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Research article

Applying fixed point techniques to solve fractional differential inclusions under new boundary conditions

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Abstract: Many scholars have lately explored fractional-order boundary value issues with a variety of conditions, including classical, nonlocal, multipoint, periodic/anti-periodic, fractional-order, and integral boundary conditions. In this manuscript, the existence and uniqueness of solutions to sequential fractional differential inclusions via a novel set of nonlocal boundary conditions were investigated. The existence results were presented under a new class of nonlocal boundary conditions, Carathéodory functions, and Lipschitz mappings. Further, fixed-point techniques have been applied to study the existence of results under convex and non-convex multi-valued mappings. Ultimately, to support our findings, we analyzed an illustrative example.

Keywords: fixed point technique; Carathéodory function; existence solution; fractional derivatives; sequential differential inclusions
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Abbreviations

The following abbreviations are used in this manuscript:

FD	Fractional differential
FDEs	Fractional differential equations
SFD	Sequential Fractional differential
l.s.c	lower semi continuous
c.c	completely continuous
u.s.c	upper semi continuous
HU	Hyers-Ulam

1. Introduction

In recent times, fractional differential (FD) equations have garnered substantial attention as a pivotal area of research. Numerous applications of fractional-order derivatives can be found in various scientific and engineering fields, where they are used to mathematically model physical and biological phenomena. A wealth of literature exists, containing a wide range of results on initial and boundary value problems related to FD equations and inclusions. For recent findings, see [1–4] and the accompanying references can be consulted. Specifically, differential inclusions have proven to be highly advantageous in the investigation of dynamic systems and stochastic processes. Instances encompass processes like sweeping phenomena, nonlinear dynamics pertaining to wheeled vehicles, granular systems, and control quandaries, among others. Comprehensive insight into pertinent topics within stochastic processes, control theory, differential games, optimization, and their practical implementation across domains such as finance, manufacturing, queuing networks, and climate control can be found in the reference [2]. For a more detailed exploration of the utilization of fractional differential inclusions in synchronization processes, reference [5] offers further information. Notably, efforts in the literature concerning boundary value problems involving FD equations and inclusions have undergone substantial expansion, encompassing a diverse array of outcomes. On the other hand, researchers have also made multiple attempts to establish sufficient conditions for the existence and uniqueness of solutions concerning different classes (inclusions) of initial and boundary value problems incorporating the Caputo fractional derivative.

Integral boundary conditions give rise to a fascinating and significant class of problems, encompassing two, three, multipoint, and nonlocal boundary value problems as distinct instances [6–8]. These boundary conditions find application in diverse fields, including population dynamics [9] and cellular systems. In the initial phase of the study, the authors employed the Bohnenblust-Karlin fixed point theorem to establish the existence of solutions pertaining to a specific category of FD inclusions characterized by separated boundary conditions. Subsequently, the endeavors extended to establishing the existence outcomes for a boundary value problem associated with FD inclusions, wherein fractional separated boundary conditions were considered. Notably, anti-periodic boundary conditions have garnered considerable attention and found utility in various domains, such as blood flow problems, chemical engineering, underground water flow, and population dynamics. Ahamad and Otero-Espinar [10] delved into fractional inclusions with anti-periodic boundary conditions, establishing certain sufficient conditions for the existence of solutions using the

Bohnenblust-Karlin fixed-point theorem. In addition to the aforementioned investigations, the study encompassed an examination of existence outcomes concerning FD inclusions of higher order, considering nonlocal boundary conditions. Moreover, the research included the reporting of existence results for FD equations of higher order, involving multi-strip Riemann-Liouville fractional integral boundary conditions.

The exploration of coupled systems of fractional differential equations (FDEs) holds immense prominence within the realm of mathematics and across various applied disciplines. These systems manifest across a diverse array of real-world predicaments, imbuing them with relevance and value in unraveling intricate phenomena. For a more profound comprehension of the significance and applications inherent to coupled systems of FDEs, one can turn to the citations [11–15]. These references furnish intricate details and exemplars that spotlight the extensive spectrum of predicaments where such systems retain relevance and practicality. Scientists spanning an array of fields, including physics, engineering, biology, and economics, have duly acknowledged the merits of delving into the study and resolution of these coupled systems to glean insights into multifaceted dynamic processes and occurrences. Inspired by these advancements, considerable headway has been achieved in delving into the existence, uniqueness, and stability of solutions for the coupled system of FD equations featuring distinct boundary conditions.

Furthermore, coupled systems of FDEs manifest in diverse applied problems. For instance, HIV, a retrovirus, specifically targets $CD4^+$ lymphocytes, the predominant white blood cells of the immune system. Perelson [16] formulated a basic model for primary HIV infection, categorizing cells into four groups: uninfected $CD4^+$ T cells, productively infected $CD4^+$ T cells, latently infected $CD4^+$ T cells and the viral population. Subsequently, Perelson et al. [17] introduced a fractional-order model to describe the infection dynamics of $CD4^+$ T-cells. The coupled system is described by the following set of fractional ordinary differential equations of order $\alpha_1, \alpha_2, \alpha_3 > 0$:

$$\begin{cases} D^{\alpha_1}(T) = s - KVT - dT + bI, \\ D^{\alpha_2}(I) = KVT - (b+\delta)I, \\ D^{\alpha_3}(I) = N\delta I - cV. \end{cases}$$

In recent times, there has been a growing emphasis on exploring the existence and uniqueness of sequential FD (SFD) equations and inclusions. This heightened attention stems from the recognition that such investigations hold significant potential in advancing our understanding of complex mathematical and applied phenomena. As a result, researchers have directed their efforts toward unraveling the intricate characteristics and properties of these sequential equations and inclusions, aiming to uncover novel insights and establish fundamental principles that contribute to the broader landscape of mathematical analysis and its applications [18]. A notable correlation exists between the SFDs as discussed in [19] and the nonsequential Riemann-Liouville derivatives documented in [11]. For recent advancements in the realm of SFD equations, readers are directed to peruse the works presented in references [2, 20–25]. Investigations undertaken in [26, 27] have delved into the analysis of SFD equations, encompassing various boundary conditions. Notably, an attempt has engaged in a discussion concerning the existence of solutions pertaining to higher-order SFD inclusions, incorporating nonlocal three-point boundary conditions. Furthermore, the examination of SFD inclusions supplemented by nonlocal Riemann-Liouville-type fractional integral boundary conditions has been documented in reference [28].

We now move to a discussion of numerous studies that have previously addressed the exploration of solution existences in scalar and coupled FD inclusions. These studies, characterized by diverse boundary conditions, have served as a significant source of motivation for undertaking the present research endeavor. For instance, in [29], the authors have considered the FD inclusions to explore existence of solutions with boundary conditions:

$$\begin{aligned}&{}^{C}\mathcal{D}^{\alpha_1}\Phi_1(\varphi)\in\Theta_1(\varphi,\Phi_1(\varphi)),\ \ \varphi\in\mathcal{J}:=[0,1],\alpha_1\in(1,2),\\&\Phi_1(0)=\alpha,\ \Phi_1(1)=\beta,\alpha,\beta\neq0,\end{aligned}$$

where ${}^{C}\mathcal{D}^{\alpha_1}\Phi_1(\varphi)$ represents the Caputo derivative and the function $\Theta_1:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ is continuous.

Subsequently, an attempt was made to investigate the existence of solutions for SFD inclusions [2]

$$({}^{C}\mathcal{D}^{\alpha_1}+\chi_1^{C}\mathcal{D}^{\alpha_1-1})\Phi_1(\varphi)\in\Theta_1(\varphi,\Phi_1(\varphi)),\ \ \tau\in[0,1],\ \ 2<\alpha\leq3,$$

under the boundary conditions

$$\Phi_1(0)=0,\ \ \Phi_1'(0)=0,\Phi_1(\zeta)=\mathfrak{a}\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\alpha_1-1}}{\Gamma(\alpha_1)}\Phi_1(\mathfrak{s})d\mathfrak{s},\quad 0<\eta<\zeta<1.$$

where χ_1, $\mathfrak{a}$, and α are positive real numbers, and the notation ${}^{C}\mathcal{D}^{\alpha_1}$ signifies the Caputo fractional derivative of order α_1. The constants $0<\eta<\zeta<1$ define the interval limits, while $\Theta_1:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ represents a multivalued mapping. The set $\mathcal{P}(\mathbb{R})$ encompasses all nonempty subsets of $\mathbb{R}$.

Ahmad et al. [13] examined the existence of solutions in the coupled Caputo-type FD inclusions

$$\begin{cases}({}^{C}\mathcal{D}^{\alpha_1})\Phi_1(\varphi)\in\Theta_1(\varphi,\Phi_1(\varphi),\Omega_1(\varphi)),\varphi\in\mathcal{J}:=[0,\mathcal{T}],\\({}^{C}\mathcal{D}^{\alpha_2})\Omega_1(\varphi)\in\Theta_2(\varphi,\Phi_1(\varphi),\Omega_1(\varphi)),\varphi\in\mathcal{J}:=[0,\mathcal{T}],\end{cases}$$

with involving the coupled boundary conditions:

$$\begin{aligned}&\Phi_1(0)=\nu_1\Omega_1(\mathcal{T}),\quad {\Phi_1}'(0)=\nu_2\Omega_1'(\mathcal{T}),\\&\Phi_1(0)=\mu_1\Omega_1(\mathcal{T}),\quad {\Phi_1}'(0)=\mu_2\Omega_1'(\mathcal{T}),\end{aligned}$$

where the notations ${}^{C}\mathcal{D}^{\alpha_1}$ and ${}^{C}\mathcal{D}^{\alpha_2}$ represent the Caputo fractional derivatives of orders α_1 and α_2, respectively. The functions $\Theta_1,\Theta_2:\mathcal{J}\times\mathbb{R}^2\to\mathbb{R}$ are multivalued mappings . Φ_1 and Ω_1 are unknown functions over an arbitrary interval (η,ζ) within the given domain $[0,\mathcal{T}]$. The set $\mathcal{P}(\mathbb{R})$ encompasses all nonempty subsets of $\mathbb{R}$. Moreover, the constants ν_1, ν_2, μ_1, and μ_2 are real values. Furthermore, in reference [11], the authors have addressed a set of boundary value problems concerning a coupled system utilizing Liouville-Caputo type fractional differential equations,

$$\begin{cases}({}^{C}\mathcal{D}^{\alpha_1})\Phi_1(\varphi)=\Theta_1(\varphi,\Phi_1(\varphi),\Omega_1(\varphi)),\varphi\in\mathcal{J}:=[0,\mathcal{T}],\\({}^{C}\mathcal{D}^{\alpha_2})\Omega_1(\varphi)=\Theta_2(\varphi,\Phi_1(\varphi),\Omega_1(\varphi)),\varphi\in\mathcal{J}:=[0,\mathcal{T}],\end{cases}$$

with the coupled boundary conditions:

$$\begin{cases}(\Phi_1+\Omega_1)(0)=-(\Phi_1+\Omega_1)(\mathcal{T}),\\\displaystyle\int_{\eta}^{\zeta}(\Phi_1-\Omega_1)(\xi)d\xi=\mathcal{A},\quad 0<\eta<\zeta<\mathcal{T},\end{cases}$$

where ${}^{C}\mathcal{D}^{\alpha_i}$ refers to the Caputo fractional derivative operator of order α_i, with $i = 1, 2$. The parameters $\alpha_1, \alpha_2 \in (1, 2]$, $0 < \eta < \zeta < \mathcal{T}$, and $\Theta_1, \Theta_2 : [0, \mathcal{T}] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ represent continuous functions and $\mathcal{P}(\mathbb{R})$ is the class of all nonempty subsets of $\mathbb{R}$.

