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Research article

A Grammian matrix and controllability study of fractional delay integro-differential Langevin systems

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Abstract: This study focused on introducing a fresh model of fractional operators incorporating multiple delays, termed fractional integro-differential Langevin equations with multiple delays. Additionally, the research evaluated the relative controllability of this model within finite-dimensional spaces. Employing fixed-point theory yields the desired outcomes, with the controllability assessment facilitated by Schauder's fixed point and the Grammian matrix defined through the Mittag-Leffler matrix function. Validation of the results was conducted through an application.

Keywords: Langevin system; integro-differential equation; fixed point technique; controllability; delay term; fractional derivatives
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Abbreviations

FDE → fractional differential equation
FIDLE → fractional integro-differential Langevin equation
FC → fractional calculus
MCT → mathematical control theory
FDSs → fractional dynamical systems
RLF → Riemann-Liouville fractional
FP → fixed point
ML → Mittag-Leffler
BS → Banach space
LTF → Laplace transformation
CFD → Caputo fractional derivative

1. Introduction

Fractional differential equations (FDEs) have gained popularity over the last three decades as they are an extremely helpful and beneficial tool for modeling the dynamics of processes via complicated media and simulating a wide range of applications, like interdisciplinary approaches. FDEs and integrals provide more accurate characterizations of the systems being researched.

A FDE is a mathematical representative that explains the hereditary properties and memory of many procedures and materials. The fractional derivative is widely used in research and engineering to represent real-world difficulties; see [1–12] for more information. Further, reference [13] contains an early monograph on fractional calculus (FC).

Recently, it is worth emphasizing that the controllability of FDEs has received great attention from various studies. Control theory is an important modeling technique with a qualitative component. The controllability problem is especially important for dynamical systems with control delays, which are described by several mathematical models. To find a solution for FDEs in dynamical control systems, controllability is a crucial qualitative quality. Fractional and classical control theories both rely on controllability, which is a fundamental concept in mathematical control theory (MCT).

Controllability has played an important role in recent MCT. In dynamical systems, controllability is used to adjust an object's behavior to achieve a certain purpose. Understanding control challenges, such as FDEs, can help solve many application problems. For more information, check [14]. In theory, a system is considered controllable if and only if it can be brought from any initial state to any other state using the input in a finite time.

For fractional semilinear dynamical systems, the controllability results are addressed in [15, 16]. Papers [17, 18] addressed the controllability of FDEs with order $1 < \alpha \leq 2$, respectively. Heping et al. [19] found that neutral integro FDEs with state-dependent delay can be precisely and continuously controlled. The study [20] investigated the qualitative features of fractional differential inclusions with nonlinearity, time-varying delays, and nonlocal boundary conditions. Sundaravadivoo et al. [21] investigated the controllability of nonlinear FDEs with state delay and delayed impulsive effects. Refer to [22–31] for further examples of fractional operators.

The controllability of fractional ordered nonlinear Langevin systems is determined by FP theorems [32]. The study [33] discussed the controllability of fractional dynamical systems (FDSs) with damping and delay. The authors in [34, 35] established the controllability for delay differentials and FDSs. The study in [36] focused on the existence of solutions to the nonlinear fractional Langevin equation with initial value problems (IVPs). Many writers have studied the relative controllability of nonlinear systems with dispersed and numerous control delays; see [37–39].

After the examination, we expect that there is currently no research on the relative controllability of nonlinear FIDLEs with various delays. This study aims to address research gaps by examining the relative controllability of nonlinear FIDLEs with numerous delays.

Studying fractional integro-differential Langevin equations with multiple delays is crucial due to their ability to model complex dynamic systems exhibiting memory effects and intricate interactions. These equations find applications in various fields such as physics, biology, and finance, offering insights into systems with delayed responses and fractional order dynamics. Understanding and analyzing these equations are essential for advancing control strategies, predicting system behavior accurately, and exploring the impact of different types of delays on system dynamics. This research area opens avenues for developing advanced numerical methods, investigating stability analyses, and enhancing our comprehension of intricate systems in diverse disciplines.

The remaining structure of this article is as follows: Section 2 covers commonly used FD operators and their special functions, including their characteristics. Section 3 provides the controllability requirement using the Grammian matrix. The controllability study is conducted with the assumption that the linear fractional ordered system is generally controllable while also taking into account the fractional nonlinear system. An application is provided in Section 4, and a conclusion is added in Section 5.

2. Basic facts

To help the reader understand our manuscript, in this section, we present the definitions, facts, and theories used to complete our task.

Definition 2.1. [1] Assume that $\vartheta(\ell)$ is a continuous function and $\rho > 0$. The Riemann-Liouville fractional (RLF) integral of order ρ can be represented as

$$I^{\rho}\vartheta(\ell) = \frac{1}{\Gamma(\rho)}\int_0^{\ell}(\ell-\nu)^{\rho-1}\vartheta(\nu)\,d\nu,$$

provided that the integral exists.

Definition 2.2. [1] Assume that the continuous function $\vartheta : (0,\infty) \to \mathbb{R}$ is integrable and $\rho > 0$. The RLF derivative of order ρ can be stated as

$$D^{\rho}\vartheta(\ell) = \frac{1}{\Gamma(u-\rho)}\left(\frac{d}{d\ell}\right)^{u}\int_0^{\ell}(\ell-\nu)^{u-\rho-1}\vartheta(\nu)\,d\nu,\ u = [\rho]+1,$$

where $[\rho]$ represents the greatest-integer that does not exceed ρ.

Definition 2.3. [1] Assume that the function ϑ is an $u-$time continuous and differentiable, the Caputo fractional derivative (CFD) with order $\rho > 0$ is defined by

$${}^{C}D^{\rho}\vartheta(\ell) = \frac{1}{\Gamma(u-\rho)}\int_0^{\ell}(\ell-\nu)^{u-\rho-1}\vartheta^{(u)}(\nu)\,d\nu,\ u = [\rho]+1,\ \rho \in (u-1,u).$$

The Laplace transformation (LTF) of CFD is stated as

$$\begin{aligned} L\left\{{}^{C}D^{\rho}\theta(\varkappa)\right\}(\varpi) &= \varpi^{\rho}\Theta(\varpi) - \sum_{\kappa=0}^{u-1}\varpi^{\rho-\kappa-1}\theta^{(\kappa)}(0),\ \rho \in (u-1,u], \\ L\{I^{\rho}\} &= \varpi^{-\rho}. \end{aligned}$$

Definition 2.4. [1] The corresponding ML functions for chosen $\mathfrak{I} \in \mathbb{C}$ are expressed as

$$\begin{aligned} \mathfrak{R}_{\rho}(\mathfrak{I}) &= \sum_{\kappa=0}^{\infty}\frac{\mathfrak{I}^{\kappa}}{\Gamma(\rho\kappa+1)},\ \rho > 0, \\ \mathfrak{R}_{\rho,\lambda}(\mathfrak{I}) &= \sum_{\kappa=0}^{\infty}\frac{\mathfrak{I}^{\kappa}}{\Gamma(\rho\kappa+\lambda)},\ \rho,\lambda > 0. \end{aligned}$$

Let Υ be a $u \times u$ matrix. The matrix extensions of the previously described ML functions are

$$\begin{aligned} \mathfrak{R}_{\rho,\lambda}(\Upsilon) &= \sum_{\kappa=0}^{\infty} \frac{\Upsilon^{\kappa}}{\Gamma(\rho\kappa+\lambda)}, \ \rho,\lambda > 0, \\ \mathfrak{R}_{\rho,1}(\Upsilon) &= \mathfrak{R}_{\rho}(\Upsilon), \ \lambda = 1. \end{aligned}$$

