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*Research article*

# Convergence results for cyclic-orbital contraction in a more generalized setting with application

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**Abstract:** In the realm of double-controlled metric-type spaces, we investigated obtaining fixed points using the application of cyclic orbital contractive conditions. Diverging from conventional approaches utilized in standard metric spaces, our technique took a unique route due to the unique features of our structure. We demonstrated the significance of our outcomes through exemplary cases, clarifying the breadth of our results through comprehensive investigations. Significantly, our work not only improved and broadened earlier findings in the literature, but also offered unique notions that were discussed in our explanatory notes. Towards the end of our inquiry, we used insights obtained from previous discoveries to develop a second-order differential equation. This equation was an effective tool for dealing with the second class of Fredholm integral problems. In conclusion, this investigation extended our examination of double-controlled metric type spaces by providing new insights on fixed point theory, expanding on prior debates and building a substantial road towards solving a class of integral equations.

**Keywords:** double controlled metric type space; fixed point; cyclic contraction; $F$-type contractive mappings

**Mathematics Subject Classification:** 34B15, 41A65, 46B20, 47H10

## 1. Introduction

A key idea in mathematics is fixed point theory, which is concerned with investigating the existence and characteristics of fixed points under particular mappings and transformations. At its heart, fixed point theory analyzes points that are unchanged after implementing a function, offering information on the behavior and structure of numerous mathematical systems. This theory has extensive applicability across different domains. In physics, it helps explain the dynamics of physical systems governed by nonlinear equations. In computer science, fixed point theory is used in algorithm design, optimization, and artificial intelligence, notably in constructing iterative approaches for solving equation systems. Furthermore, in engineering fields, fixed point theory helps with the evaluation and design of control systems, signal processing, and numerical simulations. Overall, fixed point theory is a strong tool with many applications, allowing academics to model, analyze, and solve a wide range of complicated issues from many areas.

After the revolutionary contribution of Stefan Banach [1] to fixed point theory, multiple pioneers proceeded to advance the field, extending its applications and theoretical foundations. After that, the seminal work of Stanisaw Saks [2] added to the field's richness, providing novel insights and increasing our understanding of fixed point occurrences. Their improvements to the theory of multi-valued mappings, as well as Saks' explorations into the topological features of fixed point sets, were crucial in modeling the environment of fixed point theory. Together with Banach, these forefathers opened the path for the considerable study and various applications that characterize the current state of fixed point theory.

In 1989, Bakhtin [3] and in 1993 Czerwik [4] presented the concept of $b$-metric space. In 2017, T. Kamran [5] introduced the concept of extended $b$-metric space. In 2018, Nabil Mlaiki [6], introduced the concept of controlled metric type spaces and established some fixed point results in these spaces. In his paper, Mlaiki defines a controlled metric type space as a set equipped with a control function that measures the distance between two points. This function is required to satisfy certain conditions, which generalize the properties of a metric space. After that, this work is generalized by T. Abdeljawad [7] by introducing one more function in the triangular inequality named as double controlled metric type spaces and discuss the existence and uniqueness of some fixed point results. In 2021, H. Ahmad et al. [8] explored double-controlled partial metric type spaces and convergence results. These studies underscored the evolving nature of fixed point theory, extending its reach to cutting-edge applications.

Initially, $F$-contraction was introduced by D. Wardowski [9] in 2012. The exploration of $F$-contractions in fixed point theory has garnered great attention in recent years, offering a versatile and powerful framework for studying fixed points in metric spaces. This papers lay the foundation for a deeper understanding of $F$-contractions in the context of fixed-point theory and their wide-ranging applications in mathematics and related fields. For further analysis on $F$-contractions, see [10, 11].

Inspired by the literature, we are driven by the ambition to extend fixed point theorems to $\mathfrak{F}$ type cyclic orbital contractions within the framework of double controlled metric type spaces. Unlike traditional metric spaces, this endeavor necessitates the use of specialized approaches tailored to our setting. Through symbolic examples, we not only verify our fundamental results but also showcase their practical relevance. Furthermore, our contributions expand on previous research, enhancing the theoretical environment and creating opportunities for broader applications. As a result of our work, we present a second-order differential equation that provides a powerful solution technique for the second kind of Fredholm-type integral equations. This endeavor not only broadens our understanding of fixed point theory, but also reveals promising approaches to solving complicated mathematical problems with real-world ramifications.

We will implement Definitions 1.1–1.3 in the next section to establish a cyclic orbital contraction Theorem 1 in the context of double-controlled metric type space.

**Definition 1.1.** *[12] Let* $\mathbb{T} : \Theta_1 \cup \Theta_2 \to \Theta_1 \cup \Theta_2$ *be a cyclic, if* $\mathbb{T}(\Theta_1) \subset \Theta_2$ *and* $\mathbb{T}(\Theta_2) \subset \Theta_1$*, where* $\Theta_1, \Theta_2$ *are subsets of a metric space* $(X, \mathrm{d})$ *that are nonempty.*

**Definition 1.2.** *[12] Let* $\mathbb{T} : \Theta_1 \cup \Theta_2 \to \Theta_1 \cup \Theta_2$ *know as cyclic contraction, if for all* $c_1 \in \Theta_1$*,* $c_2 \in \Theta_2$*,* $\exists\ \kappa \in (0, 1)$ *in such a way that*

$$\mathrm{d}(\mathbb{T}\hbar_1, \mathbb{T}\hbar_2) \leq \kappa \mathrm{d}(\hbar_1, \hbar_2).$$

**Definition 1.3.** *[13] Suppose* $\mathbb{T} : \Theta_1 \cup \Theta_2 \to \Theta_1 \cup \Theta_2$ *be a cyclic orbital contraction, if for all* $\hbar_1 \in \Theta_1$*,* $\exists\ \kappa \in (0, 1)$ *given that*

$$\mathrm{d}(\mathbb{T}^{2n}\hbar_1, \mathbb{T}\hbar_2) \leq \kappa \mathrm{d}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2),$$

*where* $\hbar_2 \in \Theta_1$*,* $n \in \mathbb{N}$ *and* $\Theta_1$*,* $\Theta_2$ *are closed subsets of* $X$*.*

We will exploit Definition 1.4 to develop the Theorem 2 in the next segment.

