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Research article

Topologically indistinguishable relations and separation axioms

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Abstract: This study focuses on defining separation axioms for sets without an inherent topological structure. By utilizing a mapping to relate such sets to a topological space, we first define a distinguishable relation over the universal set with respect to the neighborhood systems inspired by a topology of the co-domain set and elucidate its basic properties. To facilitate the way of discovering this distinguishable relation, we initiate a color technique for the equivalence classes inspired by a given topology. Also, we provide an algorithm to determine distinguishable members (or objects) under study. Then, we establish a framework for introducing separation properties within these structureless sets and examine their master characterizations. To better understand the obtained results and relationships, we display some illustrative instances.

Keywords: $f\mathcal{T}$-topological distinguishability; equivalence relation; separation axioms
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1. Introduction

General topology is interested in the geometric properties of an object (set) while neglecting the notion of the distance between the points (elements) of this object (set). This qualifies topology as a forceful instrument to handle some practical issues and describe their qualitative properties. In recent years, researchers and scholars have exploited topological frames to transect different types of real-life problems, which can be noted in the published monographs concerning digital topology [1], decision-making [2, 3], and information systems [4, 5]. Also, it has been benefited from some topological concepts, such as separation axioms, compactness, generalized open sets, and operators, to cope with some practical issues in diverse disciplines such as economics, incomplete information [6], medical science [7], geographic information systems [8], and algebraic structures. Moreover, extensions of topology have been applied to handle some practical issues, for example, supra topology [9], infra topology [10], minimal structures [11, 12], and bitopology [13] which enhances study of these structures from a theoretical and applied point of view. Some contributions to networks have been conducted by [14, 15].

In topology, the notion of topological distinguishability plays a fundamental role in understanding the properties and structure of spaces. It allows us to distinguish between points based on their neighborhoods and their relationships within a given topological space. This investigation falls under the study of separation axioms, which have been introduced by using diverse manners in topology such as limit points and functions. Separation axioms provide us with a tool to categorize spaces and enable us to discover the main properties of these spaces; they also reveal the behaviors of topological concepts and their properties in specific types of spaces. However, what if we are dealing with a set without any inherent structure? How can we distinguish points in such a set using only a function? This study addresses this question and presents a method to distinguish discrete points in a structureless set using a mapping from the Cartesian product of the set to a topological space. That is, we examine separation axioms for sets without an inherent topological structure.

We organize this work as follows: Section 2 is designed to recall some elementary concepts that make this manuscript self-contained. In Section 3, we display the idea of "$f\mathcal{T}$-topological distinguishability", where f is a function from the Cartesian product of a set X to a topological space $(Y, \mathcal{T})$. Also, we set up a method to discover the distinguishable elements using distinct colors for the equivalence classes inspired by a given topology. This method is described in Algorithm 1. Then, in Section 4, we display the concepts of "$f\mathcal{T} - T_i$" set ($i = 0, 1, 2$) and scrutinize their major characterizations. In addition, we reveal the relationships between them and build some counterexamples to demonstrate the invalid directions. Ultimately, we outline the main contributions of this article and suggest some future works in Section 5.

2. Preliminaries

A neighborhood of a point $x \in X$ is a subset N of X that contains an open set U containing x. The collection of all neighborhoods of x is denoted by $\mathcal{N}_{\mathcal{T}}(x)$ and is referred to as the neighborhood system of x:

$$N \in \mathcal{N}_{\mathcal{T}}(x) \Leftrightarrow x \in U \subseteq N, U \in \mathcal{T}.$$

Additionally, let $\mathcal{T}(x)$ denote the collection of all open sets in the topology $\mathcal{T}$ that contains the element x.

In a topological space, points can be distinguished based on their neighborhoods. If any two points have distinct neighborhood systems, they are said to be topologically distinguishable; otherwise, they are called topologically indistinguishable:

$$\begin{aligned} x \text{ and } y \text{ are topologically indistinguishable } &\Leftrightarrow \mathcal{N}_{\mathcal{T}}(x) = \mathcal{N}_{\mathcal{T}}(y) \\ &\Leftrightarrow \forall U, [U \in \mathcal{T}(x) \Rightarrow U \in \mathcal{T}(y)] \\ &\quad \wedge \forall V, [V \in \mathcal{T}(y) \Rightarrow V \in \mathcal{T}(x)]; \end{aligned}$$

$$x \text{ and } y \text{ are topologically distinguishable} \Leftrightarrow \mathcal{N}_{\mathcal{T}}(x) \neq \mathcal{N}_{\mathcal{T}}(y)$$
$$\Leftrightarrow \exists U \in \mathcal{T}(x), U \notin \mathcal{T}(y) \vee \exists V \in \mathcal{T}(y), V \notin \mathcal{T}(x).$$

It is apparent that topological distinguishability forms an equivalence relation, denoted by "$\sim$", where $x \sim y \Leftrightarrow x$ and y are topologically indistinguishable. We represent the equivalence class of x as

$$[x] = \{y \in X \mid x \sim y\}.$$

There are various methods to establish a topology on a set X, and one of them involves the use of a topological base. Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B} \subset 2^X$ and

$$\mathcal{B}^* = \{A \subseteq X \mid A = \bigcup_{B \in \Phi} B, \Phi \subseteq \mathcal{B}\}. \tag{2.1}$$

If $\mathcal{B}^* = \mathcal{T}$ then $\mathcal{B}$ is called a basis for the topology $\mathcal{T}$.

Lemma 2.1. *[16] Let $\mathcal{B} \subseteq P(X)$. Then the family $\mathcal{B}^*$ forms a topological structure on X, iff*

i) $X = \bigcup_{B \in \mathcal{B}} B$;

ii) $\forall A, B \in \mathcal{B}$ *and* $\forall x \in A \cap B, \exists C \in \mathcal{B}$ *such that* $x \in C \subseteq A \cap B$.

Furthermore, it is possible to construct a topological structure on a set using equivalence classes. Consider the quotient set

$$Y = X/_{\sim} = \{[x] \mid x \in X\}.$$

The quotient space Y is endowed with the quotient topology, where a subset $U \subseteq Y$ is declared open if and only if its preimage $q^{-1}(U)$, under the quotient map q: $X \to Y$ defined as

$$q(x) = [x],$$

is an open set in X.

Definition 2.2. *Let g be a function from X to Y. If E is a subset of X, then the restriction of g to E is the function g_E: $E \to Y$ given by*

$$g_E(x) = g(x)$$

for all $x \in E$.

3. $f\mathcal{T}$-distinguishability

It is feasible to establish a topology on a set that initially lacks any inherent topological structure by introducing a topological space and a function defined on it. The initial topology and the resulting topology serve as illustrative examples of this approach. Thus, by utilizing these topologies, separation axioms can be defined on such sets. However, our objective is to formulate separation axioms for a set devoid of any pre-existing topology without imposing any specific topological structure. In this study, we have developed a methodology to discern discrete points within a structureless set by employing a function from the Cartesian product of the set and a reference set to a topological space. We postulate that the two points are distinguishable if the images of these points, obtained through the Cartesian product with at least one reference point, exhibit topological distinguishability within the resulting image space.

From this point onwards, let X be a set, $\mathcal{R}$ be a reference set, and $(Y, \mathcal{T})$ be considered a topological space unless otherwise stated.