The LTF of ML function are given by

$$L\left\{\varkappa^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(+\delta\varkappa^{\rho})\right\}(\varpi) = \frac{\varpi^{-\lambda}}{(\varpi^{\rho}-\delta)}, \ Re(\rho) > 0, \ Re(\lambda) > 0, \ \text{where } \delta \in \mathbb{R}.$$

3. Supposed system and hypotheses

In this part, we will build the model under study and give hypotheses that will help in obtaining the mild solution (the controllability). Consider the following FIDLE with multiple delays:

$$\begin{cases} {}^{C}D^{\lambda}\left({}^{C}D^{\rho}+\Upsilon\right)\mathfrak{I}(\varkappa) = \sum_{\kappa=0}^{U} G_{\kappa} z(p_{\kappa}(\varkappa)) \\ \qquad +\psi\left(\varkappa, \mathfrak{I}(\varkappa), \int_0^{\varkappa} P(\varkappa,\varpi,\mathfrak{I}(\varpi))\,d\varpi, \int_0^{\varrho} Q(\varkappa,\varpi,\mathfrak{I}(\varpi))\,d\varpi\right), \\ \mathfrak{I}(0) = \mathfrak{I}_0, \ {}^{C}D^{\rho}\mathfrak{I}(\varkappa)\,|_{\varkappa=0} = q_0, \end{cases} \tag{3.1}$$

where $\rho, \lambda \in (0,1]$ with $\rho+\lambda > 1$, $\varkappa \in V = [0,\varrho]$, $\varrho > 0$ $\mathfrak{I} \in \mathbb{R}^{u}$ is the state vector, Υ is a $u \times u$ real matrix, $z \in \mathbb{R}^{v}$ is the control vector, G_{κ} are $u \times v$ real matrices for $\kappa = 1,2,...,U$, and the nonlinear functions $\psi : V \times \mathbb{R}^{u} \times \mathbb{R}^{u} \times \mathbb{R}^{u} \to \mathbb{R}^{u}$, $P, Q : V \times V \times \mathbb{R}^{u} \to \mathbb{R}^{u}$ are continuous.

Next, we suppose the following hypotheses:

(H_1) For all $\varkappa \in V$, the functions $p_{\kappa} : V \to \mathbb{R}$, $(\kappa = 1,2,...,U)$ are continuous, strictly increasing, and differentiable. Moreover, $p_{\kappa}(\varkappa) \le \varkappa$, for $\kappa = 1,2,...,U$.

(H_2) For $\varkappa \in V$, there exists time lead functions $s_{\kappa}(\varkappa) : [p_{\kappa}(0), p_{\kappa}(\varrho)] \to V$, $(\kappa = 1,2,...,U)$ such that $s_{\kappa}(p_{\kappa}(\varkappa)) = \varkappa$. Furthermore, if $p_0(\varkappa) = \varkappa$ and for $\varkappa = \varrho$, the inequality below is true

$$p_U(\varrho) \le p_{U-1}(\varrho) \le \cdots \le p_{U_{v+1}}(\varrho) \le 0 = p_v(\varrho) < p_{v-1}(\varrho) = \cdots = p_1(\varrho) = p_0(\varrho) = \varrho.$$

(H_3) For arbitrary $p > 0$, consider $z_{\varkappa}$ refer to the functions on $[-p, 0]$ described as $z : [-p,\varrho] \to \mathbb{R}^{v}$ and $\varkappa \in V$.

(H_4) The function $\psi : V \times \mathbb{R}^{u} \times \mathbb{R}^{u} \times \mathbb{R}^{u} \to \mathbb{R}^{u}$ is continuous and for all $\varkappa, \varpi \in V$, $\mathfrak{I} \in \mathbb{C}_u(V)$, there exists a positive constant α such that

$$\left|\psi\left(\varkappa, \mathfrak{I}(\varkappa), \int_0^{\varkappa} P(\varkappa,\varpi,\mathfrak{I}(\varpi))\,d\varpi, \int_0^{\varrho} Q(\varkappa,\varpi,\mathfrak{I}(\varpi))\,d\varpi\right)\right| \le \alpha.$$

(H_5) The functions $P, Q : V \times V \times \mathbb{R}^{u} \to \mathbb{R}^{u}$ are continuous and fulfill the axioms

$$\left|\int_0^{\varkappa} P(\varkappa,\varpi,\mathfrak{I}(\varpi))\,d\varpi\right| \le \sup\left(\int_0^{\varkappa} |\hbar(\varkappa,\varpi)|\,d\varpi\right)\|\varpi\|,$$

and

$$\left|\int_0^{\varrho} Q(\varkappa,\varpi,\mathfrak{I}(\varpi))\,d\varpi\right| \le \sup\left(\int_0^{\varrho} \left|\widetilde{\hbar}(\varkappa,\varpi)\right|\,d\varpi\right)\|\varpi\|,$$

such that

$$\sup\left(\int_0^{\varkappa} |\hbar(\varkappa,\varpi)|\,d\varpi\right) < 1, \ \text{and} \ \sup\left(\int_0^{\varrho} \left|\widetilde{\hbar}(\varkappa,\varpi)\right|\,d\varpi\right) < 1.$$

Definition 3.1. A complete state of the problem (3.1) at time $\varkappa$ is represented by the set $\Omega(\varkappa) = \{\Im(\varkappa), z_{\varkappa}\}$.

Definition 3.2. The model (3.1) on V is described as relatively controllability if for every $\Im_1 \in \mathbb{R}^u$ and any complete state $\Omega(0)$, there exist a control $z(\varkappa)$ defined on V such that the equation $\Im(\varrho) = \Im_1$.

Now, the linear system

$$\begin{cases} {}^{C}D^{\lambda}\left({}^{C}D^{\rho} + \Upsilon\right)\Im(\varkappa) = \sum_{\kappa=0}^{U} G_{\kappa} z(p_{\kappa}(\varkappa)), \\ \Im(0) = \Im_0, \ {}^{C}D^{\rho}\Im(\varkappa)\mid_{\varkappa=0} = q_0, \end{cases} \tag{3.2}$$

has the solution

$$\begin{aligned} \Im(\varkappa) &= \Re_{\rho}(-\Upsilon\varkappa^{\rho})\Im_0 + \varkappa^{\rho}\Upsilon\Re_{\rho,\rho+1}(-\Upsilon\varkappa^{\rho})\Im_0 + \varkappa^{\rho}\Re_{\rho,\rho+1}(-\Upsilon\varkappa^{\rho})q_0 \\ &\quad + \int_0^{\varkappa}(\varkappa-\varpi)^{\rho+\lambda-1}\Re_{\rho,\rho+\lambda}(-\Upsilon(\varkappa-\varpi)^{\rho})\sum_{\kappa=0}^{U} G_{\kappa} z(p_{\kappa}(\varpi))d\varpi. \end{aligned}$$

