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Research article

Nonlinear contractions on directed graphs with applications to boundary value problems

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Abstract: This article contains some outcomes on fixed points for a graph preserving nonlinear contraction in a metric space endued with a transitive directed graph. Our results improved, enriched, and subsumed various known fixed point theorems. To argue for reliability of our results, we presented two examples. We concluded the manuscript to investigate a unique solution of a certain first-order boundary value problem by means of our results.

Keywords: (C)-graph; Cauchy equivalence; lower solution
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1. Introduction

The classical BCP (i.e., Banach contraction principle) is a vital and crucial result of the fixed point theory in MS (i.e., metric space). In fact, BCP ensures that a contraction on a CMS (i.e., complete metric space) owns a unique fixed point. Additionally, this result provides an iterative method for computing the unique fixed point. Plenty of researchers have developed this result within the past century. Some authors improved the ordinary contraction to ψ-contraction by controlling the contraction map via a compatible self-function ψ on $[0, +\infty)$. There are so many variants of BCP under ψ-contractions using some suitable choices ψ, and there has already been a lot of writing on this particular topic. Boyd and Wong [1] and Matkowski [2] are primarily responsible for establishing two well-known and classical fixed-point outcomes under ψ-contractions. On line with Boyd and Wong [1], Ψ refers the set of functions $\psi : [0, +\infty) \to [0, +\infty)$ verifying $\psi(r) < r$ and $\limsup_{t\to r^+} \psi(t) < r$, for all $r > 0$.

Theorem 1.1. *[1] If $\mathcal{S}$ is a self-map on a CMS $(\mathbf{Y}, \sigma)$ and there exists $\psi \in \Psi$ verifying*

$$\sigma(\mathcal{S}y, \mathcal{S}z) \leq \psi(\sigma(y, z)), \quad \textit{for all } y, z \in \mathbf{Y},$$

then $\mathcal{S}$ possesses a unique fixed point.

The abovementioned contraction condition is named a ψ-contraction. In particular for $\psi(r) = \kappa \cdot r$, $0 < \kappa < 1$, ψ-contraction falls to contraction and Theorem 1.1 deduces the BCP.

Jachymski [3] presented a very intriguing approach in fixed-point theory by employing the framework of metric spaces endowed with a graph. Graphs are algebraic structures, which subsume the partial order. The key theme of Jachymski's approaches is that the contraction condition requires verifying on merely the edges of the graph. This led to the emergence of a new area in fixed-point theory, which has seen a large number of publications. Several noteworthy references from these publications include [4–16].

The results investigated in the present manuscript are fixed point results employing the $(\mathbb{G}, \psi)$-contractions in the setup of metric spaces endued with a transitive directed graph. We illustrate our findings by adopting some examples. We provide an application to a BVP (i.e., boundary value problem) verifying some additional hypotheses.

2. Directed graphs and relevant concepts

This section deals with some notions related to graph theory. Again, we refer to the paper of Jachymski [3]. A graph $\mathbb{G}$ is comprised of a nonempty set $\mathcal{V}(\mathbb{G})$ (referred to as vertex set or set of vertices) and a set $\mathcal{E}(\mathbb{G})$ (referred to as edge set or set of edges) of pair of elements of $\mathcal{V}(\mathbb{G})$. A graph is referred to as directed graph or digraph if each edge is an ordered pair of vertices. A graph $\mathbb{G}$ is represented by the pair $(\mathcal{V}(\mathbb{G}), \mathcal{E}(\mathbb{G}))$.

The conversion of a group $\mathbb{G} = (\mathcal{V}(\mathbb{G}), \mathcal{E}(\mathbb{G}))$, denoted by $\mathbb{G}^{-1}$, is a graph determined by

$$\mathcal{V}(\mathbb{G}^{-1}) = \mathcal{V}(\mathbb{G}),$$

and

$$\mathcal{E}(\mathbb{G}^{-1}) = \{(y, z) \in \mathcal{V}(\mathbb{G}) \times \mathcal{V}(\mathbb{G}) : (z, y) \in \mathcal{E}(\mathbb{G})\}.$$

Corresponding to a directed graph $\mathbb{G} = (\mathcal{V}(\mathbb{G}), \mathcal{E}(\mathbb{G}))$, we can determine an undirected graph $\tilde{\mathbb{G}}$ as follows:

$$\mathcal{V}(\tilde{\mathbb{G}}) = \mathcal{V}(\mathbb{G}) \quad \text{and} \quad \mathcal{E}(\tilde{\mathbb{G}}) = \mathcal{E}(\mathbb{G}) \cup \mathcal{E}(\mathbb{G}^{-1}).$$

In fact, we can treat $\tilde{\mathbb{G}}$ as a directed graph, whereas $\mathcal{E}(\tilde{\mathbb{G}})$ is symmetric.