Definition 3.1. *Let $g\colon X \times \mathcal{R} \to Y$ be a function. For two distinct points $a, b \in X$, if*

$$\mathcal{N}_{\mathcal{T}}(g(a, r)) = \mathcal{N}_{\mathcal{T}}(g(b, r))$$

holds for all $r \in \mathcal{R}$, then a and b are called indistinguishable via g with respect to the reference set $\mathcal{R}$ and, the topology $\mathcal{T}$ (simply, we write a and b are g_{rt}-indistinguishable). Otherwise, a and b are called g_{rt}-distinguishable. Therefore, if $g(a, r)$ and $g(b, r)$ are topologically distinguishable in $(Y, \mathcal{T})$ for at least one $r \in \mathcal{R}$, then a and b are g_{rt}-distinguishable in X.

When it is stated that two elements are g_{rt}-distinguishable, it should be noted that this distinction is made with respect to the specific reference set, the topological space involved, and the function g.

Remark 3.2. *For any distinct points $a, b \in X$, a and b are f_{rt}-distinguishable (f_{rt}-dis.) if and only if there exists an open set $U \subseteq Y$ such that $f(a, r) \in U$ and $f(b, r) \notin U$, for some $r \in \mathcal{R}$, or there exists an open set $V \subseteq Y$ such that $f(b, r) \in V$ and $f(a, r) \notin V$, for some $r \in \mathcal{R}$.*

Result 3.3. *In the context of f_{rt}-distinguishability, two distinct points a and b in X are considered f_{rt}-indistinguishable if and only if, for all $r \in \mathcal{R}$, the following condition holds: for all open sets U and V in the topology $\mathcal{T}$, either $f(a, r) \in U$ implies $f(b, r) \in U$, or $f(b, r) \in V$ implies $f(a, r) \in V$.*

Example 3.4. *Let*

$$\begin{aligned} X &= \{x_1, x_2, x_3, x_4, x_5\}, & \mathcal{R} &= \{r_1, r_2, r_3, r_4\}, \\ Y &= \{y_1, y_2, y_3, y_4, y_5, y_6\}, & \mathcal{T} &= \{\emptyset, Y, \{y_1, y_4, y_5\}\}. \end{aligned}$$

Let $f\colon X \times \mathcal{R} \to Y$ be a map defined as:

$$\begin{aligned} &f(x_1, r_1) = f(x_1, r_2) = f(x_2, r_1) = f(x_3, r_4) = f(x_4, r_1) = y_1, \\ &f(x_1, r_3) = f(x_1, r_4) = f(x_2, r_3) = f(x_2, r_4) = f(x_3, r_1) = y_2, \\ &f(x_3, r_2) = f(x_4, r_4) = f(x_5, r_1) = f(x_5, r_2) = y_3, \\ &f(x_2, r_2) = f(x_4, r_2) = f(x_5, r_3) = f(x_5, r_4) = y_4. \\ &f(x_3, r_3) = f(x_4, r_3) = y_5. \end{aligned}$$

Then x_3 and x_4 are f_{rt}-distinguishable because

$$f(x_3, r_4) \in \{y_1, y_4, y_5\} \in \mathcal{T}$$

and

$$f(x_4, r_4) \notin \{y_1, y_4, y_5\}.$$

However, x_1 and x_2 are f_{rt}−indistinguishable since for every $r \in \mathcal{R}$, $f(x_1, r)$ and $f(x_2, r)$ are in the same open set in $\mathcal{T}$.

Table 1 presents the function f with its corresponding values.

Table 1. Function f with its corresponding values.

f	r_1	r_2	r_3	r_4
x_1	y_1	y_1	y_2	y_2
x_2	y_1	y_4	y_2	y_2
x_3	y_2	y_3	y_5	y_1
x_4	y_1	y_4	y_5	y_3
x_5	y_3	y_3	y_4	y_4

To better visualize the f_{rt}-distinguishable points, we represent the topologically distinguishable points of Y in Table 2 by highlighting them with different colors.

Table 2. Different colors for topologically distinguishable points of Y.

f	r_1	r_2	r_3	r_4
x_1	y_1	y_1	y_2	y_2
x_2	y_1	y_4	y_2	y_2
x_3	y_2	y_3	y_5	y_1
x_4	y_1	y_4	y_5	y_3
x_5	y_3	y_3	y_4	y_4

Points that exhibit distinct color rows in Table 2 are considered f_{rt}-distinguishable. Table 2 reveals that the following pairs of points are f_{rt}-distinguishable:

$$(x_1,x_3),(x_1,x_4),(x_1,x_5),(x_2,x_3),(x_2,x_4),(x_2,x_5),(x_3,x_4),(x_4,x_5).$$

It is evident that the distinguishability of elements in X can vary based on the defined function and the topology on Y.

Remark 3.5. *In the subsequent section of this study, during the construction of the table for a function f, segments corresponding to topologically indistinguishable points in the co-domain set of f will be visually represented by highlighting them with the same color.*

Example 3.6. *Let us consider Example 3.4.*

i) Let g: $X \times \mathcal{R} \to (Y,\mathcal{T})$ be defined as shown in Table 3. Then the pairs of g_{rt}-distinguishable points are:

$$(x_1,x_2),(x_1,x_4),(x_1,x_5),(x_2,x_3),(x_2,x_4),(x_2,x_5),(x_3,x_4),(x_3,x_5),(x_4,x_5).$$

Table 3. Different colors for topologically distinguishable points of Y.

g	r_1	r_2	r_3	r_4
x_1	y_5	y_5	y_5	y_4
x_2	y_4	y_4	y_2	y_2
x_3	y_1	y_1	y_1	y_4
x_4	y_3	y_6	y_3	y_4
x_5	y_3	y_4	y_1	y_5

ii) Let

$$\mathcal{T}' = \{\emptyset, Y, \{y_1, y_5\}, \{y_3, y_4, y_5\}, \{y_5\}, \{y_1, y_3, y_4, y_5\}\}.$$

In this case, Table 2 transforms into Table 4. Therefore, all distinct points of X are f_{rt}-distinguishable respect to $\mathcal{T}'$.

Table 4. Function f with its corresponding values.

f	r_1	r_2	r_3	r_4
x_1	y_1	y_1	y_2	y_2
x_2	y_1	y_4	y_2	y_2
x_3	y_2	y_3	y_5	y_1
x_4	y_1	y_4	y_5	y_3
x_5	y_3	y_3	y_4	y_4

The ordering induced by the containment of topologies on Y leads to variability in the distinguishability of elements in X. The coarsening of the topology on Y restricts the distinguishability of elements in X. In other words, as the topology on Y becomes finer, a greater number of elements in X can be f_{rt}-distinguished.

Theorem 3.7. *Let $\mathcal{T} \subseteq \mathcal{T}_1$. If a and b are f_{rt}-indistinguishable with respect to $\mathcal{T}_1$, then they are also f_{rt}-indistinguishable with respect to $\mathcal{T}$.*

Proof. Straightforward. □

Result 3.8. *Let $\mathcal{T} \subseteq \mathcal{T}_1$. If a and b are f_{rt}-distinguishable with respect to $\mathcal{T}$, then they are also f_{rt}-distinguishable with respect to $\mathcal{T}_1$.*

However, the converse of Result 4.15 does not hold in general. The following example illustrates its invalidity and demonstrates that as the topology on Y becomes finer, the number of f_{rt}-indistinguishable elements in X decreases.

