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Research article

Sharp coefficient problems of functions with bounded turning subordinated to the domain of cosine hyperbolic function

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Abstract: In the current article, we consider a class of bounded turning functions associated with the cosine hyperbolic function and give some results containing coefficient functionals using the familiar Carathéodory functions. An improvement on the bound of the third-order Hankel determinant for functions in this class is provided. Furthermore, we obtain sharp estimates of the Fekete-Szegö, Krushkal, and Zalcman functionals with logarithmic coefficients as entries. All the findings are proved to be sharp.

Keywords: Hankel determinant; bounded turning functions; cosine hyperbolic function; logarithmic coefficient problems
Mathematics Subject Classification: 30C45, 30C50

1. Introduction

Complex analysis is one of the major disciplines nowadays due to its numerous applications in mathematical science and other fields. Geometric function theory is an intriguing topic of complex analysis that involves the geometrical characteristics of analytic functions. This area is crucial to applied mathematics, particularly in fields such as engineering, electronics, nonlinear integrable system theory, fluid dynamics, modern mathematical physics, and partial differential equation theory. The key problem that led to the rapid emergence of geometric function theory is the Bieberbach conjecture. It is about the coefficient bounds for functions belonging to the class $\mathcal{S}$ of univalent functions. This conjecture states that if $g \in \mathcal{S}$ with the Taylor-Maclaurin series expansion of the form

$$g(z) = z + \sum_{k=2}^{\infty} d_k z^k, \quad z \in \mathbb{D}, \tag{1.1}$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, then $|d_k| \leq k$ for all $k \geq 2$. Let $\mathcal{A}$ be the class of analytic functions with the series representation given in (1.1). The set $\mathcal{S}$ is a subclass of $\mathcal{A}$ which was first taken into account by Koebe in 1907. The above famous conjecture was proposed by Bieberbach [1] in 1916. He proved this for $k = 2$, and subsequent researchers, including Löwner [2], Garabedian and Schiffer [3], Pederson and Schiffer [4], and Pederson [5], confirmed it for $k = 3, 4, 5$, and 6, respectively. However, settling the conjecture for $k \geq 7$ remained elusive until 1985 when de Branges [6] used hypergeometric functions to prove it for every $k \geq 2$. Lawrence Zalcman proposed the inequality $\left|d_k^2 - d_{2k-1}\right| \leq (k-1)^2$ with $k \geq 2$ for $g \in \mathcal{S}$ in the late 1960's as a way of establishing the Bieberbach conjecture. Due to this, a number of articles [7–9] have been published on the Zalcman hypothesis and its generalized form $\left|\lambda d_k^2 - d_{2k-1}\right| \leq \lambda k^2 - 2k + 1$ with $\lambda \geq 0$ for different subclasses of the class $\mathcal{S}$. This conjecture has remained unsolved for a long time. Krushkal [10] established this hypothesis for $k \leq 6$. In an attempt to solve the Zalcman conjecture, Krushkal [11] investigated the inequality $\left|d_k^l - d_2^{l(k-1)}\right| \leq 2^{l(k-1)} - k^l$ with $k, l \geq 2$ for $g \in \mathcal{S}$. A broader Zalcman hypothesis for $g \in \mathcal{S}$ was proposed by Ma [12] later, in 1999, and is given by

$$\left|d_j d_k - d_{j+k-1}\right| \leq (j-1)(k-1), \quad j, k \geq 2.$$

He only proved it for a subclass of $\mathcal{S}$. The challenge is still open for the class $\mathcal{S}$.

Now, we recall the definition of subordination, which actually provides a relationship between analytic functions. We write $g_1 \prec g_2$ to illustrate that g_1 is subordinate to g_2. It is explained that for two given functions $g_1, g_2 \in \mathcal{A}$, a Schwarz function w exists such that $g_1(z) = g_2(w(z))$ for $z \in \mathbb{D}$. Once g_2 is univalent in $\mathbb{D}$, then this relation is equivalent to saying that

$$g_1(z) \prec g_2(z), \quad z \in \mathbb{D}$$

if and only if

$$g_1(0) = g_2(0) \text{ and } g_1(\mathbb{D}) \subset g_2(\mathbb{D}).$$

The three classic subclasses of univalent functions are $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{K}$, of which their functions are known respectively as convex functions, starlike functions and close-to-convex functions. These classes are defined by

$$\mathcal{C} := \left\{ g \in \mathcal{S} : \frac{(zg'(z))'}{g'(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D} \right\},$$

$$\mathcal{S}^* := \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D} \right\},$$

and

$$\mathcal{K} := \left\{ g \in \mathcal{S} : \frac{zg'(z)}{h(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D} \right\}$$

for some $h \in \mathcal{S}^*$. Taking $h(z) = z$, the class $\mathcal{K}$ reduces to the class $\mathcal{BT}$ of bounded turning functions. Further, replacing $\frac{1+z}{1-z}$ by some other special functions, various interesting subfamilies of the class $\mathcal{S}$ were studied; interested readers may refer to [13–18].

The determinant $D_{\lambda,n}(g)$, where $n, \lambda \in \mathbb{N} = \{1, 2, \ldots\}$, is known as the Hankel determinant and was presented by Pommerenke [19,20]. It is formed by the coefficients of the function $g \in \mathcal{S}$ and is defined by

$$D_{\lambda,n}(g) := \begin{vmatrix} d_n & d_{n+1} & \ldots & d_{n+\lambda-1} \\ d_{n+1} & d_{n+2} & \ldots & d_{n+\lambda} \\ \vdots & \vdots & \ldots & \vdots \\ d_{n+\lambda-1} & d_{n+\lambda} & \ldots & d_{n+2\lambda-2} \end{vmatrix}.$$

Hankel matrices are used in both pure mathematics and technological applications, including the theory of Markov processes, the theory of non-stationary signals in the Hamburger moment problem, and many other topics, see for example [21–24]. There are relatively few publications on the bounds of the Hankel determinant for functions in the general class $\mathcal{S}$. The best estimate for $g \in \mathcal{S}$ was determined by Hayman in [25], which asserted that $\left|D_{2,n}(g)\right| \leq |\eta|$, where η is a constant. Additionally, for $g \in \mathcal{S}$, it was shown in [26] that the second-order Hankel determinant $\left|D_{2,2}(g)\right| \leq \eta$ for $0 \leq \eta \leq 11/3$. The two determinants $D_{2,1}(g)$ and $D_{2,2}(g)$ have been extensively studied in the literature for various subfamilies of univalent functions. The works [27–31], in which the sharp bounds of the second-order Hankel determinant for some subclasses of $\mathcal{S}$ are determined, are particularly noteworthy.

In comparison to the second-order Hankel determinant, the sharp bound of the third-order Hankel determinant $D_{3,1}(g)$ for certain analytic univalent functions is much harder to find. The investigation on $D_{3,1}(g)$ for $\mathcal{S}$ was initiated by Babalola [32] in 2010. The exact bounds of this determinant were proved recently for the classes $\mathcal{C}$, $\mathcal{S}^*$, and $\mathcal{BT}$ in [33], [34], and [35], respectively. These bounds are given by

$$\left|D_{3,1}(g)\right| \leq \begin{cases} \frac{4}{135}, & \text{for} \quad g \in \mathcal{C}, \\ \frac{4}{9}, & \text{for} \quad g \in \mathcal{S}^*, \\ \frac{1}{4}, & \text{for} \quad g \in \mathcal{BT}. \end{cases}$$

By employing similar techniques, Khalil Ullah et al. [36] and Lecko et al. [37] derived the sharp bounds for $\left|D_{3,1}(g)\right|$ when considering functions belonging to the families $\mathcal{S}^*_{\tanh}$ and $\mathcal{S}^*(1/2)$, respectively. Additionally, the works [38–43] proved the sharp bounds for the same third-order Hankel determinant in various novel subfamilies of analytic univalent functions.

Let us consider the two function classes defined respectively by

$$\mathcal{S}^*_{\cosh} := \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec \cosh\sqrt{z} \quad (z \in \mathbb{D}) \right\}$$

and

$$\mathcal{BT}_{\cosh} := \left\{ g \in \mathcal{S} : g'(z) \prec \cosh\sqrt{z} \quad (z \in \mathbb{D}) \right\}.$$

These classes were introduced and studied by Mundalia et al. [44] and Ghaffar et al. [45], respectively. In this paper, we improved the bound of the third-order Hankel determinant $\left|D_{3,1}(g)\right|$, which was determined by Ghaffar et al. and published recently in AIMS Mathematics [45]. Furthermore, we obtain the sharp estimates of the Fekete-Szegö, Krushkal, and Zalcman functionals with logarithmic coefficients as entries.

