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Research article

The properties of generalized John domains in metric spaces

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Abstract: In this paper, we studied the properties of generalized John domains in metric space. We prove that a domain D is a φ-John domain if, and only if, $D\backslash P$ is a φ'-John domain, where P is a subset of D containing finitely many points of D. Meanwhile, we also showed that the union of φ-John domains is a φ''-John domain in metric space.

Keywords: Generalized John domain; uniform domain; decomposition property; union; quasiconvex domain
Mathematics Subject Classification: 30C20, 30C65

1. Introduction and main results

John [17] and Martio and Sarvas [24] were the first who introduced and studied John domains and uniform domains, respectively. Now, there are plenty of alternative characterizations for uniform and John domains; see [5, 6, 20, 23, 28–32]. Additionally, its importance along with some special domains throughout the function theory is well documented; see [5, 7, 13, 15, 20, 25, 26, 35–37]. Moreover, John domains and uniform domains in $\mathbb{R}^n$ enjoy with numerous geometric and function theoretic features in many areas of modern mathematical analysis, see [1–3, 6, 18, 19, 21, 22, 31, 34]. As in [10], Guo and Koskela have introduced the class of φ-John domains, which form a natural generalization of John domains. The motivation for this paper stems from the discussions in [16, 33], where the effect of the removal of a finite set of points and union of generalized John domain was examined. The main result of this paper shows that D is a φ-John domain if, and only if, $D\backslash P$ is a φ'-John domain, where P is a subset of D containing finitely many points of D, φ and φ' depend on each other, and, finally, we prove that the union of φ-John domains is φ''-John domain.

Throughout the paper, unless otherwise stated, we always assume that D is a proper subdomain of the metric space X and $B(x, r) = \{y \in X : |x - y| < r\}$ denotes the metric ball at x of radius r. For a set D in X, we use $\overline{D}$ to denote the metric completion of D, and we let $\partial D = \overline{D}\backslash D$ be its metric boundary.

We write

$$A(x; r, R) = \{y : r \le |x - y| \le R\}$$

for the closed annular ring center at x with inner and outer radii r and R, respectively.

From now on, for notational convenience, we use notation $|x - y|$ to indicate the distance between x and y in any metric space X.

Definition 1.1. A domain (open and connected) D in X is said to be a *C-uniform domain* if there exists a constant $C \ge 1$ with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc γ in D satisfying

(1) (Double cone condition) $\min\{l(\gamma[z_1, z]), l(\gamma[z_2, z])\} \le C\delta_D(z)$ for all $z \in \gamma$, and

(2) (Quasiconvex condition) $l(\gamma) \le C|z_1 - z_2|$,

where $l(\gamma)$ denotes the arc-length of γ, $\gamma[z_i, z]$ is the subcurve of γ between z_i and z, and $\delta_D(z)$ denotes the distance $\mathrm{dist}(z, \partial D)$. At this time, γ is said to be a *double C-uniform curve*.

If the condition (1) is satisfied, not necessarily (2), then D is said to be a *C-John domain* and the arc γ is called a *C-John curve*.

The classes of John domains and of uniform domains in Euclidean space enjoy an important role in many areas of modern mathematical analysis; see [14, 24, 27]. Inspired by the study on generalized quasidisks [9, 12], Guo and Koskela [11] generalized the definition of John domain as follows.

Definition 1.2. Let $D \subseteq X$ be a bounded domain, let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous, increasing function with $\varphi(0) = 0$ and $\varphi(t) \ge t$ for all $t > 0$, and let $C \ge 1$ be a constant, $z_0 \in D$. We say that D is *φ-John domain*, if for any $z \in D$, there exists a rectifiable curve $\gamma : z \curvearrowright z_0$, such that

$$l(\gamma[z, u]) \le \varphi(C\delta_D(u)),$$

for all $u \in \gamma$. The concept of φ-dist and φ-diam John domains are defined analogously. A corresponding curve is called a φ- length(dist,diam) John curve.

The notion of φ-John domains allows us to formulate a second definition of φ-John domains, and the next definition and Definition 1.2 are equivalent; please see [8].

Definition 1.3. Let $D \subseteq X$ be a bounded domain. We say that D is *φ-John domain* if there exist constant $C \ge 1$ and function φ with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc γ in D satisfying

$$\min\{l(\gamma[z_1, u]), (\gamma[u, z_2])\} \le \varphi(C\delta_D(u)),$$

for all $u \in \gamma$. Here, φ is a continuous, increasing function with $\varphi(0) = 0$ and $\varphi(t) \ge t$ for all $t > 0$.

We remark that, in general, the *generalized John domain* means the φ-John domain. Obviously, the φ-John domain is a generalization of the C-John domain since the C-John domain coincides with the φ-John domain with $\varphi(Ct) = Ct$. In this paper, we always simplify C-John domain by John domain and φ-John domain by generalized John domain.

In [16], Huang et. al. showed that a domain D in $\mathbb{R}^n$ is a John domain if, and only if, $D\backslash P$ is a John domain, where P is a subset of D containing finitely many points of D.