By using the time lead functions $s_{\kappa}(\varkappa)$ and (H_2), the above solution can be written as

$$\begin{aligned} \Im(\varkappa) &= \Re_{\rho}(-\Upsilon\varkappa^{\rho})\Im_0 + \varkappa^{\rho}\Upsilon\Re_{\rho,\rho+1}(-\Upsilon\varkappa^{\rho})\Im_0 + \varkappa^{\rho}\Re_{\rho,\rho+1}(-\Upsilon\varkappa^{\rho})q_0 \\ &\quad + \sum_{\kappa=0}^{v}\int_{p_{\kappa}(0)}^{0}(\varkappa-s_{\kappa}(\varpi))^{\rho+\lambda-1}\Re_{\rho,\rho+\lambda}(-\Upsilon(\varkappa-s_{\kappa}(\varpi))^{\rho})G_{\kappa}\dot{s}_{\kappa}(\varpi)z_0(\varpi)d\varpi \\ &\quad + \sum_{\kappa=0}^{v}\int_{0}^{\varkappa}(\varkappa-s_{\kappa}(\varpi))^{\rho+\lambda-1}\Re_{\rho,\rho+\lambda}(-\Upsilon(\varkappa-s_{\kappa}(\varpi))^{\rho})G_{\kappa}\dot{s}_{\kappa}(\varpi)z(\varpi)d\varpi \\ &\quad + \sum_{\kappa=v+1}^{U}\int_{p_{\kappa}(0)}^{p_{\kappa}(\varkappa)}(\varkappa-s_{\kappa}(\varpi))^{\rho+\lambda-1}\Re_{\rho,\rho+\lambda}(-\Upsilon(\varkappa-s_{\kappa}(\varpi))^{\rho})G_{\kappa}\dot{s}_{\kappa}(\varpi)z_0(\varpi)d\varpi. \end{aligned}$$

Assume that $C_u(V)$ is a Banach space (BS), which is the set of all continuous functions on the interval V. Clearly, $\mho = C_u(V) \times C_v(V) \times C_w(V)$ is a BS of continuous $\mathbb{R}^u \times \mathbb{R}^v \times \mathbb{R}^w$ valued functions endowed with the norm

$$\left\|(\Im, z, y)\right\|_V = \left\|\Im\right\|_V + \|z\|_V + \|y\|_V,$$

where $\left\|\Im\right\| = \sup\left\{\left|\Im(\varkappa)\right| : \varkappa \in V\right\}$, $\|z\| = \sup\{|z(\varkappa)| : \varkappa \in V\}$, and $\|y\| = \sup\{|y(\varkappa)| : \varkappa \in V\}$.

For each $\left(\widetilde{\Im}, \beta, \widetilde{\beta}\right) \in \mho$, the fractional dynamical system

$$\begin{cases} {}^{C}D^{\lambda}\left({}^{C}D^{\rho} + \Upsilon\right)\Im(\varkappa) = \sum_{\kappa=0}^{U} G_{\kappa} z(p_{\kappa}(\varkappa)) + \psi\left(\varkappa, \widetilde{\Im}, \beta, \widetilde{\beta}\right), \ \varkappa \in V, \\ \Im(0) = \Im_0, \ {}^{C}D^{\rho}\Im(\varkappa)\mid_{\varkappa=0} = q_0, \end{cases} \tag{3.3}$$

has the solution

$$\Im(\varkappa) = \Re_{\rho}(-\Upsilon\varkappa^{\rho})\Im_0 + \varkappa^{\rho}\Upsilon\Re_{\rho,\rho+1}(-\Upsilon\varkappa^{\rho})\Im_0 + \varkappa^{\rho}\Re_{\rho,\rho+1}(-\Upsilon\varkappa^{\rho})q_0$$

$$+\int_0^{\varkappa}(\varkappa-\varpi)^{\rho+\lambda-1}\,\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varkappa-\varpi)^{\rho}\right)\sum_{\kappa=0}^{U}G_{\kappa}z(p_{\kappa}(\varpi))d\varpi$$
$$+\int_0^{\varkappa}(\varkappa-\varpi)^{\rho+\lambda-1}\,\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varkappa-\varpi)^{\rho}\right)\psi\left(\varpi,\widetilde{\mathfrak{I}}(\varpi),\beta(\varpi),\widetilde{\beta}(\varpi)\right)d\varpi.$$

Using the time lead functions $s_{\kappa}(\varkappa)$ and Assumption (H_2), the solution of the above equation for $\varkappa=\varrho$ can be written as

$$\begin{aligned}\mathfrak{I}(\varrho) &= \mathfrak{R}_{\rho}\left(-\Upsilon\varkappa^{\rho}\right)\mathfrak{I}_0+\varkappa^{\rho}\Upsilon\mathfrak{R}_{\rho,\rho+1}\left(-\Upsilon\varkappa^{\rho}\right)\mathfrak{I}_0+\varkappa^{\rho}\mathfrak{R}_{\rho,\rho+1}\left(-\Upsilon\varkappa^{\rho}\right)q_0\\
&+\sum_{\kappa=0}^{v}\int_{p_{\kappa}(\varkappa)}^{0}(\varrho-s_{\kappa}(\varpi))^{\rho+\lambda-1}\,\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varrho-s_{\kappa}(\varpi))^{\rho}\right)G_{\kappa}\dot{s}_{\kappa}(\varpi)z_0(\varpi)d\varpi\\
&+\sum_{\kappa=0}^{v}\int_{0}^{\varrho}(\varrho-s_{\kappa}(\varpi))^{\rho+\lambda-1}\,\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varrho-s_{\kappa}(\varpi))^{\rho}\right)G_{\kappa}\dot{s}_{\kappa}(\varpi)z(\varpi)d\varpi\\
&+\sum_{\kappa=v+1}^{U}\int_{p_{\kappa}(0)}^{p_{\kappa}(\varrho)}(\varrho-s_{\kappa}(\varpi))^{\rho+\lambda-1}\,\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varkappa-s_{\kappa}(\varpi))^{\rho}\right)G_{\kappa}\dot{s}_{\kappa}(\varpi)z_0(\varpi)d\varpi\\
&+\int_0^{\varrho}(\varrho-\varpi)^{\rho+\lambda-1}\,\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varrho-\varpi)^{\rho}\right)\psi\left(\varpi,\widetilde{\mathfrak{I}}(\varpi),\beta(\varpi),\widetilde{\beta}(\varpi)\right)d\varpi.\end{aligned}$$

Now, for $\widetilde{\mathfrak{I}}\in C_u(V)$, our linear model (3.1) can be solved using I^{λ} on both sides of the equation, followed by the Laplace and inverse LTF and the convolution property:

$$\begin{aligned}\mathfrak{I}(\varkappa) &= \mathfrak{R}_{\rho}\left(-\Upsilon\varkappa^{\rho}\right)\mathfrak{I}_0+\varkappa^{\rho}\Upsilon\mathfrak{R}_{\rho,\rho+1}\left(-\Upsilon\varkappa^{\rho}\right)\mathfrak{I}_0+\varkappa^{\rho}\mathfrak{R}_{\rho,\rho+1}\left(-\Upsilon\varkappa^{\rho}\right)q_0\\
&+\int_0^{\varkappa}(\varkappa-\varpi)^{\rho+\lambda-1}\,\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varkappa-\varpi)^{\rho}\right)\sum_{\kappa=0}^{U}G_{\kappa}z(p_{\kappa}(\varpi))d\varpi\\
&+\int_0^{\varkappa}(\varkappa-\varpi)^{\rho+\lambda-1}\,\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varkappa-\varpi)^{\rho}\right)\\
&\times\psi\left(\varpi,\mathfrak{I}(\varpi),\int_0^{\varpi}P\left(\varpi,\xi,\widetilde{\mathfrak{I}}(\xi)\right)d\xi,\int_0^{\varrho}Q\left(\varpi,\xi,\widetilde{\mathfrak{I}}(\xi)\right)d\xi\right)d\varpi.\end{aligned}$$