2. A set of Lemmas

In the theory of univalent functions, the Carathéodory functions are well studied. They are analytic in $\mathbb{D}$ with positive real part and take series representations of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} \tau_n z^n \quad (z \in \mathbb{D}). \tag{2.1}$$

We denote by $\mathcal{P}$ the set of these functions.

To prove the main theorems, we need the following lemmas.

Lemma 2.1 [46] Let $p \in \mathcal{P}$ be the form of (2.1) with $\tau_1 \geq 0$. Then

$$2\tau_2 = \tau_1^2 + \beta\left(4 - \tau_1^2\right), \tag{2.2}$$

$$4\tau_3 = \tau_1^3 + 2\left(4 - \tau_1^2\right)\tau_1\beta - \tau_1\left(4 - \tau_1^2\right)\beta^2 + 2\left(4 - \tau_1^2\right)\left(1 - |\beta|^2\right)\eta, \tag{2.3}$$

$$\begin{aligned} 8\tau_4 = {} & \tau_1^4 + \left(4 - \tau_1^2\right)\beta\left[\tau_1^2\left(\beta^2 - 3\beta + 3\right) + 4\beta\right] - 4\left(4 - \tau_1^2\right)\left(1 - |\beta|^2\right) \\ & \cdot \left[\tau_1(\beta - 1)\eta + \overline{\beta}\eta^2 - \left(1 - |\eta|^2\right)\kappa\right] \end{aligned} \tag{2.4}$$

for some $\beta, \eta, \kappa \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

Lemma 2.2 [47] If $p \in \mathcal{P}$ is of the form (2.1) and $\vartheta \in \mathbb{C}$, we have

$$|\tau_n - \vartheta\tau_k\tau_{n-k}| \leq 2\max\{1, |2\vartheta - 1|\} \tag{2.5}$$

for all $1 \leq k \leq n - 1$.

Lemma 2.3 [48] Let μ, λ, ζ, and ς satisfy $\lambda \in (0, 1)$, $\zeta \in (0, 1)$ and

$$8\lambda(1 - \lambda)\left[(\zeta\varsigma - 2\mu)^2 + (\zeta(\lambda + \zeta) - \varsigma)^2\right] + \zeta(1 - \zeta)(\varsigma - 2\lambda\zeta)^2 \leq 4\lambda\zeta^2(1 - \zeta)^2(1 - \lambda). \tag{2.6}$$

If $p \in \mathcal{P}$ is of the form (2.1), then

$$\left|\mu\tau_1^4 + \lambda\tau_2^2 + 2\zeta\tau_1\tau_3 - \frac{3}{2}\varsigma\tau_1^2\tau_2 - \tau_4\right| \leq 2.$$

Lemma 2.4 [49] Suppose that $p \in \mathcal{P}$ is provided by (2.1). If $R \in [0, 1]$ and $R(2R - 1) \leq S \leq R$, then we have

$$\left|\tau_3 - 2R\tau_1\tau_2 + S\tau_1^3\right| \leq 2. \tag{2.7}$$

3. Coefficient results for the class $\mathcal{BT}_{\cosh}$

Theorem 3.1 If $g \in \mathcal{BT}_{\cosh}$ is of the form (1.1), then

$$|d_5 - d_2 d_4| \leq \frac{1}{10}.$$

This inequality is sharp.

Proof. Let $g \in \mathcal{BT}_{\cosh}$. Then g can be easily expressed by using the Schwarz function as

$$g'(z) = \cosh\sqrt{w(z)},\quad z \in \mathbb{D}.$$

If $p \in \mathcal{P}$, then we can write it in the form of

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + \tau_1 z + \tau_2 z^2 + \tau_3 z^3 + \cdots .$$

It follows that

$$w(z) = \frac{1}{2}\tau_1 z + \frac{1}{4}\left(2\tau_2 - \tau_1^2\right)z^2 + \frac{1}{8}\left(\tau_1^3 - 4\tau_1\tau_2 + 4\tau_3\right)z^3 + \frac{1}{16}\left(-\tau_1^4 + 6\tau_1^2\tau_2 - 8\tau_1\tau_3 - 4\tau_2^2 + 8\tau_4\right)z^4 + \cdots . \tag{3.1}$$

From (1.1), we have

$$g'(z) = 1 + 2d_2 z + 3d_3 z^2 + 4d_4 z^3 + 5d_5 z^4 + \cdots . \tag{3.2}$$

Using the series expansion of (3.1) with simple calculation, we get

$$\cosh\sqrt{w(z)} = 1 + \frac{1}{4}\tau_1 z + \left(\frac{1}{4}\tau_2 - \frac{11}{96}\tau_1^2\right)z^2 + \left(\frac{301}{5760}\tau_1^3 - \frac{11}{48}\tau_1\tau_2 + \frac{1}{4}\tau_3\right)z^3 + \left(\frac{1}{4}\tau_4 - \frac{11}{96}\tau_2^2 + \frac{301}{1920}\tau_1^2\tau_2 - \frac{15287}{645120}\tau_1^4 - \frac{11}{48}\tau_1\tau_3\right)z^4 + \cdots . \tag{3.3}$$

Comparing the coefficients in (3.2) and (3.3), we obtain

$$d_2 = \frac{1}{8}\tau_1, \tag{3.4}$$

$$d_3 = \frac{1}{12}\tau_2 - \frac{11}{288}\tau_1^2, \tag{3.5}$$

$$d_4 = \frac{301}{23040}\tau_1^3 + \frac{1}{16}\tau_3 - \frac{11}{192}\tau_1\tau_2, \tag{3.6}$$

$$d_5 = -\frac{11}{480}\tau_2^2 - \frac{15287}{3225600}\tau_1^4 + \frac{1}{20}\tau_4 + \frac{301}{9600}\tau_1^2\tau_2 - \frac{11}{240}\tau_1\tau_3. \tag{3.7}$$

Employing (3.4), (3.6) and (3.7), we may write

$$|d_5 - d_2 d_4| = \frac{1}{20}\left|\frac{13703}{107520}\tau_1^4 + \frac{11}{24}\tau_2^2 + 2\left(\frac{103}{192}\right)\tau_1\tau_3 - \frac{3}{2}\left(\frac{493}{960}\right)\tau_1^2\tau_2 - \tau_4\right|. = \frac{1}{20}\left|\mu\tau_1^4 + \lambda\tau_2^2 + 2\zeta\tau_1\tau_3 - \frac{3}{2}\varsigma\tau_1^2\tau_2 - \tau_4\right|, \tag{3.8}$$

where

$$\mu = \frac{13703}{107520},\qquad \lambda = \frac{11}{24},\qquad \zeta = \frac{103}{192},\qquad \varsigma = \frac{493}{960}.$$

These constants satisfy $\lambda \in (0,1)$, $\zeta \in (0,1)$, and

$$8\lambda(1-\lambda)\left[(\zeta\varsigma - 2\mu)^2 + (\zeta(\lambda+\zeta) - \varsigma)^2\right] + \zeta(1-\zeta)(\varsigma - 2\lambda\zeta)^2 \le 4\lambda\zeta^2(1-\zeta)^2(1-\lambda).$$

Hence, by Lemma 2.3 and (3.8), we deduce that

$$|d_5 - d_2 d_4| \leq \frac{1}{10}.$$

This required result is sharp and determined by

$$g_w(z) = \int_0^z \left(\cosh \sqrt{t^4}\right) dt = z + \frac{1}{10} z^5 + \frac{1}{216} z^9 + \cdots, \tag{3.9}$$

where we choose the branch of the square root function so that

$$\cosh \sqrt{z^4} = 1 + \frac{1}{2!} z^4 + \frac{1}{4!} z^8 + \frac{1}{6!} z^{12} + \cdots \tag{3.10}$$

□

Theorem 3.2 If $g \in \mathcal{BT}_{\cosh}$ is of the form of (1.1), then

$$\left|d_5 - d_3^2\right| \leq \frac{1}{10}.$$

This inequality is sharp.