Theorem 1.4. *(See [16], Theorem 1.4) A domain $D \subseteq \mathbb{R}^n$ $(n \geq 2)$ is a John domain if, and only if, $G = D \backslash P$ is also a John domain, where $P = \{p_1, p_2, \cdots, p_m\}$ and $p_i \in D$ $(i = 1, 2, \cdots, m)$.*

To state our results, we introduce the following definition.

Definition 1.5. Let $c \geq 1$. Let X be a rectifiable connected and locally compact metric space, and $D \subseteq X$. Then, D is called

(1) *c-quasiconvex*, if for any $x, y \in D$ there is a curve γ joining x and y in D satisfying $l(\gamma) \leq c|x-y|$. We also call this γ a c-quasiconvex curve;

(2) *c-annular quasiconvex*, if for every $x \in D$ and for all $r > 0$, each pair of points $y, z \in A(x; r, 2r) \subseteq D$ can be joined with a curve γ in $A(x; r/c, 2cr) \subseteq D$ such that $l(\gamma) \leq c|y-z|$.

Remark 1.6. It was proved by Buckley et al. in [4] that if X is connected and c-annular quasiconvex at some point $\omega \in X$, then X is $9c$-quasiconvex. Therefore, the annular quasiconvexity implies quasiconvexity.

Our first purpose is to show that a domain D in metric space X is a φ-John domain if and only if $D \backslash P$ is a φ'-John domain, where P is a subset of D containing finitely many points of D. Our proof is based on a refinement of the method of Huang et. al. [16]. We obtain a general result as follows.

Theorem 1.7. *Suppose that X is a rectifiably connected and locally compact metric space, and that domain $D \subseteq X$ is a c-annular quasiconvex. Then, the following are quantitatively equivalent:*

(1) D is a φ-John domain;

(2) $G = D \backslash P$ is a φ'-John domain, where $P = \{p_1, p_2, \cdots, p_m\}$ and $p_i \in D$ $(i = 1, 2, \cdots, m)$.

Here, φ and φ' depend on each other and c.

In [33], Väisälä studied the union of John domains in Euclidean spaces, and showed that the union of John domains also is a John domain. In [8], Guan proved that the union of John domains also is a John domain in Banach spaces. Under affable geometric conditions, we obtain a general result as follows.

Theorem 1.8. *Let X be a metric space, and let $D_1, D_2 \subseteq X$ be two c-quasiconvex domains, where $c \geq 1$. Suppose that D_1 and D_2 are two φ-John domains in X, and that $z_0 \in D_1 \cap D_2$ and $r > 0$ with*

$$B(z_0, r) \subseteq D_1 \cup D_2 \quad \textit{and} \quad \min\{\operatorname{diam}(D_1), \operatorname{diam}(D_2)\} \leq c_0 r,$$

where $c_0 \geq 1$ and $\operatorname{diam}(D_i)$ is the diameter of D_i, $i = 1, 2$. Then, $D_1 \cup D_2$ is a φ''-John domain with φ'' depending only on c, c_0 and φ. Note that the function φ'' is a continuous, increasing function with $\varphi''(0) = 0$ and $\varphi''(t) \geq t$ for all $t > 0$.

The rest of this paper is organized as follows. In Section 2, we show that D is a generalized John domain if, and only if, $D \backslash P$ is a generalized John domain, where P is a subset of D containing finitely many points of D. The goal of Section 3 is to show that the union of generalized John domains is a generalized John domain.

2. The decomposition properties of generalized John domains

In this section, we always assume that X is a rectifiably connected and locally compact metric space, and that domain $D \subseteq X$ is a c-annular quasiconvex. Furthermore, we suppose that $P = \{p_1, p_2, \cdots, p_m\}$ and $p_i \in D$ $(i = 1, 2, \cdots, m)$.

In what follows, we continue to investigate the decomposition properties of generalized John domain in metric space. The following results play a key role in the proof of Theorem 1.7. Based on [10] and [16], we will prove the following results.

Lemma 2.1. *Under the assumptions of Theorem 1.7. If D is a φ-John domain, then $G = D\backslash P$ is also a φ_1-John domain with φ_1 depending only on φ.*

Proof. By assumption, we show that $G = D\backslash P$ is also a φ_1-John domain with φ_1 depending only on φ. Without loss of generality, in order to prove Lemma 2.1, we need only to consider the case $P = \{p_1\}$. For convenience, we let

$$r = \frac{1}{2}\delta_D(p_1) \quad \text{and} \quad B_r = B(p_1, r).$$

For any points $z_1, z_2 \in G = D\backslash\{p_1\}$. Now, we divide the discussions into three cases:

Case 1. $z_1, z_2 \in D\backslash B_r$.

Since D is a φ-John domain, then there exist constant $C \geq 1$ and function φ with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc γ in D satisfying

$$\min\{l(\gamma[z_1, z]), (\gamma[z, z_2])\} \leq \varphi(C\delta_D(z)) \tag{2.1}$$

for all $z \in \gamma$.

We now consider two subcases:

Subcase 1.1. $\gamma \subseteq D\backslash B_r$.

If $\gamma \subseteq D\backslash B_r$, then we take $\beta = \gamma$, and it is clear that $\beta \subseteq G$.