Given a pair of vertices y and z in the graph $\mathbb{G}$, a path in $\mathbb{G}$ from y to z of length $p \in \mathbb{N}$ is an ordered set $\{y_0, y_1, y_2, \ldots y_p\}$ of vertices, which verifies $y_0 = y$, $y_p = z$, and $(y_{\mathsf{k}-1}, y_{\mathsf{k}}) \in \mathcal{E}(\mathbb{G})$, for every $\mathsf{k} \in \{1, 2, \ldots p\}$. Moreover, a graph $\mathbb{G}$ in which every pair of vertices admits a path is named as connected. Furthermore, if $\tilde{\mathbb{G}}$ is connected, then $\mathbb{G}$ is named as weakly connected.

If $y \in \mathcal{V}(\mathbb{G})$, then we use the symbol $[y]_{\mathbb{G}}$ defined as:

$$[y]_{\mathbb{G}} = \{z \in \mathcal{V}(\mathbb{G}) : \mathbb{G} \text{ admits a path from } y \text{ to } z\}.$$

One says that a graph $\mathbb{H} = (\mathcal{V}(\mathbb{H}), \mathcal{E}(\mathbb{H}))$ is a subgraph of $\mathbb{G} = (\mathcal{V}(\mathbb{G}), \mathcal{E}(\mathbb{G}))$ if

$$\mathcal{V}(\mathbb{H}) \subseteq \mathcal{V}(\mathbb{G}) \quad \text{and} \quad \mathcal{E}(\mathbb{H}) \subseteq \mathcal{E}(\mathbb{G}).$$

Let $\mathbb{G} = (\mathcal{V}(\mathbb{G}), \mathcal{E}(\mathbb{G}))$ be a graph in which $\mathcal{E}(\mathbb{G})$ is symmetric, then for each $y \in \mathcal{V}(\mathbb{G})$, we can determine a subgraph $\mathbb{G}_y$ whose edges and vertices are contained in a path with initial point y. Such a subgraph is named as the component of $\mathbb{G}$ containing y. Henceforth, we have $\mathcal{V}(\mathbb{G}_y) = [y]_{\mathbb{G}}$. Obviously, $\mathbb{G}_y$ is connected.

Definition 2.1. *[3] One says that a MS $(\mathbf{Y}, \sigma)$ is endued with a graph $\mathbb{G}$ if*

- $\mathcal{V}(\mathbb{G}) = \mathbf{Y}$*;*
- *all loops are contained in* $\mathcal{E}(\mathbb{G})$*;*
- $\mathbb{G}$ *admits no parallel edge.*

Definition 2.2. *[3] Assume that $(\mathbf{Y}, \sigma)$ is a MS endued with a graph $\mathbb{G}$. We say that a map $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ is orbitally $\mathbb{G}$-continuous, if for all $y, \bar{y} \in \mathbf{Y}$ and for every sequence $\{n_i\}$ of natural numbers, we have*

$$\lim_{i\to+\infty} \mathcal{S}^{n_i}(y) = \bar{y} \text{ and } (\mathcal{S}^{n_i}y, \mathcal{S}^{n_i+1}y) \in \mathcal{E}(\mathbb{G}), \ \textit{for all } \ i \in \mathbb{N} \Longrightarrow \lim_{i\to+\infty} \mathcal{S}(\mathcal{S}^{n_i}y) = \mathcal{S}(\bar{y}).$$

Definition 2.3. *[4] Let $(\mathbf{Y}, \sigma)$ be a MS endued with a graph $\mathbb{G}$. We say that $\mathbb{G}$ is a (C)-graph if any sequence $\{y_n\} \subset \mathbf{Y}$ verifying $y_n \to y$ and $(y_n, y_{n+1}) \in \mathcal{E}(\mathbb{G})$, for each $n \in \mathbb{N}$, contains a subsequence $\{y_{n_k}\}$,s which satisfies $(y_{n_k}, y) \in \mathcal{E}(\mathbb{G})$, for every $k \in \mathbb{N}$.*

Definition 2.4. *[5] Assume that $(\mathbf{Y}, \sigma)$ is a MS endued with a graph $\mathbb{G}$. One says that a map $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ is $(\mathbb{G}, \psi)$-contraction if*

(i) $(y, z) \in \mathcal{E}(\mathbb{G}) \Longrightarrow (\mathcal{S}y, \mathcal{S}z) \in \mathcal{E}(\mathbb{G})$*;*
(ii) $\psi : [0, +\infty) \to [0, +\infty)$ *is an auxiliary function which verifies*

$$\sigma(\mathcal{S}y, \mathcal{S}z) \leq \psi(\sigma(y, z)) \quad \textit{for each } \ (y, z) \in \mathcal{E}(\mathbb{G}).$$

Definition 2.5. *[9] A directed graph $\mathbb{G}$ verifying for any $y, z, w \in \mathcal{V}(\mathbb{G})$ with*

$$(y, z) \in \mathcal{E}(\mathbb{G}) \quad \textit{and} \quad (z, w) \in \mathcal{E}(\mathbb{G}) \Longrightarrow (y, w) \in \mathcal{E}(\mathbb{G}).$$

is named as transitive.