Proof. From (3.5) and (3.7), we obtain

$$\begin{aligned} \left|d_5 - d_3^2\right| &= \frac{1}{20} \left| \frac{179933}{1451520} \tau_1^4 + \frac{43}{72} \tau_2^2 + 2\left(\frac{11}{24}\right) \tau_1 \tau_3 - \frac{3}{2}\left(\frac{3259}{6480}\right) \tau_1^2 \tau_2 - \tau_4 \right| \\ &= \frac{1}{20} \left| \mu \tau_1^4 + \lambda \tau_2^2 + 2\zeta \tau_1 \tau_3 - \frac{3}{2} \varsigma \tau_1^2 \tau_2 - \tau_4 \right|, \end{aligned} \tag{3.11}$$

where

$$\mu = \frac{179933}{1451520}, \quad \lambda = \frac{43}{72}, \quad \zeta = \frac{11}{24}, \quad \varsigma = \frac{3259}{6480}.$$

These constants satisfy $\lambda \in (0, 1)$, $\zeta \in (0, 1)$, and

$$8\lambda(1-\lambda)\left[(\zeta\varsigma - 2\mu)^2 + (\zeta(\lambda+\zeta) - \varsigma)^2\right] + \zeta(1-\zeta)(\varsigma - 2\lambda\zeta)^2 \leq 4\lambda\zeta^2(1-\zeta)^2(1-\lambda).$$

Hence, by Lemma 2.3 and (3.11), we deduce that

$$\left|d_5 - d_3^2\right| \leq \frac{1}{10}.$$

This required outcome is sharp for the function g_w given in (3.9). □

Theorem 3.3 If $g \in \mathcal{BT}_{\cosh}$ is of the form of (1.1), then

$$\left|d_4 - d_2^3\right| \leq \frac{1}{8}.$$

This inequality is sharp.

Proof. Using (3.4) and (3.6), we have

$$\left|d_4 - d_2^3\right| = \frac{1}{16}\left|\tau_3 - 2\left(\frac{11}{24}\right)\tau_1\tau_2 + \left(\frac{8}{45}\right)\tau_1^3\right|.$$

Let $R = \frac{11}{24}$ and $S = \frac{8}{45}$. It is clear that

$$R(2R-1) = -\frac{11}{288} \leq S \leq R.$$

All the conditions of Lemma 2.4 are satisfied, and thus we have

$$\left|d_4 - d_2^3\right| \leq \frac{1}{8}.$$

This result is the best possible and equality is attained by

$$g_v(z) = \int_0^z \left(\cosh \sqrt{t^3}\right) dt = z + \frac{1}{8}z^4 + \frac{1}{168}z^7 + \cdots, \tag{3.12}$$

where we choose the branch of the square root function so that

$$\cosh \sqrt{z^3} = 1 + \frac{1}{2!}z^3 + \frac{1}{4!}z^6 + \frac{1}{6!}z^9 + \cdots \tag{3.13}$$

□

Theorem 3.4 If $g \in \mathcal{BT}_{\cosh}$ is of the form of (1.1), then

$$\left|d_5 - d_2^4\right| \leq \frac{1}{10}.$$

This inequality is sharp.

Proof. From (3.4) and (3.7), we obtain

$$\begin{aligned}\left|d_5 - d_2^4\right| &= \frac{1}{20}\left|\frac{32149}{322560}\tau_1^4 + \frac{11}{24}\tau_2^2 + 2\left(\frac{11}{24}\right)\tau_1\tau_3 - \frac{3}{2}\left(\frac{301}{720}\right)\tau_1^2\tau_2 - \tau_4\right| \\ &= \frac{1}{20}\left|\mu\tau_1^4 + \lambda\tau_2^2 + 2\zeta\tau_1\tau_3 - \frac{3}{2}\varsigma\tau_1^2\tau_2 - \tau_4\right|,\end{aligned} \tag{3.14}$$

where

$$\mu = \frac{32149}{322560}, \quad \lambda = \frac{11}{24}, \quad \zeta = \frac{11}{24}, \quad \varsigma = \frac{301}{720}.$$

These constants satisfy $\lambda \in (0,1)$, $\zeta \in (0,1)$, and

$$8\lambda(1-\lambda)\left[(\zeta\varsigma - 2\mu)^2 + (\zeta(\lambda+\zeta) - \varsigma)^2\right] + \zeta(1-\zeta)(\varsigma - 2\lambda\zeta)^2 \leq 4\lambda\zeta^2(1-\zeta)^2(1-\lambda).$$

Hence, by Lemma 2.3 and (3.14), we deduce that

$$\left|d_5 - d_2^4\right| \leq \frac{1}{10}.$$

This required inequality is sharp for the function g_w given in (3.9). □

Theorem 3.5 If $g \in \mathcal{BT}_{\cosh}$ has the form of (1.1), then

$$\left|D_{3,1}(g)\right| \leq \frac{1}{64}.$$

This inequality is sharp.

Proof. From the definition, we know

$$D_{3,1}(g) = 2d_2d_3d_4 - d_3^3 - d_4^2 + d_3d_5 - d_2^2d_5. \tag{3.15}$$

Let $g \in \mathcal{BT}_{\cosh}$ and $g_\theta(z) = e^{-i\theta}g\left(e^{i\theta}z\right)$ with $\theta \in \mathbb{R}$. It is noted that $g_\theta'(z) = g'\left(e^{i\theta}z\right)$ and thus $g_\theta \in \mathcal{BT}_{\cosh}$ for all $\theta \in \mathbb{R}$. Since $\left|D_{3,1}(g_\theta)\right| = \left|D_{3,1}(g)\right|$, we may choose the coefficient d_2 of g to be a non-negative real number when estimating the functional $\left|D_{3,1}(g)\right|$. Then, since $d_2 = \frac{1}{8}\tau_1$ and τ_1 is a coefficient of a function in $\mathcal{P}$, it follows that τ_1 is real and $\tau_1 = \tau \in [0, 2]$. Putting the estimations of d_i's from (3.4), (3.5), (3.6), and (3.7) into $D_{3,1}(g)$ with $\tau_1 = \tau$, we have

$$\begin{aligned} D_{3,1}(g) = \frac{1}{33443020800}&\left(513823\tau^6 - 4378896\tau^4\tau_2 + 7922880\tau^3\tau_3 + 5552064\tau^2\tau_2^2\right. \\ &\left.-89994240\tau^2\tau_4 + 155312640\tau\tau_2\tau_3 - 83220480\tau_2^3 + 139345920\tau_2\tau_4 - 130636800\tau_3^2\right). \end{aligned}$$

Let $r = 4 - \tau^2$. Then, by (2.2), (2.3), and (2.4), we obtain

$$\begin{aligned} D_{3,1}(g) = \frac{1}{33443020800}&\left\{-329\tau^6 - 10402560\beta^3r^3 + 34836480\beta^3r^2 + 10160640\tau^3r\beta\left(1-|\beta|^2\right)\eta\right. \\ &+ 10160640\tau^2r\overline{\beta}\left(1-|\beta|^2\right)\eta^2 - 10160640\tau^2r\left(1-|\beta|^2\right)\left(1-|\eta|^2\right)\kappa \\ &+ 8346240\tau r^2\beta\left(1-|\beta|^2\right)\eta - 2177280\tau r^2\beta^2\left(1-|\beta|^2\right)\eta - 34836480r^2\overline{\beta}\beta\eta^2\left(1-|\beta|^2\right) \\ &+ 34836480r^2\beta\left(1-|\beta|^2\right)\left(1-|\eta|^2\right)\kappa + 2476656\tau^2r^2\beta^2 - 10160640\tau^2r\beta^2 \\ &- 2540160\tau^4r\beta^3 - 12882240\tau^2r^2\beta^3 + 544320\tau^2r^2\beta^4 - 32659200r^2\eta^2\left(1-|\beta|^2\right)^2 \\ &\left.+2555280\tau^4r\beta^2 + 12024\tau^4r\beta - 30240\tau^3r\left(1-|\beta|^2\right)\eta\right\}. \end{aligned}$$

It is seen that we can write $D_{3,1}(g)$ in the form of

$$D_{3,1}(g) = \frac{1}{33443020800}\left[l_1(\tau,\beta) + l_2(\tau,\beta)\eta + l_3(\tau,\beta)\eta^2 + l_4(\tau,\beta,\eta)\kappa\right],$$

where $\beta, \eta, \kappa \in \overline{\mathbb{D}}$, and

$$\begin{aligned} l_1(\tau,\beta) &= -329\tau^6 + \left(4-\tau^2\right)\left[\left(4-\tau^2\right)\left(-6773760\beta^3 - 2479680\tau^2\beta^3 + 2476656\tau^2\beta^2\right.\right. \\ &\qquad \left.\left.+544320\tau^2\beta^4\right) + 2555280\tau^4\beta^2 + 12024\tau^4\beta - 10160640\tau^2\beta^2 - 2540160\tau^4\beta^3\right], \\ l_2(\tau,\beta) &= 30240\left(4-\tau^2\right)\left(1-|\beta|^2\right)\left[\left(4-\tau^2\right)\left(-72\tau\beta^2 + 276\tau\beta\right) + 336\tau^3\beta - \tau^3\right], \\ l_3(\tau,\beta) &= 725760\left(4-\tau^2\right)\left(1-|\beta|^2\right)\left[\left(4-\tau^2\right)\left(-3|\beta|^2 - 45\right) + 14\tau^2\overline{\beta}\right], \\ l_4(\tau,\beta,\eta) &= 725760\left(4-\tau^2\right)\left(1-|\beta|^2\right)\left(1-|\eta|^2\right)\left[48\beta\left(4-\tau^2\right) - 14\tau^2\right]. \end{aligned}$$