To prove this subcase, we have the following claim.

Claim 1. Let z be any point in $\beta \subseteq D\backslash B_r$, and we have

$$\delta_D(z) \leq 3\delta_G(z).$$

Since $\beta = \gamma$ and $\beta \subseteq D\backslash B_r \subseteq G$, for any $z \in \beta$, we get

$$|z - p_1| > r.$$

If $\delta_D(z) \leq |z - p_1|$, by using the definitions of $\delta_D(z)$ and $\delta_G(z)$, we have

$$\delta_D(z) = \delta_G(z). \tag{2.2}$$

If $\delta_D(z) > |z - p_1|$, it follows that

$$\delta_D(z) > \delta_G(z) = |z - p_1|.$$

According to the triangle inequality and $|z - p_1| > r$, we deduce that

$$\delta_D(z) \leq |z - p_1| + \delta_D(p_1) = \delta_G(z) + 2r \leq 3\delta_G(z). \tag{2.3}$$

So, from (2.2) and (2.3), Claim 1 is obtained.

Since D is a φ-John domain, and $\beta = \gamma \subseteq D\backslash B_r \subseteq G$, for any $z \in \beta$, from the conclusion of Claim 1 and (2.1), it follows that

$$\min\{l(\beta[z_1, z]), l(\beta[z_2, z])\} \leq \varphi(3C\delta_G(z)).$$

Subcase 1.2. $\gamma \cap B_r \neq \emptyset$.

We let z_1' be the first intersection point of γ from z_1 to z_2, with ∂B_r and z_2' as the last intersection point of γ from z_1 to z_2 with ∂B_r. Let U_r be the disk determined by z_1', z_2' and p_1 in $\overline{B_r}$ with center p_1 and radius r. Then, z_1', z_2' divide ∂U_r into subarcs, and we denote the subarc with shorter arclength by α (if they have the same arclength, then we choose one of them to be α), that is,

$$l(\alpha) \leq \pi r. \tag{2.4}$$

Set

$$\beta = \gamma[z_1, z_1'] \cup \alpha \cup \gamma[z_2', z_2].$$

Claim 2. $l(\alpha) \leq \frac{\pi}{2}|z_1' - z_2'|$.

In disk U_r, according to the chord arc formula and the properties of trigonometric function, it follows that

$$\frac{\pi}{2} \cdot 2r \cdot \sin\frac{\theta}{2} \geq \theta \cdot r \quad \text{and} \quad 2r \cdot \sin\frac{\theta}{2} = |z_1' - z_2'|. \tag{2.5}$$

where $\theta \in [0, \pi]$ is the center angle of two points z_1' and z_2'. According to (2.5) and $l(\alpha) = \theta \cdot r$, we get that

$$\frac{\pi}{2}|z_1' - z_2'| \geq l(\alpha). \tag{2.6}$$

Therefore, the proof of the Claim 2 is now complete.

By the definition of z_1', z_2' and disk U_r, we have

$$l(\gamma[z_1', z_2']) \geq |z_1' - z_2'|. \tag{2.7}$$

Together with (2.6) and (2.7), it follows that

$$l(\alpha) \leq \frac{\pi}{2} l(\gamma[z_1', z_2']).$$

If $z \in \gamma[z_1, z_1']$ or $z \in \gamma[z_2', z_2]$, by symmetry, we only prove $z \in \gamma[z_1, z_1']$; the proof of $z \in \gamma[z_2', z_2]$ uses the same argument for $z \in \gamma[z_1, z_1']$.

For any $z \in \gamma[z_1, z_1'] \subseteq D \backslash B_r$. It follows immediately from Claim 1 that $\delta_D(z) \leq 3\delta_G(z)$.

Hence, together with Claim 2 and $\delta_D(z) \leq 3\delta_G(z)$, it follows that

$$\begin{aligned}
\min\{l(\beta[z_1, z]), l(\beta[z_2, z])\} &\leq \frac{\pi}{2} \min\{l(\gamma[z_1, z]), l(\gamma[z_2, z])\} \\
&\leq \frac{\pi}{2}\varphi(C\delta_D(z)) \\
&\leq \frac{\pi}{2}\varphi(3C\delta_G(z)).
\end{aligned}$$

If $z \in \alpha \subseteq \partial U_r$, then we have

$$\delta_G(z) = r \quad \text{and} \quad \delta_G(z_2') \leq \delta_D(p_1) + r = 3r. \tag{2.8}$$

According to the inequality (2.4) and (2.8), it follows immediately from the definition of the φ-John domain that

$$\begin{aligned}\min\{l(\beta[z_1,z]), l(\beta[z_2,z])\} &\le l(\alpha) + \min\{l(\gamma[z_1,z_1']), l(\gamma[z_2',z_2])\}\\ &\le \pi r + \min\{l(\gamma[z_1,z_2']), l(\gamma[z_2',z_2])\}\\ &\le \pi r + \varphi(C\delta_D(z_2'))\\ &\le \pi\delta_G(z) + \varphi(3C\delta_G(z))\\ &\le \varphi(\pi\delta_G(z)) + \varphi(3C\delta_G(z))\\ &\le 2\varphi(4C\delta_G(z)),\end{aligned}$$

where $C \ge 1$ is a constant.