Example 3.9. *Let*

$$X = \{x_1, x_2, x_3, x_4, x_5, x_6\}, \quad Y = \{y_1, y_2, y_3, y_4, y_5\}$$

and f be defined as shown in Table 5.

Table 5. Function f with its corresponding values.

f	r_1	r_2	r_3	r_4
x_1	y_1	y_5	y_5	y_4
x_2	y_4	y_4	y_2	y_2
x_3	y_1	y_5	y_1	y_4
x_4	y_3	y_2	y_3	y_4
x_5	y_3	y_4	y_1	y_5
x_6	y_2	y_4	y_5	y_5

Let us define topologies $\mathcal{T}_2 \subseteq \mathcal{T}_1 \subseteq \mathcal{T}$ on Y such that

$$\mathcal{T}_2 = \{\emptyset, Y, \{y_1, y_5\}\},$$
$$\mathcal{T}_1 = \{\emptyset, Y, \{y_1, y_5\}, \{y_2, y_3\}, \{y_1, y_2, y_3, y_5\}\},$$
$$\mathcal{T} = \{\emptyset, Y, \{y_1, y_5\}, \{y_2, y_3\}, \{y_1, y_2, y_3, y_5\}, \{y_2, y_5\}, \{y_5\}, \{y_2\}, \{y_1, y_2, y_5\}, \{y_2, y_3, y_5\}\}.$$

Then the elements x_1 and x_3 are f_{rt}-distinguishable with respect to $\mathcal{T}$, but not with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$.

Sets of the pairs of points are f_{rt}-indistinguishable in the context of topologies $\mathcal{T}_2, \mathcal{T}_1$ and $\mathcal{T}$, respectively, as follows:

$$\{(x_1, x_3), (x_2, x_4), (x_5, x_6)\}, \{(x_1, x_3), (x_5, x_6)\}, \emptyset.$$

Thus, the finer topology on Y has increased the number of distinguishable elements.

To better visualize the distinguishable elements with respect to $\mathcal{T}_2, \mathcal{T}_1$ and $\mathcal{T}$ topologies, we present Tables 6–8, respectively.

Table 6. Topologically distinguishable points of $(Y, \mathcal{T}_2)$.

f	r_1	r_2	r_3	r_4
x_1	y_1	y_5	y_5	y_4
x_2	y_4	y_4	y_2	y_2
x_3	y_1	y_5	y_1	y_4
x_4	y_3	y_2	y_3	y_4
x_5	y_3	y_4	y_1	y_5
x_6	y_2	y_4	y_5	y_5

Table 7. Topologically distinguishable points of $(Y, \mathcal{T}_1)$.

f	r_1	r_2	r_3	r_4
x_1	y_1	y_5	y_5	y_4
x_2	y_4	y_4	y_2	y_2
x_3	y_1	y_5	y_1	y_4
x_4	y_3	y_2	y_3	y_4
x_5	y_3	y_4	y_1	y_5
x_6	y_2	y_4	y_5	y_5

Table 8. Topologically distinguishable points of $(Y, \mathcal{T})$.

f	r_1	r_2	r_3	r_4
x_1	y_1	y_5	y_5	y_4
x_2	y_4	y_4	y_2	y_2
x_3	y_1	y_5	y_1	y_4
x_4	y_3	y_2	y_3	y_4
x_5	y_3	y_4	y_1	y_5
x_6	y_2	y_4	y_5	y_5

To facilitate the way of determining distinguishable members (or objects), we provide the subsequent algorithm.

Increasing the number of reference points in the reference set may enable us to distinguish a greater number of elements in the set X. Similarly, decreasing the number of elements in the reference set may lead to a reduction in the number of distinguishable elements. The following theorem explains this situation.

Theorem 3.10. *Let $\mathcal{R}_1 \subseteq \mathcal{R}$. If any distinct points a and b are f_{rt}-distinguishable with respect to $\mathcal{R}_1$, then a and b are f_{rt}-distinguishable with respect to $\mathcal{R}$.*

Proof. Straightforward. □

The following example demonstrates that the converse of Theorem 3.10 is invalid.

Example 3.11. *Let*

$$X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}, \qquad \mathcal{R} = \{r_1, r_2, r_3, r_4\},$$
$$Y = \{y_1, y_2, y_3, y_4\}, \qquad \mathcal{T} = \{\emptyset, Y, \{y_1, y_3\}, \{y_3, y_4\}, \{y_3\}, \{y_1, y_3, y_4\}\}.$$

Algorithm 1. The algorithm for determining a f_{rt}-distinguishable relation.

Input: The universal set $X = \{x_1, x_2, ..., x_n\}$, the reference set $\mathcal{R} = \{r_1, r_2, ..., r_s\}$, and the set of topology $Y = \{y_1, y_2, ..., y_j\}$
Output: f_{rt}-distinguishable relation Ω over X.
1: Structure a topology $\mathcal{T}$ over Y;
2: **for** $y \in Y$ **do**
3: Calculate the neighborhood systems $\mathcal{N}_{\mathcal{T}}(y)$;
4: **end for**
5: Build an equivalence relation ρ over Y inspired by its topology as follows: $y_1 \rho y_2 \Leftrightarrow \mathcal{N}_{\mathcal{T}}(y_1) = \mathcal{N}_{\mathcal{T}}(y_2)$;
6: Present the equivalence classes over Y generated by ρ;
7: Ask experts of the system to define a map f from $X \times \mathcal{R}$ into Y according to the scenario under study;
8: Structure a table, consisting of elements of X as columns and elements of $\mathcal{R}$ as rows, to display the range of a map f;
9: Color each equivalence class, given in Step 6, with a distinct color;
10: Build an equivalence relation σ over X using the colored table as follows $x_1 \sigma x_2 \Leftrightarrow$ the rows of x_1 and x_2 have the same color;
11: Present the set A of equivalence classes over X generated by σ, that is, $A = \{(x_i, x_j) \in X \times X : x_i \sigma x_j\}$;
12: Structure the f_{rt}-distinguishable relation Ω as follows: $\Omega = \{(x_i, x_j) \in X \times X : (x_i, x_j) \notin A\}$;
13: Print Ω.

Let f be defined as in Table 9. Let

$$\mathcal{R}_1 = \{r_1\}, \ \ \mathcal{R}_2 = \{r_1, r_2\}, \ \ \mathcal{R}_3 = \{r_1, r_2, r_4\}.$$

Then the sets of f_{rt}-indistinguishable points with respect to $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, and $\mathcal{R}$, respectively:

$$\{(x_1, x_3), (x_1, x_4), (x_1, x_5), (x_2, x_6), (x_3, x_4), (x_3, x_5), (x_4, x_5), (x_7, x_8)\}, \{(x_1, x_3), (x_2, x_6), (x_4, x_5), (x_7, x_8)\}, \{(x_2, x_6)\}, \emptyset.$$

Table 9. Different colors for topologically distinguishable points of $(Y, \mathcal{T})$.

f	r_1	r_2	r_3	r_4
x_1	y_1	y_4	y_3	y_3
x_2	y_3	y_4	y_1	y_2
x_3	y_1	y_4	y_3	y_2
x_4	y_1	y_2	y_3	y_3
x_5	y_1	y_2	y_3	y_2
x_6	y_3	y_4	y_4	y_2
x_7	y_2	y_4	y_3	y_3
x_8	y_2	y_4	y_3	y_2

Thus, the converse of Theorem 3.10 is not valid. For instance, it can be seen that:

(i) *x_2 and x_6 are f_{rt}-distinguishable with respect to $\mathcal{R}$, but they are not f_{rt}-distinguishable with respect to $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$.*

(ii) *x_1 and x_3 are f_{rt}-distinguishable with respect to $\mathcal{R}_3$, but they are not f_{rt}-distinguishable with respect to $\mathcal{R}_1$ and $\mathcal{R}_2$.*

(iii) *x_1 and x_4 are f_{rt}-distinguishable with respect to $\mathcal{R}_2$, but they are not f_{rt}-distinguishable with respect to $\mathcal{R}_1$.*

Even if the functions we use to distinguish the elements in X are different from each other, the elements they distinguish may turn out to be the same in the end. Therefore, it would make sense to classify functions that have made the same distinction together. The following theorem gives the relationship that allows us to make this classification.