Recently, Subramanian et al. [14] investigated the coupled differential equations and inclusions involving Caputo-type sequential derivatives

$$\begin{cases} ({}^{C}\mathcal{D}^{\alpha_1} + \chi_1 {}^{C}\mathcal{D}^{\alpha_1 - 1})\Phi_1(\varphi) \in \Theta_1(\varphi, \Phi_1(\varphi), \Omega_1(\varphi)), \varphi \in \mathcal{J} := [0, \mathcal{T}], \\ ({}^{C}\mathcal{D}^{\alpha_2} + \chi_2 {}^{C}\mathcal{D}^{\alpha_2 - 1})\Omega_1(\varphi) \in \Theta_2(\varphi, \Phi_1(\varphi), \Omega_1(\varphi)), \varphi \in \mathcal{J} := [0, \mathcal{T}], \\ (\Phi_1 + \Omega_1)(0) = -(\Phi_1 + \Omega_1)(\mathcal{T}), \\ \int_\eta^\zeta (\Phi_1 - \Omega_1)(\xi) d\xi = \mathcal{A}. \end{cases}$$

where ${}^{C}\mathcal{D}^{\alpha_1}$ and ${}^{C}\mathcal{D}^{\alpha_2}$ are the Caputo derivative operator. $\alpha_1, \alpha_2 \in (0, 1]$, $\Theta_1, \Theta_2 : [0, \mathcal{T}] \times \mathbb{R}^2 \to \mathbb{R}, \Theta_1, \Theta_2 : [0, \mathcal{T}] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ are continuous functions, and $\mathcal{P}(\mathbb{R})$ is the class of all nonempty subsets of $\mathbb{R}$.

Motivated by the abovementioned studies, in this work, we intend to consider a novel category of boundary value problems to unearth the existence of Caputo-type coupled SFD inclusions

$$\begin{cases} ({}^{C}\mathcal{D}^{\alpha_1} + \chi_1 {}^{C}\mathcal{D}^{\alpha_1 - 1})\Phi_1(\varphi) \in \Theta_1(\varphi, \Phi_1(\varphi), \Omega_1(\varphi)), \varphi \in \mathcal{J} := [0, \mathcal{T}], \\ ({}^{C}\mathcal{D}^{\alpha_2} + \chi_2 {}^{C}\mathcal{D}^{\alpha_2 - 1})\Omega_1(\varphi) \in \Theta_2(\varphi, \Phi_1(\varphi), \Omega_1(\varphi)), \varphi \in \mathcal{J} := [0, \mathcal{T}], \end{cases} \tag{1.1}$$

$$\begin{cases} (\Phi_1 + \Omega_1)(0) = -(\Phi_1 + \Omega_1)(\mathcal{T}), \\ \int_\eta^\zeta (\Phi_1 - \Omega_1)(\xi) d\xi - \sum_{i=0}^{m} \varrho_i (\Phi_1 - \Omega_1)(\mathcal{U}_i) - \sum_{j=0}^{n} \vartheta_j (\Phi_1 - \Omega_1)(\mathcal{W}_j) = \mathcal{A}. \end{cases} \tag{1.2}$$

where, ${}^{C}\mathcal{D}^{\alpha_1}$ and ${}^{C}\mathcal{D}^{\alpha_2}$ are the Caputo fractional derivatives of orders $\alpha_1 \in (1, 2]$ and $\alpha_2 \in (1, 2]$, respectively. Further, for distinct indices $i = 1, \cdots, m$ and $j = 1, \cdots, n$, the inequalities $0 < \mathcal{U}i < \eta < \zeta < \mathcal{W}j < \mathcal{T}$ hold, where $\mathcal{T}$ defines a specific time scale. The functions Θ_1 and Θ_2 are continuous mappings on $[0, \mathcal{T}] \times \mathbb{R} \times \mathbb{R}$, and their values lie within the real number space $\mathbb{R}$. Within the formulation, the initial condition outlined in (1.2) is characterized by an anti-periodic property. Additionally, the second condition delineates the influence of the disparity between the unknown functions Φ_1 and Ω_1 over an arbitrary interval (η, ζ) within the given domain $[0, \mathcal{T}]$, deviating from the cumulative effect of such influences attributed to arbitrary positions at $\mathcal{U}_i, i = 1, \cdots, m$ and $\mathcal{W}_j$, $j = 1, 2, \cdots, n$, where a positive constant is involved. Further, within the framework, $\mathcal{A}$ is to be understood as nonnegative values.

The primary objective of this study is to establish criteria for solutions to the problems (1.1) and (1.2), specifically addressing both convex and non-convex valued multivalued maps denoted by Θ_1 and Θ_2. This will be achieved by employing standard fixed point theorems. The subsequent sections delineate the organization of this paper. Section 2 provides essential foundational concepts along with an auxiliary lemma that are indispensable for resolving the presented problem. The main results are developed in Section 3, wherein we leverage fixed point theorems, notably the Covitz-Nadler theorem and the nonlinear alternative to the Kakutani fixed point theorem, to establish our main findings. In Section 4, a specific illustrative example is presented, which aligns with the studied systems and serves to demonstrate the application of the fundamental theorems. We conclude our outcomes in Section 5.

2. Preliminary work

In this section, various definitions of multivalued maps and essential lemmas are explored, which are imperative for substantiating the main results [19, 30–32].

For a normed space $(\mathscr{U}, \|.\|)$, let $\mathscr{W}_{cl}(\mathscr{U}) = \{\mathcal{Y} \text{ is closed }\}$, $\mathscr{W}_{cp}(\mathscr{U}) = \{\mathcal{Y} \text{ is compact }\}$, and $\mathscr{W}_{cp,c}(\mathscr{U}) = \{\mathcal{Y} \text{ is compact and convex }\}$.

A multivalued map denoted by $\chi : \mathscr{U} \to \mathscr{W}(\mathscr{U})$ adheres to the following properties:

(a) It is categorized as convex valued if $\chi(w)$ is convex for every $w \in \mathscr{U}$.
(b) It is considered upper semicontinuous on $\mathscr{U}$ if, for each $w_0 \in \mathscr{U}$, the set $\chi(w_0)$ is a nonempty closed subset of $\mathscr{U}$ and if, for every open set $\mathcal{T}$ within $\mathscr{U}$ that encompasses $\chi(w_0)$, there exists an open neighborhood $\mathcal{T}_0$ of w_0 such that $\chi(\mathcal{T}_0) \subset \mathcal{T}$.
(c) It is termed lower semicontinuous (l.s.c.) if the set $m \in \mathscr{U} : \chi(m) \cap \mathcal{A} \neq \oslash$ remains open for any open set $\mathcal{A}$ within Θ.
(d) It is classified as completely continuous (c.c) if $\chi(\mathcal{A})$ is relatively compact (r.c) for every $\mathcal{A} \in \mathscr{W}b(\mathscr{U})$, where $\mathscr{W}b(\mathscr{U})$ signifies the ensemble of bounded multivalued maps $\mathcal{M} \in \mathscr{W}(\mathscr{U})$.

Define $\mathscr{U} = C(\mathcal{J}, \mathbb{R}) \times C(\mathcal{J}, \mathbb{R})$ as the Banach space endowed with norm $\|((\Phi_1, \Omega_1))\| = \sup_{\tau \in \mathcal{J}} |\Phi_1| + \sup_{x \in \mathcal{J}} |\Omega_1|$, for $(\Phi_1, \Omega_1) \in \mathscr{U}$.

A multivalued map $\chi : [c, d] \to \mathscr{W}_{cl}(\mathbb{R})$ is categorized as measurable if, for every $m \in \mathbb{R}$, the function $\varphi \longmapsto d(m, \chi(\varphi)) = \inf |m - k| : k \in \chi(\varphi)$ is measurable.

In the case of a multivalued map $\chi : [c, d] \times \mathbb{R} \to \mathscr{W}(\mathbb{R})$, it is designated as Caratheodory under the following conditions:

(i) The function $\varphi \longmapsto \chi(\varphi, s, m)$ is measurable for each $s, m \in \mathbb{R}$.
(ii) The mapping $(s, m) \longmapsto \chi(\varphi, s, m)$ is upper semicontinuous (u.s.c) for almost all $\varphi \in [c, d]$.

Furthermore, a Caratheodory function χ earns the title of $\mathcal{L}^1$-Caratheodory when it satisfies the subsequent criteria:

(i) For every $\epsilon > 0$, there exists a nonnegative $\psi_\epsilon \in \mathcal{L}^1([c, d], \mathbb{R}^+)$ such that $|\chi(\varphi, s, m)| = \sup |s| : s \in \chi(\varphi, s, m) \leq \psi_\epsilon(\varphi)$ for all $s, m \in \mathbb{R}$ with $|s|, |m| \leq \epsilon$, and this holds for almost every $\varphi \in [c, d]$.

Next, we proceed to revisit fundamental definitions in the realm of fractional calculus.

Definition 2.1. *[19]* The fractional integral of a function Φ_1 with a lower limit of zero, and of order α, is formally expressed as:

$$\mathcal{I}^\alpha \Phi_1(\varphi) = \frac{1}{\Gamma(\alpha)} \int_0^\varphi \frac{\Phi_1(\xi)}{(\varphi - \xi)^{1-\alpha}} d\xi. \tag{2.1}$$

This expression holds true under the condition that the right-hand side is pointwise defined over the interval $[0, \infty)$. Here, $\Gamma(\cdot)$ represents the gamma function, which can be mathematically denoted as $\Gamma(\alpha) = \int_0^\infty \varphi^{\alpha-1} e^{-\varphi} d\varphi$.

Definition 2.2. *[19]* The Caputo derivative of fractional order α for an $\Phi_1 : [0,\infty) \to \mathbb{R}$ can be written as

$$^{C}\mathcal{D}^{\alpha}_{0^+}\Phi_1(\varphi) = \mathcal{D}^{\alpha}_{0^+}\left(\Phi_1(\varphi) - \sum_{k=0}^{n-1}\frac{\varphi^k}{k!}\Phi_1^{(k)}(0)\right), \quad \Phi_1 > 0, n-1 < r < n.$$

Throughout the rest of this article, we will employ the notation $^{C}\mathcal{D}^{\alpha}$ instead of $^{C}\mathcal{D}^{\alpha}_{0^+}$ for the sake of simplicity and convenience.