Case 2. $z_1, z_2 \in \overline{B_r}\backslash\{p_1\}$.

Let z_1' be the intersection point of the ray starting from p_1 and passing through z_1 with ∂B_r, and let z_2' be the intersection point of the ray starting from p_1 and passing through z_2 with ∂B_r, then we have

$$|z_1 - z_1'| \le r \quad \text{and} \quad |z_2 - z_2'| \le r. \tag{2.9}$$

We use U_r to denote the disk determined by z_1', z_2' and p_1 in $\overline{B_r}$ with center p_1 and radius r. Then, z_1' and z_2' divide ∂U_r into two subarcs. Let α denote the subarc with the shorter arclength (if they have the same arclength, then we choose one of them to be α). We set

$$\beta = [z_1, z_1'] \cup \alpha \cup [z_2, z_2'],$$

where $[z_i, z_i']$ denotes the line segments in metric space of z_i and z_i', $i = 1, 2$.

If $z \in \alpha \subseteq \partial U_r$, it is clear that

$$\delta_G(z) = |z - p_1| = r. \tag{2.10}$$

Together with (2.9) and (2.10), it follows that

$$\begin{aligned}\min\{l(\beta[z_1,z]), l(\beta[z_2,z])\} &\le l(\alpha) + \min\{|z_1 - z_1'|, |z_2 - z_2'|\}\\ &\le \pi r + r\\ &= (\pi+1)\delta_G(z)\\ &\le \varphi((\pi+1)\delta_G(z)).\end{aligned} \tag{2.11}$$

If $z \in [z_1, z_1']$ or $z \in [z_2, z_2']$, by symmetry, it is sufficient to show that $z \in [z_1, z_1']$. For any $z \in [z_1, z_1']$, we have the desired estimate

$$\min\{l(\beta[z_1,z]), l(\beta[z_2,z])\} \le |z_1 - z| \le \delta_G(z) \le \varphi(\delta_G(z)). \tag{2.12}$$

Hence, by combing (2.11) with (2.12), for any $z \in \beta$, we deduce that

$$\min\{l(\beta[z_1,z]), l(\beta[z_2,z])\} \le \varphi((\pi+1)\delta_G(z)).$$

Case 3. $z_1 \in D\backslash\overline{B_r}$ and $z_2 \in B_r\backslash\{p_1\}$.

Since D is a φ-John domain, there must exist a curve γ joining z_1 and z_2 such that

$$\min\{l(\gamma[z_1, z]), l(\gamma[z_2, z])\} \le \varphi(C\delta_D(z))$$

for all $z \in \gamma$.

Let z_1' to be the first intersection point of γ from z_1 to z_2 with ∂B_r, and let z_2' be the intersection point of the ray starting from p_1 and passing through z_2 with ∂B_r. We use U_r to denote the disk determined by z_1', z_2' and p_1 in $\overline{B_r}$ with center p_1 and radius r. Then, z_1' and z_2' divide ∂U_r into two subarcs. Let α denote the subarc with the shorter arclength (if they have the same arclength, then we choose one of them to be α). Then, according to the description above, we have

$$l(\alpha) \le \pi r \quad \text{and} \quad |z_2 - z_2'| \le r. \tag{2.13}$$

Set

$$\beta = \gamma[z_1, z_1'] \cup \alpha \cup [z_2, z_2'],$$

where $[z_2, z_2']$ represents a straight line segment joining z_2 to z_2'.

We now consider three subcases:

Subcase 3.1. $z \in \alpha \subseteq \partial U_r$.

Using a similar argument as in Case 2, we have $\delta_G(z) = r$. According to (2.13), from the definition of φ-John domain, we get that

$$\begin{aligned}\min\{l(\beta[z_1, z]), l(\beta[z_2, z])\} &\le l(\alpha) + \min\{l(\gamma[z_1, z_1']), |z_2 - z_2'|\} \\ &\le \pi r + |z_2 - z_2'| \\ &\le \pi r + r \\ &= (\pi + 1)\delta_G(z) \\ &\le \varphi((\pi + 1)\delta_G(z)).\end{aligned}$$

Subcase 3.2. $z \in [z_2, z_2']$.

From the definition of z_2', it follows from $z \in [z_2, z_2']$ that

$$\min\{l(\beta[z_1, z]), l(\beta[z_2, z])\} \le |z_2 - z| \le \delta_G(z) \le \varphi(\delta_G(z)).$$

Subcase 3.3. $z \in \gamma[z_1, z_1']$.