By using $|\beta| = \varkappa, |\eta| = y$ and utilizing the fact $|\kappa| \le 1$, we obtain

$$\left|D_{3,1}(g)\right| \le \frac{1}{33443020800}\left[|l_1(\tau,\beta)| + |l_2(\tau,\beta)|\,y + |l_3(\tau,\beta)|\,y^2 + |l_4(\tau,\beta,\eta)|\right].$$
$$\le \frac{1}{33443020800} M(\tau,\varkappa,y), \tag{3.16}$$

where

$$M(\tau,\varkappa,y) = m_1(\tau,\varkappa) + m_2(\tau,\varkappa)\,y + m_3(\tau,\varkappa)\,y^2 + m_4(\tau,\varkappa)\left(1-y^2\right),$$

with

$$\begin{aligned} m_1(\tau,\varkappa) &= 329\tau^6 + \left(4-\tau^2\right)\Big[\left(4-\tau^2\right)\left(6773760\varkappa^3 + 2479680\tau^2\varkappa^3 + 2476656\tau^2\varkappa^2\right.\\ &\quad \left.+544320\tau^2\varkappa^4\right) + 2555280\tau^4\varkappa^2 + 12024\tau^4\varkappa + 10160640\tau^2\varkappa^2 + 2540160\tau^4\varkappa^3\Big],\\ m_2(\tau,\varkappa) &= 30240\left(4-\tau^2\right)\left(1-\varkappa^2\right)\left[\left(4-\tau^2\right)\left(72\tau\varkappa^2 + 276\tau\varkappa\right) + 336\tau^3\varkappa + \tau^3\right],\\ m_3(\tau,\varkappa) &= 725760\left(4-\tau^2\right)\left(1-\varkappa^2\right)\left[\left(4-\tau^2\right)\left(3\varkappa^2+45\right) + 14\tau^2\varkappa\right],\\ m_4(\tau,\varkappa) &= 725760\left(4-\tau^2\right)\left(1-\varkappa^2\right)\left[48\varkappa\left(4-\tau^2\right) + 14\tau^2\right]. \end{aligned}$$

Now, we have to maximize M in the closed cuboid $\Upsilon := [0,2]\times[0,1]\times[0,1]$.
In light of $(\tau,\varkappa)\in[0,2]\times[0,1]$, we observe that

$$m_3(\tau,\varkappa) \le 725760\left(4-\tau^2\right)\left(1-\varkappa^2\right)\left[\left(4-\tau^2\right)\left(3\varkappa^2+45\right) + 14\tau^2\right] =: g_3(\tau,\varkappa). \tag{3.17}$$

Taking $g_i(\tau,\varkappa) = m_i(\tau,\varkappa)$ for $i = 1,2,4$ and

$$G(\tau,\varkappa,y) = g_1(\tau,\varkappa) + g_2(\tau,\varkappa)\,y + g_3(\tau,\varkappa)\,y^2 + g_4(\tau,\varkappa)\left(1-y^2\right), \tag{3.18}$$

it is not hard to see that $M(\tau,\varkappa,y) \le G(\tau,\varkappa,y)$ in the cuboid Υ. In the following, we aim to find the maximum value of G in Υ.

By partially differentiating G with respect to y, we have

$$\frac{\partial G}{\partial y} = g_2(\tau,\varkappa) + 2\left[g_3(\tau,\varkappa) - g_4(\tau,\varkappa)\right] y. \tag{3.19}$$

In view of $g_2(\tau,\varkappa) \ge 0$ and

$$g_3(\tau,\varkappa) - g_4(\tau,\varkappa) = 725760\left(4-\tau^2\right)\left(1-\varkappa^2\right)\left[\left(3\varkappa^2 - 48\varkappa + 45\right)\left(4-\tau^2\right)\right] \ge 0 \tag{3.20}$$

on $[0,2]\times[0,1]$, we have $\frac{\partial G}{\partial y} \ge 0$ for all $y\in[0,1]$. It follows that

$$G(\tau,\varkappa,y) \le G(\tau,\varkappa,1), \tag{3.21}$$

where

$$\begin{aligned} G(\tau,\varkappa,1) &= g_1(\tau,\varkappa) + g_2(\tau,\varkappa) + g_3(\tau,\varkappa)\\ &= 329\tau^6 + 72(4-\tau^2)\left[q_4(\tau)\varkappa^4 + q_3(\tau)\varkappa^3 + q_2(\tau)\varkappa^2 + q_1(\tau)\varkappa + q_0(\tau)\right] \end{aligned}$$

$$=: Q(\tau,\varkappa),$$

where

$$\begin{aligned}
q_4(\tau) &= 7560(4-\tau^2)(\tau^2-4\tau-4),\\
q_3(\tau) &= 840(\tau^4-30\tau^3+52\tau^2-552\tau+448),\\
q_2(\tau) &= 84(13\tau^4-365\tau^3+6678\tau^2+1440\tau-20160),\\
q_1(\tau) &= \tau(167\tau^3+25200\tau^2+463680),\\
q_0(\tau) &= 420(\tau^3-744\tau^2+4320).
\end{aligned}$$

Then the problem reduces to finding the maximum value of Q on $[0,2]\times[0,1]$. By noting that $q_4(\tau)\le 0$ for all $\tau\in[0,2]$, we obtain that

$$Q(\tau,\varkappa)\le 329\tau^6+72(4-\tau^2)\left[q_3(\tau)\varkappa^3+q_2(\tau)\varkappa^2+q_1(\tau)\varkappa+q_0(\tau)\right]=: W(\tau,\varkappa). \tag{3.22}$$

Setting $\tau=0$, we obtain

$$\begin{aligned}
W(0,\varkappa) &= 108380160\varkappa^3-487710720\varkappa^2+522547200\\
&= 108380160\varkappa^2\left(\varkappa-\frac{9}{2}\right)+522547200\\
&\le 522547200\approx 5.2255\times 10^8
\end{aligned}$$

for all $\varkappa\in[0,1]$. Setting $\tau=2$, we get

$$W(2,\varkappa)\equiv 21056,\quad \varkappa\in[0,1].$$

It is left to consider the case of $\tau\in(0,2)$. For the system of equations

$$\frac{\partial W}{\partial\tau}=0 \qquad\text{and}\qquad \frac{\partial W}{\partial\varkappa}=0$$

with $(\tau,\varkappa)\in(0,2)\times(0,1)$, a numerical computation indicates that all the real approximate solutions are listed as $(1.6125,-1.0547)$, $(158.7578,-0.6573)$, $(2.0982,-0.4927)$, $(2.1274,0.3361)$, $(1.0709,0.9834)$, $(0,0)$, and $(2,2.1500)$. Thus, the only critical point of W that lies in $(0,2)\times(0,1)$ is about $(1.0709,0.9834)$. For this point, we have $W(1.0709,0.9834)\approx 1.9621\times 10^8$.

Thus, from above cases, we conclude that

$$M(\tau,\varkappa,y)\le G(\tau,\varkappa,y)\le G(\tau,\varkappa,1)\le Q(\tau,\varkappa)\le W(\tau,\varkappa)\le 522547200$$

on $[0,2]\times[0,1]\times[0,1]$. From (3.16) we get that

$$\left|D_{3,1}(g)\right|\le\frac{1}{33443020800}\left[M(\tau,\varkappa,y)\right]\le\frac{522547200}{33443020800}=\frac{1}{64}.$$

If $g\in\mathcal{BT}_{\cosh}$, then the sharp bound for this Hankel determinant is determined by

$$\left|D_{3,1}(g)\right|=\frac{1}{64}\approx 0.01562,$$

with an extremal function g_v given in (3.12) . □

4. Logarithmic coefficient problems

The logarithmic coefficients ξ_k of $g \in \mathcal{S}$ are given by

$$G_g(z) := \log\left(\frac{g(z)}{z}\right) = 2\sum_{k=1}^{\infty} \xi_k z^k, \quad z \in \mathbb{D}.$$

These coefficients contribute significantly in many estimations to the theory of univalent functions. In 1985, de Branges [6] completed the proof of the Milin conjecture [50], which asserted that for all positive integers $k \geq 1$,

$$\sum_{l=1}^{k} l(k-l+1)|\xi_k|^2 \leq \sum_{l=1}^{k} \frac{k-l+1}{l},$$

and equality holds if and only if g takes the form $z/\left(1-e^{i\varphi}z\right)^2$ for some $\varphi \in \mathbb{R}$. This inequality leads to the famous Bieberbach–Robertson–Milin conjectures. In 2005, Kayumov [51] was able to solve the Brennans conjecture for conformal mappings by considering the logarithmic coefficients. For some recent works on the study of logarithmic coefficients, see, for example, [52–56].