Lemma 2.1. *[31] In the event that $\mathcal{G} : X \to \mathcal{P}_{cl}(Y)$ is u.s.c, then $Gr(\mathcal{G})$ is established as a closed subset within $X \times Y$. This implies that for any given sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if x_n converges to x_* and y_n converges to y_* as n approaches infinity, and if $y_n \in \mathcal{G}(x_n)$, then it holds that $y_* \in \mathcal{G}(x_*)$. Conversely, in the scenario where $\mathcal{G}$ is both completely continuous and exhibits a closed graph, the function is demonstrated to be u.s.c.*

Lemma 2.2. *[8] Consider a separable Banach space X. Let $\mathcal{G} : [0,T] \times \mathbb{R}^2 \to \mathcal{P}_{cp,c}(\mathbb{R})$ be an $\mathcal{L}^1$- Carathedory multivalued map and let $\mathcal{X}$ be a linear operator from $\mathcal{L}^1([0,T],\mathbb{R})$ to $C([0,T],\mathbb{R})$. Under these conditions, it can be established that the mapping forms a closed graph within the space $C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$.*

Lemma 2.3. *(Nonlinear alternative for kakutani maps.p.no.14) [33] Let $\mathcal{E}$ be a closed convex subset of of a Banach space $\mathcal{H}$ and Θ_1 be an open subset of $\mathcal{E}$ with $0 \in \Theta_1$. In addition, $\Theta_2 : \mathcal{E} \to \mathcal{J}_{c,cp}(\mathcal{E})$ is an u.s.c compact map. Then, either*

- *Θ_2 has fixed point in $\mathcal{E}$ or*
- *$\exists\, u \in \partial\Theta_1$ and $\mu \in (0,1)$ such that $u \in \mu\Theta_2(u)$.*

Definition 2.3. *[2]* A multivalued $\chi : \mathscr{U} \to \mathscr{W}(\mathscr{U})$ mapping is called

(i) δ-Lipschitz if $\exists\, \delta > 0 \ni \Pi_d(\chi(c),\chi(d)) \leq \delta d(c,d)$ for each $c,d \in \mathscr{U}$; and
(ii) a contraction if it is δ-Lipschitz with $\delta < 1$.

Lemma 2.4. *[20] Let $(\mathcal{H},d)$ be a complete metric space. If $\chi : \mathcal{H} \to \mathcal{J}_{cl}(\mathcal{H})$ is a contraction, then adopt χ to have at least one fixed point.*

Lemma 2.5. *Let $q > 0$ and $f(\tau) \in \mathcal{AC}^n[0,\infty)$ or $C^{[0,\infty)}$. Then*

$$(\mathcal{I}^{q\,C}\mathcal{D}^q f)(\tau) = f(\tau) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}\tau^k, \tau > 0, n-1 < q < n. \tag{2.2}$$

The following lemma deals with the linear variant of the problem.

Lemma 2.6. *Let $\alpha_1,\alpha_2 \in (1.2], \varrho_i, \vartheta_j > 0$ and $\Psi_1,\Psi_2 \in C(\mathcal{J},\mathbb{R})$, then the solution of the following system:*

$$\begin{cases} (^{C}\mathcal{D}^{\alpha_1} + \chi_1{}^{C}\mathcal{D}^{\alpha_1-1})\Phi_1(\varphi) = \Psi_1(\varphi), \quad \varphi \in \mathcal{J} := [0,\mathcal{T}], \\ (^{C}\mathcal{D}^{\alpha_2} + \chi_2{}^{C}\mathcal{D}^{\alpha_2-1})\Omega_1(\varphi) = \Psi_2(\varphi), \quad \varphi \in \mathcal{J} := [0,\mathcal{T}], \\ (\Phi_1 + \Omega_1)(0) = -(\Phi_1 + \Omega_1)(\mathcal{T}), \\ \displaystyle\int_\eta^\zeta (\Phi_1 - \Omega_1)(\xi)d\xi - \sum_{i=0}^{m}\varrho_i(\Phi_1 - \Omega_1)(\mathcal{U}_i) - \sum_{j=0}^{n}\vartheta_j(\Phi_1 - \Omega_1)(\mathcal{W}_j) = \mathcal{A}. \end{cases} \tag{2.3}$$

is defined by

$$
\begin{aligned}
\Phi_1(\varphi) =& \frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg) \\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg)\Bigg] \\
&+\int_0^{\varphi} e^{-\chi_1(\varphi-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi,
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
\Omega_1(\varphi) =& \frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg) \\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg)\Bigg]
\end{aligned}
$$

$$+\int_0^{\varphi} e^{-\chi_2(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi. \tag{2.5}$$

where

$$\begin{cases}\Delta_1 = (1+e^{-\chi_1\mathcal{T}})\\ \Delta_2 = \int_\eta^\zeta e^{-\chi_1\xi}d\xi - \sum_{i=0}^{m}\varrho_i e^{-\chi_1(\mathcal{U}_i)} - \sum_{j=0}^{n}\vartheta_j e^{-\chi_1(\mathcal{W}_j)} \neq 0.\\ \mathcal{I}_1 = \left(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\right)d\xi\right.\\ \left.-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi\right)\\ \mathcal{I}_2 = \left(\mathcal{A}-\int_\eta^\zeta\left(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\right.\\ +\int_\eta^\zeta\left(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\ +\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\right)d\xi\\ -\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi\\ +\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\right)d\xi\\ \left.-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi\right)\end{cases} \tag{2.6}$$

Proof. Applying the operator $\mathcal{I}_1^\alpha$ and $\mathcal{I}_2^\alpha$ on both sides of FDEs in (2.3), respectively, and using Lemma 2.5, we obtain

$$\begin{cases}\Phi_1(\varphi) = \mathfrak{c}_0 e^{-\chi_1\varphi} + \int_0^{\varphi} e^{-\chi_1(\varphi-\xi)}\left(\int_0^{\varphi}\frac{(s-\mathfrak{a})^{(\alpha_1-2)}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\right)d\xi,\\ \Omega_1(\varphi) = \mathfrak{d}_0 e^{-\chi_2\varphi} + \int_0^{\varphi} e^{-\chi_2(\varphi-\xi)}\left(\int_0^{\varphi}\frac{(s-\mathfrak{u})^{(\alpha_2-2)}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi,\end{cases} \tag{2.7}$$

where $\mathfrak{c}_0, \mathfrak{d}_0 \in \mathbb{R}$. Using the boundary conditions $(\Phi_1+\Omega_1)(0) = -(\Phi_1+\Omega_1)(\mathcal{T})$, $\int_\eta^\zeta(\Phi_1-\Omega_1)(\xi)d\xi - \sum_{i=0}^{m}\varrho_i(\Phi_1-\Omega_1)(\mathcal{U}_i) - \sum_{j=0}^{n}\vartheta_j(\Phi_1-\Omega_1)(\mathcal{W}_j) = \mathcal{A}$. in () and () we obtain

$$\mathfrak{c}_0 + \mathfrak{d}_0 = \mathcal{I}_1 \tag{2.8}$$

$$\mathfrak{c}_0 - \mathfrak{d}_0 = \mathcal{I}_2. \tag{2.9}$$

Solving (2.8) and (2.9) together for $\mathfrak{c}_0$ and $\mathfrak{d}_0$, it is found that

$$\mathfrak{c}_0 = \frac{1}{2}\left\{\frac{1}{\Delta_1}\left(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\right)d\xi\right.\right.$$
$$\left.-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi\right)$$

$$
\begin{aligned}
&+\frac{1}{\Delta_2}\left(\mathcal{A}-\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\right.\\
&+\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^n\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&\left.-\sum_{j=0}^n\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi\right)\Bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{d}_0=&\frac{1}{2}\left\{\frac{1}{\Delta_1}\left(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\right)d\xi\right.\right.\\
&\left.-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi\right)\\
&-\frac{1}{\Delta_2}\left(\mathcal{A}-\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\right.\\
&+\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^n\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Psi_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&\left.\left.-\sum_{j=0}^n\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Psi_2(\mathscr{J})d\mathscr{J}\right)d\xi\right)\right\}.
\end{aligned}
$$

By inserting the values of $\mathfrak{c}_0$ and $\mathfrak{d}_0$ from (2.7), we derive the solutions (2.4) and (2.5), respectively.

3. Main results

Assume that $(\Phi_1,\Omega_1)\in C(\mathcal{J},\mathbb{R})\times C(\mathcal{J},\mathbb{R})$ satisfying

$$
\int_\eta^\zeta(\Phi_1-\Omega_1)(\xi)d\xi-\sum_{i=0}^m\varrho_i(\Phi_1-\Omega_1)(\mathcal{U}_i)-\sum_{j=0}^n\vartheta_j(\Phi_1-\Omega_1)(\mathcal{W}_j)=\mathcal{A},
$$

and there exist functions

$$\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{L}^1(\mathcal{J}, \mathbb{R}) \ni \mathcal{H}_1(\varphi) \in \Theta_1(\varphi, \Phi_1(\varphi), \Omega_1(\varphi)$$
$$\mathcal{H}_2(\varphi) \in \Theta_2(\varphi, \Phi_1(\varphi), \Omega_1(\varphi)),$$

a.e on $\varphi \in \mathcal{J}$ and

$$\begin{aligned}
\Phi_1(\varphi) = & \frac{e^{-\chi_1 \varphi}}{2} \Bigg[\frac{1}{\Delta_1} \Bigg(- \int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \mathcal{H}_1(\mathscr{J}) d\mathscr{J} \right) d\xi \\
& - \int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{H}_2(\mathscr{J}) d\mathscr{J} \right) d\xi \Bigg) \\
& + \frac{1}{\Delta_2} \Bigg(\mathcal{A} - \int_{\eta}^{\zeta} \left(\int_0^{\xi} e^{-\chi_1(\xi - \mathscr{J})} \left(\int_0^{\mathscr{J}} \frac{(\mathscr{J} - \mathfrak{m})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \mathcal{H}_1(\mathfrak{m}) d\mathfrak{m} \right) d\mathscr{J} \right) d\xi \\
& + \int_{\eta}^{\zeta} \left(\int_0^{\xi} e^{-\chi_2(\xi - \mathscr{J})} \left(\int_0^{\mathscr{J}} \frac{(\mathscr{J} - \mathfrak{m})^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{H}_2(\mathfrak{m}) d\mathfrak{m} \right) d\mathscr{J} \right) d\xi \\
& + \sum_{i=0}^{m} \varrho_i \int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i - \xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \mathcal{H}_1(\mathscr{J}) d\mathscr{J} \right) d\xi \\
& - \sum_{i=0}^{m} \varrho_i \int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i - \xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{H}_2(\mathscr{J}) d\mathscr{J} \right) d\xi \\
& + \sum_{i=0}^{n} \mathfrak{q}_i \int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j - \xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \mathcal{H}_1(\mathscr{J}) d\mathscr{J} \right) d\xi \\
& - \sum_{j=0}^{n} \vartheta_j \int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j - \xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{H}_2(\mathscr{J}) d\mathscr{J} \right) d\xi \Bigg) \Bigg] \\
& + \int_0^{\varphi} e^{-\chi_1(\varphi - \xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \mathcal{H}_1(\mathscr{J}) d\mathscr{J} \right) d\xi,
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\Omega_1(\varphi) = & \frac{e^{-\chi_1 \varphi}}{2} \Bigg[\frac{1}{\Delta_1} \Bigg(- \int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \mathcal{H}_1(\mathscr{J}) d\mathscr{J} \right) d\xi \\
& - \int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{H}_2(\mathscr{J}) d\mathscr{J} \right) d\xi \Bigg) \\
& - \frac{1}{\Delta_2} \Bigg(\mathcal{A} - \int_{\eta}^{\zeta} \left(\int_0^{\xi} e^{-\chi_1(\xi - \mathscr{J})} \left(\int_0^{\mathscr{J}} \frac{(\mathscr{J} - \mathfrak{m})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \mathcal{H}_1(\mathfrak{m}) d\mathfrak{m} \right) d\mathscr{J} \right) d\xi \\
& + \int_{\eta}^{\zeta} \left(\int_0^{\xi} e^{-\chi_2(\xi - \mathscr{J})} \left(\int_0^{\mathscr{J}} \frac{(\mathscr{J} - \mathfrak{m})^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{H}_2(\mathfrak{m}) d\mathfrak{m} \right) d\mathscr{J} \right) d\xi \\
& + \sum_{i=0}^{m} \varrho_i \int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i - \xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \mathcal{H}_1(\mathscr{J}) d\mathscr{J} \right) d\xi \\
& - \sum_{i=0}^{m} \varrho_i \int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i - \xi)} \left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{H}_2(\mathscr{J}) d\mathscr{J} \right) d\xi
\end{aligned}$$

$$+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi$$
$$-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]$$
$$+\int_0^{\varphi}e^{-\chi_2(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi, \tag{3.2}$$

is referred to as a coupled solution for a system (1.2).