3. Auxiliary results

We'll utilize the following notions:

$$\mathbf{Y}_{\mathcal{S}} = \{y \in \mathbf{Y} : (y, \mathcal{S}y) \in \mathcal{E}(\mathbb{G})\},$$

and

$$\mathrm{Fix}(\mathcal{S}) = \{y \in \mathbf{Y} : \mathcal{S}(y) = y\}.$$

Proposition 3.1. *Assume that $(\mathbf{Y}, \sigma)$ is a CMS endued with a directed graph $\mathbb{G}$. If there exists $\psi \in \Psi$ such that $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ is a $(\mathbb{G}, \psi)$-contraction, then $\mathcal{S}$ is both a $(\mathbb{G}^{-1}, \psi)$-contraction as well as a $(\tilde{\mathbb{G}}, \psi)$-contraction.*

Proof. The result is a direct consequence of symmetric property of σ and the fact $\mathcal{E}(\tilde{\mathbb{G}}) = \mathcal{E}(\mathbb{G}) \cup \mathcal{E}(\mathbb{G}^{-1})$.

Definition 3.1. *[17] A self-map $\mathcal{S}$ on a MS $(\mathbf{Y}, \sigma)$ is termed as asymptotically regular at point $y \in \mathbf{Y}$ if*

$$\lim_{n\to+\infty} \sigma(\mathcal{S}^n y, \mathcal{S}^{n+1} y) = 0.$$

Lemma 3.1. *Assume that $(\mathbf{Y}, \sigma)$ is a CMS endued with a directed graph $\mathbb{G}$. If there exists $\psi \in \Psi$ such that $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ is a $(\mathbb{G}, \psi)$-contraction and $\mathbf{Y}_{\mathcal{S}} \neq \emptyset$, then $\mathcal{S}$ is asymptotically regular at every $y \in \mathbf{Y}_{\mathcal{S}}$.*

Proof. Choose an arbitrary $y \in \mathbf{Y}_{\mathcal{S}}$, then, one has $(y, \mathcal{S}y) \in \mathcal{E}(\mathbb{G})$. By $(\mathbb{G}, \psi)$-contraction condition of $\mathcal{S}$ and easy induction, we obtain $(\mathcal{S}^n y, \mathcal{S}^{n+1} y) \in \mathcal{E}(\mathbb{G})$. Denote $y_n := \mathcal{S}^n(y)$, for each $n \in \mathbb{N}$. Hence, we conclude

$$(y_n, y_{n+1}) \in \mathcal{E}(\mathbb{G}), \quad \text{for every } n \in \mathbb{N}. \tag{3.1}$$

Define $\sigma_n := \sigma(y_n, y_{n+1})$, for every $n \in \mathbb{N}$. Employing $(\mathbb{G}, \psi)$-contraction condition of $\mathcal{S}$ for (3.1), we get

$$\sigma(y_n, y_{n+1}) = \sigma(\mathcal{S}y_{n-1}, \mathcal{S}y_n) \leq \psi(\sigma(y_{n-1}, y_n)),$$

so that

$$\sigma_n \leq \psi(\sigma_{n-1}) \quad \text{for every } n \in \mathbb{N}. \tag{3.2}$$

If $\sigma_n > 0$ for all $n \in \mathbb{N}$, then using the definition of ψ in (3.2), we get

$$\sigma_n \leq \psi(\sigma_{n-1}) < \sigma_{n-1}, \quad \text{for every } n \in \mathbb{N}.$$

If $\sigma_n = 0$ for some $n \in \mathbb{N}$, then $0 = \sigma_n \leq \sigma_{n-1}$. Thus, in both the cases, $\{\sigma_n\}$ is a decreasing sequence in $[0, +\infty)$, which is bounded below also; so there exists $p \geq 0$ enjoying

$$\lim_{n\to+\infty} \sigma_n = p. \tag{3.3}$$

Let $p > 0$. With the upper limit in (3.2) and by using (3.3) and the property of ψ, we obtain

$$p = \limsup_{n\to+\infty} \sigma_n \leq \limsup_{n\to+\infty} \psi(\sigma_{n-1}) = \limsup_{\sigma_n\to p^+} \psi(\sigma_{n-1}) < p,$$

which arises a contradiction. Therefore, $p = 0$ and, hence, we have

$$\lim_{n\to+\infty} \sigma_n = \lim_{n\to+\infty} \sigma(\mathcal{S}^n y, \mathcal{S}^{n+1} y) = 0. \tag{3.4}$$

Thus, $\mathcal{S}$ is asymptotically regular at every $y \in \mathbf{Y}_{\mathcal{S}}$.

Now, we indicate the following classical and well-known result.