If $g \in \mathcal{S}$ is in the form of (1.1), then its logarithmic coefficients are given by

$$\xi_1 = \frac{1}{2}d_2, \tag{4.1}$$

$$\xi_2 = \frac{1}{2}\left(d_3 - \frac{1}{2}d_2^2\right), \tag{4.2}$$

$$\xi_3 = \frac{1}{2}\left(d_4 - d_2d_3 + \frac{1}{3}d_2^3\right), \tag{4.3}$$

$$\xi_4 = \frac{1}{2}\left(d_5 - d_2d_4 + d_2^2d_3 - \frac{1}{2}d_3^2 - \frac{1}{4}d_2^4\right). \tag{4.4}$$

Plugging (3.4), (3.5), (3.6), and (3.7) into (4.1), (4.2), (4.3), and (4.4), we get

$$\xi_1 = \frac{1}{16}\tau_1, \tag{4.5}$$

$$\xi_2 = \frac{1}{24}\tau_2 - \frac{53}{2304}\tau_1^2, \tag{4.6}$$

$$\xi_3 = \frac{71}{7680}\tau_1^3 + \frac{1}{32}\tau_3 - \frac{13}{384}\tau_1\tau_2, \tag{4.7}$$

$$\xi_4 = -\frac{19}{1440}\tau_2^2 - \frac{1802099}{464486400}\tau_1^4 + \frac{1}{40}\tau_4 + \frac{14861}{691200}\tau_1^2\tau_2 - \frac{103}{3840}\tau_1\tau_3. \tag{4.8}$$

Define

$$D_{2,1}\left(G_g/2\right) := \xi_1\xi_3 - \xi_2^2, \tag{4.9}$$

$$D_{2,2}\left(G_g/2\right) := \xi_2\xi_4 - \xi_3^2. \tag{4.10}$$

It is observed that $D_{2,1}\left(G_g/2\right)$ resembles the well-known functional $D_{2,1}(g) = d_1d_3 - d_2^2$ over the class $\mathcal{S}$ or its subclasses.

Theorem 4.1 If $g \in \mathcal{BT}_{\cosh}$ is of the form (1.1) , then

$$\left|\xi_2 - \vartheta\xi_1^2\right| \leq \max\left\{\frac{1}{12}, \frac{|9\vartheta + 5|}{576}\right\}, \quad \vartheta \in \mathbb{C}.$$

This inequality is sharp.

Proof. By employing (4.5) and (4.6), we may write

$$\left|\xi_2 - \vartheta\xi_1^2\right| = \frac{1}{24}\left|\tau_2 - \frac{9\vartheta + 53}{96}\tau_1^2\right|.$$

An application of Lemma 2.2 leads to

$$\left|\xi_2 - \vartheta\xi_1^2\right| \leq \max\left\{\frac{1}{12}, \frac{|9\vartheta + 5|}{576}\right\}.$$

The bound $\frac{1}{12}$ is achieved by the function g_y given as

$$g_y(z) = \int_0^z \left(\cosh \sqrt{t^2}\right) dt = z + \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots . \tag{4.11}$$

The bound $\frac{|9\vartheta+5|}{576}$ for $\vartheta \in \mathbb{C}$ is attained by the function g_g given as

$$g_g(z) = \int_0^z \left(\cosh \sqrt{t}\right) dt = z + \frac{1}{4}z^2 + \frac{1}{72}z^3 + \cdots . \tag{4.12}$$

Here, we choose the branch of the square root function so that

$$\cosh \sqrt{z^2} = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \frac{1}{6!}z^6 + \cdots \tag{4.13}$$

and

$$\cosh \sqrt{z} = 1 + \frac{1}{2!}z + \frac{1}{4!}z^2 + \frac{1}{6!}z^3 + \cdots . \tag{4.14}$$

□

Substituting $\vartheta = 1$, we deduce the corollary stated below.

Corollary 4.1 If the function $g \in \mathcal{BT}_{\cosh}$ has the form of (1.1), then

$$\left|\xi_2 - \xi_1^2\right| \leq \frac{1}{12}.$$

This bound is achieved by the function g_y given in (4.11).

Theorem 4.2 If $g \in \mathcal{BT}_{\cosh}$ has the form of (1.1), then

$$|\xi_1\xi_2 - \xi_3| \leq \frac{1}{16}.$$

This inequality is sharp.

Proof. Using (4.5), (4.6), and (4.7), we have

$$|\xi_1\xi_2 - \xi_3| = \frac{1}{32}\left|\tau_3 - 2\left(\frac{7}{12}\right)\tau_1\tau_2 + \frac{1969}{5760}\tau_1^3\right|.$$

Let $R = \frac{7}{12}$ and $S = \frac{1969}{5760}$. It is clear that

$$R(2R-1) = \frac{7}{72} \leq S \leq R.$$

By Lemma 2.4, we obtain

$$|\xi_1\xi_2 - \xi_3| \leq \frac{1}{16}.$$

The equality is attained by the function g_v given in (3.12). □

Theorem 4.3 Let $g \in \mathcal{BT}_{\cosh}$ have the series expansion (1.1). Then

$$|\xi_4 - \xi_1\xi_3| \leq \frac{1}{20}.$$

The inequality is sharp.

Proof. From (4.5), (4.7), and (4.8), we have

$$\begin{aligned} |\xi_4 - \xi_1\xi_3| &= \frac{1}{40}\left|\frac{2070479}{11612160}\tau_1^4 + \frac{19}{36}\tau_2^2 + 2\left(\frac{221}{384}\right)\tau_1\tau_3 - \frac{3}{2}\left(\frac{32647}{51840}\right)\tau_1^2\tau_2 - \tau_4\right| \\ &= \frac{1}{40}\left|\mu\tau_1^4 + \lambda\tau_2^2 + 2\zeta\tau_1\tau_3 - \frac{3}{2}\varsigma\tau_1^2\tau_2 - \tau_4\right|, \end{aligned} \tag{4.15}$$

where

$$\mu = \frac{2070479}{11612160}, \qquad \lambda = \frac{19}{36}, \qquad \zeta = \frac{221}{384}, \qquad \varsigma = \frac{32647}{51840}.$$

These constants satisfy $\lambda \in (0,1)$, $\zeta \in (0,1)$, and

$$8\lambda(1-\lambda)\left[(\zeta\varsigma - 2\mu)^2 + (\zeta(\lambda+\zeta) - \varsigma)^2\right] + \zeta(1-\zeta)(\varsigma - 2\lambda\zeta)^2 \leq 4\lambda\zeta^2(1-\zeta)^2(1-\lambda).$$

Hence, by Lemma 2.3 and (4.15), we deduce that

$$|\xi_4 - \xi_1\xi_3| \leq \frac{1}{20}.$$

This equality is achieved by the function g_w given in (3.9). □

Theorem 4.4 Let $g \in \mathcal{BT}_{\cosh}$ be in the form of (1.1). Then

$$\left|\xi_4 - \xi_2^2\right| \leq \frac{1}{20}.$$

This inequality is sharp.

Proof. From (4.6) and (4.8), we obtain

$$\begin{aligned}\left|\xi_4-\xi_2^2\right| &= \frac{1}{40}\left|\frac{4095773}{23224320}\tau_1^4+\frac{43}{72}\tau_2^2+2\left(\frac{103}{192}\right)\tau_1\tau_3-\frac{3}{2}\left(\frac{8093}{12960}\right)\tau_1^2\tau_2-\tau_4\right|.\\ &= \frac{1}{40}\left|\mu\tau_1^4+\lambda\tau_2^2+2\zeta\tau_1\tau_3-\frac{3}{2}\varsigma\tau_1^2\tau_2-\tau_4\right|, \end{aligned} \tag{4.16}$$

where

$$\mu=\frac{4095773}{23224320},\qquad \lambda=\frac{43}{72},\qquad \zeta=\frac{103}{192},\qquad \varsigma=\frac{8093}{12960}.$$

These constants satisfy $\lambda\in(0,1)$, $\zeta\in(0,1)$, and

$$8\lambda(1-\lambda)\left[(\zeta\varsigma-2\mu)^2+(\zeta(\lambda+\zeta)-\varsigma)^2\right]+\zeta(1-\zeta)(\varsigma-2\lambda\zeta)^2\leq 4\lambda\zeta^2(1-\zeta)^2(1-\lambda).$$

Hence, by Lemma 2.3 and (4.16), we deduce that

$$\left|\xi_4-\xi_2^2\right|\leq\frac{1}{20}.$$

The required inequality is sharp and determined by the function g_w given in (3.9). □

Theorem 4.5 If $g\in\mathcal{BT}_{\cosh}$ has the form of (1.1), then

$$\left|\xi_3-\xi_1^3\right|\leq\frac{1}{16}.$$

This inequality is sharp.