For the purpose of simplifying calculations, we introduce the following notation:

$$v=\frac{e^{-\chi_1\varphi}}{2},$$

$$\Upsilon_1=v\Bigg[\frac{1}{(1+e^{-\chi_1\mathcal{T}})}\left(\frac{\mathcal{T}^{\alpha_1-1}}{\chi_1\Gamma(\alpha_1)}(1-e^{-\chi_1\mathcal{T}})\right)+\frac{1}{\Delta_2}\Bigg\{\left(\frac{\zeta^{\alpha_1-1}-\eta^{\alpha_1-1}}{\chi_1^2\Gamma(\alpha_1)}\right)(\zeta\chi_1+e^{-\chi_1\zeta}-\chi_1\eta-e^{-\chi_1\eta}) \tag{3.3}$$
$$+\sum_{i=0}^{m}\varrho_i\left(\frac{\mathcal{U}_i^{\alpha_1-1}}{\chi_1\Gamma(\alpha_1)}(1-e^{-\chi_1\mathcal{U}_i})\right)+\sum_{i=0}^{m}\mathfrak{q}_i\left(\frac{\mathcal{W}_i^{\alpha_1-1}}{\chi_1\Gamma(\alpha_1)}(1-e^{-\chi_1\mathcal{W}_i})\right)\Bigg\}\Bigg],$$

$$\Upsilon_2=v\Bigg[\frac{1}{(1+e^{-\chi_2\mathcal{T}})}\left(\frac{\mathcal{T}^{\alpha_2-1}}{\chi_2\Gamma(\alpha_2)}(1-e^{-\chi_2\mathcal{T}})\right)+\frac{1}{\Delta_2}\Bigg\{\left(\frac{\zeta^{\alpha_2-1}-\eta^{\alpha_2-1}}{\chi_2^2\Gamma(\alpha_2)}\right)(\zeta\chi_2+e^{-\chi_2\zeta}-\chi_2\eta-e^{-\chi_2\eta}) \tag{3.4}$$
$$+\sum_{i=0}^{m}\varrho_i\left(\frac{\mathcal{U}_i^{\alpha_2-1}}{\chi_2\Gamma(\alpha_2)}(1-e^{-\chi_2\mathcal{U}_i})\right)+\sum_{i=0}^{m}\mathfrak{q}_i\left(\frac{\mathcal{W}_i^{\alpha_2-1}}{\chi_2\Gamma(\alpha_2)}(1-e^{-\chi_2\mathcal{W}_i})\right)\Bigg\}\Bigg],$$

Let

$$\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}=\{\mathcal{H}_1\in\mathcal{L}^1(\mathcal{J},\mathbb{R}):\mathcal{H}_1(\varphi)\in\Theta_1(\varphi,\Phi_1(\varphi),\Omega_1(\varphi)),\ for\ a.e\ \varphi\in\mathcal{J}\},$$

and

$$\mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}=\{\mathcal{H}_2\in\mathcal{L}^1(\mathcal{J},\mathbb{R}):\mathcal{H}_2(\varphi)\in\Theta_2(\varphi,\Phi_1(\varphi),\Omega_1(\varphi)),\ for\ a.e\ \varphi\in\mathcal{J}\}$$

describe the sets of Θ_1,Θ_2 selections for each $(\Phi_1,\Omega_1)\in\mathscr{U}\times\mathscr{U}$. By using Lemma 2.6, the following operators $\Lambda_1,\Lambda_2:\mathscr{U}\times\mathscr{U}\to\mathscr{W}(\mathscr{U}\times\mathscr{U})$ by:

$$\Lambda_1(\Phi_1,\Omega_1)(\varphi)=\{\mathfrak{g}_1\in\mathscr{U}\times\mathscr{U}:\exists\,\mathcal{H}_1\in\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)},\mathcal{H}_2\in\mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}\ \ni\ \mathfrak{g}_1(\Phi_1,\Omega_1)(\varphi)=\mathcal{P}_1(\Phi_1,\Omega_1)(\varphi),\ \forall\,\varphi\in\mathcal{J}\} \tag{3.5}$$

and

$$\Lambda_2(\Phi_1,\Omega_1)(\varphi)=\{\mathfrak{g}_2\in\mathscr{U}\times\mathscr{U}:\exists\,\mathcal{H}_1\in\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)},\mathcal{H}_2\in\mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}\ \ni\ \mathfrak{g}_2(\Phi_1,\Omega_1)(\varphi)=\mathcal{P}_2(\Phi_1,\Omega_1)(\varphi),\ \forall\,\varphi\in\mathcal{J}\}, \tag{3.6}$$

where

$$\mathcal{P}_1(\Phi_1,\Omega_1)(\varphi)=\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi$$

$$
\begin{aligned}
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\
&+\int_0^{\varphi} e^{-\chi_1(\varphi-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi, \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{P}_2(\Phi_1,\Omega_1)(\varphi)=\frac{e^{-\chi_1\varphi}}{2}\Bigg[&\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\
&+\int_0^{\varphi} e^{-\chi_2(\varphi-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi. \qquad (3.8)
\end{aligned}
$$

Following that, the operator $\Lambda : \mathscr{U} \times \mathscr{U} \to \mathscr{W}(\mathscr{U} \times \mathscr{U})$ is described by

$$\Lambda(\Phi_1, \Omega_1)(\varphi) = \begin{pmatrix} \Lambda_1(\Phi_1, \Omega_1)(\varphi) \\ \Lambda_2(\Phi_1, \Omega_1)(\varphi) \end{pmatrix},$$

where Λ_1 and Λ_2 are defined in (3.5) and (3.6), respectively.

4. Existence results via the Carathéodory function

Now, we establish the existence of solutions for the BVPs (1.1) and (1.2) by utilizing the nonlinear alternative of Leray-Schauder. Subsequently, we introduce the assumptions that form the basis for illustrating the main findings of this study.
(Q_1) $\Theta_1, \Theta_2 : \mathcal{J} \times \mathbb{R}^2 \to \mathscr{W}(\mathbb{R})$ are convex and $\mathcal{L}^1$-Carathéodory functions.
(Q_2) There exist continuous increasing functions $\gamma_1, \gamma_2, k_1, k_2 : [0, \infty) \to [0, \infty)$ and functions $l_1, l_2 \in C(\mathcal{J}, \mathbb{R}^+)$, such that

$$\|\Theta_1(\varphi, \Phi_1, \Omega_1)\|_{\mathscr{W}} := \sup\{|\mathcal{H}_1| : \mathcal{H}_1 \in \Theta_1(\varphi, \Phi_1, \Omega_1)\} \leq l_1(\varphi)[\gamma_1(\|\Phi_1\|) + k_1(\|\Omega_1\|)] \quad \text{for each } (\varphi, \Phi_1, \Omega_1) \in \mathcal{J} \times \mathbb{R}^2,$$
$$\|\Theta_2(\varphi, \Phi_1, \Omega_1)\|_{\mathscr{W}} := \sup\{|\mathcal{H}_2| : \mathcal{H}_2 \in \Theta_2(\varphi, \Phi_1, \Omega_1)\} \leq l_2(\varphi)[\gamma_2(\|\Phi_1\|) + k_2(\|\Omega_1\|)] \quad \text{for each } (\varphi, \Phi_1, \Omega_1) \in \mathcal{J} \times \mathbb{R}^2.$$

(Q_3) There exists a constant $\mathcal{Z} > 0$ such that

$$\frac{\mathcal{Z}}{(2\Upsilon_1)\|l_1\|(\gamma_1(\mathcal{Z}) + k_1(\mathcal{Z})) + (2\Upsilon_2)\|l_2\|(\gamma_2(\mathcal{Z}) + k_2(\mathcal{Z}))} > 1,$$

where Υ_1, Υ_2 are defined by (3.3) and (3.4).
(Q_4) $\Theta_1, \Theta_2 : \mathcal{J} \times \mathbb{R}^2 \to \mathscr{W}_{cp}(\mathbb{R})$ are such that $\Theta_1(\cdot, \Phi_1, \Omega_1) : \mathcal{J} \to \mathscr{W}_{cp}(\mathbb{R}^2)$ and $\Theta_2(\cdot, \Phi_1, \Omega_1) : \mathcal{J} \to \mathscr{W}_{cp}(\mathbb{R}^2)$ are measurable for each $\Phi_1, \Omega_1 \in \mathbb{R}$.
(Q_5)

$$\Pi_d(\Theta_1(\varphi, \Phi_1, \Omega_1), \Theta_1(\varphi, \hat{\Phi}_1, \hat{\Omega}_1)) \leq s_1(\varphi)(|\Phi_1 - \hat{\Phi}_1| + |\Omega_1 - \hat{Y}|)$$

and

$$\Pi_d(\Theta_2(\varphi, \Phi_1, \Omega_1), \Theta_2(\varphi, \hat{\Phi}_1, \hat{\Omega}_1)) \leq s_2(\varphi)(|\Phi_1 - \hat{\Phi}_1| + |\Omega_1 - \hat{Y}|)$$

$\forall\, \varphi \in \mathcal{J}$ and $\Phi_1, \Omega_1, \hat{\Phi}_1, \hat{\Omega}_1 \in \mathbb{R}$ with $s_1, s_2 \in C(\mathcal{J}, \mathbb{R}^+)$ and $d(0, \Theta_1(\varphi, 0, 0)) \leq s_1(\varphi)$, $d(0, \Theta_2(\varphi, 0, 0)) \leq s_2(\varphi)$ $\forall\, \varphi \in \mathcal{J}$.

Theorem 4.1. *Under the assumptions (Q_1)–(Q_3), it can be asserted that the systems (1.1) and (1.2) possesses at least one solution within the interval $\mathcal{J}$.*

Proof. Consider $\Lambda_1, \Lambda_2 : \mathscr{U} \times \mathscr{U} \to \mathscr{W}(\mathscr{U} \times \mathscr{U})$ the operators which are given by (3.5) and (3.6), respectively. Using the assumption (Q_1), the sets $\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$ and $\mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}$ are nonempty for each $(\Phi_1, \Omega_1) \in \mathscr{U} \times \mathscr{U}$. Then, for $\mathcal{H}_1 \in \mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$ and $\mathcal{H}_2 \in \mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}$ for $(\Phi_1, \Omega_1) \in \mathscr{U} \times \mathscr{U}$, we have

$$\mathfrak{g}_1(\Phi_1, \Omega_1)(\varphi) = \frac{e^{-\chi_1\varphi}}{2}\left[\frac{1}{\Delta_1}\left(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi} \frac{(\xi - \mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1 - 1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\right.\right.$$

$$
\begin{aligned}
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\
&+\int_0^{\varphi}e^{-\chi_1(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi, \qquad (4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{g}_2(\Phi_1,\Omega_1)(\varphi)=\frac{e^{-\chi_1\varphi}}{2}\Bigg[&\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\
&+\int_0^{\varphi}e^{-\chi_2(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi. \qquad (4.2)
\end{aligned}
$$

where $\mathfrak{g}_1 \in \Lambda_1(\Phi_1, \Omega_1)$, $\mathfrak{g}_2 \in \Lambda_2(\Phi_1, \Omega_1)$, and so $(\mathfrak{g}_1, \mathfrak{g}_2) \in \Lambda(\Phi_1, \Omega_1)$.

