{(s+2)(2)}$.

Case 2. Either w_1 or w_2 are adjacent to vertices in $L \cup R$.

If w_i is only adjacent to vertices in $L \cup R$, then we can choose L and R such that w_i is adjacent to some vertices in M for $i = 1, 2$. Owing to Case 1, we only need to consider three subcases in the following.

Subcase 2.1. Only one of w_1 or w_2, say, w_1 is adjacent to vertices in $L \cup R$.

We only consider the case that w_1 is adjacent to vertices in both L and M, and w_2 is not adjacent to any vertices in $L \cup R$.

Since $P = x_s x_{s+1} \cdots x_{n-s-1}$ is the shortest path connecting L and R, we obtain that $N_G(w_1) \cap \{x_{s+1}, \ldots, x_{n-s}\} \subseteq \{x_{s+1}, x_{s+2}\}$. Let $x' = x_{s+2}$ if $x_{s+2} \in N_G(w_1)$, and let $x' = x_{s+1}$ otherwise. If $x' = x_{s+2}$, then by a similar argument as in the proof of Case 2 in Theorem 3.2, we can change $G[L \cup \{x_{s+1}, x_{s+2}, w\}]$ to a path such that x_{s+2} is still adjacent to vertices x_{s+1} and w, and one of x_{s+1} and w_1 is an endvertex of this path. If $x' = x_{s+1}$, then by a similar argument we can change $G[L \cup \{x_{s+1}, w\}]$ to a path such that x_{s+1} is still adjacent to vertices x_s and w_1, and one of x_s and w_1 is an endvertex of this path.

Suppose w_2 is adjacent to some vertices in M. Then delete all edges incident with w_2 except for one edge joining w_2 to a vertex in M. Then, G is changed to a graph isomorphic to $T_n^{i,j}$. Suppose w_2 is only adjacent to w_1. Then, G is changed to a graph isomorphic to $T_n^{i(2)}$.

Subcase 2.2. Both w_1 and w_2 are adjacent to vertices in L (or R).

We only consider the case that w_i is adjacent to vertices in both L and M for $i = 1, 2$.

Since $P = x_s x_{s+1} \cdots x_{n-s-1}$ is the shortest path connecting L and R, we obtain that $N_G(w_i) \cap \{x_{s+1}, \ldots, x_{n-s}\} \subseteq \{x_{s+1}, x_{s+2}\}$ for $i = 1, 2$. Let $x' = x_{s+2}$ if $x_{s+2} \in N_G(w_1)$, and let $x' = x_{s+1}$ otherwise. Let $x'' = x_{s+2}$ if $x_{s+2} \in N_G(w_2)$, and let $x'' = x_{s+1}$ otherwise. Here we only give the proof when $x' = x_{s+2}$ and $x'' = x_{s+2}$. Other cases can be proved similarly. We consider the induced subgraph $G[L \cup \{x_{s+1}, x_{s+2}, w_1, w_2\}]$. First, we change it to a tree by removing some edges in $E(G[L \cup \{x_{m+1}, x_{m+2}, w_1, w_2\}]) \setminus \{x_{s+1}x_{s+2}, x_{s+2}w_1, x_{s+2}w_2\}$. Then, we transform it to a tree such that x_{s+2} is adjacent to two pendent vertices as follows: we take one of the longest paths from x_{s+2} and gradually enlarge it to an even longer path by appending the rest of the vertices in L to the current endvertex on the other side of this path, one after another. Note that x_{s+2} is still adjacent to vertices x_{s+1}, w_1, and w_2, and two of x_{s+1}, w_1, and w_2 are pendent vertices adjacent to x_{s+2}. Then, G is changed to a graph isomorphic to $T_n^{i,j}$.

Subcase 2.3. One of w_1 or w_2, say w_1, is adjacent to vertices in L and w_2 is adjacent to vertices in R.

Since $P = x_s x_{s+1} \cdots x_{n-s-1}$ is the shortest path connecting L and R, we obtain that $N_G(w_1) \cap \{x_{s+1}, \ldots, x_{n-s}\} \subseteq \{x_{s+1}, x_{s+2}\}$. Let $x' = x_{s+2}$ if $x_{s+2} \in N_G(w_1)$, and let $x' = x_{s+1}$ otherwise. If $x' = x_{s+2}$, then by a similar argument as in the proof of Case 2 in Theorem 3.2, we can change $G[L \cup \{x_{s+1}, x_{s+2}, w_1\}]$ to a path such that x_{s+2} is still adjacent to vertices x_{s+1} and w_1, and one of x_{s+1} and w_1 is an endvertex of this path. If $x' = x_{s+1}$, then by a similar argument we can change $G[L \cup \{x_{s+1}, w_1\}]$ to a path such that x_{s+1} is still adjacent to vertices x_s and w_1, and one of x_s and w_1 is an endvertex of this path. Similarly, if $w_2 x_{n-s-3} \in E(G)$, we can change $G[R \cup \{x_{n-s-3}, x_{n-s-2}, w_2\}]$ to a path such that x_{n-s-3} is still adjacent to vertices x_{n-s-2} and w_2, and one of x_{n-s-2} and w_2 is an endvertex of this path. If $w_2 x_{n-s-3} \notin E(G)$, we can change $G[R \cup \{x_{n-s-2}, w_2\}]$ to a path such that x_{n-s-2} is still adjacent to vertices x_{n-s-1} and w_2, and one of x_{n-s-1} and w_2 is an endvertex of this path. Thus, G is changed to a graph isomorphic to $T_n^{i,j}$.