Theorem 3.12. *Let $F(X \times \mathcal{R}, Y)$ represent the set of all functions from $X \times \mathcal{R}$ to $(Y, \mathcal{T})$. We define a relation on $F(X \times \mathcal{R}, Y)$ for any $f, g \in F(X \times \mathcal{R}, Y)$ as follows: $f \sim g$ if and only if the elements that are f_{rt}-indistinguishable and the elements that are g_{rt}-indistinguishable are the same. Then $\sim$ is an equivalence relation.*

Proof. The proof of the theorem is straightforward. □

Example 3.13. *Let us consider the sets and a function mentioned in Example 3.4. Let g be defined as shown in Table 10.*

Table 10. Function g with its corresponding values.

g	r_1	r_2	r_3	r_4
x_1	y_2	y_2	y_1	y_4
x_2	y_2	y_2	y_4	y_4
x_3	y_2	y_3	y_3	y_4
x_4	y_4	y_2	y_2	y_3
x_5	y_2	y_3	y_2	y_4

In view of the fact that all pairs of points that are f_{rt}-distinguishable and g_{rt}-distinguishable are identical, it follows that f and g are equivalent, denoted by $f \sim g$.

It is clear that f_{rt}-indistinguishability defines an equivalence relation on X. Thus, by utilizing equivalence classes, it is possible to classify the elements in a set without any topological structure by examining the individual relations of different pairs of elements with other elements of the set.

The following theorem demonstrates that a topology can be constructed on the set X using the generated equivalence classes.

Theorem 3.14. *[16] Let ρ be an equivalence relation on X, and $\mathcal{B}$ be the set of equivalence classes:*

$$\mathcal{B} = \{[x] \mid x \in X\}.$$

Then the set $\mathcal{B}^*$ defined in (2.1) forms a topological structure on X.

Theorem 3.14 states that a topology can be constructed on X using the equivalence relation ρ defined as:

$$x\rho y \Leftrightarrow x \text{ and } y \text{ are } f_{\mathrm{rt}}\text{-indistinguishable}.$$

Let us refer to this topology as f_{rt}−topology and denote it as $f\mathcal{T}$.

Furthermore, it should be noted that in this particular classification, there exists no one-to-one correspondence between the quotient set resulting from the topology indistinguishable relation between the value set and the quotient set in X. Specifically, in example 3.4, the equivalence classes formed in $(Y, \mathcal{T})$ are $\{[y_1], [y_2]\}$, where

$$[y_1] = \{y_1, y_4, y_5\}$$

and

$$[y_2] = \{y_2, y_3, y_6\}.$$

On the other hand, the equivalence classes in X with respect to $\mathcal{T}$ are $\{[x_1], [x_3], [x_4]\}$, where

$$[x_1] = \{x_1, x_2\}, \quad [x_3] = \{x_3, x_5\} \quad \text{and} \quad [x_4] = \{x_4\},$$

where $[x]$ refers to the respective equivalence class of an element x.

The equivalence relation established in Theorem 3.12 divides the set $F(X \times R, Y)$ into equivalence classes. Let $[f]$ denote the equivalence class of any function f, and let $[X \times R, Y]$ represent the set of all equivalence classes in $F(X \times R, Y)$. The following theorem demonstrates that a partial order relation can be defined on the set $[X \times R, Y]$.

The following result is immediate, so we omit its proof.

Theorem 3.15. *Let us define*

$$[f] \preceq [g] \Leftrightarrow f\mathcal{T} \subseteq g\mathcal{T}$$

for any $[f], [g] \in [X \times R, Y]$. *Then* $([X \times R, Y], \preceq)$ *is a partial ordered set.*

Proof. The proof of the theorem is evident. □

Example 3.16. *If* $f\mathcal{T}$ *is the indiscrete topology on X, then* $[f]$ *is the minimum element of* $([X \times R, Y], \preceq)$. *Also, if* $f\mathcal{T}$ *is the discrete topology, then* $[f]$ *is the maximum element.*

The following example shows that "$\preceq$" is not a total ordering relation.

Example 3.17. *Let*

$$X = \{x_1, x_2, x_3, x_4, x_5\}, \quad \mathcal{R} = \{r_1, r_2, r_3, r_4, r_5, r_6\}, \quad Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\},$$
$$\mathcal{T} = \{\emptyset, Y, \{y_1, y_5\}, \{y_2, y_4\}, \{y_3\}, \{y_1, y_3, y_5\}, \{y_2, y_3, y_4\}, \{y_1, y_2, y_4, y_5\}\}.$$

Let us define $f, g \in F(X \times R, Y)$ *as shown in Tables 11 and 12, respectively.*

Table 11. Function f with its corresponding values.

f	r_1	r_2	r_3	r_4	r_5	r_6
x_1	y_1	y_1	y_5	y_2	y_2	y_2
x_2	y_5	y_3	y_3	y_4	y_4	y_4
x_3	y_1	y_1	y_3	y_2	y_4	y_3
x_4	y_5	y_5	y_3	y_4	y_4	y_3
x_5	y_5	y_1	y_1	y_4	y_2	y_2

Table 12. Function g with its corresponding values.

g	r_1	r_2	r_3	r_4	r_5	r_6
x_1	y_3	y_3	y_3	y_2	y_2	y_2
x_2	y_3	y_3	y_3	y_4	y_4	y_2
x_3	y_3	y_3	y_3	y_4	y_2	y_2
x_4	y_5	y_5	y_1	y_3	y_3	y_3
x_5	y_1	y_1	y_1	y_3	y_3	y_1

Then we have

$$f\mathcal{T} = \{\emptyset, X, \{x_2\}, \{x_1, x_5\}, \{x_3, x_4\}, \{x_1, x_2, x_5\}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4, x_5\}\},$$
$$g\mathcal{T} = \{\emptyset, X, \{x_1, x_2, x_3\}, \{x_4\}, \{x_5\}, \{x_4, x_5\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_5\}\}.$$

As a result, $[f] \npreceq [g]$ *and* $[g] \npreceq [f]$.

4. f_{rt}-separation axioms

Separation axioms play a fundamental role in the study of topological spaces and are crucial for determining and analyzing topological properties. Hausdorff (or T_2) axiom, introduced by Felix Hausdorff in 1914, defines a topological space where distinct points can be separated. This axiom represents the strongest form of separability in topology, requiring points to be completely separated from each other.