We will establish that the operator Λ meets the criteria of the Leray-Schauder nonlinear alternative through a series of steps. Initially, we will demonstrate that $\Lambda(\Phi_1, \Omega_1)$ exhibits a convex valued property. Let $(\mathfrak{g}_i, \hat{\mathfrak{g}}_i) \in (\Lambda_1, \Lambda_2)$, $i = 1, 2$. Then, $\exists\ \mathcal{H}_{1i} \in \mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$, $\mathcal{H}_{2i} \in \mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}$, $i = 1, 2$, $\ni$ for each $\varphi \in \mathcal{J}$, and we achieve

$$\begin{aligned}
\mathfrak{g}_i(\varphi) =& \frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg) \\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg)\Bigg] \\
&+\int_0^{\varphi} e^{-\chi_1(\varphi-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi,
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
\hat{\mathfrak{g}}_i(\varphi) =& \frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg) \\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi
\end{aligned}$$

$$+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi$$
$$-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]$$
$$+\int_0^{\varphi}e^{-\chi_2(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi. \tag{4.4}$$

Let $0\le\nu\le 1$. Then, for each $\varphi\in\mathcal{J}$, we arrive at

$$\begin{aligned}
&\Big[\nu\mathfrak{g}_1+(1-\nu)\mathfrak{g}_2\Big](\varphi)\\
=&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}[\nu\mathcal{H}_{11}(\mathscr{J})+(1-\nu)\mathcal{H}_{11}(\mathscr{J})]d\mathscr{J}\right)d\xi\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}[\nu\mathcal{H}_{21}(\mathscr{J})+(1-\nu)\mathcal{H}_{21}(\mathscr{J})]d\mathscr{J}\right)d\xi\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}[\nu\mathcal{H}_{11}(\mathfrak{m})+(1-\nu)\mathcal{H}_{11}(\mathfrak{m})]d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}[\nu\mathcal{H}_{21}(\mathfrak{m})+(1-\nu)\mathcal{H}_{21}(\mathfrak{m})]d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}[\nu\mathcal{H}_{11}(\mathscr{J})+(1-\nu)\mathcal{H}_{11}(\mathscr{J})]d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}[\nu\mathcal{H}_{21}(\mathscr{J})+(1-\nu)\mathcal{H}_{21}(\mathscr{J})]d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}[\nu\mathcal{H}_{11}(\mathscr{J})+(1-\nu)\mathcal{H}_{11}(\mathscr{J})](\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}[\nu\mathcal{H}_{21}(\mathscr{J})+(1-\nu)\mathcal{H}_{21}(\mathscr{J})]d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\
&+\int_0^{\varphi}e^{-\chi_1(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}[\nu\mathcal{H}_{21}(\mathscr{J})+(1-\nu)\mathcal{H}_{21}(\mathscr{J})](\mathscr{J})d\mathscr{J}\right)d\xi,
\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}
&\Big[\nu\hat{\mathfrak{g}}_1+(1-\nu)\hat{\mathfrak{g}}_2\Big](\varphi)\\
=&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Big[\nu\hat{\mathcal{H}}_{11}(\mathscr{J})+(1-\nu)\hat{\mathcal{H}}_{12}(\mathscr{J})\Big]d\mathscr{J}\right)d\xi\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Big[\nu\hat{\mathcal{H}}_{21}(\mathscr{J})+(1-\nu)\hat{\mathcal{H}}_{22}(\mathscr{J})\Big]d\mathscr{J}\right)d\xi\Bigg)\\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Big[\nu\hat{\mathcal{H}}_{11}(\mathfrak{m})+(1-\nu)\hat{\mathcal{H}}_{12}(\mathfrak{m})\Big]d\mathfrak{m}\right)d\mathscr{J}\right)d\xi
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\eta}^{\zeta}\left(\int_{0}^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\left(\int_{0}^{\mathscr{J}} \frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Big[\nu\hat{\mathcal{H}}_{21}(\mathfrak{m})+(1-\nu)\hat{\mathcal{H}}_{22}(\mathfrak{m})\Big]d\mathfrak{m}\right)d\mathscr{J}\right)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_{0}^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Big[\nu\hat{\mathcal{H}}_{11}(\mathscr{J})+(1-\nu)\hat{\mathcal{H}}_{12}(\mathscr{J})\Big]d\mathscr{J}\right)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_{0}^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Big[\nu\hat{\mathcal{H}}_{21}(\mathscr{J})+(1-\nu)\hat{\mathcal{H}}_{22}(\mathscr{J})\Big]d\mathscr{J}\right)d\xi \\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_{0}^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\Big[\nu\hat{\mathcal{H}}_{11}(\mathscr{J})+(1-\nu)\hat{\mathcal{H}}_{12}(\mathscr{J})\Big](\mathscr{J})d\mathscr{J}\right)d\xi \\
&-\sum_{j=0}^{n}\vartheta_j\int_{0}^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Big[\nu\hat{\mathcal{H}}_{21}(\mathscr{J})+(1-\nu)\hat{\mathcal{H}}_{22}(\mathscr{J})\Big](\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg] \\
&+\int_{0}^{\varphi} e^{-\chi_2(\varphi-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\Big[\nu\hat{\mathcal{H}}_{21}(\mathscr{J})+(1-\nu)\hat{\mathcal{H}}_{22}(\mathscr{J})\Big](\mathscr{J})d\mathscr{J}\right)d\xi. \qquad (4.6)
\end{aligned}
$$

By virtue of the convex values associated with Θ_1 and Θ_2, it follows that both $\mathcal{V}_{\Theta 1(\Phi_1,\Omega_1)}$ and $\mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}$ inherently possess convex values as well. Clearly, for $\nu \in [0,1]$, we have $\nu\mathfrak{g}_1 + (1-\nu)\mathfrak{g}_2 \in \Lambda_1$, $\nu\hat{\mathfrak{g}_1} + (1-\nu)\hat{\mathfrak{g}_2} \in \Lambda_2$, and, thus, $\nu(\mathfrak{g}_1,\hat{\mathfrak{g}_1}) + (1-\nu)(\mathfrak{g}_2,\hat{\mathfrak{g}_2}) \in \Lambda$. For a nonnegative number $\mathfrak{r}$, let $\mathcal{B}_{\mathfrak{r}} = \{(\Phi_1,\Omega_1) \in \mathscr{U}\times\mathscr{U} : \|\Phi_1,\Omega_1\| \le \mathfrak{r}\}$ be a bounded set in $\mathscr{U}\times\mathscr{U}$. Then $\exists\ \mathcal{H}_1 \in \mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$, $\mathcal{H}_2 \in \mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}$ such that

$$
\begin{aligned}
\mathfrak{g}_1(\Phi_1,\Omega_1)(\varphi) = &\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_{0}^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi \\
&-\int_{0}^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg) \\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_{\eta}^{\zeta}\left(\int_{0}^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\left(\int_{0}^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi \\
&+\int_{\eta}^{\zeta}\left(\int_{0}^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\left(\int_{0}^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_{0}^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_{0}^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi \\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_{0}^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi \\
&-\sum_{j=0}^{n}\vartheta_j\int_{0}^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg] \\
&+\int_{0}^{\varphi} e^{-\chi_1(\varphi-\xi)}\left(\int_{0}^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi, \qquad (4.7)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{g}_2(\Phi_1,\Omega_1)(\varphi) = \frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(&-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi \\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg) \\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\left(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi \\
&+\int_\eta^{\zeta}\left(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi \\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi \\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg] \\
&+\int_0^{\varphi} e^{-\chi_2(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi. \qquad (4.8)
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
|\mathfrak{g}_1(\Phi_1,\Omega_1)(\varphi)| \le \frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(&-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi \\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi\Bigg) \\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\left(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathfrak{m}\right)d\mathscr{J}\right)d\xi \\
&+\int_\eta^{\zeta}\left(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathfrak{m}\right)d\mathscr{J}\right)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi \\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi \\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi\Bigg)\Bigg]
\end{aligned}
$$

$$+\int_0^{\varphi} e^{-\chi_1(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi,$$
$$\leq \Upsilon_1\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))+\Upsilon_2\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))$$

and

$$|\mathfrak{g}_2(\Phi_1,\Omega_1)(\varphi)|\leq \Upsilon_1\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))+\Upsilon_2\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r})).$$

Thus we get

$$\begin{aligned}\|\mathfrak{g}_1,\mathfrak{g}_2\| &= \|\mathfrak{g}_1(\Phi_1,\Omega_1)\|+\|\mathfrak{g}_1(\Phi_1,\Omega_1)\|\\ &\leq 2\Upsilon_1\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))+2\Upsilon_2\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r})).\end{aligned}$$

Subsequently, we proceed to establish the equicontinuity of the operator Λ. Let $\varphi_1,\varphi_2\in\mathcal{J}$ with $\varphi_1<\varphi_2$. Then, $\exists\,\mathcal{H}_1\in\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$, $\mathcal{H}_2\in\mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}$ such that

$$\begin{aligned}\mathfrak{g}_1(\Phi_1,\Omega_1)(\varphi)=&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\
&+\int_0^{\varphi}e^{-\chi_1(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi,\end{aligned}$$

and

$$\begin{aligned}\mathfrak{g}_2(\Phi_1,\Omega_1)(\varphi)=&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\Delta_2}\left(\mathcal{A}-\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\right.\\
&+\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&\left.-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\right)\Bigg]\\
&+\int_0^\varphi e^{-\chi_2(\varphi-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi.
\end{aligned}
$$

$$
\begin{aligned}
&|\mathfrak{g}_1(\Phi_1,\Omega_1)(\varphi_2)-\mathfrak{g}_1(\Phi_1,\Omega_1)(\varphi_1)|\\
\leq&\left|\frac{e^{-\chi_1\varphi_2}-e^{-\chi_1\varphi_1}}{2}\left[\frac{1}{\Delta_1}\left(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi\right.\right.\right.\\
&\left.-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi\right)\\
&+\frac{1}{\Delta_2}\left(\mathcal{A}-\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\right.\\
&+\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi\\
&\left.\left.\left.-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi\right)\right]\right\}\Bigg|\\
&+\left|\int_0^{\varphi_1}(e^{-\chi_1(\varphi_2-\xi)}-e^{-\chi_1(\varphi_1-\xi)})\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi\right.\\
&\left.+\int_{\varphi_1}^{\varphi_2}e^{-\chi_1(\varphi_2-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi\right|,
\end{aligned}
$$