If $l(\gamma[z_1, z]) \le l(\gamma[z_2, z])$. Since D is a φ-John domain, by using the conclusion of Claim 1, it follows that

$$\min\{l(\beta[z_1, z]), l(\beta[z_2, z])\} \le l(\gamma[z_1, z]) \le \varphi(C\delta_D(z)) \le \varphi(3C\delta_G(z)).$$

If $l(\gamma[z_1, z]) > l(\gamma[z_2, z])$, by the conclusion of Subcase 3.1, we deduce that

$$l(\beta[z_2, z_1']) = l(\alpha) + |z_2 - z_2'| \le (\pi + 1)r. \tag{2.14}$$

Now, for any $z \in \gamma[z_1, z_1']$ with $l(\gamma[z_1', z]) < r/2$, then we have

$$\delta_G(z) \ge \delta_D(z_1') - |z_1' - z| \ge \delta_D(z_1') - l(\gamma[z_1', z]) \ge r - \frac{r}{2} = \frac{r}{2}. \tag{2.15}$$

Together with (2.14) and (2.15), it follows that

$$\begin{aligned}\min\{l(\beta[z_1, z]), l(\beta[z_2, z])\} &= l(\beta[z_2, z_1']) + l(\gamma[z_1', z])\\ &\le \left(\pi + \frac{3}{2}\right) r\\ &\le (2\pi + 3)\delta_G(z)\\ &\le \varphi((2\pi + 3)\delta_G(z)).\end{aligned}$$

If $l(\gamma[z_1', z]) \ge r/2$, for any $z \in \gamma[z_1, z_1'] \subseteq D\backslash\overline{B_r}$, by Claim 1, we know that $\delta_D(z) \le 3\delta_G(z)$. According to inequality (2.14) and the definition of the φ-John domain, we get

$$\begin{aligned}\min\{l(\beta[z_1, z]), l(\beta[z_2, z])\} &= l(\beta[z_2, z_1']) + l(\gamma[z_1', z])\\ &\le (\pi + 1)r + l(\gamma[z_1', z])\\ &\le (2\pi + 3)l(\gamma[z_1', z])\\ &\le (2\pi + 3)\varphi(3C\delta_G(z)).\end{aligned}$$

Therefore, as discussed above, it follows that $G = D\backslash P$ is a φ_1-John domain. Here,

$$\varphi_1(C_1 t) = \max\{2\varphi(4Ct),\ \varphi((2\pi + 3)t),\ (2\pi + 3)\varphi(3Ct)\}.$$

Hence, this completes the proof of Lemma 2.1. □

Lemma 2.2. *Under the assumptions of Theorem 1.7. If $G = D\backslash P$ is a φ-John domain, then D is also a φ_2-John domain, where φ_2 depends only on φ and c.*

Proof. We always assume that $D \subseteq X$ is a c-annular quasiconvex, and that $G = D\backslash P$ is a domain, where

$$P = \{p_1, p_2, \cdots, p_m\} \quad \text{and} \quad p_i \in D \ \ (i = 1, 2, \cdots, m).$$

Let φ be a continuous, increasing function with $\varphi(0) = 0$ and $\varphi(t) \ge t$ for all $t > 0$. For any pair of points $z_1, z_2 \in D$, we divide the discussions into three cases:

Case 1. $z_1, z_2 \in G$.

According to the assumptions of Theorem 1.7, we know that $G = D\backslash P$ is a φ-John domain, that is, there is a rectifiable curve $\gamma \subset G$ connecting z_1 and z_2 such that

$$\min\{l(\gamma[z_1, z]), l(\gamma[z_2, z])\} \le \varphi(C\delta_G(z))$$

for all $z \in \gamma$, where $C \ge 1$ is a constant. Since $G = D\backslash P \subseteq D$, we have

$$\delta_G(z) \le \delta_D(z).$$

According to φ being a continuous, increasing function, it follows that

$$\min\{l(\gamma[z_1, z]), l(\gamma[z_2, z])\} \le \varphi(C\delta_D(z)) \tag{2.16}$$

for all $z \in \gamma$.

Case 2. $z_1, z_2 \in D\backslash G$.

Let $z_1, z_2 \in P = D\backslash G$, and $P = \{p_1, p_2, \cdots, p_m\}$, $p_i \in D$ $(i = 1, 2, \cdots, m)$. Set

$$\min\{\delta_D(z_1), \delta_D(z_2)\} = 12cr \quad \text{and} \quad s = \min\{|p_i - p_j| : i \neq j\}.$$

We choose

$$z_1' \in B(z_1, \tau) \cap G \quad \text{and} \quad z_2' \in B(z_2, \tau) \cap G,$$

where

$$\tau = \min\left\{r, \frac{s}{12c}\right\}.$$

According to the definition of the φ-John domain, there must exist a curve $\gamma \subseteq G$ joining z_1' with z_2' such that

$$\min\{l(\gamma[z_1', z]), l(\gamma[z_2', z])\} \leq \varphi(C\delta_G(z)).$$

for all $z \in \gamma$.