However, separation axioms are not limited to just the T_2 axiom. Mathematicians such as Alexandroff and Smirnov have defined other separability axioms, such as T_0 and T_1. The T_0 axiom states that distinct points must have at least one point that distinguishes them topologically, while the T_1 axiom requires that every pair of distinct points has disjoint neighborhoods. Therefore, separation axioms are considered essential concepts in topology and play a crucial role in various mathematical studies. The works of Felix Hausdorff, contributions by Alexandroff and Smirnov, and the research of other mathematicians have supported the development of separation axioms in topology. Understanding and applying separation axioms is a fundamental step in the advancement of topological analysis and the better understanding of mathematical structures.

We will now define and investigate certain separation axioms using the concept of f_{rt}-distinguishability on the space X and analyze their fundamental properties. Subsequently, we define the concept of $f_{\text{rt}} - T_0$ set, which represents an extension of the distinguishability concept to a higher level of rigor. Finally, we present the concept of the $f_{\text{rt}} - T_1$ set, which imposes further conditions on distinct elements, facilitating their definitive separation based on specific criteria. Each successive separation axiom strengthens the distinguishability attribute compared to its predecessor and provides a more robust capacity to discriminate between distinct elements.

Definition 4.1. *A set in which all distinct points are f_{rt}-distinguishable is referred to as an $f_{rt} - T_0$ set.*

Definition 4.2. *If, for any two distinct points $x_1, x_2 \in X$, and for some $r_1, r_2 \in \mathcal{R}$, there exist neighborhoods U and V in $(Y, \mathcal{T})$ such that $f(x_1, r_1) \in U$ and $f(x_2, r_1) \notin U$, and $f(x_2, r_2) \in V$ and $f(x_1, r_2) \notin V$, then X is referred to as an $f_{rt} - T_1$ set. In other words, X satisfies the conditions:*

$$\exists U \in \mathcal{N}_{\mathcal{T}}\left(f(x_1, r_1)\right), f(x_2, r_1) \notin U$$

and

$$\exists V \in \mathcal{N}_{\mathcal{T}}(f(x_2, r_2)), f(x_1, r_2) \notin V.$$

Conversely, for all $r \in \mathcal{R}$, if for all $U \in \mathcal{N}_{\mathcal{T}}(f(x_1, r))$, we have $f(x_2, r) \in U$, or for all $V \in \mathcal{N}_{\mathcal{T}}(f(x_2, r))$, we have $f(x_1, r) \in V$, then X is not an $f_{rt} - T_1$ set.

Definition 4.3. *Let $f\colon X \times \mathcal{R} \to Y$ be a function. If for any two distinct points $a, b \in X$, there exists $r \in \mathcal{R}$ such that there exist neighborhoods of each which are disjoint from each other, then X is called as $f_{rt} - T_2$ set, i.e.,*

$$\exists U \in \mathcal{N}_{\mathcal{T}}\left(f(a, r)\right), \exists V \in \mathcal{N}_{\mathcal{T}}(f(b, r)), U \cap V \neq \emptyset.$$

The following result is a natural consequence of the definition of the f_{rt} separation axioms.

Result 4.4. *The following propositions are valid:*

- *i) X is an $f_{rt} - T_0$ set if and only if, for all distinct points $x_1, x_2 \in X$, there exists an open set $U \in \mathcal{T}(f(x_1, r))$ such that $f(x_2, r) \notin U$ or there exists an open set $V \in \mathcal{T}(f(x_2, r))$ such that $f(x_1, r) \notin V$, for some $r \in \mathcal{R}$.*
- *ii) X is not an $f_{rt} - T_0$ set if and only if there exist distinct points $x_1, x_2 \in X$, and for all open sets $U \in \mathcal{T}$, the condition*

$$f(x_1, r) \in U \Leftrightarrow f(x_2, r) \in U$$

  holds for all $r \in \mathcal{R}$.
- *iii) X is an $f_{rt} - T_1$ set if and only if for all distinct points $x_1, x_2 \in X$, there exist open sets $U \in \mathcal{T}(f(x_1, r))$ and $V \in \mathcal{T}(f(x_2, r))$, such that $f(x_2, r) \notin U$ and $f(x_1, r) \notin V$, for some $r \in \mathcal{R}$.*
- *iv) X is not an $f_{rt} - T_1$ set if and only if for some distinct points $x_1, x_2 \in X$, there exists an open set $U \in \mathcal{T}$, containing both $f(x_1, r)$ and $f(x_2, r)$, which is different from Y for all $r \in \mathcal{R}$.*

v) X is an $f_{rt} - T_2$ set if and only if for all distinct points $x_1, x_2 \in X$, there exist open sets $U \in \mathcal{T}(f(x_1, r))$ and $V \in \mathcal{T}(f(x_2, r))$, such that $U \cap V = \emptyset$, for some $r \in \mathcal{R}$.

vi) X is not an $f_{rt} - T_2$ set if and only if there exist distinct points $x_1, x_2 \in X$ such that, for all open sets $U \in \mathcal{T}(f(x_1, r))$ and $V \in \mathcal{T}(f(x_2, r))$, we have $U \cap V \neq \emptyset$, for all $r \in \mathcal{R}$.

Theorem 4.5. *X is an $f_{rt} - T_0$ set if and only if, for any distinct points $x_1, x_2 \in X$ and for some $r \in \mathcal{R}$, the closures of the sets $\{f(x_1, r)\}$ and $\{f(x_2, r)\}$ are not equal.*

Proof. Let $\overline{A}$ denote the closure of the subset $A \subset X$. Consider X to be an $f_{\mathrm{rt}} - T_0$ space. We assume that

$$\overline{\{f(x_1, r)\}} = \overline{\{f(x_2, r)\}}$$

for all $r \in \mathcal{R}$. Since

$$f(x_1, r) \in \overline{\{f(x_1, r)\}},$$

it follows that

$$f(x_1, r) \in \overline{\{f(x_2, r)\}}.$$

Therefore, any open set containing $f(x_1, r)$ in $\mathcal{T}$, must also contain $f(x_2, r)$. Similarly, for any open set containing $f(x_2, r)$ in $\mathcal{T}$, it must contain $f(x_1, r)$. However, this contradicts the fact that X is an $f_{\mathrm{rt}} - T_0$ set. Hence,

$$\overline{\{f(x_1, r)\}} \neq \overline{\{f(x_2, r)\}}.$$

Now, to prove the converse, assume X is not an $f_{\mathrm{rt}} - T_0$ set. Then there exist distinct points x_1 and x_2 such that

$$\mathcal{N}_{\mathcal{T}}(f(x_1, r)) = \mathcal{N}_{\mathcal{T}}(f(x_2, r)) \tag{4.1}$$

for all $r \in \mathcal{R}$. Let

$$c \in \overline{\{f(x_1, r)\}}.$$

In this case, every open set in $(Y, \mathcal{T})$ that contains c also contains $f(x_1, r)$. From (4.1), we have that every open set containing c also contains $f(x_2, r)$. Thus

$$c \in \overline{\{f(x_2, r)\}}.$$

This implies

$$\overline{\{f(x_1, r)\}} \subseteq \overline{\{f(x_2, r)\}}.$$

Similarly, we can show that

$$\overline{\{f(x_2, r)\}} \subseteq \overline{\{f(x_1, r)\}}.$$

Therefore

$$\overline{\{f(x_1, r)\}} = \overline{\{f(x_2, r)\}}.$$

However, this contradicts the assumption. Hence X is an $f_{\mathrm{rt}} - T_0$ set. □

Theorem 4.6. *Let f be an injective map, and for all $x \in X$ and some $r \in \mathcal{R}$, $\{f(x, r)\}$ be closed in $(Y, \mathcal{T})$. Then X is an $f_{rt} - T_0$ set.*

Proof. For any distinct points $x_1, x_2 \in X$, and for any $r \in \mathcal{R}$, the injective property of the function f ensures that

$$\overline{\{f(x_1, r)\}} = \{f(x_1, r)\} \neq \{f(x_2, r)\} = \overline{\{f(x_2, r)\}}.$$

Consequently, by virtue of Theorem 4.5, we deduce that X is an $f_{\rm rt} - T_0$ set. □

Theorem 4.6 also holds for the topology $\mathcal{T}$ that considers every finite set as a closed set, where f is injective.