For the sake of convenience, let

$$\mathfrak{I}(\varpi;\mathfrak{I},q_0)=\mathfrak{R}_{\rho}\left(-\Upsilon\varkappa^{\rho}\right)\mathfrak{I}_0+\varkappa^{\rho}\Upsilon\mathfrak{R}_{\rho,\rho+1}\left(-\Upsilon\varkappa^{\rho}\right)\mathfrak{I}_0+\varkappa^{\rho}\mathfrak{R}_{\rho,\rho+1}\left(-\Upsilon\varkappa^{\rho}\right)q_0.$$

Thus, $\mathfrak{I}(\varkappa)$ reduces to

$$\begin{aligned}\mathfrak{I}(\varkappa) &= \mathfrak{I}(\varpi;\mathfrak{I},q_0)+\int_0^{\varkappa}(\varkappa-\varpi)^{\rho+\lambda-1}\,\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varkappa-\varpi)^{\rho}\right)\sum_{\kappa=0}^{U}G_{\kappa}z(p_{\kappa}(\varpi))d\varpi\\
&+\int_0^{\varkappa}(\varkappa-\varpi)^{\rho+\lambda-1}\,\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varkappa-\varpi)^{\rho}\right)\\
&\times\psi\left(\varpi,\widetilde{\mathfrak{I}}(\varpi),\int_0^{\varpi}P\left(\varpi,\xi,\widetilde{\mathfrak{I}}(\xi)\right)d\xi,\int_0^{\varrho}Q\left(\varpi,\xi,\widetilde{\mathfrak{I}}(\xi)\right)d\xi\right)d\varpi.\end{aligned}$$

Using the time lead functions $s_{\kappa}(\varkappa)$ and Assumption (H_2), the equation above has a solution, for $\varkappa=\varrho$,

$$\begin{aligned}\mathfrak{I}(\varrho) &= \mathfrak{R}_{\rho}(-\Upsilon\varkappa^{\rho})\mathfrak{I}_0 + \varkappa^{\rho}\Upsilon\mathfrak{R}_{\rho,\rho+1}(-\Upsilon\varkappa^{\rho})\mathfrak{I}_0 + \varkappa^{\rho}\mathfrak{R}_{\rho,\rho+1}(-\Upsilon\varkappa^{\rho})q_0 \\ &+ \sum_{\kappa=0}^{v}\int_{p_\kappa(0)}^{0}(\varrho - s_\kappa(\varpi))^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varrho - s_\kappa(\varpi))^{\rho})G_\kappa\dot{s}_\kappa(\varpi)z_0(\varpi)d\varpi \\ &+ \sum_{\kappa=0}^{v}\int_{0}^{\varrho}(\varrho - s_\kappa(\varpi))^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varrho - s_\kappa(\varpi))^{\rho})G_\kappa\dot{s}_\kappa(\varpi)z(\varpi)d\varpi \\ &+ \sum_{\kappa=v+1}^{U}\int_{p_\kappa(0)}^{p_\kappa(\varrho)}(\varrho - s_\kappa(\varpi))^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varrho - s_\kappa(\varpi))^{\rho})G_\kappa\dot{s}_\kappa(\varpi)z_0(\varpi)d\varpi \\ &+ \int_0^{\varrho}(\varrho-\varpi)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varrho-\varpi)^{\rho})\psi\left(\varpi,\widetilde{\mathfrak{I}}(\varpi),\beta(\varpi),\widetilde{\beta}(\varpi)\right)d\varpi.\end{aligned}$$

To keep things simple, assume that

$$\begin{aligned}\Psi(\varkappa) &= \sum_{\kappa=0}^{v}\int_{p_\kappa(0)}^{0}(\varkappa - s_\kappa(\varpi))^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varkappa - s_\kappa(\varpi))^{\rho})G_\kappa\dot{s}_\kappa(\varpi)z_0(\varpi)d\varpi \\ &+ \sum_{\kappa=v+1}^{U}\int_{p_\kappa(0)}^{p_\kappa(\varkappa)}(\varkappa - s_\kappa(\varpi))^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varkappa - s_\kappa(\varpi))^{\rho})G_\kappa\dot{s}_\kappa(\varpi)z_0(\varpi)d\varpi.\end{aligned}$$

To be brief, consider

$$\begin{aligned}\phi\left(\Omega(0),\mathfrak{I}_1;\widetilde{\mathfrak{I}},\beta,\widetilde{\beta}\right) &= \mathfrak{I}_1 - \mathfrak{I}(\varrho;\mathfrak{I}_0,q_0) - \Psi(\varrho) \\ &- \int_0^{\varrho}(\varrho-\varpi)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varrho-\varpi)^{\rho})\psi\left(\varpi,\widetilde{\mathfrak{I}}(\varpi),\beta(\varpi),\widetilde{\beta}(\varpi)\right)d\varpi,\end{aligned}$$

for an arbitrary complete state $\Omega(0)$, where $\mathfrak{I}_1 \in \mathbb{R}^u$.

Furthermore, the Grammian controllability matrix is supplied by

$$\begin{aligned}K &= \sum_{\kappa=0}^{v}\int_0^{\varrho}(\varrho - s_\kappa(\varpi))^{2(\rho+\lambda-1)}\left[\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varrho - s_\kappa(\varpi))^{\rho})\right] \\ &\times\left[\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon)(\varrho - s_\kappa(\varpi))^{\rho})\right]^{*}d\varpi,\end{aligned}$$

where $\left[\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon)(\varrho - s_\kappa(\varpi))^{\rho})\right]^{*}$ represents the matrix transpose. Similar to [40], the linear system is relatively controllability on $V = [0,\varrho]$ iff the controllability Grammian matrix is positive definite for some $\varrho > 0$.

Now, we can present our first main theorem in this section.

Theorem 3.3. *The linear model (3.1) is controllability on V, provided that the linear form (3.2) is controllability and the hypothesis* (H_2) *is true.*

Proof. Describe the operator $\Phi : C_u(V) \to C_u(V)$ as

$$\begin{aligned}\Phi\widetilde{\mathfrak{I}}(\varkappa) &= \mathfrak{R}_{\rho}(-\Upsilon\varkappa^{\rho})\mathfrak{I}_0 + \varkappa^{\rho}\Upsilon\mathfrak{R}_{\rho,\rho+1}(-\Upsilon\varkappa^{\rho})\mathfrak{I}_0 + \varkappa^{\rho}\mathfrak{R}_{\rho,\rho+1}(-\Upsilon\varkappa^{\rho})q_0 \\ &+ \int_0^{\varkappa}(\varkappa-\varpi)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varkappa-\varpi)^{\rho})\sum_{\kappa=0}^{U}G_{\kappa}z(p_{\kappa}(\varpi))d\varpi \\ &+ \int_0^{\varkappa}(\varkappa-\varpi)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varkappa-\varpi)^{\rho}) \\ &\times\psi\left(\varpi, \mathfrak{I}(\varpi), \int_0^{\varpi}P\left(\varpi,\xi,\widetilde{\mathfrak{I}}(\xi)\right)d\xi, \int_0^{\varrho}Q\left(\varpi,\xi,\widetilde{\mathfrak{I}}(\xi)\right)d\xi\right)d\varpi,\end{aligned}$$

where the control $z(\varkappa)$ is given by

$$\begin{aligned}z(\varkappa) &= \left((\varrho - s_{\kappa}(\varkappa))^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varrho-\varkappa)^{\rho})G_{\kappa}\dot{s}_{\kappa}(\varpi)\right)^{*}K^{-1} \\ &\left[\mathfrak{I}_1 - \mathfrak{I}(\varrho;\mathfrak{I}_0,q_0) - \Psi(\varrho) - \int_0^{\varrho}(\varrho-\varpi)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}(-\Upsilon(\varrho-\varpi)^{\rho}) - \right. \\ &\left.\times\,\psi\left(\varpi,\widetilde{\mathfrak{I}}(\varpi), \int_0^{\varpi}P\left(\varpi,\xi,\widetilde{\mathfrak{I}}(\xi)\right)d\xi, \int_0^{\varpi}Q\left(\varpi,\xi,\widetilde{\mathfrak{I}}(\xi)\right)d\xi\right)d\varpi\right].\end{aligned}$$