**Definition 1.4.** *[9] Assume that* $\mathbb{T} : G \to G$ *be a mapping on a metric space* $(X, \mathrm{d})$ *is known as* $\mathfrak{F}$*-contraction, whenever there is* $\Omega > 0$ *in such a way that*

$$\Omega + \mathfrak{F}(\mathrm{d}(\mathbb{T}\hbar_1, \mathbb{T}\hbar_2)) \leq \mathfrak{F}(\mathrm{d}(\hbar_1, \hbar_2)),$$

*for all* $\hbar_1, \hbar_2 \in X$ *with* $\mathrm{d}(\hbar_1, \hbar_2) > 0$*, in relation to the function* $\mathfrak{F} : [0, \infty) \to \mathbb{R}$ *that adhering the subsequent axioms:*

*(I)* $\mathfrak{F}$ *is sharply increasing;*
*(II) across every sequence* $\{a_n\}_{n\in\mathbb{N}}$ *encompassing non-negative real numbers,* $\lim_{n\to\infty} a_n = 0$ *is true in both directions* $\lim_{n\to\infty} \mathfrak{F}(a_n) = -\infty$*;*
*(III) within each sequence* $\{a_n\}_{n\in\mathbb{N}}$ *characterized by non-negative real numbers,* $\lim_{n\to\infty} a_n = 0$*,* $\exists\ \kappa \in (0, 1)$ *in such a way that* $\lim_{n\to\infty} (a_n)^\kappa \mathfrak{F}(a_n) = 0$*.*
*We symbolize* $\mathcal{F}$ *the in adherence to the collection of all functions* $\mathfrak{F}$ *pleasing* $(I)$*–*$(III)$*.*

**Definition 1.5.** *[7] Let* $G \neq \varphi$ *and consider* $\Phi, \Psi : G \times G \to [1, \infty)$ *be functions. Let* $\mathcal{S} : G \times G \to [0, \infty)$ *pleasing*

(1) $\mathcal{S}(\hbar_1, \hbar_2) = 0$ if and only if $\hbar_1 = \hbar_2$,
(2) $\mathcal{S}(\hbar_1, \hbar_2) = \mathcal{S}(\hbar_2, \hbar_1)$,
(3) $\mathcal{S}(\hbar_1, \hbar_2) \leq \Phi(\hbar_1, \hbar_3)\mathcal{S}(\hbar_1, \hbar_3) + \Psi(\hbar_3, \hbar_2)\mathcal{S}(\hbar_3, \hbar_2)$,

for all $\hbar_1, \hbar_2, \hbar_3 \in G$, then $(G, \mathcal{S})$ is known as double controlled metric type space.

**Remark 1.** *Every double controlled metric type space $(G, \mathcal{S})$ is controlled metric type space, an alternative view of $\Phi(\hbar_1, \hbar_2) = \Psi(\hbar_1, \hbar_2) \geq 1$. However, the contrary statement is not accurate (see Example 1).*

**Example 1.** *[7] Let $G = [0, \infty)$ and $\mathcal{S}$ be defined as*

$$\mathcal{S}(\hbar, \varsigma) = \begin{cases} 0, & \text{iff } \hbar = \varsigma, \\ \frac{1}{\hbar} & \text{if } \hbar \geq 1 \text{ and } \varsigma \in [0, 1), \\ \frac{1}{\varsigma} & \text{if } \varsigma \geq 1 \text{ and } \hbar \in [0, 1), \\ 1 & \text{if not.} \end{cases}$$

*Consider $\Phi, \Psi : G \times G \to [1, \infty)$ be functions*

$$\Phi(\hbar, \varsigma) = \begin{cases} \hbar, & \text{if } \hbar, \varsigma \geq 1, \\ 1, & \text{if not,} \end{cases}$$

*and*

$$\Psi(\hbar, \varsigma) = \begin{cases} 1, & \text{if } \hbar, \varsigma < 1, \\ \max\{\hbar, \varsigma\}, & \text{if not.} \end{cases}$$

*Clearly $(G, \mathcal{S})$ is double controlled metric type space is not necessarily a controlled metric type space because $\Phi(\hbar_1, \hbar_2) \neq \Psi(\hbar_1, \hbar_2)$.*

Definitions 1.6–1.8 used to prove the Cauchy sequences in 1,2.

**Definition 1.6.** *[7] A sequence $\{\hbar_n\}$ in $(G, \mathcal{S})$ converges to some $\hbar$ in $G$, whenever under each positive $\epsilon$, there may exists a positive $N_\epsilon$ in such a way that $\mathcal{S}(\hbar_n, \hbar) < \epsilon$ for each $n \geq N_\epsilon$. It might be articulated as*

$$\lim_{n\to\infty} \hbar_n = \hbar.$$

**Definition 1.7.** *[7] The sequence $\{\hbar_n\}$ in $(G, \mathcal{S})$ is labeled a Cauchy sequence, whenever given each $\epsilon > 0, \mathcal{S}(\hbar_n, \hbar_m) < \epsilon$ for all $m, n \geq N_\epsilon$ where $N_\epsilon \in \mathbb{N}$.*

**Definition 1.8.** *[7] Since $(G, \mathcal{S})$ is known as complete under the circumstance that each Cauchy sequence converges in $G$.*

## 2. Main results

In this segment, we elucidate the concept of cyclic orbital contractions within the environment of double controlled metric type space.

**Definition 2.1.** *Let $(G, \mathcal{S})$ be a double controlled metric type space characterized by two nonempty subsets $\Theta_1$ and $\Theta_2$. Then $\mathbb{T} : \Theta_1 \cup \Theta_2 \to \Theta_1 \cup \Theta_2$ be a cyclic orbital contraction, if there is $\hbar_1 \in \Theta_1$, then $\exists\, \lambda \in (0, 1)$ in such a way that*

$$\mathcal{S}(\mathbb{T}^{2n}\hbar_1, \mathbb{T}\hbar_2) \leq \lambda \mathcal{S}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2), \tag{2.1}$$

*where $\hbar_2 \in \Theta_1$ and $n \in \mathbb{N}$.*

**Theorem 1.** *Let $(G, \mathcal{S})$ double controlled metric type space endowed with two nonempty subsets $\Theta_1$ and $\Theta_2$ in such a way that $\mathcal{S}$ is a continuous functional. If $\mathbb{T} : \Theta_1 \cup \Theta_2 \to \Theta_1 \cup \Theta_2$ linked with cyclic orbital contraction following that for some $\hbar_0 \in G$, such that*

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\Phi(\hbar_{i+1}, \hbar_{i+2})}{\Phi(\hbar_i, \hbar_{i+1})} \Psi(\hbar_i, \hbar_m) < \frac{1}{\lambda}, \text{ where } \lambda \in (0, 1) \tag{2.2}$$

*and $\hbar_n = \mathbb{T}^n \hbar_0$, $n = 1, 2, \ldots$. Then $\Theta_1 \cap \Theta_2 \neq \phi$ and $\mathbb{T}$ concedes at most one fixed point.*

*Proof.* Let $\hbar_0 \in \Theta_1$ be any element satisfying Eq (2.1). Now, we generate an iterative sequence $\{\hbar_n\}$ starting from $\hbar_0$ as follows:

$$\hbar_1 = \mathbb{T}\hbar_0,\ \hbar_2 = \mathbb{T}\hbar_1 = \mathbb{T}(\mathbb{T}\hbar_0) = \mathbb{T}^2\hbar_0,\ \ldots,\ \hbar_n = \mathbb{T}^n\hbar_0,\ \ldots$$