The following relation between separation axioms is evident:

$$f_{\rm rt} - T_2 \Rightarrow f_{\rm rt} - T_1 \Rightarrow f_{\rm rt} - T_0. \tag{4.2}$$

Example 4.7. *If $(Y, \mathcal{T})$ is a discrete topological space, then X becomes a $f_{rt} - T_2$ set, thereby implying that it is also either a $f_{rt} - T_1$ or a $f_{rt} - T_0$ set, considering all $f \in F(X \times R, Y)$. Conversely, if $(Y, \mathcal{T})$ is an indiscrete topological space, then X ceases to be a $f_{rt} - T_0$ set, leading to its inability to qualify as a $f_{rt} - T_1$ or a $f_{rt} - T_2$ set for any $f \in F(X \times R, Y)$.*

However, it should be noted that the converse of (4.2) is not universally valid.

Example 4.8. *Let $X = \mathcal{R} = \mathbb{Z}$ and $Y = \mathbb{R}$ with the right ray topology*

$$\mathcal{D}^{-} = \{A_\lambda \mid A = \emptyset \;\; or \;\; A_\lambda =]\lambda, \infty[\}.$$

Consider the function f: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, defined as

$$f(z, r) = (z + r)^2.$$

Let z_1 and z_2 be distinct points in $\mathbb{Z}$. Then, for $r \in \mathbb{Z}$, if $U \in \mathcal{D}^{-}(f(z_1, r))$, there exists $\epsilon > 0$ such that

$$U =](z_1 + r)^2 - \epsilon, \infty[.$$

Similarly, if $V \in \mathcal{D}^{-}(f(z_2, r))$, there exists $\delta > 0$ such that

$$V =](z_2 + r)^2 - \delta, \infty[.$$

It is evident that

$$U \cap V =](z_1 + r)^2 - \epsilon, \infty[\cap](z_2 + r)^2 - \delta, \infty[\neq \emptyset.$$

Thus, $\mathbb{Z}$ is not an $f_{rt} - T_2$ set.

Let

$$max\{f(z_1, r), f(z_2, r)\} = f(z_1, r).$$

Then, there exists $\gamma > 0$ such that

$$f(z_2, r) \notin](z_1 + r)^2 - \gamma, \infty[.$$

Therefore, $\mathbb{Z}$ is an $f_{rt} - T_0$ set.

Example 4.9. *Let*

$$X = \{x_1, x_2, x_3, x_4, x_5\}, \quad \mathcal{R} = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\},$$
$$Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}.$$

Consider the topological base

$$\mathcal{B} = \{\{y_1, y_5, y_6\}, \{y_2, y_3\}, \{y_4\}, \{y_7, y_8\}, \{y_4, y_9\}\}$$

for the topology $\mathcal{T}$, and let $f: X \times \mathcal{R} \to Y$ be defined as in Table 13.

Table 13. Function f with its corresponding values.

f	r_1	r_2	r_3	r_4	r_5	r_6	r_7
x_1	y_2	y_3	y_2	y_4	y_4	y_7	y_9
x_2	y_6	y_1	y_2	y_2	y_4	y_1	y_6
x_3	y_3	y_2	y_2	y_4	y_4	y_8	y_4
x_4	y_1	y_1	y_2	y_3	y_4	y_8	y_6
x_5	y_5	y_7	y_7	y_7	y_5	y_7	y_3

Since all distinct points of X are f_{rt}-distinguishable, X is an $f_{rt} - T_0$ set. However, since $x_1 \neq x_3$ but there does not exist any $U \in \mathcal{T}$ containing $f(x_1, r)$ and not containing $f(x_3, r)$ for all $r \in \mathcal{R}$, X is not an $f_{rt} - T_1$ set. Consequently, X is also not an $f_{rt} - T_2$ set.

The following examples demonstrate that there is no necessary and sufficient condition between X being a $f_{\text{rt}} - T_i$ set and Y being a T_i-space, where $i = 0, 1, 2$.

Example 4.10. *Let*

$$X = \{x_1, x_2, x_3, x_4\}, Y = \{0, 1\}$$

and

$$\mathcal{R} = \{r_1, r_2, r_3, r_4, r_5\}.$$

Consider the topological space $(Y, \{\emptyset, Y, \{1\}\})$ and let $f: X \times \mathcal{R} \to Y$ be defined as in Table 14.

Table 14. Function f with its corresponding values.

f	r_1	r_2	r_3	r_4	r_5
x_1	1	1	1	1	0
x_2	1	1	1	1	0
x_3	0	0	1	1	0
x_4	1	1	0	0	0

Thus, X is not an $f_{rt} - T_0$ set, whereas Y is a T_0-space.

Example 4.11. *Consider*

$$Y = \{y_1, y_2, y_3\}, \quad \mathcal{T} = \{\emptyset, Y, \{y_1\}\}$$

and let $g\colon X \times \mathcal{R} \to Y$ be defined as in Table 15.

Table 15. Function g with its corresponding values.

g	r_1	r_2	r_3	r_4	r_5
x_1	y_1	y_1	y_2	y_2	y_3
x_2	y_2	y_1	y_1	y_3	y_3
x_3	y_2	y_2	y_3	y_3	y_1
x_4	y_3	y_1	y_3	y_2	y_2

Consequently, X is a $g_{rt} - T_0$ set, whereas $(Y, \mathcal{T})$ is not a T_0-space.

Example 4.12. *Let*

$$X = \{x_1, x_2, x_3, x_4\}, \quad \mathcal{R} = \{r_1, r_2, r_3\}, \quad Y = \{0, 1, 2\}, \quad \text{and} \quad \mathcal{T} = \{\emptyset, Y, \{0\}, \{1\}, \{0, 1\}\}.$$

Consider the function $f\colon X \times \mathcal{R} \to Y$, defined as in Table 16.

Table 16. Function f with its corresponding values.

f	r_1	r_2	r_3
x_1	0	1	0
x_2	1	0	1
x_3	1	1	2
x_4	2	0	0

Then X is an $f_{rt} - T_2$ set, while $(Y, \mathcal{T})$ is not an T_2-space.