Likewise, one can construct

$$
\begin{aligned}
&|\mathfrak{g}_2(\Phi_1,\Omega_1)(\varphi_2)-\mathfrak{g}_2(\Phi_1,\Omega_1)(\varphi_1)|\\
\leq&\left|\frac{e^{-\chi_1\varphi_2}-e^{-\chi_1\varphi_1}}{2}\left[\frac{1}{\Delta_1}\left(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi\right.\right.\right.\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi\Bigg)\\
&-\frac{1}{\Delta_2}\left(\mathcal{A}-\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\right.\\
&+\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|l_1\|(\gamma_1(\mathfrak{r})+k_1(\mathfrak{r}))d\mathscr{J}\right)d\xi\\
&\left.\left.\left.-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi\right)\right]\right\}\Bigg|\\
&+\left|\int_0^{\varphi_1}(e^{-\chi_1(\varphi_2-\xi)}-e^{-\chi_1(\varphi_1-\xi)})\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi\right.\\
&\left.+\int_{\varphi_1}^{\varphi_2}e^{-\chi_1(\varphi_2-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\|l_2\|(\gamma_2(\mathfrak{r})+k_2(\mathfrak{r}))d\mathscr{J}\right)d\xi\right|.
\end{aligned}
$$

Consequently, it follows that the operator $\Lambda(\Phi_1,\Omega_1)$ is equicontinuous. Therefore, in accordance with the Ascoli-Arzelá theorem, the operator $\Lambda(\Phi_1,\Omega_1)$ can be classified as a completely continuous operator. As stated in [26], a c.c operator possesses a closed graph when it is also u.s.c. Consequently, our task is to illustrate that Λ indeed possesses a closed graph. Let $(\Phi_{1n},\Omega_{1n})\to(\Phi_{1*},\Omega_{1*})$, $(\mathfrak{g}_n,\hat{\mathfrak{g}_n})\in\Lambda(\Phi_{1n},\Omega_{1n})$, and $(\mathfrak{g}_n,\hat{\mathfrak{g}_n})\to(\mathfrak{g}_*,\mathfrak{g}_*)$, then we must demonstrate $(\mathfrak{g}_*,\hat{\mathfrak{g}_*})\in\Lambda(\Phi_{1*},\Omega_{1*})$. Remember that $(\mathfrak{g}_n,\mathfrak{g}_n)\in\Lambda(\Phi_{1n},\Omega_{1n})$ implies that $\exists\,\mathcal{H}_{1n}\in\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$, and $\mathcal{H}_{2n}\in\mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}$ such that

$$
\begin{aligned}
\mathfrak{g}_n(\Phi_{1n},\Omega_{1n})(\varphi)=&\frac{e^{-\chi_1\varphi}}{2}\left[\frac{1}{\Delta_1}\left(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{1n}(\mathscr{J})d\mathscr{J}\right)d\xi\right.\right.\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{2n}(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\
&+\frac{1}{\Delta_2}\left(\mathcal{A}-\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{1n}(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\right.\\
&+\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{2n}(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{1n}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{2}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{1n}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{2n}(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\
&+\int_0^{\varphi}e^{-\chi_1(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{1n}(\mathscr{J})d\mathscr{J}\right)d\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\mathfrak{g}}_n(\Phi_{1n},\Omega_{1n})(\varphi)=\frac{e^{-\chi_1\varphi}}{2}\Bigg[&\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{1n}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{2n}(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_{\eta}^{\zeta}\left(\int_0^{\xi}e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{1n}(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\int_{\eta}^{\zeta}\left(\int_0^{\xi}e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{2n}(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{1n}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{2n}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{1n}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{2n}(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\
&+\int_0^{\varphi}e^{-\chi_2(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{2n}(\mathscr{J})d\mathscr{J}\right)d\xi.
\end{aligned}
$$

Consider the $\Pi_1, \Pi_2 : \mathcal{L}^1(\mathcal{J}, \mathscr{U}\times\mathscr{U}) \to C(\mathcal{J}, \mathscr{U}\times\mathscr{U})$ continuous linear operators given by

$$
\begin{aligned}
\Pi_1(\Phi_1,\Omega_1)(\varphi)=\frac{e^{-\chi_1\varphi}}{2}\Bigg[&\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{1}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{2}(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Delta_2}\left(\mathcal{A}-\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\right.\\
&+\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^n\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&\left.-\sum_{j=0}^n\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\right)\Bigg]\\
&+\int_0^\varphi e^{-\chi_1(\varphi-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi, \qquad (4.9)
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi_2(\Phi_1,\Omega_1)(\varphi)=\frac{e^{-\chi_1\varphi}}{2}&\left[\frac{1}{\Delta_1}\left(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\right.\right.\\
&\left.-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\right)\\
&-\frac{1}{\Delta_2}\left(\mathcal{A}-\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\right.\\
&+\int_\eta^\zeta\left(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^n\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\
&\left.\left.-\sum_{j=0}^n\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\right)\right]\\
&+\int_0^\varphi e^{-\chi_2(\varphi-\xi)}\left(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi. \qquad (4.10)
\end{aligned}
$$

It can be inferred from [8] that $(\Phi_1,\Omega_1)\circ(\mathcal{V}_{\Theta_1},\mathcal{V}_{\Theta_2})$ constitutes a closed graph operator. Furthermore, we have $(\mathfrak{g}_n,\hat{\mathfrak{g}}_n)\in(\Phi_1,\Omega_1)\circ(\mathcal{V}_{\Theta_1(\Phi_{1n},\Omega_{1n})},\mathcal{V}_{\Theta_2(\Phi_{1n},\Omega_{1n})})$ for all n. Since

$(\Phi_{1n}, \Omega_{1n}) \to (\Phi_{1*}, \Omega_{1*})$, $(\mathfrak{g}_n, \hat{\mathfrak{g}_n}) \to \mathfrak{g}_*, \hat{\mathfrak{g}_*})$, it follows that $\mathcal{H}_{1n} \in \mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$, $\mathcal{H}_{2n} \in \mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}$ such that

$$\begin{aligned}
\mathfrak{g}_*(\Phi_{1*}, \Omega_{1*})(\varphi) =& \frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg) \\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_{\eta}^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1*})(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\int_{\eta}^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2*})(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg)\Bigg] \\
&+\int_0^{\varphi} e^{-\chi_1(\varphi-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi,
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathfrak{g}}_*(\Phi_{1*}, \Omega_{1*})(\varphi) =& \frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg) \\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_{\eta}^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1*})(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\int_{\eta}^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2*})(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1*})(\mathscr{J})d\mathscr{J}\Bigg)d\xi
\end{aligned}$$

$$-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2*})(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]$$
$$+\int_0^{\varphi}e^{-\chi_2(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2*})(\mathscr{J})d\mathscr{J}\right)d\xi,$$

(i.e)., $(\mathfrak{g}_n,\mathfrak{g}_n)\in\Lambda(\Phi_{1*},\Omega_{1*})$.

Finally, for a priori, let $(\Phi_1,\Omega_1)\in\upsilon\Lambda(\Phi_1,\Omega_1)$. Then $\exists\,\mathcal{H}_1\in\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$, $\mathcal{H}_2\in\mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}$ such that

$$\begin{aligned}\Phi_1(\varphi)=&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\&+\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\\&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\&+\int_0^{\varphi}e^{-\chi_1(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi,\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}\Omega_1(\varphi)=&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\\&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\&+\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=0}^{m} \varrho_i \int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)} \left(\int_0^{\xi} \frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \mathcal{H}_2(\mathscr{J}) d\mathscr{J} \right) d\xi \\
&+ \sum_{i=0}^{n} \mathfrak{q}_i \int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)} \left(\int_0^{\xi} \frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \mathcal{H}_1(\mathscr{J}) d\mathscr{J} \right) d\xi \\
&- \sum_{j=0}^{n} \vartheta_j \int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)} \left(\int_0^{\xi} \frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \mathcal{H}_2(\mathscr{J}) d\mathscr{J} \right) d\xi \Bigg) \Bigg] \\
&+ \int_0^{\varphi} e^{-\chi_2(\varphi-\xi)} \left(\int_0^{\xi} \frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \mathcal{H}_2(\mathscr{J}) d\mathscr{J} \right) d\xi. \qquad (4.12)
\end{aligned}
$$

For each $\varphi \in \mathcal{J}$, we achieve

$$
\begin{aligned}
\|\Phi_1, \Omega_1\| &= \|\Phi_1\| + \|\Omega_1\| \\
&\leq 2\Upsilon_1 l_1(\gamma_1(\|\Phi_1\|) + k_1(\|\Omega_1\|)) + 2\Upsilon_2 l_2(\gamma_2(\|\Phi_1\|) + k_2(\|\Omega_g\|)),
\end{aligned}
$$

which signifies that

$$
\frac{\|(\Phi_1, \Omega_1)\|}{2\Upsilon_1 l_1(\gamma_1(\|\Phi_1\|) + k_1(\|\Omega_1\|)) + 2\Upsilon_2 l_2(\gamma_2(\|\Phi_1\|) + k_2(\|\Omega_g\|))} \leq 1.
$$

According to (Q_3), $\mathcal{Z}$ exists such that $\|(\Phi_1, \Omega_1)\| \neq \mathcal{Z}$. Let us adopt

$$
\mathcal{E} = \{(\Phi_1, \Omega_1) \in \mathscr{U} \times \mathscr{U} : \|(\Phi_1, \Omega_1)\| < \mathcal{Z}\}.
$$

The operator $\Lambda : \overline{\mathcal{E}} \to \mathscr{W}_{cv,cp}(\mathscr{U}) \times \mathscr{W}_{cv,cp}(\mathscr{U})$ is c.c and u.s.c. There is no $(\Phi_1, \Omega_1) \in \partial\mathcal{E} \ni (\Phi_1, \Omega_1) \in \upsilon\Lambda(\Phi_1, \Omega_1)$ for some $\upsilon \in (0, 1)$ by $\mathcal{E}$ selection. Consequently, based on the Leray–Schauder nonlinear alternative [33], we can infer that Λ possesses a fixed point $(\Phi_1, \Omega_1) \in \overline{\mathcal{E}}$, thereby serving as a solution to the system (1.2).

5. Existence results under Lipschitz mapping

The subsequent outcome leverages Covitz and Nadler's theorem for multivalued maps as presented in [34].

Let $(\mathscr{U}, d)$ represent a metric space generated by the normed space $(\mathscr{U}, \|\cdot\|)$, and let $\Pi_d : \mathscr{W}(\mathscr{U}) \times \mathscr{W}(\mathscr{U}) \to \mathbb{R} \cup \{\infty\}$ be described by

$$
\Pi_d(\Phi_1, \Omega_1) = \max\{\sup_{x \in \Phi_1} d(x, \Omega_1), \sup_{y \in \Omega_1} d(\Phi_1, y)\},
$$

where $d(\Phi_1, y) = \inf_{x \in \Phi_1} d(x, y)$ and $d(x, \Omega_1) = \inf y \in \Omega_1 d(x, y)$.

Then, $(\mathscr{W}_{cl,b}(\mathscr{U}), \Pi_d)$ is a metric space and $(\mathscr{W}_{cl}(\mathscr{U}), \Pi_d)$ is a generalized metric space; see [2].

Theorem 5.1. *Under the assumptions (Q_4) and (Q_5), it can be affirmed that the systems (1.1) and (1.2) possess at least one solution within the interval $\mathcal{J}$, provided that:*

$$
(2\Upsilon_1)\|s_1\| + (2\Upsilon_2)\|s_2\| < 1. \qquad (5.1)
$$

Proof. Assuming (Q_4) that the sets $\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$ and $\mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}$ are nonempty for each $(\Phi_1,\Omega_1) \in \mathscr{U}\times\mathscr{U}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ have measurable selections (see Theorem III.6 in [35]). Subsequently, we proceed to establish that the operator Λ satisfies the theorem of Covitz and Nadler [34].