Lemma 3.2. *[1] Assume that $\{y_n\}$ is a sequence in a MS $(\mathbf{Y}, \sigma)$. If $\{y_n\}$ is not Cauchy, then $\exists \varepsilon > 0$ and two subsequences $\{y_{n_k}\}$ and $\{y_{m_k}\}$ of $\{y_n\}$ verifying*

$$k \leq m_k < n_k, \ \sigma(y_{m_k}, y_{n_k}) > \varepsilon \geq \sigma(y_{m_k}, y_{n_{k-1}}), \quad \textit{for every } k \in \mathbb{N}. \tag{3.5}$$

Further, if $\lim_{n\to+\infty} \sigma(y_n, y_{n+1}) = 0$, then

(i) $\lim\limits_{k\to+\infty} \sigma(y_{m_k}, y_{n_k}) = \varepsilon$;
(ii) $\lim\limits_{k\to+\infty} \sigma(y_{m_k}, y_{n_k+1}) = \varepsilon$;
(iii) $\lim\limits_{k\to+\infty} \sigma(y_{m_k+1}, y_{n_k}) = \varepsilon$;
(iv) $\lim\limits_{k\to+\infty} \sigma(y_{m_k+1}, y_{n_k+1}) = \varepsilon$.

Lemma 3.3. *Assume that $(\mathbf{Y}, \sigma)$ is a CMS endued with a transitive directed graph $\mathbb{G}$. If there exists $\psi \in \Psi$ for which $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ is a $(\mathbb{G}, \psi)$-contraction and $\mathbf{Y}_{\mathcal{S}} \neq \emptyset$, then for each $y \in \mathbf{Y}_{\mathcal{S}}$, there is $y^*(y) \in \mathbf{Y}$ enjoying $\mathcal{S}^n(y) \longrightarrow y^*(y)$ as $n \to +\infty$.*

Proof. Let $y \in \mathbf{Y}_{\mathcal{S}}$ and define $y_n := \mathcal{S}^n(y)$, for every $n \in \mathbb{N}$. We'll establish that $\{y_n\}$ is Cauchy. On the contrary, let $\{y_n\}$ be not Cauchy. By Lemma 3.2, $\exists\varepsilon > 0$ and subsequences $\{y_{n_k}\}$ and $\{y_{m_k}\}$ of $\{y_n\}$, for which (3.5) holds. Using the transitivity of $\mathbb{G}$ and (3.1), we have $(y_{m_k}, y_{n_k}) \in \mathcal{E}(\mathbb{G})$. Employing $(\mathbb{G}, \psi)$-contractivity of $\mathcal{S}$, we conclude

$$\sigma(y_{m_k+1}, y_{n_k+1}) = \sigma(\mathcal{S}y_{m_k}, \mathcal{S}y_{n_k}) \leq \psi(\sigma(y_{m_k}, y_{n_k})),$$

so that

$$\sigma(y_{m_k+1}, y_{n_k+1}) \leq \psi(\sigma(y_{m_k}, y_{n_k})). \tag{3.6}$$

Employing Lemma 3.1, we get $\lim\limits_{n\to+\infty} \sigma(y_n, y_{n+1}) = 0$. Therefore, by Lemma 3.2, we obtain

$$\lim_{k\to+\infty} \sigma(y_{m_k}, y_{n_k}) = \lim_{k\to+\infty} \sigma(y_{m_k+1}, y_{n_k+1}) = \varepsilon. \tag{3.7}$$

Using limit superior in (3.6) and by (3.7), we conclude

$$\varepsilon = \limsup_{k\to+\infty} \sigma(y_{m_k+1}, y_{n_k+1}) \leq \limsup_{k\to+\infty} \psi(\sigma(y_{m_k}, y_{n_k})),$$

which by using the property of ψ, yields that

$$\varepsilon \leq \limsup_{k\to+\infty} \psi(\sigma(y_{m_k}, y_{n_k})) = \limsup_{s\to\varepsilon^+} \psi(s) < \varepsilon,$$

which arises a contradiction so that $\{y_n\}$ is Cauchy. By completeness of $(\mathbf{Y}, \sigma)$, we can find $y^*(y) \in \mathbf{Y}$ verifying $y_n \xrightarrow{\sigma} y^*(y)$.

Definition 3.2. *[3] Two Cauchy sequences $\{y_n\}$ and $\{z_n\}$ in a MS $(\mathbf{Y}, \sigma)$ are called Cauchy equivalent if*

$$\lim_{n\to+\infty} \sigma(y_n, z_n) = 0.$$

Lemma 3.4. *Assume that $(\mathbf{Y}, \sigma)$ is a MS endued with a graph $\mathbb{G}$, then the following are equivalent:*

(i) *$\mathbb{G}$ is weakly connected;*
(ii) *if for some $\psi \in \Psi$, $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ is a $(\mathbb{G}, \psi)$-contraction, then for every $y, z \in \mathbf{Y}$, $\{\mathcal{S}^n y\}$ and $\{\mathcal{S}^n z\}$ are Cauchy equivalent sequences;*
(iii) *if for some $\psi \in \Psi$, $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ is a $(\mathbb{G}, \psi)$-contraction, then* $\mathrm{card}(\mathrm{Fix}(\mathcal{S})) \leq 1$.