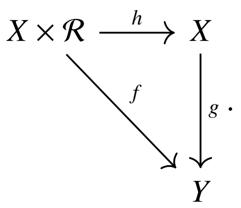
Example 4.13. *Let $X = \mathbb{Z}$, $\mathcal{R} = \mathbb{Z} - 0$, and $Y = \mathbb{R}$ with the usual topology $\mathcal{U}$. Consider the function $f\colon \mathbb{Z} \times \mathcal{R} \to (\mathbb{R}, \mathcal{U})$ defined as*

$$f(z, r) = \frac{\sin z}{r}.$$

For distinct points z_1 and $z_2 = z_1 + 2\pi$, we observe that $f(z_1, r) = f(z_2, r)$ for all $r \in \mathcal{R}$. Consequently, $\mathbb{Z}$ fails to satisfy the $f_{rt} - T_0$ property, while $(\mathbb{R}, \mathcal{U})$ is a T_2-space (Hausdorff space). Therefore, $\mathbb{Z}$ is not even an $f_{rt} - T_1$ or $f_{rt} - T_2$ set.

Theorem 4.14. *Let $(Y, \mathcal{T})$ be a discrete topological space, and consider a bijective map $g\colon X \to Y$. In this context, there exists a function $f\colon X \times \mathcal{R} \to Y$ such that it renders the set X an $f_{rt} - T_2$ set.*

Proof. Let h: $X \times \mathcal{R} \to X$ be a function defined as $h(x, r) = x$. Let us show that the following diagram is commutative:



Let us consider the composition of functions, $f = g \circ h$. For any distinct points x_1 and x_2 in X, we have

$$f(x_1, r) = (g \circ h)(x_1, r) = g(x_1)$$

and

$$f(x_2, r) = (g \circ h)(x_2, r) = g(x_2)$$

for all $r \in \mathcal{R}$. Since g is an injective map, it follows that $g(x_1) \neq g(x_2)$. Additionally, since $(Y, \mathcal{T})$ is a discrete topological space, we observe that

$$U = \{g(x_1)\} \in \mathcal{N}_{\mathcal{T}}(f(x_1, r))$$

and

$$V = \{g(x_2)\} \in \mathcal{N}_{\mathcal{T}}(f(x_2, r)), \ U \cap V = \emptyset.$$

Thus, X is an $f_{\mathrm{rt}} - T_2$ set. □

The following result can be derived as a consequence of Theorem 3.7.

Result 4.15. *Let X be a $f_{rt} - T_0$ set with respect to $\mathcal{T}$ and $\mathcal{P}$ is a finer topology than $\mathcal{T}$ (i.e., $\mathcal{T} \subset \mathcal{P}$). Then X is a $f_{rt} - T_0$ set with respect to $\mathcal{P}$.*

The result stated in 4.15 holds true for $f_{\mathrm{rt}} - T_2$ as well.

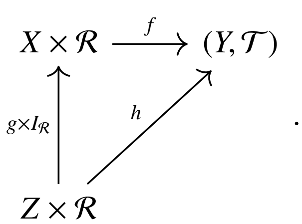
Theorem 4.16. *Let X be a $f_{rt} - T_2$ set with respect to $\mathcal{T}$, and $\mathcal{P}$ be a finer topology than $\mathcal{T}$. Then X is a $f_{rt} - T_2$ set with respect to $\mathcal{P}$.*

Proof. Let x_1 and x_2 be distinct points in X. As X is an $f_{\mathrm{rt}} - T_2$ set, there exist open sets $U \in \mathcal{T}(f(x_1, r))$ and $V \in \mathcal{T}(f(x_2, r))$ for some $r \in \mathcal{R}$, such that $U \cap V = \emptyset$. Additionally, since $\mathcal{T} \subseteq \mathcal{P}$, it follows that U and V are in $\mathcal{P}$. Hence, X is also an $f_{\mathrm{rt}} - T_2$ set with respect to $\mathcal{P}$. □

Let us state the theorem that proves the preservation of the $f_{\mathrm{rt}} - T_2$ property under bijective functions:

Theorem 4.17. *Let g: $Z \to X$ be a bijective map. If X is an $f_{rt} - T_2$ set, then there exists a map h: $Z \times \mathcal{R} \to (Y, \mathcal{T})$ such that Z becomes an $h_{rt} - T_2$ set.*

Proof. Let us show that the following diagram is commutative:



We define the map

$$h : Z \times \mathcal{R} \to (Y, \mathcal{T})$$

as

$$h = f \circ (g \times I_{\mathcal{R}}),$$

where $I_{\mathcal{R}}$ represents the identity function on the reference set $\mathcal{R}$. Now, consider two distinct points, z_1 and z_2, in Z. Then

$$h(z_1, r) = f(g(z_1), I_{\mathcal{R}}(r)) = f(g(z_1), r),$$

$$h(z_2, r) = f(g(z_2), I_{\mathcal{R}}(r)) = f(g(z_2), r).$$

Since g is bijective, we have

$$(g(z_1), r) \neq (g(z_2), r)$$

for all $r \in \mathcal{R}$. Additionally, as X is an $f_{\mathrm{rt}} - T_2$ set, there exist open sets

$$U \in \mathcal{T}(f(g(z_1), r)) \text{ and } V \in \mathcal{T}(f(g(z_2), r))$$

such that $U \cap V = \emptyset$. Therefore,

$$U \in \mathcal{T}(h(z_1, r)) \text{ and } V \in \mathcal{T}(h(z_2, r))$$

such that $U \cap V = \emptyset$. Thus, we have shown that Z is also an $h_{\mathrm{rt}} - T_2$ set. □

Result 4.18. *Let g: $Z \to X$ be a bijective map. The results obtained from the relationship* (4.2) *and Theorem 3.14 are as follows:*

i) If X is an $f_{rt} - T_1$ set, then there exists a map h: $Z \times \mathcal{R} \to (Y, \mathcal{T})$ such that Z becomes an $h_{rt} - T_1$ set.
ii) If X is an $f_{rt} - T_0$ set, then there exists a map h: $Z \times \mathcal{R} \to (Y, \mathcal{T})$ such that Z becomes an $h_{rt} - T_0$ set.

The following theorem demonstrates that the property of being an $f_{\mathrm{rt}} - T_2$ set is a hereditary property.

Theorem 4.19. *Let X be an $f_{rt} - T_2$ set, and $A \subset X$. Then A is also an $g_{rt} - T_2$ set.*

Proof. Let X be an $f_{\mathrm{rt}} - T_0$ set, and $A \subset X$. Let g: $A \times R \to Y$ be defined as $g(a, r) = f(a, r)$. Consider any distinct points $a, b \in A$. Since $a, b \in X$ and X is an $f_{\mathrm{rt}} - T_2$, there exist open sets U and V in $\mathcal{T}$ such that

$$g(a, r) = f(a, r) \in U, g(b, r) = f(b, r) \in V \text{ and } U \cap V = \emptyset.$$

Therefore A is an $g_{\mathrm{rt}} - T_2$ set. □

Result 4.20. *Let f: $X \times R \to Y$, $A \subseteq X$ and g: $A \times R \to Y$, $g(a, r) = f(a, r)$. Then*

$$X \textit{ is an } f_{rt} - T_0 \Rightarrow A \textit{ is an } g_{rt} - T_0,$$

$$X \textit{ is an } f_{rt} - T_1 \Rightarrow A \textit{ is an } g_{rt} - T_1.$$

The following example demonstrates that the converses of Theorem 4.19 and Result 4.20 are not true.

Example 4.21. *Let us consider Example 4.10. Let*

$$A = \{x_1, x_3, x_4\} \subseteq X.$$

Then A is an $g_{rt} - T_0$ set, while X is not an $f_{rt} - T_0$ set, where g is the restriction of f to A.

Similar examples can be found for $f_{rt} - T_1$ and $f_{rt} - T_2$ sets as well.