Proof. Using (4.5) and (4.7), we have

$$\left|\xi_3-\xi_1^3\right|=\frac{1}{32}\left|\tau_3-2\left(\frac{13}{24}\right)\tau_1\tau_2+\frac{553}{1920}\tau_1^3\right|.$$

Let $R=\frac{13}{24}$ and $S=\frac{553}{1920}$. It is clear that

$$R(2R-1)=\frac{13}{288}\leq S\leq R.$$

By Lemma 2.4, it follows that

$$\left|\xi_3-\xi_1^3\right|\leq\frac{1}{16}.$$

This result is the best possible and the extremal function is g_v as given in (3.12). □

Theorem 4.6 If $g\in\mathcal{BT}_{\cosh}$ is of the form (1.1), then

$$\left|\xi_4-\xi_1^4\right|\leq\frac{1}{20}.$$

This inequality is sharp.

Proof. From (4.5) and (4.8), we obtain

$$\begin{aligned}\left|\xi_4-\xi_1^4\right| &= \frac{1}{40}\left|\frac{3618373}{23224320}\tau_1^4+\frac{19}{36}\tau_2^2+2\left(\frac{103}{192}\right)\tau_1\tau_3-\frac{3}{2}\left(\frac{14861}{25920}\right)\tau_1^2\tau_2-\tau_4\right|. \\ &= \frac{1}{40}\left|\mu\tau_1^4+\lambda\tau_2^2+2\zeta\tau_1\tau_3-\frac{3}{2}\varsigma\tau_1^2\tau_2-\tau_4\right|, \end{aligned} \tag{4.17}$$

where

$$\mu=\frac{3618373}{23224320},\qquad \lambda=\frac{19}{36},\qquad \zeta=\frac{103}{192},\qquad \varsigma=\frac{14861}{25920}.$$

These constants satisfy $\lambda\in(0,1)$, $\zeta\in(0,1)$, and

$$8\lambda(1-\lambda)\left[(\zeta\varsigma-2\mu)^2+(\zeta(\lambda+\zeta)-\varsigma)^2\right]+\zeta(1-\zeta)(\varsigma-2\lambda\zeta)^2\le 4\lambda\zeta^2(1-\zeta)^2(1-\lambda).$$

Hence, by Lemma 2.3 and (4.17), we deduce that

$$\left|\xi_4-\xi_1^4\right|\le\frac{1}{20}.$$

This required inequality is sharp and determined by using (4.1), (4.4) and (3.9). □

Theorem 4.7 If $g\in\mathcal{BT}_{\cosh}$ has the form of (1.1), then

$$\left|D_{2,2}\left(G_g/2\right)\right|\le\frac{1}{256}.$$

This inequality is sharp.

Proof. Suppose that $g\in\mathcal{BT}_{\cosh}$ and $g_\theta(z)=e^{-i\theta}g\left(e^{i\theta}z\right)$ with $\theta\in\mathbb{R}$. Since $\left|D_{2,2}\left(G_{g_\theta}/2\right)\right|=\left|D_{2,2}\left(G_g/2\right)\right|$ for all $\theta\in\mathbb{R}$, we still assume that $\tau_1=\tau\in[0,2]$. Putting (4.6), (4.7), and (4.8) into (4.10) with $\tau_1=\tau$, we obtain

$$\begin{aligned}D_{2,2}\left(G_g/2\right)=\frac{1}{1070176665600}&\left(4047343\tau^6-32414400\tau^4\tau_2+41973120\tau^3\tau_3+56996352\tau^2\tau_2^2\right.\\&-615444480\tau^2\tau_4+1068318720\tau\tau_2\tau_3-588349440\tau_2^3+1114767360\tau_2\tau_4\\&\left.-1045094400\tau_3^2\right).\end{aligned}$$

Let $r=4-\tau^2$. Then, by (2.2), (2.3), and (2.4), we obtain

$$\begin{aligned}D_{2,2}\left(G_g/2\right)=\frac{1}{1070176665600}&\left\{-2237760\tau^3r\left(1-|\beta|^2\right)\eta-73543680\beta^3r^3+278691840\beta^3r^2\right.\\&+29030400\tau^3r\beta\left(1-|\beta|^2\right)\eta+29030400\tau^2r\overline{\beta}\eta^2\left(1-|\beta|^2\right)\\&-29030400\tau^2r\left(1-|\beta|^2\right)\left(1-|\eta|^2\right)\kappa+23224320\tau\beta r^2\eta\left(1-|\beta|^2\right)\\&-17418240\tau\beta^2r^2\left(1-|\beta|^2\right)\eta-278691840\beta r^2\left(1-|\beta|^2\right)\overline{\beta}\eta^2\\&+278691840\beta r^2\left(1-|\beta|^2\right)\left(1-|\eta|^2\right)\kappa-261273600r^2\left(1-|\beta|^2\right)^2\kappa^2\\&+8376480\tau^4r\beta^2-107424\tau^4\beta r+8443008\tau^2\beta^2r^2-29030400r\tau^2\beta^2\end{aligned}$$

$$-7257600\tau^4 r\beta^3 - 81285120\beta^3 r^2\tau^2 + 4354560\beta^4 r^2\tau^2 + 2671\tau^6\Big\}.$$

It is observed that we can write $D_{2,2}\left(G_g/2\right)$ in the form of

$$D_{2,2}\left(G_g/2\right) = \frac{1}{1070176665600}\left[k_1(\tau,\beta) + k_2(\tau,\beta)\eta + k_3(\tau,\beta)\eta^2 + k_4(\tau,\beta,\eta)\kappa\right],$$

where $\beta,\eta,\kappa \in \overline{\mathbb{D}}$, and

$$\begin{aligned}
k_1(\tau,\beta) &= 2671\tau^6 + \left(4-\tau^2\right)\Big[\left(4-\tau^2\right)\Big(8443008\tau^2\beta^2 - 15482880\beta^3 - 7741440\tau^2\beta^3 \\
&\quad +4354560\tau^2\beta^4\Big) - 29030400\tau^2\beta^2 + 8376480\tau^4\beta^2 - 107424\tau^4\beta - 7257600\tau^4\beta^3\Big],\\
k_2(\tau,\beta) &= 60480\left(4-\tau^2\right)\left(1-|\beta|^2\right)\left[\left(4-\tau^2\right)\left(-288\tau\beta^2 + 384\tau\beta\right) + 480\tau^3\beta - 37\tau^3\right],\\
k_3(\tau,\beta) &= 5806080\left(4-\tau^2\right)\left(1-|\beta|^2\right)\left[\left(4-\tau^2\right)\left(-3|\beta|^2 - 45\right) + 5\tau^2\overline{\beta}\right],\\
k_4(\tau,\beta,\eta) &= 5806080\left(4-\tau^2\right)\left(1-|\beta|^2\right)\left(1-|\eta|^2\right)\left[-5\tau^2 + 48\beta\left(4-\tau^2\right)\right].
\end{aligned}$$