By using Eq (2.1), we have

$$\mathcal{S}(\mathbb{T}^2\hbar_0, \mathbb{T}\hbar_0) \leq \lambda \mathcal{S}(\mathbb{T}\hbar_0, \hbar_0).$$

Following the above inequality same, we have

$$\mathcal{S}(\mathbb{T}^3\hbar_0, \mathbb{T}^2\hbar_0) \leq \lambda \mathcal{S}(\mathbb{T}^2\hbar_0, \mathbb{T}\hbar_0) \leq \lambda^2 \mathcal{S}(\mathbb{T}\hbar_0, \hbar_0).$$

Continue the same process under any circumstances $n \in \mathbb{N}$, so we attain

$$\mathcal{S}(\mathbb{T}^{n+1}\hbar_0, \mathbb{T}^n\hbar_0) \leq \lambda^n \mathcal{S}(\mathbb{T}\hbar_0, \hbar_0),$$

that is,

$$\mathcal{S}(\hbar_{n+1}, \hbar_n) \leq \lambda^n \mathcal{S}(\hbar_1, \hbar_0). \tag{2.3}$$

We need to demonstrate that the sequence $\{\hbar_n\}$ is a Cauchy sequence for all $n, m \in \mathbb{N}$ with $m > n$, we gain

$$\begin{aligned}
\mathcal{S}(\hbar_n, \hbar_m) &\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \Psi(\hbar_{n+1}, \hbar_m)\mathcal{S}(\hbar_{n+1}, \hbar_m) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \Phi(\hbar_{n+1}, \hbar_{n+2})\Psi(\hbar_{n+1}, \hbar_m)\mathcal{S}(\hbar_{n+1}, \hbar_{n+2}) \\
&+ \Psi(\hbar_{n+1}, \hbar_m)\Psi(\hbar_{n+2}, \hbar_m)\mathcal{S}(\hbar_{n+2}, \hbar_m) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \Phi(\hbar_{n+1}, \hbar_{n+2})\Psi(\hbar_{n+1}, \hbar_m)\mathcal{S}(\hbar_{n+1}, \hbar_{n+2}) \\
&+ \Phi(\hbar_{n+2}, \hbar_{n+3})\Psi(\hbar_{n+1}, \hbar_m)\Psi(\hbar_{n+2}, \hbar_m)\mathcal{S}(\hbar_{n+2}, \hbar_{n+3}) \\
&+ \Psi(\hbar_{n+1}, \hbar_m)\Psi(\hbar_{n+2}, \hbar_m)\Psi(\hbar_{n+3}, \hbar_m)\mathcal{S}(\hbar_{n+3}, \hbar_m) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \sum_{i=n+1}^{m-2} \left( \prod_{j=n+1}^{i} \Psi(\hbar_j, \hbar_m) \right) \Phi(\hbar_i, \hbar_{i+1})\mathcal{S}(\hbar_i, \hbar_{i+1}) \\
&+ \prod_{k=n+1}^{m-1} \Psi(\hbar_k, \hbar_m)\mathcal{S}(\hbar_{m-1}, \hbar_m) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\lambda^n \mathcal{S}(\hbar_0, \hbar_1) + \sum_{i=n+1}^{m-2} \left( \prod_{j=n+1}^{i} \Psi(\hbar_j, \hbar_m) \right) \Phi(\hbar_i, \hbar_{i+1})\lambda^i \mathcal{S}(\hbar_0, \hbar_1) \\
&\quad + \prod_{k=n+1}^{m-1} \Psi(\hbar_k, \hbar_m)\lambda^{m-1}\mathcal{S}(\hbar_0, \hbar_1)
\end{aligned}$$

$$
\begin{aligned}
&\leq \Phi(\hbar_n,\hbar_{n+1})\lambda^n\mathcal{S}(\hbar_0,\hbar_1)+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\Psi(\hbar_j,\hbar_m)\right)\Phi(\hbar_i,\hbar_{i+1})\lambda^i\mathcal{S}(\hbar_0,\hbar_1)\\
&+\prod_{k=n+1}^{m-1}\Psi(\hbar_k,\hbar_m)\Phi(\hbar_{m-1},\hbar_m)\lambda^{m-1}\mathcal{S}(\hbar_0,\hbar_1)\\
&= \Phi(\hbar_n,\hbar_{n+1})\lambda^n\mathcal{S}(\hbar_0,\hbar_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\Psi(\hbar_j,\hbar_m)\right)\Phi(\hbar_i,\hbar_{i+1})\lambda^i\mathcal{S}(\hbar_0,\hbar_1)\\
&\leq \Phi(\hbar_n,\hbar_{n+1})\lambda^n\mathcal{S}(\hbar_0,\hbar_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\Psi(\hbar_j,\hbar_m)\right)\Phi(\hbar_i,\hbar_{i+1})\lambda^i\mathcal{S}(\hbar_0,\hbar_1)\\
&\leq \Phi(\hbar_n,\hbar_{n+1})\lambda^n\mathcal{S}(\hbar_0,\hbar_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\Psi(\hbar_j,\hbar_m)\right)\Phi(\hbar_i,\hbar_{i+1})\lambda^i\mathcal{S}(\hbar_0,\hbar_1).
\end{aligned}
$$

Assume that

$$C_p=\sum_{i=0}^{p}\left(\prod_{j=0}^{i}\Psi(\hbar_j,\hbar_m)\right)\Phi(\hbar_i,\hbar_{i+1})\lambda^i\mathcal{S}(\hbar_0,\hbar_1).$$

Then, we obtain

$$\mathcal{S}(\hbar_n,\hbar_m)\leq\mathcal{S}(\hbar_0,\hbar_1)[\lambda^n(\mathcal{S}(\hbar_0,\hbar_1))\Phi(\hbar_n,\hbar_{n+1})+(C_{m-1}-C_n)]. \tag{2.4}$$

Using ratio test, we have

$$a_i=\left(\prod_{j=0}^{i}\Psi(\hbar_j,\hbar_m)\right)\Phi(\hbar_i,\hbar_{i+1})\lambda^i(\mathcal{S}(\hbar_0,\hbar_1)),\text{ where }\frac{a_{i+1}}{a_i}<\frac{1}{\lambda},$$

taking $\lim_{n,m\to\infty}$, so (2.4) becomes

$$\lim_{n,m\to\infty}\mathcal{S}(\hbar_n,\hbar_m)=0.$$

This implies that $\{\hbar_n\}$ is a Cauchy sequence in a complete $(G,\mathcal{S})$. Therefore, as a consequence, there exists $\rho\in\Theta_1\cup\Theta_2$ in such a way that $\hbar_n\to\rho$. Now, it's worth noting that the sequences $\{\hbar_{2n}\}=\{\mathbb{T}^{2n}\hbar_0\}$ in $\Theta_1$ and $\{\hbar_{2n-1}\}=\{\mathbb{T}^{2n-1}\hbar_0\}$ in $\Theta_2$ so the pair converge to $\rho$. As though the sets $\Theta_1$, $\Theta_2$ are closed in $G$ and $\rho\in\Theta_1\cap\Theta_2$, thus $\Theta_1\cap\Theta_2\neq\phi$. Following this, we establish that $\mathbb{T}$ concedes a fixed point $\rho$. Utilizing the continuity of $\mathcal{S}$ leads to

$$\mathcal{S}(\rho,\mathbb{T}\rho)=\lim_{n\to\infty}\mathcal{S}(\mathbb{T}^{2n}\hbar,\mathbb{T}\rho)\leq\lambda\lim_{n\to\infty}\mathcal{S}(\mathbb{T}^{2n-1}\hbar,\rho)=0.$$

Therefore, $\mathbb{T}$ concedes $\rho$ as fixed point.