The following example establishes that the property of being $f_{rt} - T_0$ sets with respect to the same topological space, is not preserved under the Cartesian product.

Example 4.22. *Let*

$$Y = \{y_1, y_2, y_3\}, \quad \mathcal{T} = \{\emptyset, Y, \{y_1, y_2\}\}.$$

Consider $X_1 = \{x_1, x_2\}$, $\mathcal{R}_1 = \{r_1, r_2\}$, and f be defined as

$$f(x_1, r_1) = f(x_1, r_2) = y_1, \quad f(x_2, r_1) = f(x_2, r_2) = y_2.$$

Let

$$X_2 = \{x'_1, x'_2, x'_3\}, \quad \mathcal{R}_2 = \{r'_1, r'_2\},$$

and g be defined as

$$g(x'_1, r'_1) = g(x'_2, r'_2) = g(x'_3, r'_1) = g(x'_3, r'_2) = y_3, \quad g(x'_1, r'_2) = y_2, g(x'_2, r'_1) = y_1.$$

Let us define

$$h : (X_1 \times X_2) \times (\mathcal{R}_1 \times \mathcal{R}_2) \to ((Y \times Y), \mathcal{P}),$$

such that

$$h((x, x'), (r, r')) = (f(x, r), g(x', r')),$$

where $\mathcal{P}$ is the product topology on $Y \times Y$. The table for h is given below in Table 17.

Table 17. Function h with its corresponding values.

h	$r_1 \times r'_1$	$r_1 \times r'_2$	$r_2 \times r'_1$	$r_2 \times r'_2$
(x_1, x'_1)	(y_1, y_3)	(y_1, y_2)	(y_1, y_3)	(y_1, y_2)
(x_1, x'_2)	(y_1, y_1)	(y_1, y_3)	(y_1, y_1)	(y_1, y_3)
(x_1, x'_3)	(y_1, y_3)	(y_1, y_3)	(y_1, y_3)	(y_1, y_3)
(x_2, x'_1)	(y_2, y_3)	(y_2, y_2)	(y_2, y_3)	(y_2, y_2)
(x_2, x'_2)	(y_2, y_1)	(y_2, y_3)	(y_2, y_1)	(y_2, y_3)
(x_2, x'_3)	(y_2, y_3)	(y_2, y_3)	(y_2, y_3)	(y_2, y_3)

Therefore, $X_1 \times X_2$ is not an $h_{rt} - T_0$ set, while X_1 and X_2 are $-_{rt} - T_0$ sets.

Theorem 4.23. *Let X be an $f_{rt} - T_2$ set with respect to the topological space (X, τ), and $(Y, \mathcal{T})$ be homeomorphic to $(Z, \mathcal{V})$. Then X is an $h_{rt} - T_2$ set with respect to the topological space $(Z, \mathcal{V})$.*

Proof. Since X is an $f_{\mathrm{rt}} - T_2$ set, for any distinct points $x_1, x_2 \in X$, there exist open sets $U, V \in \mathcal{T}$ such that $f(x_1, r) \in U, f(x_2, r) \in V$, and $U \cap V = \emptyset$. Let g be the homeomorphism between Y and Z. Define $h = g \circ f$.

$$\begin{array}{ccc} X\times\mathcal{R} & \xrightarrow{f} & (Y,\mathcal{T}) \\ & \overset{h}{\searrow} & \big\downarrow g \\ & & (Z,\mathcal{V}) \end{array} .$$

Then we have

$$h(x_1, r) = g(f(x_1, r)) \in g(U) \in \mathcal{V},$$

$$h(x_2, r) = g(f(x_1, r)) \in g(U) \in \mathcal{V}$$

and

$$g(U \cap V) = g(U) \cap g(V) = \emptyset.$$

Consequently, X is an $(g \circ f)_{\mathrm{rt}} - T_2$ set. □

Similar to the expression in Theorem 4.19, the validity extends to reference sets. The subsequent theorem establishes that the separation axioms of $f_{\mathrm{rt}} - T_i$ provided by X remain preserved in the event of a one-to-one correspondence between two reference sets.

Theorem 4.24. *Let X be an $f_{rt} - T_i$ set, h: $\mathcal{R} \to \mathcal{R}_1$ is a bijective map. then X is a $g_{rt} - T_i$ set, where $i = 0, 1, 2$.*

Proof. Let us define g: $X \times \mathcal{R}_1 \to (Y, \mathcal{T})$ as the composition of f and $(1_X, h)$:

$$\begin{array}{ccc} X\times\mathcal{R} & \xrightarrow{f} & (Y,\mathcal{T}) \\ {\scriptstyle (1_X,h)}\big\uparrow & \overset{g}{\nearrow} & \\ X\times\mathcal{R}_1 & & \end{array} .$$

Consider distinct points x_1 and x_2 in X:

i) Suppose X is an $f_{\mathrm{rt}} - T_0$ set. Then there exists $r \in \mathcal{R}$ such that

$$\mathcal{N}_{\mathcal{T}}(f(x_1, r)) \neq \mathcal{N}_{\mathcal{T}}(f(x_2, r)).$$

Since h is a bijective map, there exists $r' \in \mathcal{R}_1$ such that $h(r') = r$. Therefore,

$$g(x_1, r') = (f \circ (1_X, h))(x_1, r') = f(x_1, h(r')) = f(x_1, r), \tag{4.3}$$

$$g(x_2, r') = (f \circ (1_X, h))(x_2, r') = f(x_2, h(r')) = f(x_2, r). \tag{4.4}$$

Hence,

$$\mathcal{N}_{\mathcal{T}}(g(x_1, r')) \neq \mathcal{N}_{\mathcal{T}}(g(x_2, r')).$$

ii) Suppose X is an $f_{\text{rt}} - T_1$ set. Then there exist an open set U and a reference point $r \in \mathcal{R}$ such that $f(x_1, r) \in U$ and $f(x_2, r) \notin U$, or there exist an open set V and a reference point r_0 such that $f(x_2, r_0) \in V$ and $f(x_1, r_0) \notin V$. Since h is a bijective map, there exists $r', r'_0 \in \mathcal{R}_1$ such that

$$h(r') = r \quad \text{and} \quad h(r'_0) = h(r_0).$$

Thus, from (4.3) and (4.4), the result follows.

iii) The argument can be extended similarly to case (ii).

□

5. Conclusions and future work

In this article, we have introduced a novel approach to studying separation axioms using functions instead of topological spaces. That is, we have categorized elements of a set without an inherent topological structure. To familiarize ourselves with this approach, we have presented the concept of distinguishable relation over the universal set, which is defined by a topology over another set. We have discussed the main properties of this relationship and provided some elucidative examples. To facilitate the way of discovering distinguishable relations, we provide a color technique for the equivalences classes inspired by a given topology. Then, we have defined a novel set of separation axioms within these structureless sets and revealed the interrelations between them with the aid of some illustrative instances.

In future work, we will plan to display separation axioms corresponding to regularity and normality by using the current approach. Also, we shall examine how the proposed separation axioms behave when the structure of the topology is relaxed, that is, it is replaced by supra topology, infra topology, and weak structure. Moreover, we expect that utilizing elements from the reference set to determine distinguishable elements in the desired context (where the chosen function defines the nature of the differentiation) would be highly advantageous in practical applications. Therefore, we look for some real applications in the upcoming work.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interests

The authors declare that they have no competing interests.

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