Since $D \subseteq X$ is a c-annular quasiconvex, by Remark 1.6, we have the D is $9c$-quasiconvex, that is, there exists a rectifiable curve γ_1 in D joining z_1 to z_1' with $l(\gamma_1) \leq 9c|z_1 - z_1'|$, and there exists a rectifiable curve γ_2 in D joining z_2 to z_2' with $l(\gamma_2) \leq 9c|z_2 - z_2'|$, where $c \geq 1$. Hence, we have

$$\max\{l(\gamma_1), l(\gamma_2)\} \leq 9c\tau \leq 9cr.$$

Let

$$\beta = \gamma_1 \cup \gamma \cup \gamma_2.$$

For any $z \in \beta$, if $z \in \gamma_1$ or $z \in \gamma_2$, by symmetry, we assume that $z \in \gamma_1$. From the definition of quasiconvexity, it follows that

$$\begin{aligned}\min\{l(\beta[z_1, z]), l(\beta[z_2, z])\} &\leq l(\beta[z_1, z]) = l(\gamma_1[z_1, z]) \\ &\leq 9c|z_1 - z| \\ &\leq \frac{9c}{12c-1}\delta_D(z) \\ &\leq \varphi\left(\frac{9c}{12c-1}\delta_D(z)\right) \\ &< \varphi(\delta_D(z)).\end{aligned} \tag{2.17}$$

If $z \in \gamma$ and $\min\{l(\gamma[z_1', z]), l(\gamma[z_2', z])\} \leq cr$, we deduce that

$$\begin{aligned}\delta_D(z) &\geq \min\{\delta_D(z_1), \delta_D(z_2)\} - \min\{l(\beta[z_1, z]), l(\beta[z_2, z])\} \\ &= 12cr - \min\{l(\beta[z_1, z]), l(\beta[z_2, z])\} \\ &= 12cr - (\min\{l(\gamma_1) + l(\gamma[z_1', z]), l(\gamma_2) + l(\gamma[z_2', z])\}) \\ &\geq 12cr - (9cr + \min\{l(\gamma[z_1', z]), l(\gamma[z_2', z])\}) \\ &\geq 2cr.\end{aligned} \tag{2.18}$$

Hence, by inequality (2.18) and the definition of function φ, it is clear that

$$\begin{aligned}\min\{l(\beta[z_1,z]),l(\beta[z_2,z])\} &= \min\{l(\gamma_1)+l(\gamma[z_1',z]),l(\gamma_2)+l(\gamma[z_2',z])\}\\ &\le 9cr+\min\{l(\gamma[z_1',z]),l(\gamma[z_2',z])\}\\ &\le 10cr\\ &\le 5\delta_D(z)\\ &\le \varphi(5\delta_D(z)).\end{aligned}\tag{2.19}$$

If $z\in\gamma$ and $\min\{l(\gamma[z_1',z]),l(\gamma[z_2',z])\}>cr$, it follows from the definition of the φ-John domain that

$$\begin{aligned}\min\{l(\beta[z_1,z]),l(\beta[z_2,z])\} &= \min\{l(\gamma_1)+l(\gamma[z_1',z]),l(\gamma_2)+l(\gamma[z_2',z])\}\\ &\le 9cr+\min\{l(\gamma[z_1',z]),l(\gamma[z_2',z])\}\\ &\le 10\cdot\min\{l(\gamma[z_1',z]),l(\gamma[z_2',z])\}\\ &\le 10\varphi(C\delta_D(z)).\end{aligned}\tag{2.20}$$

Together with (2.17), (2.19) and (2.20), which shows that in this subcase, the lemma holds with

$$\varphi_2(C_2t)=\max\{\varphi(5t),10\varphi(Ct)\}.\tag{2.21}$$

Case 3. $z_1\in G$ and $z_2\in D\backslash G$.

Using a similar argument as in Case 2, we can show that there is a rectifiable curve $\gamma\subseteq D$ connecting z_1 and z_2 such that for any $z\in\gamma$,

$$\min\{l(\gamma[z_1,z]),l(\gamma[z_2,z])\}\le\varphi_2(C_2\delta_D(z)).\tag{2.22}$$

By combining (2.16), (2.21) and (2.22), we get that D is a φ_2-John domain, that is

$$\min\{l(\beta[z_1,z]),l(\beta[z_2,z])\}\le\varphi_2(C_2\delta_D(z)).$$

where $C_2\ge 1$ is a constant, and

$$\varphi_2(C_2t)=\max\{\varphi(5t),10\varphi(Ct)\}.$$

Hence, Lemma 2.2 is proved. □

The proof of Theorem 1.7. Under the assumptions of Theorem 1.7, Theorem 1.7 follows from Lemmas 2.1 and 2.2. □

3. The union of generalized John domains

The proof of Theorem 1.8. The assumption implies that $D_1,D_2\subseteq X$ are two c-quasiconvex and φ-John domains. Furthermore, we suppose that $z_0\in D_1\cap D_2$ and $r>0$ with

$$B(z_0,r)\subseteq D_1\cup D_2\quad\text{and}\quad\min\{\mathrm{diam}(D_1),\mathrm{diam}(D_2)\}\le c_0r.$$

Let $D = D_1 \cup D_2$. Under these assumptions, in order to prove Theorem 1.8, we need only to show that there exist constant $C'' \geq 1$ and function φ'' with the property that each pair of points a, b in D can be joined by a rectifiable arc γ in D satisfying

$$\min\{l(\gamma[a,z]), l(\gamma[z,b])\} \leq \varphi''(C''\delta_D(z))$$

for all $z \in \gamma$. Here, φ'' is a continuous, increasing function with $\varphi''(0) = 0$ and $\varphi''(t) \geq t$ for all $t > 0$.