Now, by using $|\beta| = \varkappa$, $|\eta| = y$ and utilizing the fact $|\kappa| \le 1$, we get

$$\begin{aligned}
\left|D_{2,2}\left(G_g/2\right)\right| &\le \frac{1}{1070176665600}\left[|k_1(\tau,\beta)| + |k_2(\tau,\beta)|y + |k_3(\tau,\beta)|y^2 + |k_4(\tau,\beta,\eta)|\right].\\
&\le \frac{1}{1070176665600}\Lambda(\tau,\varkappa,y),
\end{aligned} \tag{4.18}$$

where

$$\Lambda(\tau,\varkappa,y) = t_1(\tau,\varkappa) + t_2(\tau,\varkappa)y + t_3(\tau,\varkappa)y^2 + t_4(\tau,\varkappa)\left(1-y^2\right),$$

with

$$\begin{aligned}
t_1(\tau,\varkappa) &= 2671\tau^6 + \left(4-\tau^2\right)\Big[\left(4-\tau^2\right)\Big(8443008\tau^2\varkappa^2 + 15482880\varkappa^3 + 7741440\tau^2\varkappa^3 \\
&\quad +4354560\tau^2\varkappa^4\Big) + 29030400\tau^2\varkappa^2 + 8376480\tau^4\varkappa^2 + 107424\tau^4\varkappa + 7257600\tau^4\varkappa^3\Big],\\
t_2(\tau,\varkappa) &= 60480\left(4-\tau^2\right)\left(1-\varkappa^2\right)\left[\left(4-\tau^2\right)\left(288\tau\varkappa^2 + 384\tau\varkappa\right) + 480\tau^3\varkappa + 37\tau^3\right],\\
t_3(\tau,\varkappa) &= 5806080\left(4-\tau^2\right)\left(1-\varkappa^2\right)\left[\left(4-\tau^2\right)\left(3\varkappa^2 + 45\right) + 5\tau^2\varkappa\right],\\
t_4(\tau,\varkappa) &= 5806080\left(4-\tau^2\right)\left(1-\varkappa^2\right)\left[5\tau^2 + 48\varkappa\left(4-\tau^2\right)\right].
\end{aligned}$$

Now, we have to maximize Λ in the closed cuboid Υ.

In view of

$$t_3(\tau,\varkappa) \le 5806080\left(4-\tau^2\right)\left(1-\varkappa^2\right)\left[\left(4-\tau^2\right)\left(3\varkappa^2 + 45\right) + 5\tau^2\right] =: u_3(\tau,\varkappa) \tag{4.19}$$

for all $(\tau,\varkappa) \in [0,2]\times[0,1]$, by setting $u_i(\tau,\varkappa) = t_i(\tau,\varkappa)\,(i = 1,2,4)$ and

$$\Theta(\tau,\varkappa,y) = u_1(\tau,\varkappa) + u_2(\tau,\varkappa)y + u_3(\tau,\varkappa)y^2 + u_4(\tau,\varkappa)\left(1-y^2\right),$$

it is not hard to see that $\Lambda(\tau,\varkappa,y) \le \Theta(\tau,\varkappa,y)$ on Υ. In the following, we aim to discuss the maximum value of Θ on Υ.

By partially differentiating Θ with respect to y, we get

$$\frac{\partial \Theta}{\partial y} = u_2(\tau,\varkappa) + 2\left[u_3(\tau,\varkappa) - u_4(\tau,\varkappa)\right] y.$$

Because $u_2(\tau,\varkappa) \geq 0$ and

$$u_3(\tau,\varkappa) - u_4(\tau,\varkappa) = 5806080\left(4-\tau^2\right)\left(1-\varkappa^2\right)\left[\left(3\varkappa^2 - 48\varkappa + 45\right)\left(4-\tau^2\right)\right] \geq 0$$

on $[0,2]\times[0,1]$, we have $\frac{\partial \Theta}{\partial y} \geq 0$ for all $y \in [0,1]$. Hence, we obtain

$$\Theta(\tau,\varkappa,y) \leq \Theta(\tau,\varkappa,1), \tag{4.20}$$

where

$$\begin{aligned}\Theta(\tau,\varkappa,1) &= u_1(\tau,\varkappa) + u_2(\tau,\varkappa) + u_3(\tau,\varkappa)\\ &= 2671\tau^6 + 288(4-\tau^2)\left[v_4(\tau)\varkappa^4 + v_3(\tau)\varkappa^3 + v_2(\tau)\varkappa^2 + v_1(\tau)\varkappa + v_0(\tau)\right]\\ &=: V(\tau,\varkappa),\end{aligned}$$

where

$$v_4(\tau) = 15120(4-\tau^2)(\tau^2 - 4\tau - 4), \tag{4.21}$$

$$v_3(\tau) = -1680(\tau^4 + 12\tau^3 - 32\tau^2 + 192\tau - 128), \tag{4.22}$$

$$v_2(\tau) = -21(11\tau^4 + 3250\tau^3 - 45904\tau^2 - 11520\tau + 161280), \tag{4.23}$$

$$v_1(\tau) = \tau(373\tau^3 + 20160\tau^2 + 322560), \tag{4.24}$$

$$v_0(\tau) = 30(259\tau^3 - 26880\tau^2 + 120960). \tag{4.25}$$

Taking the fact of $v_4(\tau) \leq 0$ for all $\tau \in [0,2]$, we obtain

$$V(\tau,\varkappa) \leq 2671\tau^6 + 288(4-\tau^2)\left[v_3(\tau)\varkappa^3 + v_2(\tau)\varkappa^2 + v_1(\tau)\varkappa + v_0(\tau)\right] =: K(\tau,\varkappa). \tag{4.26}$$

Now, we need to find the maximum value of K on $[0,2]\times[0,1]$. For $\tau = 0$, we have

$$K(0,\varkappa) = 247726080\varkappa^2\left(\varkappa - \frac{63}{4}\right) + 4180377600 \leq 4180377600 \tag{4.27}$$

for all $\varkappa \in [0,1]$. If $\tau = 2$, it is calculated that $K(2,\varkappa) \equiv 170944$ with $\varkappa \in [0,1]$. Hence, it is left to discuss the case of $(\tau,\varkappa) \in (0,2)\times(0,1)$.

For the system of equations

$$\frac{\partial K}{\partial \tau} = 0 \qquad \text{and} \qquad \frac{\partial K}{\partial \varkappa} = 0$$

with $(\tau,\varkappa) \in (0,2)\times(0,1)$, a numerical computation indicates that the approximate solutions are $(0,0)$, $(1.7758,-1.2237)$, $(2.0405,-0.4933)$, $(52.7638,0.2085)$, $(-212.4757,0.2665)$, $(-2.0293,0.3246)$, $(-1.5447,1.0839)$, $(5.1393,2.5129)$, $(2,2.1720)$, and $(-2,-1.1906)$. It is found that there are no critical points of K that lie in $(0,2)\times(0,1)$.

From the above cases we conclude that

$$\Lambda(\tau, \varkappa, y) \leq \Theta(\tau, \varkappa, y) \leq \Theta(\tau, \varkappa, 1) = V(\tau, \varkappa) \leq K(\tau, \varkappa) \leq 4180377600$$

on $[0, 2] \times [0, 1] \times [0, 1]$. Using (4.18), we have

$$\left| D_{2,2}\left(G_g/2\right) \right| \leq \frac{1}{1070176665600} \Lambda(\tau, \varkappa, y) \leq \frac{4180377600}{1070176665600} = \frac{1}{256} \approx 0.003906.$$

If $g \in \mathcal{BT}_{\cosh}$, then the sharp bound for this second-order Hankel determinant is determined by using (4.2), (4.3), (4.4), and (3.12). □

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

Prof. Dr. Nak Eun Cho is the Guest Editor of special issue "Geometric Function Theory and Special Functions" for AIMS Mathematics. Prof. Dr. Nak Eun Cho was not involved in the editorial review and the decision to publish this article.

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References

1. L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Eintheitskreises vermiteln, *Sitzungsberichte Preussische Akademie der Wissenschaften*, **138** (1916), 940–955.
2. K. Löwner, Untersuchungen iiber schlichte konforme Abbildungen des Einheitskreises, *Math. Ann.*, **89** (1923), 103–121.
3. P. R. Garabedian, M. Schiffer, A proof of the Bieberbach conjecture for the fourth coefficient, *J. Ration. Mech. Anal.*, **4** (1955), 427–465.
4. R. N. Pederson, M. Schiffer, A proof of the Bieberbach conjecture for the fifth coefficient, *Arch. Ration. Mech. Anal.*, **45** (1972), 161–193.
5. R. N. Pederson, A proof of the Bieberbach conjecture for the sixth coefficient, *Arch. Ration. Mech. Anal.* **31** (1968), 331–351.
6. L. De Branges, A proof of the Bieberbach conjecture, *Acta Math.*, **154** (1985), 137–152.
7. J. E. Brown, A. Tsao, On the Zalcman conjecture for starlike and typically real functions, *Math. Z.*, **191** (1986), 467–474.