Let $\mho(s) = \left\{\widetilde{\mathfrak{I}} \in C_u(V) : \left\|\widetilde{\mathfrak{I}}\right\| \leq s\right\}$ be a closed convex subset such that

$$s = u_1 + \frac{u_0\varrho^{\lambda+\rho}\|G\|\widetilde{K}}{\lambda+\rho} + \frac{u_0\varrho^{\lambda+\rho}\varrho}{\lambda+\rho},$$

and

$$\widetilde{K} = u_0(\varrho-\varkappa)^{\lambda+\rho}\|G\|^{*}\left\|K^{-1}\right\|\left(\left|\mathfrak{I}_1\right| + u_1 + \frac{\hbar_1\varrho^{\lambda+\rho}U}{\lambda+\rho}\right).$$

The operator Φ maps $\mho(s)$ into itself and is completely continuous, making it easy to illustrate. According to Schauder's FP theorem, there exists a FP $\widetilde{\mathfrak{I}} \in \mho(s)$ such that

$$\Phi\widetilde{\mathfrak{I}} = \widetilde{\mathfrak{I}} = \mathfrak{I}.$$

Adding the value of $p(\varkappa)$ to the foregoing equation, yields $\mathfrak{I}(\varrho) = \mathfrak{I}_1$. Thus, the integro-differential system on V is controllability. □

Now, consider the following FIDLE:

$$\begin{cases} {}^{C}D^{\lambda}\left({}^{C}D^{\rho}+\Upsilon\right)\mathfrak{I}(\varkappa) = \sum_{\kappa=0}^{U}G_{\kappa}z(p_{\kappa}(\varkappa)) \\ \qquad +\psi\left(\varkappa,\mathfrak{I}(\varkappa),\int_0^{\varkappa}P(\varkappa,\varpi,\mathfrak{I}(\varpi))d\varpi,\int_0^{\varrho}Q(\varkappa,\varpi,\mathfrak{I}(\varpi))d\varpi,z(\varkappa)\right), \\ \mathfrak{I}(0) = \mathfrak{I}_0,\ {}^{C}D^{\rho}\mathfrak{I}(\varkappa)\mid_{\varkappa=0} = q_0, \end{cases} \tag{3.4}$$

where $\lambda, \rho, \mathfrak{I}(\varkappa), \Upsilon, G_{\kappa}, z, p(\varkappa), P,$ and Q are defined after model (3.1) in the previous section and $\psi : V \times \mathbb{R}^u \times \mathbb{R}^u \times \mathbb{R}^u \times \mathbb{R}^u \to \mathbb{R}^u$ is continuous function.

The solution of the nonlinear model (3.4) takes the form

$$\begin{aligned}\mathfrak{I}(\varkappa) &= \mathfrak{R}_{\rho}\left(-\Upsilon\varkappa^{\rho}\right)\mathfrak{I}_0 + \varkappa^{\rho}\Upsilon\mathfrak{R}_{\rho,\rho+1}\left(-\Upsilon\varkappa^{\rho}\right)\mathfrak{I}_0 + \varkappa^{\rho}\mathfrak{R}_{\rho,\rho+1}\left(-\Upsilon\varkappa^{\rho}\right)q_0 \\ &+ \sum_{\kappa=0}^{v}\int_{p_\kappa(0)}^{0}\left(\varkappa - s_\kappa(\varpi)\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa - s_\kappa(\varpi)\right)^{\rho}\right)G_\kappa\dot{s}_\kappa(\varpi)\,z_0(\varpi)\,d\varpi \\ &+ \sum_{\kappa=0}^{v}\int_{0}^{\varkappa}\left(\varkappa - s_\kappa(\varpi)\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa - s_\kappa(\varpi)\right)^{\rho}\right)G_\kappa\dot{s}_\kappa(\varpi)\,z(\varpi)\,d\varpi \\ &+ \sum_{\kappa=v+1}^{U}\int_{p_\kappa(0)}^{p_\kappa(\varkappa)}\left(\varkappa - s_\kappa(\varpi)\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa - s_\kappa(\varpi)\right)^{\rho}\right)G_\kappa\dot{s}_\kappa(\varpi)\,z_0(\varpi)\,d\varpi \\ &+ \int_0^{\varkappa}\left(\varkappa - \varpi\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa - \varpi\right)^{\rho}\right) \\ &\times\psi\left(\varpi, \mathfrak{I}(\varpi), \int_0^{\varpi}P\left(\varpi,\xi,\mathfrak{I}(\xi)\right)d\xi, \int_0^{\varpi}Q\left(\varpi,\xi,\mathfrak{I}(\xi)\right)d\xi, z(\varkappa)\right)d\varpi.\end{aligned}$$

Now, we present our results on the controllability of the FIDLE. To accomplish this, we employ, for $\varkappa \in [0, \varrho]$

$$g(\varkappa) = \int_0^{\varkappa}P\left(\varkappa,\varpi,\mathfrak{I}(\varpi)\right)d\varpi \text{ and } \widetilde{g}(\varkappa) = \int_0^{\varrho}Q\left(\varkappa,\varpi,\mathfrak{I}(\varpi)\right)d\varpi.$$

For ease of presenting and summarizing the results, we, consider the following notations:

$$\begin{aligned}\hbar_\kappa &= \sup\left\|\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varrho - s_\kappa(\varpi)\right)^{\rho}\right)\right\|,\ \widetilde{\hbar}_\kappa = \sup\left\|\dot{s}_\kappa(\varpi)\right\|,\ \kappa = 0,1,\cdots,U, \\ \widehat{\hbar} &= \sup\left\|z_0(\varpi)\right\|,\ u_0 = \sup\left\|\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varrho - \varpi\right)^{\rho}\right)\right\|,\ u_1 = \sup\left\|\mathfrak{I}\left(\varrho;\mathfrak{I}_0,q_0\right)\right\|, \\ \hbar &= \max\left\{\widetilde{\hbar}\varrho^{\lambda+\rho}\left\|G_\kappa\right\|(\lambda+\rho)^{-1}, 1\right\},\ \widetilde{\hbar} = \sum_{\kappa=0}^{v}\hbar_\kappa\widetilde{\hbar}_\kappa M_\kappa, \\ \varphi &= \sum_{\kappa=0}^{v}\hbar_\kappa\widetilde{\hbar}_\kappa\left\|G_\kappa\right\|\widetilde{M}_\kappa + \sum_{\kappa=v+1}^{U}\hbar_\kappa\widetilde{\hbar}_\kappa\left\|G_\kappa\right\|\widehat{M}_\kappa, \\ \widetilde{M}_\kappa &= \int_{p_\kappa(0)}^{0}\left(\varrho - s_\kappa(\varpi)\right)^{\rho+\lambda-1}d\varpi,\ \kappa = 0,1,\cdots,v,\ M_\kappa = \int_0^{\varrho}\left(\varrho - s_\kappa(\varpi)\right)^{\rho+\lambda-1}d\varpi, \\ \widehat{M}_\kappa &= \int_{g_\kappa(0)}^{p_\kappa(\varrho)}\left(\varrho - s_\kappa(\varpi)\right)^{\rho+\lambda-1}d\varpi + \int_{\widetilde{g}_\kappa(0)}^{p_\kappa(\varrho)}\left(\varrho - s_\kappa(\varpi)\right)^{\rho+\lambda-1}d\varpi,\ \kappa = v+1, v+2,\cdots,U, \\ \widehat{c_\kappa} &= 8\hbar_\kappa\widetilde{\hbar}_\kappa u_0\varrho^{\lambda+\rho}\left\|G_\kappa^*\right\|\left\|K^{-1}\right\|(\lambda+\rho)^{-1},\ c_1 = 8u_0\varrho^{\lambda+\rho}(\lambda+\rho)^{-1}, \\ \widehat{d_\kappa} &= 8\hbar_\kappa\widetilde{\hbar}_\kappa\left\|G_\kappa^*\right\|\left\|K^{-1}\right\|\left(\left|\mathfrak{I}_1 + u_1 + \varphi\right|\right),\ c_1 = 8\left(\mathfrak{I}_1 + \varphi\widetilde{\hbar}\right), \\ c &= \max\left\{\widehat{c_\kappa}, c_1\right\},\ d = \max\left\{\widehat{d_\kappa}, d_1\right\},\ \kappa = 0,1,\cdots,v, \\ \sup|\psi| &= \sup_{\varpi\in V}\left\{\psi\left(\varpi, \widetilde{\mathfrak{I}}(\varpi), g(\varkappa), \widetilde{g}(\varpi), \widetilde{\beta}(\varpi)\right)\right\}.\end{aligned}$$

The second main theorem here is as follows:

Theorem 3.4. *Let the assertion* (H_2) *holds and the linear system (3.2) be a controllability on V. Then, the FIDLE (3.1) is a controllability on V, provided that the continuous function* ψ *satisfies the below condition*

$$\lim_{\left|\mathfrak{I},g,\widetilde{g},p\right|\to\infty}\frac{\left|\psi\left(\mathfrak{I},g,\widetilde{g},p\right)\right|}{\left|\mathfrak{I},g,\widetilde{g},p\right|}=0,\ \textit{uniformly in}\ \varkappa\in V.$$

Proof. Describe the operator $\Xi:\mho\to\mho$ as $\Xi\left(\widetilde{\mathfrak{I}},\widetilde{\beta}\right)=\Xi\left(\mathfrak{I},p\right)$, where

$$\begin{aligned} z\left(\varkappa\right) &= \left(\left(\varrho-s_{\kappa}(\varkappa)\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varrho-\varkappa\right)^{\rho}\right)G_{\kappa}\dot{s}_{\kappa}\left(\varpi\right)\right)^{*}K^{-1} \\ &\quad \left[\mathfrak{I}_1-\mathfrak{I}\left(\varrho;\mathfrak{I}_0,q_0\right)-\Psi\left(\varrho\right)-\int_0^{\varrho}\left(\varrho-\varpi\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varrho-\varpi\right)^{\rho}\right)\right. \\ &\quad \left.\times\psi\left(\varpi,\widetilde{\mathfrak{I}}\left(\varpi\right),g\left(\varpi\right),\widetilde{g}\left(\varpi\right),\widetilde{\beta}\left(\varpi\right)\right)d\varpi\right], \end{aligned}$$

and

$$\begin{aligned} \mathfrak{I}\left(\varkappa\right) &= \mathfrak{R}_{\rho}\left(-\Upsilon\varkappa^{\rho}\right)\mathfrak{I}_0+\varkappa^{\rho}\Upsilon\mathfrak{R}_{\rho,\rho+1}\left(-\Upsilon\varkappa^{\rho}\right)\mathfrak{I}_0+\varkappa^{\rho}\mathfrak{R}_{\rho,\rho+1}\left(-\Upsilon\varkappa^{\rho}\right)q_0 \\ &\quad +\sum_{\kappa=0}^{v}\int_{p_{\kappa}(0)}^{0}\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho}\right)G_{\kappa}\dot{s}_{\kappa}\left(\varpi\right)z_0\left(\varpi\right)d\varpi \\ &\quad +\sum_{\kappa=0}^{v}\int_{0}^{\varkappa}\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho}\right)G_{\kappa}\dot{s}_{\kappa}\left(\varpi\right)z\left(\varpi\right)d\varpi \\ &\quad +\sum_{\kappa=v+1}^{U}\int_{p_{\kappa}(0)}^{p_{\kappa}(\varkappa)}\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho}\right)G_{\kappa}\dot{s}_{\kappa}\left(\varpi\right)z_0\left(\varpi\right)d\varpi \\ &\quad +\int_0^{\varkappa}\left(\varkappa-\varpi\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa-\varpi\right)^{\rho}\right)\psi\left(\varpi,\widetilde{\mathfrak{I}}\left(\varpi\right),g\left(\varpi\right),\widetilde{g}\left(\varpi\right),\widetilde{\beta}\left(\varkappa\right)\right)d\varpi, \end{aligned}$$

that is,

$$\begin{aligned} \mathfrak{I}\left(\varkappa\right) &= \mathfrak{I}\left(\varpi;\mathfrak{I},q_0\right)+\sum_{\kappa=0}^{v}\int_{p_{\kappa}(0)}^{0}\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho}\right)G_{\kappa}\dot{s}_{\kappa}\left(\varpi\right)z_0\left(\varpi\right)d\varpi \\ &\quad +\sum_{\kappa=0}^{v}\int_{0}^{\varkappa}\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho}\right)G_{\kappa}\dot{s}_{\kappa}\left(\varpi\right) \\ &\quad \times\left(\left(\varrho-s_{\kappa}(\varkappa)\right)^{\rho+\lambda-1}\right)\left(G_{\kappa}^{*}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon^{*}\left(\varrho-s_{\kappa}(\varkappa)\right)^{\rho}\right)\right)\dot{s}_{\kappa}\left(\varpi\right)^{*}K^{-1}\phi\left(\Omega\left(0\right),\mathfrak{I}_1;\widetilde{\mathfrak{I}},\widetilde{\beta}\right)d\varpi \\ &\quad +\sum_{\kappa=v+1}^{U}\int_{p_{\kappa}(0)}^{p_{\kappa}(\varkappa)}\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa-s_{\kappa}(\varpi)\right)^{\rho}\right)G_{\kappa}\dot{s}_{\kappa}\left(\varpi\right)z_0\left(\varpi\right)d\varpi \\ &\quad +\int_0^{\varkappa}\left(\varkappa-\varpi\right)^{\rho+\lambda-1}\mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa-\varpi\right)^{\rho}\right)\psi\left(\varpi,\widetilde{\mathfrak{I}}\left(\varpi\right),g\left(\varpi\right),\widetilde{g}\left(\varpi\right),\widetilde{\beta}\left(\varkappa\right)\right)d\varpi. \end{aligned}$$