Without loss of generality, we assume that $\operatorname{diam}(D_1) \leq \operatorname{diam}(D_2)$. Let $a \in D_1$ and $b \in D_2$, and we can choose φ-John curves $\alpha : a \curvearrowright z_0$ and $\beta : b \curvearrowright z_0$ in D_1 and D_2, respectively. The continuum $\alpha \cup \beta$ contains a curve $\gamma : a \curvearrowright b$. It suffices to show that γ is a φ''-John curve in $D = D_1 \cup D_2$.

We choose two points $a_1 \in \alpha$ and $b_1 \in \beta$ dividing α and β to subarcs of equal length, respectively. Let

$$a_2 = \sup\left\{u \in \alpha \;:\; \alpha[u, z_0] \subseteq \overline{B}\left(z_0, \frac{r}{2}\right)\right\},$$

and

$$b_2 = \sup\left\{v \in \beta \;:\; \beta[v, z_0] \subseteq \overline{B}\left(z_0, \frac{r}{2}\right)\right\}.$$

In what follows, we will divide the proof into two steps.

Step 1. *For all $x \in \alpha$, we prove that*

$$\min\{l(\alpha[a,x]), l(\alpha[x,z_0] \cup \beta)\} \leq \varphi_3(C_3\delta_D(x)) \tag{3.1}$$

with constant $C_3 \geq 1$.

Since D_1 is φ-John domain and $\alpha : a \curvearrowright z_0$ is φ-John curve in D_1, thus, we have

$$\min\{l(\alpha[a,x]), l(\alpha[x,z_0])\} \leq \varphi(C\delta_{D_1}(x)). \tag{3.2}$$

Let $x \in \alpha$. Now, to prove inequality (3.1), we divide the discussions into three cases:

Case 1. $x \in \alpha[a, a_1]$.

Since $D = D_1 \cup D_2$, it is clear that

$$\delta_{D_1}(x) \leq \delta_D(x).$$

From the definitions of a_1 and the φ-John domain, for any $x \in \alpha[a, a_1]$, by (3.2), it follows that

$$\begin{aligned}\min\{l(\alpha[a,x]), l(\alpha[x,z_0] \cup \beta)\} &\leq \min\{l(\alpha[a,x]), l(\alpha[x,z_0])\}\\ &\leq \varphi(C\delta_{D_1}(x))\\ &\leq \varphi(C\delta_D(x)).\end{aligned}$$

Case 2. $x \in \alpha[a_2, z_0]$.

Now that $\operatorname{diam}(D_1) \leq c_0 r$, according to the definition of a_2, it is clear that

$$\delta_D(x) \geq \frac{r}{2} \geq \frac{\operatorname{diam}(D_1)}{2c_0}. \tag{3.3}$$

Since D_1 is a c-quasiconvex domain, we have

$$l(\alpha) \leq c|a - z_0| \leq c \cdot \operatorname{diam}(D_1), \tag{3.4}$$

where $c \geq 1$ is a constant. Therefore, according to (3.3), (3.4) and the definition of function φ, it follows that

$$\min\{l(\alpha[a,x]), l(\alpha[x,z_0] \cup \beta)\} \leq l(\alpha) \leq c \cdot \operatorname{diam}(D_1) \leq 2cc_0\delta_D(x) \leq 2cc_0\varphi(\delta_D(x)).$$

Case 3. $x \in \alpha[a_1, a_2]$ and $a_2 \in \alpha[a_1, z_0]$.

This case may be empty. From the construction of a_2, it is obvious that

$$\min\{l(\alpha[a,x]), l(\alpha[x,z_0])\} \geq \frac{r}{2}. \tag{3.5}$$

From the definition of c-quasiconvex domain and inequality $\operatorname{diam}(D_1) \leq c_0 r$, we obtain that

$$l(\alpha) \leq c|a - z_0| \leq c \cdot \operatorname{diam}(D_1) \leq cc_0 r. \tag{3.6}$$

Hence, Combing (3.2), (3.5) and (3.6), it follows that

$$\begin{aligned}\min\{l(\alpha[a,x]), l(\alpha[x,z_0] \cup \beta)\} &\leq l(\alpha[a,x]) \leq l(\alpha) \leq cc_0 r \\ &\leq 2cc_0 \cdot \min\{l(\alpha[a,x]), l(\alpha[x,z_0])\} \\ &\leq 2cc_0\varphi(C\delta_{D_1}(x)) \\ &\leq 2cc_0\varphi(C\delta_D(x)).\end{aligned}$$

Therefore, for all $x \in \alpha$, we have

$$\min\{l(\alpha[a,x]), l(\alpha[x,z_0] \cup \beta)\} \leq \varphi_3(C_3\delta_D(x)),$$

where

$$\varphi_3(C_3 t) = \max\{2cc_0\varphi(t), 2cc_0\varphi(Ct)\} = 2cc_0\varphi(Ct),$$

and $C_3 \geq 1$ is a constant.