Next, we illustrate that $\Lambda(\Phi_1,\Omega_1) \in \mathscr{W}_{cl}(\mathscr{U})\times\mathscr{W}_{cl}(\mathscr{U})$ for each $(\Phi_1,\Omega_1) \in \mathscr{U}\times\mathscr{U}$. Let $(\mathfrak{g}_n,\hat{\mathfrak{g}}_n) \in \Lambda(\Phi_{1n},\Omega_{1n})$ such that $(\mathfrak{g}_n,\hat{\mathfrak{g}}_n) \to (\mathfrak{g},\hat{\mathfrak{g}})$ in $\mathscr{U}\times\mathscr{U}$. Then $(\mathfrak{g},\hat{\mathfrak{g}}) \in \mathscr{U}\times\mathscr{U}$, $\exists\, \mathcal{H}_{1n} \in \mathcal{V}_{\Theta_1(\Phi_{1n},\Omega_{1n})}$ and $\mathcal{H}_{2n} \in \mathcal{V}_{\Theta_1(\Phi_{1n},\Omega_{1n})}$ such that

$$
\begin{aligned}
\mathfrak{g}_n(\Phi_{1n},\Omega_{1n})(\varphi) =& \frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg) \\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1n})(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2n})(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg)\Bigg] \\
&+\int_0^{\varphi} e^{-\chi_1(\varphi-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\mathfrak{g}}_n(\Phi_{1n},\Omega_{1n})(\varphi) =& \frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg) \\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1n})(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\int_\eta^{\zeta}\Bigg(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2n})(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi \\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi \\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2n})(\mathscr{J})d\mathscr{J}\Bigg)d\xi
\end{aligned}
$$

$$+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}(\mathcal{H}_{1n})(\mathscr{J})d\mathscr{J}\right)d\xi$$
$$-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2n})(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]$$
$$+\int_0^{\varphi}e^{-\chi_2(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}(\mathcal{H}_{2n})(\mathscr{J})d\mathscr{J}\right)d\xi.$$

As a consequence of the compact values associated with Θ_1 and Θ_2, we proceed by selecting subsequences (referred to as sequences) to guarantee the convergence of $\mathcal{H}_{1n}$ and $\mathcal{H}_{2n}$ to $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, in the space $\mathcal{L}^1(\mathcal{J},\mathbb{R})$. Hence, $\mathcal{H}_1\in\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$ and $\mathcal{H}_2\in\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$ for each $\varphi\in\mathcal{J}$ such that

$$\mathfrak{g}_n(\Phi_{1n},\Omega_{1n})(\varphi)\to\mathfrak{g}(\Phi_1,\Omega_1)(\varphi)$$
$$=\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi$$
$$-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)$$
$$+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_{\eta}^{\zeta}\left(\int_0^{\xi}e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi$$
$$+\int_{\eta}^{\zeta}\left(\int_0^{\xi}e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi$$
$$+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi$$
$$-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi$$
$$+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi$$
$$-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]$$
$$+\int_0^{\varphi}e^{-\chi_1(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi,$$

and

$$\hat{\mathfrak{g}}_n(\Phi_{1n},\Omega_{1n})(\varphi)\to\hat{\mathfrak{g}}(\Phi_1,\Omega_1)(\varphi)$$
$$=\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\right)d\xi$$
$$-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)$$

$$
\begin{aligned}
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^\zeta\Bigg(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi\\
&+\int_\eta^\zeta\Bigg(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi\\
&+\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi\\
&-\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi\\
&+\sum_{i=0}^n\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathscr{J})d\mathscr{J}\Bigg)d\xi\\
&-\sum_{j=0}^n\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg)\Bigg]\\
&+\int_0^\varphi e^{-\chi_2(\varphi-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathscr{J})d\mathscr{J}\Bigg)d\xi.
\end{aligned}
$$

As a consequence, $(\mathfrak{g},\hat{\mathfrak{g}})\in\Lambda$, such that Λ is closed. Following that, we demonstrate that one gets from (5.1)

$$
\Pi_d(\Lambda(\Phi_1,\Omega_1),\Lambda(\hat{\Phi}_1,\hat{\Omega}_1))\leq\hat{\rho}(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega_1-\hat{\Omega}_1\|)\text{ for each }\Phi_1,\hat{\Phi}_1,\Omega_1,\hat{\Omega}_1\in\mathscr{U}.
$$

Let $(\Phi_1,\hat{\Phi}_1),(\Omega_1,\hat{\Omega}_1)\in\mathscr{U}\times\mathscr{U}$ and $(\mathfrak{g}_1,\hat{\mathfrak{g}}_1)\in\Lambda(\Phi_1,\Omega_1)$. Then, $\exists\ \mathcal{H}_{11}\in\mathcal{V}_{\Theta_1(\Phi_1,\Omega_1)}$ and $\mathcal{H}_{21}\in\mathcal{V}_{\Theta_2(\Phi_1,\Omega_1)}\ni$, for each $\varphi\in\mathcal{J}$, and we have

$$
\begin{aligned}
\mathfrak{g}_1(\Phi_{1n},\Omega_{1n})(\varphi)=&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\Bigg)d\xi\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^\zeta\Bigg(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi\\
&+\int_\eta^\zeta\Bigg(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi\\
&+\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\Bigg)d\xi\\
&-\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\Bigg)d\xi\\
&+\sum_{i=0}^n\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\Bigg)d\xi\\
&-\sum_{j=0}^n\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\Bigg(\int_0^\xi\frac{(\xi-)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg)\Bigg]
\end{aligned}
$$

$$+\int_0^{\varphi} e^{-\chi_1(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\right)d\xi,$$

and

$$\begin{aligned}\hat{\mathfrak{g}}_1(\Phi_{1n},\Omega_{1n})(\varphi)=&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}} e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\right)d\xi\\&-\int_0^{\mathcal{T}} e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_{\eta}^{\zeta}\left(\int_0^{\xi} e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\&+\int_{\eta}^{\zeta}\left(\int_0^{\xi} e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\right)d\xi\\&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i} e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\right)d\xi\\&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j} e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\right)d\xi\\&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j} e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\&+\int_0^{\varphi} e^{-\chi_2(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\right)d\xi.\end{aligned}$$

Utilizing $(\mathcal{Q}_5)$, we acquire

$$\Pi_d(\Theta_1(\varphi,\Phi_1,\Omega_1),\Theta_1(\varphi,\hat{\Phi}_1,\hat{\Omega}_1))\leq s_1(\varphi)(|\Phi_1(\varphi)-\hat{\Phi}_1(\varphi)|+|\Omega_1(\varphi)-\hat{\Omega}_1(\varphi)|),$$

and

$$\Pi_d(\Theta_2(\varphi,\Phi_1,\Omega_1),\Theta_2(\varphi,\hat{\Phi}_1,\hat{\Omega}_1))\leq s_2(\varphi)(|\Phi_1(\varphi)-\hat{\Phi}_1(\varphi)|+|\Omega_1(\varphi)-\hat{\Omega}_1(\varphi)|).$$

So, $\exists\,\mathcal{H}_1\in\Theta_1(\varphi,\Phi_1(\varphi),\Omega_1(\varphi))$ and $\mathcal{H}_2\in\Theta_2(\varphi,\Phi_1(\varphi),\Omega_1(\varphi))$ such that

$$|\mathcal{H}_{11}(\varphi)-u|\leq s_1(\varphi)(|\Phi_1(\varphi)-\hat{\Phi}_1(\varphi)|+|\Omega_1(\varphi)-\hat{\Omega}_1(\varphi)|),$$

and

$$|\mathcal{H}_{21}(\varphi)-v|\leq s_2(\varphi)(|\Phi_1(\varphi)-\hat{\Phi}_1(\varphi)|+|\Omega_1(\varphi)-\hat{\Omega}_1(\varphi)|).$$

Define $\Omega_{11},\Omega_{12}:\mathcal{J}\to\mathscr{W}(\mathbb{R})$ by

$$\Omega_{11}(\varphi)=\{\mathcal{H}_1\in\mathcal{L}^1(\mathcal{J},\mathbb{R}):s_1(\varphi)(|\Phi_1(\varphi)-\hat{\Phi}_1(\varphi)|+|\Omega_1(\varphi)-\hat{\Omega}_1(\varphi)|)\},$$

and

$$\Omega_{12}(\varphi)=\{\mathcal{H}_2\in\mathcal{L}^1(\mathcal{J},\mathbb{R}):s_2(\varphi)(|\Phi_1(\varphi)-\hat{\Phi}_1(\varphi)|+|\Omega_1(\varphi)-\hat{\Omega}_1(\varphi)|)\}.$$

There are functions $\mathcal{H}_{12}(\varphi)$,$\mathcal{H}_{22}(\varphi)$ that are an observable selection for Ω_{11},Ω_{12} because the multivalued operators $\Omega_{11}\cap\Theta_1(\varphi,\Phi_1(\varphi),\Omega_1(\varphi))$ and $\Omega_{12}\cap\Theta_2(\varphi,\Phi_1(\varphi),\Omega_1(\varphi))$ are measurable (Proposition III.4 in [26]). Also, $\mathcal{H}_{12}(\varphi)\in\Theta_1(\varphi,\Phi_1(\varphi),\Omega_1(\varphi))$, $\mathcal{H}_{22}(\varphi)\in\Theta_2(\varphi,\Phi_1(\varphi),\Omega_1(\varphi))$ such that $\forall\ \varphi\in\mathcal{J}$, and we arrive at

$$|\mathcal{H}_{11}(\varphi)-\mathcal{H}_{12}(\varphi)|\le s_1(\varphi)(|\Phi_1(\varphi)-\hat{\Phi}_1(\varphi)|+|\Omega_1(\varphi)-\hat{\Omega}_1(\varphi)|),$$

and

$$|\mathcal{H}_{21}(\varphi)-\mathcal{H}_{22}(\varphi)|\le s_2(\varphi)(|\Phi_1(\varphi)-\hat{\Phi}_1(\varphi)|+|\Omega_1(\varphi)-\hat{\Omega}_1(\varphi)|).$$