So, it is straightforward to establish that

$$\begin{aligned}|p(\varkappa)| &\leq \left\|G_\kappa^*\right\| \hbar_\kappa \widetilde{\hbar}_\kappa \left\|K^{-1}\right\| \left[\left\|\Im_1\right\| + u_1 + \varphi\right] + \hbar_\kappa \widetilde{\hbar}_\kappa \left\|K^{-1}\right\| u_0 (\lambda+\rho)^{-1} \varrho^{\lambda+\rho} \sup|\psi|, \\ &\leq \left(\frac{\widehat{d_\kappa}}{8\hbar} + \frac{\widehat{c_\kappa}}{8\hbar} \sup|\psi|\right) \leq \frac{1}{8\hbar}(d + c\sup|\psi|),\end{aligned}$$

and

$$\begin{aligned}\left|\Im(\varkappa)\right| &\leq u_1 + \varphi\widehat{\hbar} + \frac{1}{8\hbar}\left(\sum_{\kappa=0}^{v} \hbar_\kappa \widetilde{\hbar}_\kappa \left\|G_\kappa\right\| M_\kappa (\lambda+\rho)^{-1} \varrho^{\lambda+\rho}\right)(d + c\sup|\psi|) \\ &\quad + u_0 (\lambda+\rho)^{-1} \varrho^{\lambda+\rho} \sup|\psi| \\ &\leq \frac{d}{8} + \frac{1}{8}(d + c\sup|\psi|) + \frac{c}{8}\sup|\psi| \\ &= \frac{1}{4}d + \frac{1}{4}c\sup|\psi|.\end{aligned}$$

Hypothetically, the function ψ meets the conditions outlined in [40]. Thus, for every $c, d > 0$, there is a positive constant $\widetilde{r}$ such that

$$c\left|\psi(\varkappa, \Im, g, \widetilde{g}, p)\right| + d \leq \widetilde{r}, \text{ for all } \varkappa \in V, \tag{3.5}$$

provided that $\left|(\Im, p)\right| \leq \widetilde{r}$. Further, for any $\widetilde{r}_1 > 0$ such that $\widetilde{r}_1 < \widetilde{r}$ fulfills also Eq (3.5). Hence, under the same assumptions of c, d and $\widetilde{r}$, if $\left\|\widetilde{\Im}\right\| \leq \frac{\widetilde{r}}{4}$ and $\left\|\widetilde{\beta}\right\| \leq \frac{\widetilde{r}}{4}$, then

$$\left|\widetilde{\Im}(\varpi)\right| + |g(\varpi)| + |\widetilde{g}(\varpi)| + \left|\widetilde{\beta}(\varpi)\right| \leq \widetilde{r}.$$

Hence, one has

$$d + c\sup|\psi| \leq r.$$

Accordingly, for each $\varpi \in V$, $|z(\varpi)| \leq \frac{\widetilde{r}}{8\hbar}$. Hence, $\|z\| \leq \frac{\widetilde{r}}{8\hbar}$. Thus, $\left\|\Im\right\| \leq \frac{\widetilde{\widetilde{r}}}{4}$. This proves that if

$$\mho(\widetilde{r}) = \left\{\left(\widetilde{\Im}, \beta, \widetilde{\beta}\right) \in \mho : \left\|\widetilde{\Im}\right\| \leq \frac{\widetilde{r}}{4},\ \|\beta\| \leq \frac{\widetilde{r}}{4}, \text{ and } \left\|\widetilde{\beta}\right\| \leq \frac{\widetilde{r}}{4}\right\},$$

then Ξ maps $\mho(\widetilde{r})$ into itself. The continuity of ψ implies to the continuity of Ξ, and hence completely continuous based on Arzela-Ascoli theorem. Since $\mho(\widetilde{r})$ is closed, bounded, and convex, then by Schauder FP, the operator Ξ has a FP $\left(\widetilde{\Im}, \beta, \widetilde{\beta}\right) \in \mho(\widetilde{r})$ such that $\Xi\left(\widetilde{\Im}, \beta, \widetilde{\beta}\right) = \left(\widetilde{\Im}, \beta, \widetilde{\beta}\right) = \left(\Im, \beta, \widetilde{\beta}\right)$. As a result,

$$\begin{aligned}\Im(\varkappa) &= \Im(\varpi; \Im, q_0) + \sum_{\kappa=0}^{v} \int_{p_\kappa(0)}^{0} (\varkappa - s_\kappa(\varpi))^{\rho+\lambda-1} \mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varkappa - s_\kappa(\varpi))^\rho\right) G_\kappa \dot{s}_\kappa(\varpi) z_0(\varpi) d\varpi \\ &\quad + \sum_{\kappa=0}^{v} \int_{0}^{\varkappa} (\varkappa - s_\kappa(\varpi))^{\rho+\lambda-1} \mathfrak{R}_{\rho,\rho+\lambda}\left(-\Upsilon(\varkappa - s_\kappa(\varpi))^\rho\right) G_\kappa \dot{s}_\kappa(\varpi)\end{aligned}$$

$$+\sum_{\kappa=\nu+1}^{U}\int_{p_\kappa(0)}^{p_\kappa(\varkappa)}(\varkappa-s_\kappa(\varpi))^{\rho+\lambda-1}\,\Re_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa-s_\kappa(\varpi)\right)^{\rho}\right)G_\kappa\dot{s}_\kappa(\varpi)\,z_0(\varpi)\,d\varpi$$

$$+\int_0^{\varkappa}(\varkappa-\varpi)^{\rho+\lambda-1}\,\Re_{\rho,\rho+\lambda}\left(-\Upsilon\left(\varkappa-\varpi\right)^{\rho}\right)\psi\left(\varpi,\widetilde{\Im}(\varpi),g(\varpi),\widetilde{g}(\varpi),\widetilde{\beta}(\varkappa)\right)d\varpi.$$

Thus, it is easy to show that $\Im(\varrho)=\Im_1$, and the solution of the FIDLE (3.1) is $\Im(\varkappa)$. Therefore, the model is CA on V. □

4. An application

Consider the following model:

$$\begin{cases} {}^{C}D^{\lambda}\left({}^{C}D^{\rho}+\Upsilon\right)\Im(\varkappa)=G_1z(\varkappa)+G_2z(\varkappa-1) \\ \qquad\qquad +\psi\left(\varkappa,\Im(\varkappa),\int_0^{\varkappa}P(\varkappa,\varpi,\Im(\varpi))\,d\varpi,\int_0^{\varrho}Q(\varkappa,\varpi,\Im(\varpi))\,d\varpi\right), \\ \Im(0)=\Im_0,\ {}^{C}D^{\rho}\Im(\varkappa)\mid_{\varkappa=0}=q_0, \end{cases} \tag{4.1}$$

where $\rho,\lambda\in(0,1]$, $\varrho=6$, $\varkappa\in[0,6]$, $\rho=\frac{3}{4}$, $\lambda=\frac{1}{3}$, $\rho+\lambda>1$, $\Im_0=\begin{bmatrix}0 & 1\end{bmatrix}$, $\Im_1=\begin{bmatrix}1 & 0\end{bmatrix}$, $q_0=\begin{bmatrix}0 & 0\end{bmatrix}$, $G_1=\begin{bmatrix}1 & 0\end{bmatrix}$, $G_2=\begin{bmatrix}0 & 1\end{bmatrix}$, $\Upsilon=\begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}$, $\Im(\varkappa)=\begin{bmatrix}\Im_2(\varkappa) & \Im_1(\varkappa)\end{bmatrix}$, and the nonlinear function ψ is suggested as

$$\psi\left(\varkappa,\Im(\varkappa),\int_0^{\varkappa}P(\varkappa,\varpi,\Im(\varpi))\,d\varpi,\int_0^{\varrho}Q(\varkappa,\varpi,\Im(\varpi))\,d\varpi,z(\varpi)\right)=\begin{bmatrix}1 & \frac{\int_0^{\varkappa}\exp\left(-\Im_1(\varkappa)\right)d\varpi}{1+\Im^2(\varkappa)+z^2(\varkappa)}\end{bmatrix}.$$