**Uniqueness.** Presume the existence of $\varrho\in\Theta_1\cap\Theta_2$, $\rho\neq\varrho$ in such a way that $\mathbb{T}\varrho=\varrho$. So,

$$\mathcal{S}(\rho,\varrho)=\mathcal{S}(\mathbb{T}\rho,\mathbb{T}\varrho)\leq\lambda\mathcal{S}(\rho,\varrho)<\mathcal{S}(\rho,\varrho),$$

this inequality contradicts the previous assertion. Consequently, $\rho=\varrho$ and $\mathbb{T}$ concedes $\rho$ as fixed point that is unique. □

**Remark 2.** *Either $(G, \mathcal{S})$ is not in predominantly a controlled metric type space, so Theorem 1 is different from the Theorem 1 of N. Alamgir [14]. If we take $\Phi(\hbar_1, \hbar_2) = \Psi(\hbar_1, \hbar_2) \geq 1$, then theorem presented above can be simplified as controlled metric type space. For $\Phi(\hbar_1, \hbar_2) \geq 1$, depends on the inequality of left hand sides then above Theorem shrinks to extended b-metric space [15] and for $\Phi(\hbar_1, \hbar_2) = s \geq 1$, the preceding theorem condenses to the b-metric space. If $s = 1$, then theorem outlined earlier can be shrinks to the main results of fixed points in metric space presented by Karpagam et al. [13].*

**Example 2.** *Let $G = \mathbb{R}$ and $\mathcal{S} : G \times G \to [0, \infty)$ is defined as*

$$\mathcal{S}(\hbar_1, \hbar_2) = (\hbar_1 - \hbar_2)^2.$$

*Suppose that $\Phi, \Psi : G \times G \to [1, \infty)$, where $\Phi(\hbar_1, \hbar_2) = 3\hbar_1 + 4\hbar_2 + 5$ and $\Psi(\hbar_1, \hbar_2) = 5\hbar_1 + 6\hbar_2 + 7$. It is easy to see that $(G, \mathcal{S})$ is a complete double controlled metric type space. Assume $\Theta_1 = [0, \frac{1}{4}]$, $\Theta_2 = [\frac{1}{5}, 1]$ and define a mapping $\mathbb{T} : \Theta_1 \cup \Theta_2 \to \Theta_1 \cup \Theta_2$ as:*

$$\mathbb{T}\hbar_1 = \begin{cases} \frac{1}{4}, & 0 \leq \hbar_1 \leq \frac{1}{5}, \\ \frac{1}{3}(1 - \hbar_1), & \frac{1}{5} < \hbar_1 \leq 1. \end{cases}$$

*First, we discuss some cases to indicate that $\mathbb{T}$ is a cyclic map:*

*(a) If $\hbar_1 = 0 \in \Theta_1$, then $\mathbb{T}0 = \frac{1}{4} \in \Theta_2$.*
*(b) If $\hbar_1 = \frac{1}{4} \in \Theta_1$, then $\mathbb{T}\frac{1}{4} = \frac{1}{4} \in \Theta_2$.*
*(c) If $\hbar_1 = \frac{1}{5} \in \Theta_2$, then $\mathbb{T}\frac{1}{5} = \frac{1}{4} \in \Theta_1$.*
*(d) If $\hbar_1 = 1 \in \Theta_2$, then $\mathbb{T}1 = 0 \in \Theta_1$.*

*Transparently, $\mathbb{T}(\Theta_1) \subseteq \Theta_2$, $\mathbb{T}(\Theta_2) \subseteq \Theta_1$ and $\mathbb{T}$ is a cyclic map. Now, we fix any $\hbar_1 = 0 \in \Theta_1$, then we have*

$$\mathbb{T}\hbar_1 = \frac{1}{4},\ \mathbb{T}^2\hbar_1 = \mathbb{T}(\mathbb{T}\hbar_1) = \frac{1}{4},\ \ldots .$$

*Thus, $\mathbb{T}^n\hbar_1 = \frac{1}{4}$, therefore $\mathbb{T}^{2n}\hbar_1 = \frac{1}{4}$ and $\mathbb{T}^{2n-1}\hbar_1 = \frac{1}{4}$.*
*For $\hbar_2$, we will examine the following scenarios:*
***Case 1.*** *If $\hbar_2 = 0$, $\mathbb{T}0 = \frac{1}{4}$, then*

$$\begin{aligned} \mathcal{S}(\mathbb{T}^{2n}\hbar_1, \mathbb{T}\hbar_2) &\leq \lambda \mathcal{S}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2), \\ \mathcal{S}(\frac{1}{4}, \frac{1}{4}) &\leq \lambda \mathcal{S}(\frac{1}{4}, 0), \\ 0 &\leq \lambda(\frac{1}{16}). \end{aligned}$$

***Case 2.*** *For $\hbar_2 = \frac{1}{4}$, $\mathbb{T}\frac{1}{4} = \frac{1}{4}$*

$$\begin{aligned} \mathcal{S}(\mathbb{T}^{2n}\hbar_1, \mathbb{T}\hbar_2) &\leq \lambda \mathcal{S}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2), \\ \mathcal{S}(\frac{1}{4}, \frac{1}{4}) &\leq \lambda \mathcal{S}(\frac{1}{4}, \frac{1}{4}), \\ 0 &\leq \lambda(0). \end{aligned}$$

***Case 3.*** *When $0 < \hbar_2 < \frac{1}{4}$, we will take subcases:*

***Subcase A.*** *If* $0 < \hbar_2 \le \frac{1}{5}$, $\mathbb{T}\hbar_2 = \frac{1}{4}$, *then we have*

$$\begin{aligned} \mathcal{S}(\mathbb{T}^{2n}\hbar_1, \mathbb{T}\hbar_2) &\le \lambda\mathcal{S}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2), \\ \mathcal{S}(\frac{1}{3}, \frac{1}{3}) &\le \lambda\mathcal{S}(\frac{1}{3}, \hbar_2), \\ 0 &\le \lambda(\frac{1}{3} - \hbar_2)^2. \end{aligned}$$