Let

$$\begin{aligned}
g_2(\Phi_{1n},\Omega_{1n})(\varphi)=&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_1(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\int_\eta^{\zeta}\left(\int_0^{\xi}e^{-\chi_2(\xi-\mathscr{J})}\left(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathfrak{m})d\mathfrak{m}\right)d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{i=0}^{m}\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&+\sum_{i=0}^{n}\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\sum_{j=0}^{n}\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)\Bigg]\\
&+\int_0^{\varphi}e^{-\chi_1(\varphi-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\right)d\xi,
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathfrak{g}}_2(\Phi_{1n},\Omega_{1n})(\varphi)=&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\right)d\xi\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\left(\int_0^{\xi}\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\right)d\xi\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^\zeta\Bigg(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi\\
&+\int_\eta^\zeta\Bigg(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathfrak{m})d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi\\
&+\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\Bigg)d\xi\\
&-\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\Bigg)d\xi\\
&+\sum_{i=0}^n\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_{11}(\mathscr{J})d\mathscr{J}\Bigg)d\xi\\
&-\sum_{j=0}^n\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\Bigg)d\xi\Bigg)\Bigg]\\
&+\int_0^\varphi e^{-\chi_2(\varphi-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_{21}(\mathscr{J})d\mathscr{J}\Bigg)d\xi.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|\mathfrak{g}_1(\Phi_1,\Omega_1)(\varphi)-\mathfrak{g}_2(\Phi_1,\Omega_1)(\varphi)|\\
\le&\frac{e^{-\chi_1\varphi}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathcal{T}}e^{-\chi_1(\mathcal{T}-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}s_1(\mathscr{J})(|\Phi_1(\mathscr{J})-\hat{\Phi}_1(\mathscr{J})|+|\Omega_1(\mathscr{J})-\hat{\Omega}_1(\mathscr{J})|)d\mathscr{J}\Bigg)d\xi\\
&-\int_0^{\mathcal{T}}e^{-\chi_2(\mathcal{T}-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}s_2(\mathscr{J})(|\Phi_1(\mathscr{J})-\hat{\Phi}_1(\mathscr{J})|+|\Omega_1(\mathscr{J})-\hat{\Omega}_1(\mathscr{J})|)d\mathscr{J}\Bigg)d\xi\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{A}-\int_\eta^\zeta\Bigg(\int_0^\xi e^{-\chi_1(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}s_1(\mathfrak{m})(|\Phi_1(\mathfrak{m})-\hat{\Phi}_1(\mathfrak{m})|+|\Omega_1(\mathfrak{m})-\hat{\Omega}_1(\mathfrak{m})|)d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi\\
&+\int_\eta^\zeta\Bigg(\int_0^\xi e^{-\chi_2(\xi-\mathscr{J})}\Bigg(\int_0^{\mathscr{J}}\frac{(\mathscr{J}-\mathfrak{m})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}s_2(\mathfrak{m})(|\Phi_1(\mathfrak{m})-\hat{\Phi}_1(\mathfrak{m})|+|\Omega_1(\mathfrak{m})-\hat{\Omega}_1(\mathfrak{m})|)d\mathfrak{m}\Bigg)d\mathscr{J}\Bigg)d\xi\\
&+\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_1(\mathcal{U}_i-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}s_1(\mathscr{J})(|\Phi_1(\mathscr{J})-\hat{\Phi}_1(\mathscr{J})|+|\Omega_1(\mathscr{J})-\hat{\Omega}_1(\mathscr{J})|)d\mathscr{J}\Bigg)d\xi\\
&-\sum_{i=0}^m\varrho_i\int_0^{\mathcal{U}_i}e^{-\chi_2(\mathcal{U}_i-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}s_2(\mathscr{J})(|\Phi_1(\mathscr{J})-\hat{\Phi}_1(\mathscr{J})|+|\Omega_1(\mathscr{J})-\hat{\Omega}_1(\mathscr{J})|)d\mathscr{J}\Bigg)d\xi\\
&+\sum_{i=0}^n\mathfrak{q}_i\int_0^{\mathcal{W}_j}e^{-\chi_1(\mathcal{W}_j-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}s_1(\mathscr{J})(|\Phi_1(\mathscr{J})-\hat{\Phi}_1(\mathscr{J})|+|\Omega_1(\mathscr{J})-\hat{\Omega}_1(\mathscr{J})|)d\mathscr{J}\Bigg)d\xi\\
&-\sum_{j=0}^n\vartheta_j\int_0^{\mathcal{W}_j}e^{-\chi_2(\mathcal{W}_j-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}s_2(\mathscr{J})(|\Phi_1(\mathscr{J})-\hat{\Phi}_1(\mathscr{J})|+|\Omega_1(\mathscr{J})-\hat{\Omega}_1(\mathscr{J})|)d\mathscr{J}\Bigg)d\xi\Bigg)\Bigg]\\
&+\int_0^\varphi e^{-\chi_1(\varphi-\xi)}\Bigg(\int_0^\xi\frac{(\xi-\mathscr{J})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}s_1(\mathscr{J})(|\Phi_1(\mathscr{J})-\hat{\Phi}_1(\mathscr{J})|+|\Omega_1(\mathscr{J})-\hat{\Omega}_1(\mathscr{J})|)d\mathscr{J}\Bigg)d\xi\\
&\le\Upsilon_1\|s_1\|(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega-\hat{\Omega}_1\|)+\Upsilon_2\|s_2\|(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega-\hat{\Omega}_1\|).
\end{aligned}
$$

Thus,

$$\|\mathcal{H}_1(\Phi_1,\Omega_1)-\mathcal{H}_2(\Phi_1,\Omega_1)\| \le \Upsilon_1\|s_1\|(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega_1-\hat{\Omega}_1\|)+\Upsilon_2\|s_2\|(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega_1-\hat{\Omega}_1\|).$$

Similarly, we can define that

$$\|\hat{\mathcal{H}}_1(\Phi_1,\Omega_1)-\hat{\mathcal{H}}_2(\Phi_1,\Omega_1)\| \le \Upsilon_1\|s_1\|(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega-\hat{\Omega}_1\|)+\Upsilon_2\|s_2\|(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega-\hat{\Omega}_1\|).$$

Therefore,

$$\|(\mathcal{H}_1,\hat{\mathcal{H}}_1),(\mathcal{H}_2,\hat{\mathcal{H}}_2)\| \le 2\Upsilon_1\|s_1\|(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega-\hat{\Omega}_1\|)+2\Upsilon_2\|s_2\|(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega-\hat{\Omega}_1\|).$$

Likewise, by interchanging the positions of (Φ_1,Ω_1) and $(\hat{\Phi}_1,\hat{\Omega}_1)$, we can acquire

$$\|\Pi_d(\mathcal{P}(\Phi_1,\Omega_1),\mathcal{P}(\hat{\Phi}_1,\hat{\Omega}_1))\| \le 2\Upsilon_1\|s_1\|(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega-\hat{\Omega}_1\|)+2\Upsilon_2\|s_2\|(\|\Phi_1-\hat{\Phi}_1\|+\|\Omega-\hat{\Omega}_1\|).$$

Given the provided assumption, it can be established that Λ satisfies the contraction condition (5.1). Consequently, by virtue of Nadler's fixed point theorem, Λ possesses a fixed point (Φ_1,Ω_1), which serves as a solution to the system (1.2).

6. An application

Aligned with the formulations of systems (1.1) and (1.2), and in accordance with the primary theorems, we present illustrative examples within this section.

Example 6.1. Let us examine the following system:

$$\begin{cases} ({}^C\mathcal{D}^{\alpha_1}+\chi_1{}^C\mathcal{D}^{\alpha_1-1})\Phi_1(\varphi)\in\Theta_1(\varphi,\Phi_1(\varphi),\Omega_1(\varphi)), \varphi\in\mathcal{J}:=[0,\mathcal{T}],\\ ({}^C\mathcal{D}^{\alpha_2}+\chi_2{}^C\mathcal{D}^{\alpha_2-1})\Omega_1(\varphi)\in\Theta_2(\varphi,\Phi_1(\varphi),\Omega_1(\varphi)), \varphi\in\mathcal{J}:=[0,\mathcal{T}],\\ (\Phi_1+\Omega_1)(0)=-(\Phi_1+\Omega_1)(\mathcal{T}),\\ \int_\eta^\zeta(\Phi_1-\Omega_1)(\xi)d\xi-\sum_{i=0}^{m}\varrho_i(\Phi_1-\Omega_1)(\mathcal{U}_i)-\sum_{j=0}^{n}\vartheta_j(\Phi_1-\Omega_1)(\mathcal{W}_j)=\mathcal{A}. \end{cases} \tag{6.1}$$

where $\alpha_1=3/2,\beta=4/3,\eta=3/4,\zeta=3/2,\mathcal{T}=2,\mathcal{A}=1,\mathfrak{z}_1=1,\mathfrak{z}_2=1/5,\mathfrak{q}_1=26/100,\mathfrak{q}_2=6/25,\mathcal{U}_1=1/4,\mathcal{U}_2=1/3,\mathcal{W}_1=7/4,\mathcal{W}_2=47/25$, $\mathcal{H}_1(\varphi,\Phi_1,\Omega_1)=\left[\frac{-1}{16}\frac{\Phi_1}{1+|\Phi_1|},0\right]\cup\left[0,\frac{1}{16}\frac{|\sin(\Omega_1)|}{1+|\sin(\Omega_1)|}\right]$, and $\mathcal{H}_2(\varphi,\Phi_1,\Omega_1)=\left[\frac{-1}{16}\frac{|\Omega_1|}{1+|\Omega_1|},0\right]\cup\left[0,\frac{1}{16}\frac{|\cos(\Phi_1)|}{1+|\cos(\Phi_1)|}\right]$, and on the other hand,

$$\Pi_d(\mathcal{H}_1(\varphi,\Phi_1,\Omega_1),\mathcal{H}_1(\varphi,\widehat{\Phi_1},\widehat{\Omega_1}))\le\frac{1}{16}|\Phi_1-\widehat{\Phi_1}|+\frac{1}{16}|\Omega_1-\widehat{\Omega_1}|,\ \forall\ \Phi_1,\widehat{\Phi_1},\Omega_1,\widehat{\Omega_1}\in\mathbb{R},$$
$$\Pi_d(\mathcal{H}_2(\varphi,\Phi_1,\Omega_1),\mathcal{H}_2(\varphi,\widehat{\Phi_1},\widehat{\Omega_1}))\le\frac{1}{16}|\Phi_1-\widehat{\Phi_1}|+\frac{1}{16}|\Omega_1-\widehat{\Omega_1}|,\ \forall\ \Phi_1,\widehat{\Phi_1},\Omega_1,\widehat{\Omega_1}\in\mathbb{R}.$$

Implementing the abovesaid data, we calculate $\Upsilon_1=0.200876$ $\Upsilon_2=0.0.156334$ and $(2\Upsilon_1)\Phi_{11}+(2\Upsilon_2)\Phi_{12}\approx 0.04465125<1$. Every assumption outlined in Theorem 5.1 is satisfied, thereby leading to the conclusion that $\exists$ is a solution to the system (6.1).

7. Conclusions and future work

In this study, our focus has been on introducing a novel category of coupled nonlocal boundary conditions as a means to explore coupled nonlinear SFD inclusions of the Caputo type. Through the utilization of multivalued maps, we have successfully established the existence of solutions. By harnessing established fixed point theorems tailored to cater to multivalued maps, we have obtained intriguing solutions for the specified problem under conditions encompassing both convex and non-convex values within the multivalued maps. The implications of the results presented in this paper hold substantial importance for the scientific community. As an example, if we take $\xi = (\mathcal{H})\xi$ in $\int_{\eta}^{\zeta}(\Phi_1 - \Omega_1)(\xi)d(\mathcal{H})\xi - \sum_{i=0}^{m}\varrho_i(\Phi_1 - \Omega_1)(\mathcal{U}_i) - \sum_{j=0}^{n}\vartheta_j(\Phi_1 - \Omega_1)(\mathcal{W}_j) = \mathcal{A}$ in (1.1) and (1.2), our findings align with those for novel coupled Stieltjes boundary conditions. Furthermore, this set allows us to derive new existence results. We believe that the results discussed in this paper are of great significance to the scientific audience. As a potential avenue for future research, the examination of controllability and the dependency of solutions for a system of coupled SFD equations incorporating a combination of Caputo derivatives could be considered as real applied problems. Also what are the real applications of the considered model?

Author contributions

All authors contributed equally and significantly in writing this article.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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