Step 2. *For all $y \in \beta$, we prove that*

$$\min\{l(\alpha \cup \beta[z_0, y]), l(\beta[y,b])\} \leq \varphi_4(C_4\delta_D(y)) \tag{3.7}$$

with constant $C_4 \geq 1$.

Let $y \in \beta$. To prove inequality (3.7), our proof consists of three parts. For the first part, if $y \in \beta[b, b_1]$, Since D_2 is φ-John domain and $\beta : b \curvearrowright z_0$ is a φ-John curve in D_2, thus, we get that

$$\min\{l(\beta[z_0, y]), l(\beta[y,b])\} \leq \varphi(C\delta_{D_2}(y)). \tag{3.8}$$

By the inequality (3.8), we deduce that

$$\begin{aligned}\min\{l(\alpha \cup \beta[z_0,y]), l(\beta[y,b])\} &\leq \min\{l(\beta[z_0,y]), l(\beta[y,b])\} \\ &\leq \varphi(C\delta_{D_2}(y)) \\ &\leq \varphi(C\delta_D(y)).\end{aligned}$$

For the second part, if $y \in \beta[b_2, z_0]$, according to the definition of b_2, we obtain that

$$\delta_D(y) \geq \frac{r}{2} \geq \frac{\text{diam}(D_1)}{2c_0}. \tag{3.9}$$

Since D_2 is a c-quasiconvex domain, and β is a φ-John curve, from the definitions of a_2 and b_2, it is clear that

$$l(\beta[z_0, y]) \leq c|y - z_0| \leq c|a_2 - z_0| \leq c \cdot \text{diam}(D_1). \tag{3.10}$$

Combing (3.4), (3.9) and (3.10), it follows that

$$\begin{aligned}\min\{l(\alpha \cup \beta[z_0, y]), l(\beta[y, b])\} &\leq l(\alpha \cup \beta[z_0, y]) \\ &= l(\alpha) + l(\beta[z_0, y]) \\ &\leq 2c \cdot \text{diam}(D_1) \\ &\leq 4cc_0\delta_D(y) \\ &\leq 4cc_0\varphi(\delta_D(y)).\end{aligned}$$

For the final part, if $b_2 \in \beta[b_1, z_0]$ and $y \in \beta[b_1, b_2]$, this case may again be empty. Since D_1 is a c-quasiconvex domain, and $\text{diam}(D_1) \leq c_0 r$, we get that

$$l(\alpha) \leq c|a - z_0| \leq c \cdot \text{diam}(D_1) \leq cc_0 r, \tag{3.11}$$

and since $D_2 \subseteq D$ is a φ-John domain, we have

$$\min\{l(\beta[z_0, y]), l(\beta(y, b))\} \leq \varphi(C\delta_{D_2}(y)) \leq \varphi(C\delta_D(y)). \tag{3.12}$$

In addition, from the definition of b_2, we deduce that

$$\min\{l(\beta[z_0, y]), l(\beta(y, b))\} \geq \frac{r}{2}. \tag{3.13}$$

According to (3.11)–(3.13), we get

$$l(\alpha) \leq cc_0 r \leq 2cc_0 \cdot \min\{l(\beta[z_0, y]), l(\beta(y, b))\} \leq 2cc_0\varphi(C\delta_D(y)). \tag{3.14}$$

Now, it follows immediately from the inequality (3.14) that

$$\begin{aligned}\min\{l(\alpha \cup \beta[z_0, y]), l(\beta[y, b])\} &\leq l(\alpha) + \min\{l(\beta[z_0, y]), l(\beta[y, b])\} \\ &\leq 2cc_0\varphi(C\delta_D(y)) + \varphi(C\delta_D(y)) \\ &\leq 4cc_0\varphi(C\delta_D(y)).\end{aligned}$$

Therefore, for all $y \in \beta$, we have

$$\min\{l(\alpha \cup \beta[z_0, y]), l(\beta[y, b])\} \leq \varphi_4(C_4\delta_D(y))$$

where

$$\varphi_4(C_4 t) = \max\{4cc_0\varphi(t), 4cc_0\varphi(Ct)\} = 4cc_0\varphi(Ct),$$

and $C_4 \geq 1$ is a constant.

Hence, we verified all the cases and our conclusion holds, that is, $D_1 \cup D_2$ is a φ''-John domain with

$$\varphi''(C''t) = \max\{\varphi_3(C_3 t), \varphi_4(C_4 t)\} = \varphi_4(C_4 t) = 4cc_0\varphi(Ct),$$

where $C'' \geq 1$ is a constant. □

4. Conclusions

In summary, we investigated the removability and union of generalized John domain, that is, the main result of this paper showed that D is a φ-John domain if, and only if, $D\backslash P$ is a φ'-John domain, where P is a subset of D containing finitely many points of D, φ and φ' depend on each other, and finally we prove the union of φ-John domains is φ''-John domain.

Given the Theorem 1.7 of the paper, it is natural to ask the following question:

Question 4.1. Let X be a rectifiably connected, locally compact and c-annular quasiconvex metric space, and let P be a countable subset of X. Is X φ-John metric space if and only if $X\backslash P$ φ'-John metric space?

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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