***Subcase B.*** *For* $\frac{1}{5} < \hbar_2 \le \frac{1}{4}$, $\mathbb{T}\hbar_2 = \frac{1}{2}(1 - \hbar_2)$, *then*

$$\begin{aligned} \mathcal{S}(\mathbb{T}^{2n}\hbar_1, \mathbb{T}\hbar_2) &\le \lambda\mathcal{S}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2), \\ \mathcal{S}(\frac{1}{4}, \frac{1}{3}(1 - \hbar_2))^2 &\le \lambda\mathcal{S}(\frac{1}{4}, \hbar_2), \\ \frac{1}{9}(\hbar_2 - \frac{1}{4})^2 &\le \lambda(\hbar_2 - \frac{1}{4})^2. \end{aligned}$$

*If we choose* $\lambda = \frac{1}{2} \in (0, 1)$, *then all the cases and their subcases are satisfied. Hence, all the conditions of Theorem 1 are satisfied, and* $\frac{1}{4}$ *is the unique fixed point of the cyclic mapping* $\mathbb{T}$.

**Definition 2.2.** *Let* $(G, \mathcal{S})$ *consist of two nonempty subsets* $\Theta_1$ *and* $\Theta_2$. *Then* $\mathbb{T} : \Theta_1 \cup \Theta_2 \to \Theta_1 \cup \Theta_2$ *be a cyclic orbital* $\mathfrak{F}$*-contraction, where* $\mathfrak{F} \in \mathcal{F}$, *if for some* $\hbar_1 \in \Theta_1$, $\exists\, \Omega > 0$ *in such a way that for all* $\hbar_1, \hbar_2 \in G$ *with* $\mathcal{S}(\mathbb{T}\hbar_1, \mathbb{T}\hbar_2) > 0$, *then*

$$\Omega + \mathfrak{F}(\lambda\mathcal{S}(\mathbb{T}^{2n}\hbar_1, \mathbb{T}\hbar_2)) \le \mathfrak{F}(\mathcal{S}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2)), \tag{2.5}$$

*where* $\lambda > 1$, $\hbar_2 \in \Theta_1$ *and* $n \in \mathbb{N}$.

**Theorem 2.** *Let* $(G, \mathcal{S})$ *double controlled metric type space having two nonempty subsets* $\Theta_1$ *and* $\Theta_2$. *Suppose* $\mathbb{T} : \Theta_1 \cup \Theta_2 \to \Theta_1 \cup \Theta_2$ *represents* $\mathfrak{F}$*-contraction that is cyclic orbital and continuous. Then for some* $\hbar_0 \in G$, *such that*

$$\sup_{m\ge1} \lim_{i\to\infty} \frac{\Phi(\hbar_{i+1}, \hbar_{i+2})}{\Phi(\hbar_i, \hbar_{i+1})} \Psi(\hbar_i, \hbar_m) < \frac{1}{\lambda}, \tag{2.6}$$

*where* $\hbar_n = \mathbb{T}^n\hbar_0$. *Then* $\Theta_1 \cap \Theta_2$ *is not empty and* $\mathbb{T}$ *concedes a fixed point.*

*Proof.* Assume a random element $\hbar_0 \in \Theta_1$ fulfilling Eq (2.5) for all $n \in \mathbb{N}$. Consequently, we obtain

$$\mathfrak{F}(\lambda^n\mathcal{S}(\mathbb{T}^{n+1}\hbar, \mathbb{T}^n\hbar)) \le \mathfrak{F}(\mathcal{S}(\mathbb{T}\hbar, \hbar)) - n\Omega. \tag{2.7}$$

Taking $n \to \infty$ in Eq (2.7), we gain $\lim_{n\to\infty} \mathfrak{F}(\mathcal{S}(\hbar_n, \hbar_{n+1})) = -\infty$. Implementing the prerequisite $(II)$ of the Definition 1.4, we gain

$$\lim_{n\to\infty} \lambda^n\mathcal{S}(\hbar_n, \hbar_{n+1}) = 0.$$

Additionally, from the constraint $(III)$, $\exists\, l \in (0, 1)$ in a manner that

$$\lim_{n\to\infty} (\lambda^n\mathcal{S}(\hbar_n, \hbar_{n+1}))^l \mathfrak{F}(\lambda^n\mathcal{S}(\hbar_n, \hbar_{n+1})) = 0.$$

Derived from Eq (2.7), for all $n \in \mathbb{N}$, the subsequent valid:

$$\begin{bmatrix} \lim_{n\to\infty}(\lambda^n\mathcal{S}(\hbar_n, \hbar_{n+1}))^l \mathfrak{F}(\mathcal{S}(\hbar_n, \hbar_{n+1})) \\ -\lim_{n\to\infty}(\lambda^n\mathcal{S}(\hbar_n, \hbar_{n+1}))^l \mathfrak{F}(\mathcal{S}(\mathbb{T}\hbar, \hbar)) \end{bmatrix} \le \lim_{n\to\infty} -(\lambda^n\mathcal{S}(\hbar_n, \hbar_{n+1}))^l n\Omega \le 0. \tag{2.8}$$

Taking $n \to \infty$ in (2.8), we acquire

$$\lim_{n\to\infty} n(\lambda^n \mathcal{S}(\hbar_n, \hbar_{n+1}))^l = 0. \tag{2.9}$$

Utilizing (2.9), $\exists\, n_1 \in \mathbb{N}$ in such way that $n\mathcal{S}(\hbar_n, \hbar_{n+1})^l \leq 1$ throughout $n \geq n_1$. So, for each $n \geq n_1$, we achieve

$$\mathcal{S}(\hbar_n, \hbar_{n+1}) \leq \frac{1}{n^{\frac{1}{l}}}. \tag{2.10}$$

We need to demonstrate that the sequence $\{\hbar_n\}$ is a Cauchy sequence across all $n, m \in \mathbb{N}$ with $m > n$, we gain