All cases lead to $W(G) \leq W(T_n^{s+1,n-s-2})$ or $W(G) \leq W(T_n^{(s+2)(2)})$. So, we only need to compare $W(T_n^{s+1,n-s-2})$ and $W(T_n^{(s+2)(2)})$. Since $W(T_n^{s+1,n-s-2}) - W(T_n^{(s+2)(2)}) = \frac{1}{2}n^2 - (s + \frac{3}{2})n + s^2 + 7s > 0$, we obtain that $W(G) \leq W(T_n^{s+1,n-s-2})$, and equality holds if and only if $G \cong T_n^{s+1,n-s-2}$. The proof is completed. □

4. Conclusions

In this paper, we characterize the graphs with the maximum Wiener index among all graphs with conditional diameter $D(G; s) = n - 2s - c$ ($-1 \leq c \leq 1$), which extends partial results in [18].

Author contributions

Supervision, Y.T.; writing—original draft preparation, J.A.; writing—review and editing, Y.T. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research is supported by National Natural Science Foundation of China (12261086).

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. M. L. Bai, Y. Z. Tian, Wiener index of the direct product of a path and a generalized petersen graph, *J. Xinjiang Univ.*, **41** (2024), 218–227. https://doi.org/10.13568/j.cnki.651094.651316.2023.04.13.0002
2. J. A. Bondy, U. S. R. Murty, *Graph theory*, Berlin: Springer, 2008. https://doi.org/10.1007/978-1-84628-970-5
3. Q. Cai, T. Li, Y. Shi, H. Wang, Sum of weighted distances in trees, *Discrete Appl. Math.*, **257** (2019), 67–84. https://doi.org/10.1016/j.dam.2018.08.033
4. S. Cambie, An asymptotic resolution of a problem of Plesník, *J. Comb. Theory. Ser. B*, **145** (2020), 341–358. https://doi.org/10.1016/j.jctb.2020.06.003
5. S. Cambie, Extremal total distance of graphs of given radius I, *J. Graph Theory*, **97** (2021), 104–122. https://doi.org/10.1002/jgt.22644
6. S. Cambie, Corrigendum on Wiener index, Zagreb Indices and Harary index of Eulerian graphs, *Discrete Appl. Math.*, **347** (2024), 139–142. https://doi.org/10.1016/j.dam.2024.01.011
7. K. C. Das, M. J. Nadjafi-Arani, On maximum Wiener index of trees and graphs with given radius, *J. Comb. Optim.*, **34** (2017), 574–587. https://doi.org/10.1007/s10878-016-0092-y
8. E. DeLaViña, B. Waller, Spanning trees with many leaves and average distances, *Electron. J. Comb.*, **15** (2008). https://doi.org/10.37236/757

9. A. A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: Theory and applications, *Acta Appl. Math.*, **66** (2001), 211–249. https://doi.org/10.1023/a:1010767517079

10. Y. L. Jin, X. D. Zhang, On the two conjectures of the Wiener index, *Arxiv*, 2013. https://doi.org/10.48550/arXiv.1304.0873

11. M. Knor, R. Škrekovski, A. Tepeh, Mathematical aspects of Wiener index, *Arxiv*, 2015, 1–28. https://doi.org/10.48550/arXiv.1510.00800

12. M. Knor, R. Škrekovski, A. Tepeh, Selected topics on Wiener index, *Arxiv*, 2023, 1–34. https://doi.org/10.48550/arXiv.2303.11405

13. H. Liu, X. Pan, On the Wiener index of trees with fixed diameter, *MATCH Commun. Math. Comput. Chem.*, **60** (2008), 85–94.

14. S. Mukwembi, T. Vertík, Wiener index of trees of given order and diameter at most 6, *Bull. Aust. Math. Soc.*, **89** (2014), 379–396. https://doi.org/10.1017/S0004972713000816

15. J. Plesník, On the sum of all distances in a graph or digraph, *J. Graph Theory*, **8** (1984), 1–21. https://doi.org/10.1002/jgt.3190080102

16. A. V. Sills, H. Wang, On the maximal Wiener index and related questions, *Discrete Appl. Math.*, **160** (2012), 1615–1623. https://doi.org/10.1016/j.dam.2012.03.002

17. D. Stevanović, Maximizing Wiener index of graphs with fixed maximum degree, *MATCH Commun. Math. Comput. Chem.*, **60** (2008), 71–83.

18. Q. Sun, B. Ikica, R. Škrekovski, V. Vukašinović, Graphs with a given diameter that maximise the Wiener index, *Appl. Math. Comput.*, **356** (2019), 438–448. https://doi.org/10.1016/j.amc.2019.03.025

19. S. Wang, X. Guo, Trees with extremal Wiener indices, *MATCH Commun. Math. Comput. Chem.*, **60** (2008), 609–622.

20. H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.*, **69** (1947), 17–20. https://doi.org/10.1021/ja01193a005

21. K. Xu, M. Liu, K. C. Das, I. Gutman, B. Furtula, A survey on graphs extremal with respect to distance-based topological indices, *MATCH Commun. Math. Comput. Chem.*, 2014. https://scidar.kg.ac.rs/handle/123456789/17378

22. X. D. Zhang, Y. Liu, M. X. Han, Maximum Wiener index of trees with given degree sequence, *Arxiv*, 2009, 1–19. https://doi.org/10.48550/arXiv.0907.3772