8. L. Li, S. Ponnusamy, J. Qiao, Generalized Zalcman conjecture for convex functions of order α, *Acta Math. Hung.*, **150** (2016), 234–246.
9. W. C. Ma, The Zalcman conjecture for close-to-convex functions, *Proc. Amer. Math. Soc.*, **104** (1988), 741–744.
10. S. L. Krushkal, Proof of the Zalcman conjecture for initial coefficients, *Georgian Math. J.*, **17** (2010), 663–681.
11. S. L. Krushkal, A short geometric proof of the Zalcman and Bieberbach conjectures, 2014. https://doi.org/10.48550/arXiv.1408.1948
12. W. Ma, Generalized Zalcman conjecture for starlike and typically real functions, *J. Math. Anal. Appl.*, **234** (1999), 328–339.
13. R. Mendiratta, S. Nagpal, V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function, *Bull. Malays. Math. Sci. Soc.*, **38** (2015), 365–386.
14. R. K. Raina, P. Sharma, J. Sokół, Certain classes of analytic functions related to the crescent-shaped regions, *J. Contemp. Math. Anal.*, **53** (2018), 355–362.
15. L. A. Wani, A. Swaminathan, Radius problems for functions associated with a nephroid domain, *RACSAM*, **114** (2020), 178. https://doi.org/10.1007/s13398-020-00913-4
16. P. Goel, S. S. Kumar, Certain class of starlike functions associated with modified sigmoid function, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 957–991.
17. K. Ullah, H. M. Srivastava, A. Rafiq, M. Darus, M. Shutaywi, Radius problems for starlike functions associated with the tan hyperbolic function, *J. Funct. Space.*, **2021** (2021), 9967640.
18. S. Gandhi, P. Gupta, S. Nagpal, V. Ravichandran, Starlike functions associated with an Epicycloid, *Hacet. J. Math. Stat.*, **51** (2022), 1637–1660.
19. C. Pommerenke, On the coefficients and Hankel determinants of univalent functions, *J. London Math. Soc.*, **1** (1966), 111–122.
20. C. Pommerenke, On the Hankel determinants of univalent functions, *Mathematika*, **14** (1967), 108–112.
21. P. Dienes, *The Taylor series: An introduction to the theory of functions of a complex variable*, New York: Dover, 1957.
22. D. G. Cantor, Power series with integral coefficients, *Bull. Am. Math. Soc.*, **69** (1963), 362–366.
23. M. A. Hasan, A. A. Hasan, Hankel matrices of finite rank with applications to signal processing and polynomials, *J. Math. Anal. Appl.*, **208** (1997), 218–242.
24. R. Wilson, Determinantal criteria for meromorphic functions, *Proc. Lond. Math. Soc.*, **4** (1954), 357–374.
25. W. K. Hayman, On second Hankel determinant of mean univalent functions, *Proc. London Math. Soc.*, **3** (1968), 77–94.
26. M. Obradović, N. Tuneski, Hankel determinants of second and third order for the class $\mathcal{S}$ of univalent functions, *Math. Slovaca*, **71** (2021), 649–654.
27. A. Janteng, S. A. Halim, M. Darus, Hankel determinant for starlike and convex functions, *Int. J. Math. Anal.*, **1** (2007), 619–625.

28. S. K. Lee, V. Ravichandran, S. Supramaniam, Bounds for the second Hankel determinant of certain univalent functions, *J. Inequal. Appl.*, **2013** (2013), 281. https://doi.org/10.1186/1029-242X-2013-281

29. A. Ebadian, T. Bulboacă, N. E. Cho, E. A. Adegani, Coefficient bounds and differential subordinations for analytic functions associated with starlike functions, *RACSAM*, **114** (2020), 128. https://doi.org/10.1007/s13398-020-00871-x

30. N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko, Y. J. Sim, The bound of the Hankel detrminant for strongly starlike functions of order alpha, *J. Math. Ineq.*, **11** (2017), 429–439.

31. Y. J. Sim, D. K. Thomas, P. Zaprawa, The second Hankel determinant for starlike and convex functions of order alpha, *Complex Var. Elliptic Equations*, **67** (2022), 2423–2443.

32. K. O. Babalola, On $H_3(1)$ Hankel determinant for some classes of univalent functions, *Inequality Theory Appl.*, **6** (2010), 1–7.

33. B. Kowalczyk, A. Lecko, Y. J. Sim, The sharp bound for the Hankel determinant of the third kind for convex functions, *Bull. Aust. Math. Soc.*, **97** (2018), 435–445.

34. B. Kowalczyk, A. Lecko, D. K. Thomas, The sharp bound of the third Hankel determinant for starlike functions, *Forum Math.*, **34** (2022), 1249–1254.

35. B. Kowalczyk, A. Lecko, The sharp bound of the third Hankel determinant for functions of bounded turning, *Bol. Soc. Mat. Mex.*, **27** (2021), 69. https://doi.org/10.1007/s40590-021-00383-7

36. K. Ullah, H. M. Srivastava, A. Rafiq, M. Arif, S. Arjika, A study of sharp coefficient bounds for a new subfamily of starlike functions, *J. Inequal. Appl.*, **2021** (2021), 194.

37. A. Lecko, Y. J. Sim, B. Śmiarowska, The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2, *Complex Anal. Oper. Theory*, **13** (2019), 2231–2238. https://doi.org/10.1007/s11785-018-0819-0

38. M. Arif, O. M. Barukab, S. A. Khan, M. Abbas, The sharp bounds of Hankel Determinants for the families of three-leaf-type analytic functions, *Fractal Fract.*, **6** (2022), 291.

39. O. M. Barukab, M. Arif, M. Abbas, S. A. Khan, Sharp bounds of the coefficient results for the family of bounded turning functions associated with petal shaped domain, *J. Funct. Space.*, **2021** (2021), 5535629.

40. Z. G. Wang, M. Raza, M. Arif, K. Ahmad, On the third and fourth Hankel determinants for a subclass of analytic functions, *Bull. Malays. Math. Sci. Soc.*, **45** (2022), 323–359.

41. B, Kowalczyk, A. Lecko, D. K. Thomas, The sharp bound of the third Hankel determinant of convex functions of order -1/2, *J. Math. Inequa.*, **17** (2023), 191–204.

42. L. Shi, M. Arif, Sharp bounds on the third Hankel determinant for the Ozaki close-to-convex and convex functions, *Lith. Math. J.*, **63** (2023), 487–504.

43. S. Banga, S. S. Kumar, The sharp bounds of the second and third Hankel determinants for the class $\mathcal{SL}$, *Math. Slovaca*, **70** (2020), 849–862.

44. M. Mundula, S. S. Kumar, On subfamily of starlike functions related to hyperbolic cosine function, *J. Anal.*, **31** (2023), 2043–2062.

45. M. G. Khan, W. K. Mashwani, L. Shi, S. Araci, B. Ahmad, B. Khan, Hankel inequalities for bounded turning functions in the domain of cosine hyperbolic function, *AIMS Mathematics*, **8** (2023), 21993–22008. http://doi.org/10.3934/math.20231121

46. O. S. Kwon, A. Lecko, Y. J. Sim, On the fourth coefficient of functions in the Carathéodory class, *Comput. Methods Funct. Theory*, **18** (2018), 307–314.

47. I. Efraimidis, A generalization of Livingston's coefficient inequalities for functions with positive real part, *J. Math. Anal. Appl.*, **435** (2016), 369–379.

48. V. Ravichandran, S. Verma, Bound for the fifth coefficient of certain starlike functions, *C. R. Math. Acad. Sci. Paris*, **353** (2015), 505–510.

49. R. J. Libera, E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivatives in $\mathcal{P}$, *Proc. Am. Math. Soc.*, **87** (1983), 251–257.

50. I. M. Milin, *Univalent Functions and Orthonormal Systems (Nauka, Moscow 1971), English Translation*, Providence: American Mathematical Society, 1977.

51. I. P. Kayumov, On Brennan's conjecture for a special class of functions, *Math. Notes*, **78** (2005), 498–502.

52. P. Sunthrayuth, I. Aldawish, M. Arif, M. Abbas, S. El-Deeb, Estimation of the second-order Hankel determinant of logarithmic coefficients for two subclasses of starlike functions, *Symmetry*, **14** (2022), 2039.

53. Z. Ye, The logarithmic coefficients of close-to-convex functions, *Bull. Inst. Math. Acad. Sin.(NS)*, **3** (2008), 445–452.

54. D. Girela, Logarithmic coefficients of univalent functions, *Ann. Acad. Sci. Fenn.*, **25** (2000), 337–350.

55. L. Shi, M. Arif, A. Rafiq, M. Abbas, J. Iqbal, Sharp Bounds of Hankel Determinant on Logarithmic Coefficients for Functions of Bounded Turning Associated with Petal-Shaped Domain, *Mathematics*, **10** (2022), 1939.

56. O. Roth, A sharp inequality for the logarithmic coefficients of univalent functions, *Proc. Am. Math. Soc.*, **135** (2007), 2051–2054.