Proof. (i) $\Rightarrow$ (ii) Assume that $\mathcal{S}$ is a $(\mathbb{G},\psi)$-contraction and $y,z\in\mathbf{Y}$. By (i), $[y]_{\tilde{\mathbb{G}}}=\mathbf{Y}$ and, hence, $z\in[y]_{\tilde{\mathbb{G}}}$. There exists a path $\{y_0,y_1,y_2,\dots,y_p\}$ in $\tilde{\mathbb{G}}$ from y to z, enjoying

$$y_0=y,\ y_p=z \text{ and } (y_\mathsf{k},y_{\mathsf{k}+1})\in\mathcal{E}(\tilde{\mathbb{G}}) \text{ for each } \mathsf{k}\ (0\le\mathsf{k}\le p-1).$$

Using $(\mathbb{G},\psi)$-contraction condition and by induction, we get

$$(\mathcal{S}^\mathsf{n}y_\mathsf{k},\mathcal{S}^\mathsf{n}y_{\mathsf{k}+1})\in\mathcal{E}(\tilde{\mathbb{G}}) \text{ for each } \mathsf{k}\ (0\le\mathsf{k}\le p-1) \text{ and for each } \mathsf{n}\in\mathbb{N}.$$

Now, for each k $(0\le\mathsf{k}\le p-1)$, define $t_\mathsf{n}^\mathsf{k}=:\sigma(\mathcal{S}^\mathsf{n}y_\mathsf{k},\mathcal{S}^\mathsf{n}y_{\mathsf{k}+1})$ for every $\mathsf{n}\in\mathbb{N}$. We'll establish that

$$\lim_{\mathsf{n}\to+\infty}t_\mathsf{n}^\mathsf{k}=0. \tag{3.8}$$

For each fixed k, we have two cases. To begin, assume that $t_{\mathsf{n}_0}^\mathsf{k}=\sigma(\mathcal{S}^{\mathsf{n}_0}y_\mathsf{k},\mathcal{S}^{\mathsf{n}_0}y_{\mathsf{k}+1})=0$ for some $\mathsf{n}_0\in\mathbb{N}$, *i.e.*, $\mathcal{S}^{\mathsf{n}_0}(y_\mathsf{k})=\mathcal{S}^{\mathsf{n}_0}(y_{\mathsf{k}+1})$, which yields that $\mathcal{S}^{\mathsf{n}_0+1}(y_\mathsf{k})=\mathcal{S}^{\mathsf{n}_0+1}(y_{\mathsf{k}+1})$. It follows that $t_{\mathsf{n}_0+1}^\mathsf{k}=d(\mathcal{S}^{\mathsf{n}_0+1}y_\mathsf{k},\mathcal{S}^{\mathsf{n}_0+1}y_{\mathsf{k}+1})=0$. Using easy induction, we obtain $t_\mathsf{n}^\mathsf{k}=0$ for all $\mathsf{n}\ge\mathsf{n}_0$, thereby yielding $\lim\limits_{\mathsf{n}\to+\infty}t_\mathsf{n}^\mathsf{k}=0$. In either case, we have $t_\mathsf{n}>0$ for all $\mathsf{n}\in\mathbb{N}$, then by $(\mathbb{G},\psi)$-contraction condition of $\mathcal{S}$ and Proposition 3.1, we get

$$\begin{aligned} t_{\mathsf{n}+1}^\mathsf{k} &= \sigma(\mathcal{S}^{\mathsf{n}+1}y_\mathsf{k},\mathcal{S}^{\mathsf{n}+1}y_{\mathsf{k}+1})\\ &\le \psi(\sigma(\mathcal{S}^\mathsf{n}y_\mathsf{k},\mathcal{S}^\mathsf{n}y_{\mathsf{k}+1}))\\ &= \psi(t_\mathsf{n}^\mathsf{k}), \end{aligned}$$

thereby yielding

$$t_{\mathsf{n}+1}^\mathsf{k}\le\psi(t_\mathsf{n}^k).$$

This yields that $\lim\limits_{\mathsf{n}\to+\infty}t_\mathsf{n}^\mathsf{k}=0$. Thus, in both the cases, (3.8) is proved for each k $(0\le\mathsf{k}\le p-1)$. Employing (3.8) and the triangular inequality, we get

$$\sigma(\mathcal{S}^\mathsf{n}y,\mathcal{S}^\mathsf{n}z)=\sigma(\mathcal{S}^\mathsf{n}y_0,\mathcal{S}^\mathsf{n}y_p)\le t_\mathsf{n}^0+t_\mathsf{n}^1+\cdots+t_\mathsf{n}^{p-1}\to0 \ \text{ as } \ \mathsf{n}\to+\infty.$$

In the same way, there is a path $\{z_0,z_1,z_2,\dots,z_l\}$ in $\tilde{\mathbb{G}}$ from y to $\mathcal{S}(y)$, so

$$z_0=y,\ y_l=\mathcal{S}(y) \text{ and } (z_\mathsf{k},z_{\mathsf{k}+1})\in\mathcal{E}(\tilde{\mathbb{G}}) \text{ for each } \mathsf{k}\ (0\le\mathsf{k}\le l-1).$$

Thus, we have

$$\sigma(\mathcal{S}^\mathsf{n}y,\mathcal{S}^{\mathsf{n}+1}y)\to0 \ \text{ as } \ \mathsf{n}\to+\infty.$$

Likewise the proof of Lemma 3.1, we conclude that the sequences $\{\mathcal{S}^\mathsf{n}y\}$ and $\{\mathcal{S}^\mathsf{n}z\}$ are Cauchy. Therefore, the sequences $\{\mathcal{S}^\mathsf{n}\mathsf{y}\}$ and $\{\mathcal{S}^\mathsf{n}\mathsf{z}\}$ are Cauchy equivalent.