$$\begin{aligned}
\mathcal{S}(\mathbb{T}^n\hbar, \mathbb{T}^m\hbar) &= \mathcal{S}(\hbar_n, \hbar_m) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \Psi(\hbar_{n+1}, \hbar_m)\mathcal{S}(\hbar_{n+1}, \hbar_m) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \Phi(\hbar_{n+1}, \hbar_{n+2})\Psi(\hbar_{n+1}, \hbar_m)\mathcal{S}(\hbar_{n+1}, \hbar_{n+2}) \\
&+ \Psi(\hbar_{n+1}, \hbar_m)\Psi(\hbar_{n+2}, \hbar_m)\mathcal{S}(\hbar_{n+2}, \hbar_m) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \Phi(\hbar_{n+1}, \hbar_{n+2})\Psi(\hbar_{n+1}, \hbar_m)\mathcal{S}(\hbar_{n+1}, \hbar_{n+2}) \\
&+ \Phi(\hbar_{n+2}, \hbar_{n+3})\Psi(\hbar_{n+1}, \hbar_m)\Psi(\hbar_{n+2}, \hbar_m)\mathcal{S}(\hbar_{n+2}, \hbar_{n+3}) \\
&+ \Psi(\hbar_{n+1}, \hbar_m)\Psi(\hbar_{n+2}, \hbar_m)\Psi(\hbar_{n+3}, \hbar_m)\mathcal{S}(\hbar_{n+3}, \hbar_m) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\Psi(\hbar_j, \hbar_m)\right)\Phi(\hbar_i, \hbar_{i+1})\mathcal{S}(\hbar_i, \hbar_{i+1}) \\
&+ \prod_{k=n+1}^{m-1}\Psi(\hbar_k, \hbar_m)\mathcal{S}(\hbar_{m-1}, \hbar_m) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\Psi(\hbar_j, \hbar_m)\right)\Phi(\hbar_i, \hbar_{i+1})\mathcal{S}(\hbar_i, \hbar_{i+1}) \\
&+ \prod_{k=n+1}^{m-1}\Psi(\hbar_k, \hbar_m)\mathcal{S}(\hbar_i, \hbar_m) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\Psi(\hbar_j, \hbar_m)\right)\Phi(\hbar_i, \hbar_{i+1})\mathcal{S}(\hbar_i, \hbar_{i+1}) \\
&+ \prod_{k=n+1}^{m-1}\Psi(\hbar_k, \hbar_m)\Phi(\hbar_{m-1}, \hbar_m)\mathcal{S}(\hbar_{m-1}, \hbar_m) \\
&= \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\Psi(\hbar_j, \hbar_m)\right)\Phi(\hbar_i, \hbar_{i+1})\mathcal{S}(\hbar_i, \hbar_{i+1}) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\Psi(\hbar_j, \hbar_m)\right)\Phi(\hbar_i, \hbar_{i+1})\mathcal{S}(\hbar_i, \hbar_{i+1}) \\
&\leq \Phi(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_{n+1}) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\Psi(\hbar_j, \hbar_m)\right)\Phi(\hbar_i, \hbar_{i+1})\mathcal{S}(\hbar_i, \hbar_{i+1})
\end{aligned}$$

$$\leq \quad \Phi(\hbar_n, \hbar_{n+1})\frac{1}{n^{\frac{1}{l}}} + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\Psi(\hbar_j, \hbar_m)\right)\Phi(\hbar_i, \hbar_{i+1})\frac{1}{i^{\frac{1}{l}}}.$$

Assume that

$$C_p = \sum_{i=0}^{p}\left(\prod_{j=0}^{i}\Psi(\hbar_j, \hbar_m)\right)\Phi(\hbar_i, \hbar_{i+1})\frac{1}{i^{\frac{1}{l}}}.$$

Therefore, for $m > n$, we have

$$\mathcal{S}(\hbar_n, \hbar_m) \leq \Phi(\hbar_n, \hbar_{n+1})\frac{1}{n^{\frac{1}{l}}} + S_{m-1} - S_n. \tag{2.11}$$

Using ratio test, we have

$$a_i = \left(\prod_{j=0}^{i}\Psi(\hbar_j, \hbar_m)\right)\Phi(\hbar_i, \hbar_{i+1})\frac{1}{i^{\frac{1}{l}}}, \text{ where } \frac{a_{i+1}}{a_i} < \frac{1}{\lambda},$$

taking $\lim\limits_{n,m\to\infty}$, so (2.11) becomes

$$\lim_{n,m\to\infty}\mathcal{S}(\hbar_n, \hbar_m) = 0.$$

This implies that $\{\hbar_n\}$ is Cauchy sequence in complete $(G, \mathcal{S})$. Therefore, as a consequence there exists $\rho \in \Theta_1 \cup \Theta_2$ in such a way that $\hbar_n \to \rho$. Currently, it is worth noting that the sequences $\{\hbar_{2n}\} = \{\mathbb{T}^{2n}\hbar_0\}$ in $\Theta_1$ and $\{\hbar_{2n-1}\} = \{\mathbb{T}^{2n-1}\hbar_0\}$ in $\Theta_2$ so both converge to $\rho$. In light of that sets $\Theta_1$ and $\Theta_2$ are closed in $G$ and $\rho \in \Theta_1 \cap \Theta_2$, which assures that $\Theta_1 \cap \Theta_2 \neq \phi$. Following this, we demonstrate that $\mathbb{T}$ concedes $\rho$ as a fixed point. Let $\rho \neq \mathbb{T}\rho$, then

$$\mathcal{S}(\rho, \mathbb{T}\rho) \leq \Phi(\rho, \mathbb{T}^{2n}\hbar_0)\mathcal{S}(\rho, \mathbb{T}^{2n}\hbar_0) + \Psi(\mathbb{T}^{2n}\hbar_0, \mathbb{T}\rho)\mathcal{S}(\mathbb{T}^{2n}\hbar_0, \mathbb{T}\rho). \tag{2.12}$$

Since $\mathbb{T}^{2n-1}\hbar_0 \to \rho$ as $n \to \infty$, and utilizing the continuity of $\mathbb{T}$, we achieve $\lim_{n\to\infty}\mathcal{S}(\mathbb{T}^{2n}\hbar_0, \mathbb{T}\rho) = 0$. Thus, from (2.12), we gain $\mathcal{S}(\rho, \mathbb{T}\rho) = 0$ as $n \to \infty$ and $\mathbb{T}$ contains a fixed point that is $\rho$ . □

**Example 3.** *Let* $G = \{\frac{1}{3^n} : n \in \mathbb{N}\} \cup \{0\}$ *and* $\mathcal{S} : G \times G \to [0, \infty)$ *is defined as*

$$\mathcal{S}(\hbar_1, \hbar_2) = (\hbar_1 - \hbar_2)^2.$$

*Suppose that* $\Phi, \Psi : G \times G \to [1, \infty)$, *where* $\Phi(\hbar_1, \hbar_2) = 3\hbar_1 + 4\hbar_2 + 5$ *and* $\Psi(\hbar_1, \hbar_2) = 5\hbar_1 + 6\hbar_2 + 7$. *It is easy to see that* $(G, \mathcal{S})$ *is a complete double controlled metric type space. Assume that* $\Theta_1 = \{\frac{1}{3^{2n-1}} : n \in \mathbb{N}\} \cup \{0\}$ *and* $\Theta_2 = \{\frac{1}{3^{2n}} : n \in \mathbb{N}\} \cup \{0\}$. *Define a mapping* $\mathbb{T} : \Theta_1 \cup \Theta_2 \to \Theta_1 \cup \Theta_2$ *as*

$$\mathbb{T}\hbar_1 = \begin{cases} \frac{1}{3^{n+1}}, & \hbar_1 \in \{\frac{1}{3^n} : n \in \mathbb{N}\}, \\ 0, & \hbar_1 = 0. \end{cases}$$