The solution of the model (4.1) is

$$\begin{aligned}\Im(\varkappa) &= \Re_{\frac{3}{4}}\left(-\Upsilon\varkappa^{\frac{3}{4}}\right)\Im_0+\varkappa^{\frac{3}{4}}\Upsilon\Re_{\frac{3}{4},\frac{7}{4}}\left(-\Upsilon\varkappa^{\frac{3}{4}}\right)\Im_0+\varkappa^{\frac{3}{4}}\Re_{\frac{3}{4},\frac{7}{4}}\left(-\Upsilon\varkappa^{\frac{3}{4}}\right)q_0 \\ &\quad+\int_0^{\varkappa}(\varkappa-\varpi)^{\frac{1}{12}}\,\Re_{\frac{3}{4},\frac{13}{12}}\left(-\Upsilon(\varkappa-\varpi)^{\frac{3}{4}}\right)G_1z(\varpi)\,d\varpi \\ &\quad+\int_0^{\varkappa}(\varkappa-\varpi+1)^{\frac{1}{12}}\,\Re_{\frac{3}{4},\frac{13}{12}}\left(-\Upsilon(\varkappa-\varpi+1)^{\frac{3}{4}}\right)G_2z(\varpi)\,d\varpi \\ &\quad+\int_0^{\varkappa}(\varkappa-\varpi)^{\frac{1}{12}}\,\Re_{\frac{3}{4},\frac{13}{12}}\left(-\Upsilon(\varkappa-\varpi)^{\frac{3}{4}}\right) \\ &\quad\times\psi\left(\varpi,\widetilde{\Im}(\varpi),\int_0^{\varpi}P(\varpi,\xi,\Im(\xi))\,d\xi,\int_0^{\varrho}Q(\varpi,\xi,\Im(\xi))\,d\xi,z(\varkappa)\right)d\varpi.\end{aligned}$$

By performing a simple calculation on the matrix, we have $K>0$, that is, K is a positive definite. Consequently, the linear model is AC on $[0,6]$. Further, the continuous function ψ fulfills the hypothesis (H_2), hence the fractional model is relatively controllability. Thus, the nonlinear model guidance from initial state $\Im_0=\begin{bmatrix}0 & 1\end{bmatrix}$ to the desirable state $\Im_1=\begin{bmatrix}1 & 0\end{bmatrix}$ on the interval $[0,6]$. It can be estimated as follows:

$$
\begin{aligned}
z_u(\varkappa) &= \Big[(6-\varkappa)^{\frac{1}{12}}\,\mathfrak{R}_{\frac{3}{4},\frac{13}{12}}\left(-\Upsilon(6-\varkappa)^{\frac{3}{4}}G_1\right)^{*} \\
&\quad + (6-\varkappa+1)^{\frac{1}{12}}\,\mathfrak{R}_{\frac{3}{4},\frac{13}{12}}\left(-\Upsilon(6-\varkappa+1)^{\frac{3}{4}}G_2\right)^{*}\Big] \\
&\quad \times K^{-1}\left[\mathfrak{I}_1 - \mathfrak{I}(6;\mathfrak{I}_0,q_0) - \Psi(6) - \int_0^6 (6-\varpi)^{\frac{1}{12}}\,\mathfrak{R}_{\frac{3}{4},\frac{13}{12}}\left(-\Upsilon(6-\varpi)^{\rho}\right)\right. \\
&\quad \left.\times\,\psi\left(\varkappa,\mathfrak{I}(\varkappa),\int_0^{\varkappa}P(\varkappa,\varpi,\mathfrak{I}(\varpi))\,d\varpi,\int_0^{\varrho}Q(\varkappa,\varpi,\mathfrak{I}(\varpi))\,d\varpi,z_u(\varpi)\right)d\varpi\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{I}_{u+1}(\varkappa) &= \mathfrak{R}_{\frac{3}{4}}\left(-\Upsilon\varkappa^{\frac{3}{4}}\right)\mathfrak{I}_0 + \varkappa^{\frac{3}{4}}\Upsilon\mathfrak{R}_{\frac{3}{4},\frac{7}{4}}\left(-\Upsilon\varkappa^{\frac{3}{4}}\right)\mathfrak{I}_0 + \varkappa^{\frac{3}{4}}\mathfrak{R}_{\frac{3}{4},\frac{7}{4}}\left(-\Upsilon\varkappa^{\frac{3}{4}}\right)q_0 \\
&\quad + \int_0^{\varkappa}(\varkappa-\varpi)^{\frac{1}{12}}\,\mathfrak{R}_{\frac{3}{4},\frac{13}{12}}\left(-\Upsilon(\varkappa-\varpi)^{\frac{3}{4}}\right)G_1 z_u(\varpi)\,d\varpi \\
&\quad + \int_0^{\varkappa}(\varkappa-\varpi+1)^{\frac{1}{12}}\,\mathfrak{R}_{\frac{3}{4},\frac{13}{12}}\left(-\Upsilon(\varkappa-\varpi+1)^{\frac{3}{4}}\right)G_2 z_u(\varpi)\,d\varpi \\
&\quad + \int_0^{\varkappa}(\varkappa-\varpi)^{\frac{1}{12}}\,\mathfrak{R}_{\frac{3}{4},\frac{13}{12}}\left(-\Upsilon(\varkappa-\varpi)^{\frac{3}{4}}\right) \\
&\quad \times\psi\left(\varpi,\mathfrak{I}_u(\varpi),\int_0^{\varpi}P(\varpi,\xi,\mathfrak{I}(\xi))\,d\xi,\int_0^{\varrho}Q(\varpi,\xi,\mathfrak{I}(\xi))\,d\xi,z_u(\varkappa)\right)d\varpi,
\end{aligned}
$$

for all $\varkappa \in [0,6]$, where $u = 0,1,2,\cdots$ with $\mathfrak{I}_{(0)}(\varkappa) = \mathfrak{I}_0$. Hence, all requirements of Theorem 3.4 are satisfied. Therefore, the nonlinear model (4.1) is controllability on $[0,6]$.

5. Conclusions and future work

The study of fractional differential equations stands out as a captivating research domain. This work primarily introduces a novel model of fractional operators featuring multiple delays, termed fractional integro-differential Langevin equations with multiple delays. The research also delves into estimating the relative controllability of this model within finite-dimensional spaces. By employing fixed-point theory, the study achieves its objectives effectively. The controllability assessment utilizes Schauder's FP, and the Grammian matrix is defined by the ML matrix function. Validation of the findings is conducted through an application. Future research directions regarding fractional integro-differential Langevin equations with multiple delays could entail exploring advanced numerical techniques tailored to these intricate equations, analyzing stability under various conditions, and investigating applications spanning physics, biology, and finance. Furthermore, investigating the impact of different delay types on system behavior, developing control strategies, and considering stochastic elements within the equation framework could offer promising avenues for further exploration.

Author contributions

All authors contributed equally to the writing of this article. All authors have accepted responsibility for entire content of the manuscript and approved its submission.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

All authors confirm that they have no conflict of interest.

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