(ii) $\Rightarrow$ (iii) Assume that $\mathcal{S}$ is $(\mathbb{G},\psi)$-contraction and $y,z\in\mathrm{Fix}(\mathcal{S})$. In view of (ii), the sequences $\{\mathcal{S}^\mathsf{n}\mathsf{y}\}$ and $\{\mathcal{S}^\mathsf{n}\mathsf{z}\}$ are Cauchy equivalent, thereby implying $y=z$.

(iii) $\Rightarrow$ (i) Assume that $\mathbb{G}$ is not weakly connected, i.e., $\tilde{\mathbb{G}}$ is not connected. Take $y_0\in\mathbf{Y}$, then, $[y_0]_{\tilde{\mathbb{G}}}\ne\emptyset$ and $\mathbf{Y}-[y_0]_{\tilde{\mathbb{G}}}\ne\emptyset$. Let $z_0\in\mathbf{Y}-[y_0]_{\tilde{\mathbb{G}}}$. Define the operator $\theta:\mathbf{Y}\to\mathbf{Y}$ by

$$\theta(y)=\begin{cases} y_0, & \text{if } y\in[y_0]_{\tilde{\mathbb{G}}},\\ z_0, & \text{otherwise.}\end{cases}$$

Thus, $Fix\,(\theta) = \{y_0, z_0\}$. Further, θ is a $(\mathbb{G}, \psi)$-contraction. Indeed, if $(y, z) \in \mathcal{E}(\mathbb{G})$, then $[y]_{\tilde{\mathbb{G}}} = [z]_{\tilde{\mathbb{G}}}$. Therefore, $y, z \in [y_0]_{\tilde{\mathbb{G}}}$ or $y, z \in \mathbf{Y} - [y_0]_{\tilde{\mathbb{G}}}$. In both the cases, we conclude that $\theta(y) = \theta(z)$. Hence, $(\theta y, \theta z) \in \triangle \subset \mathcal{E}(\mathbb{G})$. Moreover, $\sigma(\theta y, \theta z) = 0 \leq \psi(\sigma(y, z))$. Therefore, θ admits two fixed points, which contradicts to (iii).

Lemma 3.5. *Assume that $(\mathbf{Y}, \sigma)$ is a CMS endued with a directed graph $\mathbb{G}$. If there exists $\psi \in \Psi$ such that $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ is $(\mathbb{G}, \psi)$-contraction for which there is some $y_0 \in \mathbf{Y}$ such that $\mathcal{S}(y_0) \in [y_0]_{\tilde{\mathbb{G}}}$, then*

(i) $[y_0]_{\tilde{\mathbb{G}}}$ *is $\mathcal{S}$-invariant;*
(ii) $\mathcal{S}|_{[y_0]_{\tilde{\mathbb{G}}}}$ *is a $(\tilde{\mathbb{G}}_{y_0}, \psi)$-contraction;*
(iii) *for any $y, z \in [y_0]_{\tilde{\mathbb{G}}}$, the sequences $\{\mathcal{S}^n y\}$ and $\{\mathcal{S}^n z\}$ are Cauchy equivalent.*

Proof. (i) Take $y \in [y_0]_{\tilde{\mathbb{G}}}$, then $\tilde{\mathbb{G}}$ admits a path $\{y_0, y_1, y_2, \ldots y_p\}$ such that $y_p = y$ and $(y_{k-1}, y_k) \in \mathcal{E}(\mathbb{G})$, for each $k \in \{1, 2, \ldots p\}$. By Proposition 3.1, $\mathcal{S}$ is a $(\tilde{\mathbb{G}}, \psi)$-contraction. This implies that $(\mathcal{S}y_{k-1}, \mathcal{S}y_k) \in \mathcal{E}(\mathbb{G})$, for all $k \in \{1, 2, \ldots p\}$. Hence, $\{\mathcal{S}y_0, \mathcal{S}y_1, \mathcal{S}y_2, \ldots \mathcal{S}y_p\}$ forms a path in $\tilde{\mathbb{G}}$ from $\mathcal{S}(y_0)$ to $\mathcal{S}(y)$. Thus, we conclude $\mathcal{S}(y) \in [\mathcal{S}y_0]_{\tilde{\mathbb{G}}}$. By hypothesis, we have $\mathcal{S}(y_0) \in [y_0]_{\tilde{\mathbb{G}}}$, i.e., $[\mathcal{S}y_0]_{\tilde{\mathbb{G}}} = [y_0]_{\tilde{\mathbb{G}}}$ thereby yielding $\mathcal{S}(y) \in [y_0]_{\tilde{\mathbb{G}}}$. Hence $[y_0]_{\tilde{\mathbb{G}}}$ is $\mathcal{S}$-invariant.