*Clearly* $\mathbb{T}(\Theta_1) \subseteq \Theta_2$, $\mathbb{T}(\Theta_2) \subseteq \Theta_1$ *and* $\mathbb{T}$ *is a cyclic map. Following that, pick any* $\hbar_1 = \frac{1}{3^{2n-1}} \in \Theta_1$, *then we gain*

$$\mathbb{T}\hbar_1 = \frac{1}{3^{2n}},\ \mathbb{T}^2\hbar_1 = \mathbb{T}(\mathbb{T}\hbar_1) = \frac{1}{3^{2n+1}}, \ldots$$

*Thus,* $\mathbb{T}^{2n}\hbar_1 = \frac{1}{3^{2n+2n-1}} = \frac{1}{3^{4n-1}}$ *and* $\mathbb{T}^{2n-1}\hbar_1 = \frac{1}{3^{4n-2}}$*. For* $\hbar_2$*, we will consider the following scenarios:*
***Case 1.*** *If* $\hbar_2 \in \Theta_1/\{0,1\}$*, let*

$$\hbar_2 = \frac{1}{3^{2m-1}}, \quad (m > n \geq 1),$$

*since* $\mathbb{T}\hbar_2 = \frac{1}{3^{2m}}$*, then we gain*

$$\mathcal{S}(\mathbb{T}^{2n}\hbar_1, \mathbb{T}\hbar_2) = \mathcal{S}\left(\frac{1}{3^{4n-1}}, \frac{1}{3^{2m}}\right) = \left(\frac{1}{3^{4n-1}} - \frac{1}{3^{2m}}\right)^2 = \left(\frac{3^{2m} - 3^{4n-1}}{3^{4n+2m-1}}\right)^2,$$

*and*

$$\mathcal{S}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2) = \mathcal{S}\left(\frac{1}{3^{4n-2}}, \frac{1}{3^{2m-1}}\right) = \left(\frac{1}{3^{4n-2}} - \frac{1}{3^{2m-1}}\right)^2 = \left(\frac{3^{2m-1} - 3^{4n-2}}{3^{4n+2m-3}}\right)^2.$$

*Now,* $\mathfrak{F}(t) = \ln t$ *be a function from* $\mathfrak{F} : [0,\infty) \to \mathbb{R}$ *, for each* $t \in [0,\infty)$ *and* $\Omega > 0$*.*

$$\begin{aligned}
\mathfrak{F}(2d(\mathbb{T}^{2n}\hbar_1, \hbar_2)) - \mathfrak{F}(\mathcal{S}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2)) &= \ln 2 + 2\left(\ln \frac{3^{2m} - 3^{4n-1}}{3^{4n+2m-1}})^2 - \ln \frac{3^{2m-1} - 3^{4n-2}}{3^{4n+2m-3}}\right) \\
&= \ln 2 + 2\ln\left(\frac{3^{2m} - 3^{4n-1}}{3^{4n+2m-1}})^2 \times \frac{3^{4n+2m-3}}{3^{2m-1} - 3^{4n-2}}\right) \\
&= \ln 2 + 2\ln\left(\frac{3^{2m} - 3^{4n-1}}{3^{2m-1} - 3^{4n-2}} \times 3^{-2}\right) \\
&= \ln 2 + 2\ln\left(\frac{3^{2m} - 3^{4n-1}}{3^{2m+1} - 3^{4n}}\right) \\
&= \ln 2 + 2\ln\left(\frac{3^{2m} - 3^{4n-1}}{3(3^{2m-1} - 3^{4n-1})}\right) \\
&= \ln 2 + 2\ln\frac{1}{3} \\
&< -\frac{1}{4}.
\end{aligned}$$

***Case 2.*** *For* $\hbar_2 = 0$ *and* $\mathbb{T}0 = 0$*, then*

$$\mathcal{S}(\mathbb{T}^{2n}\hbar_1, \mathbb{T}\hbar_2) = \mathcal{S}\left(\frac{1}{3^{4n-1}}, 0\right) = \left(\frac{1}{3^{4n-1}}\right)^2,$$

*and*

$$\mathcal{S}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2) = \mathcal{S}\left(\frac{1}{3^{4n-2}}, 0\right) = \left(\frac{1}{3^{4n-2}}\right)^2.$$

*Thus, we have*

$$\begin{aligned}
&\mathfrak{F}(2\mathcal{S}(\mathbb{T}^{2n}\hbar_1, \hbar_2)) - \mathfrak{F}(\mathcal{S}(\mathbb{T}^{2n-1}\hbar_1, \hbar_2)) \\
&= \ln 2 + 2\left(\ln \frac{1}{3^{4n-1}} - \ln \frac{1}{3^{4n-2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \ln 2 + 2\left(\frac{1}{3^{4n-1}} \times \ln \frac{3^{4n-2}}{1}\right) \\
&= \ln 2 + \ln\left(3^{-1}\right) \\
&= \ln 2 + \ln \frac{1}{3} \\
&< -\frac{1}{4}.
\end{aligned}
$$

*Thus, for $\Omega = \frac{1}{4}$ and $\mathbb{T}$ is cyclic orbital F-contraction. For this reason, all the requirements of Theorem 2 are fulfilled, and $\mathbb{T}$ concedes 0 as a fixed point.*

**Remark 3.** *Since $(G, \mathcal{S})$ is not usually a controlled metric type space, so Theorem 2 contrasts from the Theorem* 1 *of N. Alamgir [14]. If we take $\Phi(\hbar_1, \hbar_2) = \Psi(\hbar_1, \hbar_2) \geq 1$, then the above theorem simplifies to controlled metric type space. For $\Phi(\hbar_1, \hbar_2) \geq 1$, depends on the inequality of left hand sides then the preceding theorem condenses to extended b-metric space [15] and for $\Phi(\hbar_1, \hbar_2) = s \geq 1$, the above theorem shrinks to the b-metric space. If $s = 1$, the above theorem simplifies the main results of fixed points in metric space presented by Karpagam et al. [13].*

## 3. Application of second order differential equation into Fredholm integral equation

In this section, we implemented cyclic orbital contraction in a double controlled metric type space structure, we gained the solution for a second-order differential equation linked with a Fredholm integral equation. This technique, motivated by the strategy described in K. H. Hussain's [16], supported the proficiency of existence and uniqueness results for our problem domain. Let a set of all real valued continuous function $G = \mathbb{C}[[0, 1], \mathbb{R}]$ on $[[0, 1], \mathbb{R}]$ and $\mathcal{S} : G \times G \to \mathbb{R}$ is defined as

$$\mathcal{S}(\hbar_1, \hbar_2) = \sup_{t \in [0,1]} |\hbar_1(t) - \hbar_2(t)|^2 .$$