(ii) Take $(y, z) \in \mathcal{E}(\tilde{\mathbb{G}})$, then $\tilde{\mathbb{G}}$ admits a path $\{y_0, y_1, y_2, \ldots y_{p-1} = y, y_p = z\}$ such that $(y_{k-1}, y_k) \in \mathcal{E}(\tilde{\mathbb{G}})$, for all $k \in \{1, 2, \ldots p\}$. By $(\mathbb{G}, \psi)$-contraction condition, we get $(\mathcal{S}y_{k-1}, \mathcal{S}y_k) \in \mathcal{E}(\tilde{\mathbb{G}})$, for all $k \in \{1, 2, \ldots p\}$. Let $\{z_0, z_1, z_2, \ldots z_{l-1}, z_l\}$ be a path between y_0 and $\mathcal{S}(y_0)$. Hence

$$\{y_0 = z_0, z_1, z_2, \ldots z_{l-1}, z_l = \mathcal{S}y_0, \mathcal{S}y_1, \mathcal{S}y_2, \ldots \mathcal{S}y_{p-1} = \mathcal{S}y, \mathcal{S}y_p = \mathcal{S}z\}$$

forms a path in $\tilde{\mathbb{G}}$ from y_0 to $\mathcal{S}(z)$ enjoying $(\mathcal{S}y, \mathcal{S}z) \in \mathcal{E}(\tilde{\mathbb{G}})$. Since $\mathcal{E}(\tilde{\mathbb{G}}_{y_0}) \subset \mathcal{E}(\tilde{\mathbb{G}})$ and $\mathcal{S}$ is a $(\mathbb{G}, \psi)$-contraction, $\mathcal{S}|_{[y_0]_{\tilde{\mathbb{G}}}}$ is a $(\tilde{\mathbb{G}}_{y_0}, \psi)$-contraction.

(iii) As $\tilde{\mathbb{G}}_{y_0}$ is connected, the conclusion is immediate in view of items (i) and (iii) of Lemma 3.4.

We conclude this section to revisit the following notions of existing literature.

Definition 3.3. *[18] One says that a self-map $\mathcal{S}$ on a MS $(\mathbf{Y}, \sigma)$ is*

- *a PO (i.e., Picard operator) if $\mathcal{S}$ enjoys a unique fixed point y^* and $\mathcal{S}^n(y) \to y^*$, for all $y \in \mathbf{Y}$;*
- *a WPO (i.e., weakly Picard operator) if* $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$ *and $\{\mathcal{S}^n y\}$ converges to a fixed point of $\mathcal{S}$, for all $y \in \mathbf{Y}$.*

4. Main results

We'll present two fixed point theorems for a $(\mathbb{G}, \psi)$-contraction self-map in a CMS endued with a transitive graph.

Theorem 4.1. *Assume that $(\mathbf{Y}, \sigma)$ is a CMS endued with a transitive directed graph $\mathbb{G}$, which is also* (C)*-graph. If there exists $\psi \in \Psi$ such that $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ is a $(\mathbb{G}, \psi)$-contraction, then*

(I) $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$ *if and only if $\mathbf{Y}_{\mathcal{S}} \neq \emptyset$;*
(II) *$\mathcal{S}$ is a PO whenever $\mathbf{Y}_{\mathcal{S}} \neq \emptyset$ and $\mathbb{G}$ remains weakly connected;*
(III) *for any $y \in \mathbf{Y}_{\mathcal{S}}$, $\mathcal{S}|_{[y]_{\tilde{\mathbb{G}}}}$ is a PO;*
(IV) *$\mathcal{S}$ is a WPO whenever $\mathbf{Y} = \mathbf{Y}_{\mathcal{S}}$.*

Proof. We'll first prove the statement (III). Take $y \in \mathbf{Y}_{\mathcal{S}}$, then $\mathcal{S}(y) \in [y]_{\tilde{\mathbb{G}}}$. By Lemma 3.3, we can determine $y^* \in \mathbf{Y}$ enjoying $\lim_{n\to+\infty} \mathcal{S}^n(y) = y^*$. Now, take $z \in [y]_{\tilde{\mathbb{G}}}$, then owing to Lemma 3.5, $\{\mathcal{S}^n y\}$ and $\{\mathcal{S}^n z\}$ are Cauchy equivalent. It follows that $\lim_{n\to+\infty} \mathcal{S}^n(z) = y^*$.