Let $\Phi, \Psi : G \times G \to [1, \infty)$ defined as

$$
\begin{aligned}
\Phi(\hbar_1, \hbar_2) &= 3\hbar_1(t) + 4\hbar_2(t) + 5, \\
\Psi(\hbar_1, \hbar_2) &= 5\hbar_1(t) + 6\hbar_2(t) + 7,
\end{aligned}
$$

for all $\hbar_1, \hbar_2 \in G$ and $t \in [a, b]$. Obviously $(G, \mathcal{S})$ is complete double controlled metric type space. Now, we will assume second order differential equation as

$$
\begin{cases}
\hbar''(\hbar) = f(t, \hbar(t)), \\
\hbar(0) = \hbar_0, \ \hbar(1) = \hbar_1,
\end{cases} \tag{3.1}
$$

for all $t \in [0, 1]$ and $f : [0, 1] \times \mathbb{R}$, is a continuous function. The problem defined in (3.1) is equivalent to second kind Fredholm integral equation

$$\hbar(t) = L(t) + \gamma \int_0^1 \dot{G}(t, s)\hbar(s)ds, \tag{3.2}$$

where $t \in [0,1]$ and $L(t) = u_0 + t(u_1 - u_0)$. In (3.2), $\dot{G}(t,s)$ is Green's function that is

$$\dot{G}(t,s) = \begin{cases} s(1-s), & 0 \le s \le t, \\ t(1-s), & t \le s \le 1, \end{cases}$$

and if $\hbar \in G$ is a fixed point of $\mathbb{T}$ then $\hbar$ is a solution of (3.1).

**Theorem 3.** *Let* $\mathbb{T} : \Theta_1 \cup \Theta_2 \to \Theta_1 \cup \Theta_2$ *be a continuous nonlinear integral operator defined by*

$$\hbar(t) = L(t) + \gamma \int_0^1 \dot{G}(t,s)\hbar(s)ds,$$

*for all* $t \in [0,1]$. *Assume that following conditions holds*

(1) $\sup_{m\ge1} \lim_{i\to\infty} \frac{\Phi(\mathbb{T}^{i+1}\hbar_0,\mathbb{T}^{i+1}\hbar_1)\Psi(\mathbb{T}^{i}\hbar_0,\mathbb{T}^{m}\hbar_0)}{\Phi(\mathbb{T}^{i}\hbar_0,\mathbb{T}^{i+1}\hbar_0)} < \frac{1}{\kappa}$.

(2) For any $\hbar, \varsigma \in G$ and $\gamma > 0$, we have

$$\sup_{t\in[0,1]} \left(\frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{2}\right) = \frac{1}{2\gamma}.$$

Then second order differential Eq (3.1) have solution in $G$.

*Proof.* Let $\Theta_1 = \Theta_2 = G = C[[0,1],\mathbb{R}]$, clearly $\Theta_1$ and $\Theta_2$ are closed subsets of $G$. Since $\mathbb{T}(\Theta_1) \subset \Theta_2$ and $\mathbb{T}(\Theta_2) \subset \Theta_1$, which shows that $\mathbb{T}$ is cyclic map on $\Theta_1 \cup \Theta_2$. For any $\hbar_0 \in G$, we define a sequence $\{\hbar_n\}$ in $G$, by $\hbar_{n+1} = \mathbb{T}\hbar_n = \mathbb{T}^{n+1}\hbar_0$, $n \ge 1$ then we obtain

$$\hbar_{n+1}(t) = \mathbb{T}\hbar_n(t) = L(t) + \gamma \int_0^1 \dot{G}(t,s)\hbar_n(s)ds,$$

for $\hbar_1, \hbar_2 \in G$, we have

$$\begin{aligned} |\mathbb{T}(\mathbb{T}\hbar_1(t) - \mathbb{T}\hbar_2(t)| &= \left| L(t) + \gamma \int_0^1 \dot{G}(t,s)\mathbb{T}\hbar_1(s)ds - L(\hbar) + \gamma \int_0^1 \dot{G}(t,s)\hbar_2(s)ds \right| \\ &\le \gamma \int_0^1 \dot{G}(t,s)\left|\mathbb{T}x_1(s) - \hbar_2(s)\right| ds \\ &\le \gamma \sup_{t\in[0,1]} |\mathbb{T}\hbar_1(t) - \hbar_2(t)| \int_0^1 \dot{G}(t,s)ds \\ &\le \gamma \sup_{\hbar\in[0,1]} \left(\frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{2}\right) \sup_{t\in[0,1]} \left|\mathbb{T}\hbar_1(t) - \hbar_2(t)\right|, \end{aligned}$$

which implies that

$$\sup_{t\in[0,1]} \left|\mathbb{T}^2\hbar_1(t) - \mathbb{T}\hbar_2(t)\right|^2 \le \frac{1}{4} \sup_{t\in[0,1]} \left|\mathbb{T}\hbar_1(t) - \hbar_2(t)\right|^2.$$

Taking $\lambda = \frac{1}{4} \in (0,1)$, we can write

$$\mathcal{S}(\mathbb{T}^2\hbar_1, \mathbb{T}\hbar_2) \le \lambda \mathcal{S}(\mathbb{T}\hbar_1, \hbar_2).$$

Thus all the conditions of Theorem 1 are satisfied. Hence, $\mathbb{T}$ has a fixed point and Fredholm integral Eq (3.2) has a solution. □

## 4. Conclusions

In this research, we explored the realm of double-controlled metric-type spaces to examine the attainment of fixed points via the application of cyclic orbital contractive conditions. Moving from typical techniques spotted in standard metric spaces, our strategy takes a distinctive path, utilizing the unique characteristics of our structure. We have highlighted the significance of our findings by presenting illustrative situations and detailed examinations, revealing the breadth of our discoveries. Significantly, our research not only improves and broadens the scope of previous findings in the literature, but it also introduces unique concepts that are explained in our explanatory notes. Furthermore, using insights from prior studies, we created a second-order differential equation that is a valuable tool for solving the second class of Fredholm integral problems. In conclusion, our research adds to the growth of learning in double-controlled metric-type spaces by providing new insights into fixed point theory, enhancing previous discussions, and building a significant path approaching the completion of a class of integral equation. For future practical implementations of fixed point results, the optimization approach [17] and the complex standard Eigenvalue Problem [18] may be of interest. Furthermore, this theory is applicable to spacecraft reorientation [19] as well as hyperbolic and parabolic PDEs [20].
**Open Problems.** Can we obtained fixed point results using these type of orbital contraction mappings in the double controlled quasi-metric type spaces and in double controlled partial metric type spaces? In graphically controlled metric type space? In graphically double controlled metric type space? Is there interest to find serious applications to integral equations and dynamical systems?

## Author contributions

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## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

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