By (C)-graph property of $\mathbb{G}$, $\{y_n\}$ contains a subsequence $\{y_{n_k}\}$ enjoying $(y_{n_k}, y^*) \in \mathcal{E}(\mathbb{G})$, for all $k \in \mathbb{N}$. Owing to $(\mathbb{G}, \psi)$-contraction condition of $\mathcal{S}$, one gets

$$\sigma(y_{n_k+1}, \mathcal{S}y^*) = \sigma(\mathcal{S}y_{n_k}, \mathcal{S}y^*) \leq \psi(\sigma(y_{n_k}, y^*)).$$

Now, we claim that

$$\sigma(y_{n_k+1}, \mathcal{S}y^*) \leq \sigma(y_{n_k}, y^*). \tag{4.1}$$

If there is $k_0 \in \mathbb{N}$ for which $\sigma(y_{n_{k_0}}, y^*) = 0$, then one gets $\sigma(\mathcal{S}y_{n_{k_0}}, \mathcal{S}y^*) = 0$, i.e., $\sigma(y_{n_{k_0}+1}, \mathcal{S}y^*) = 0$ and therefore (4.1) holds for these $k_0 \in \mathbb{N}$. Otherwise, we have $\sigma(y_{n_k}, y^*) > 0$, for all $k \in \mathbb{N}$. Utilizing the definition of ψ, one gets $\psi(\sigma(y_{n_k}, y^*)) < \sigma(y_{n_k}, y^*)$, for every $k \in \mathbb{N}$. Thus (4.1) holds for every $k \in \mathbb{N}$. Using the limit in (4.1) and by $y_{n_k} \xrightarrow{\sigma} y^*$, we get $y_{n_k+1} \xrightarrow{\sigma} \mathcal{S}(y^*)$. This yields that $\mathcal{S}(y^*) = y^*$. Thus, $\mathcal{S}|_{[y]_{\tilde{\mathbb{G}}}}$ is a PO. Hence the conclusion (I) is verified. By weakly connectedness of $\mathbb{G}$, one has $[y]_{\tilde{\mathbb{G}}} = \mathbf{Y}$ and hence (II) follows from (III).

From (III), it follows that $\text{Fix}(\mathcal{S}) \neq \emptyset$ if $\mathbf{Y}_{\mathcal{S}} \neq \emptyset$. Now assume that $\text{Fix}(\mathcal{S}) \neq \emptyset$. Due to $\triangle \subseteq \mathcal{E}(\mathbb{G})$, we conclude that $\mathbf{Y}_{\mathcal{S}} \neq \emptyset$. Therefore, the conclusion (I) holds.

If $\mathbf{Y} = \mathbf{Y}_{\mathcal{S}}$, then in view of (III), we conclude that $\lim_{n\to+\infty} \mathcal{S}^n(y) \in \text{Fix}(\mathcal{S})$, for any $y \in \mathbf{Y}$. Consequently, $\mathcal{S}$ is a WPO and (IV) is proved.

Theorem 4.2. *Assume that $(\mathbf{Y}, \sigma)$ is a CMS endued with a transitive directed graph $\mathbb{G}$ and $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ is an orbitally $\mathbb{G}$-continuous mapping. If for some $\psi \in \Psi$, $\mathcal{S}$ is $(\mathbb{G}, \psi)$-contraction, then*

(I) $\text{Fix}(\mathcal{S}) \neq \emptyset$ *if and only if* $\mathbf{Y}_{\mathcal{S}} \neq \emptyset$*;*
(II) *$\mathcal{S}$ is a PO whenever $\mathbf{Y}_{\mathcal{S}} \neq \emptyset$ and $\mathbb{G}$ remains weakly connected;*
(III) *for any $y \in \mathbf{Y}_{\mathcal{S}}$ and $z \in [y]_{\tilde{\mathbb{G}}}$, $\lim_{n\to\infty} \mathcal{S}^n(z) \in \text{Fix}(\mathcal{S})$ and, $\lim_{n\to+\infty} \mathcal{S}^n(z)$ does not depend on z;*
(IV) *$\mathcal{S}$ is a WPO whenever $\mathbf{Y} = \mathbf{Y}_{\mathcal{S}}$.*

Proof. We'll first prove the conclusion (III). Take $y \in \mathbf{Y}_{\mathcal{S}}$ and $z \in [y]_{\tilde{\mathbb{G}}}$. Due to Lemma 3.5, we conclude that $\{\mathcal{S}^n y\}$ and $\{\mathcal{S}^n z\}$ converge to the same point x^*. Also, we have $(\mathcal{S}^n y, \mathcal{S}^{n+1} y) \in \mathcal{E}(\mathbb{G})$, for every $n \in \mathbb{N}$. Employing the orbitally $\mathbb{G}$-continuity of $\mathcal{S}$, we obtain $\mathcal{S}^{n+1}(y) = \mathcal{S}(\mathcal{S}^n y) \xrightarrow{\sigma} \mathcal{S}(y^*)$. Consequently, we have $\mathcal{S}(y^*) = y^*$.

The conclusion (I) follows from (III) and $\triangle \subset \mathcal{E}(\mathbb{G})$. (IV) remains an immediate consequence of (III). To prove (II), let us assume that $y_0 \in \mathbf{Y}_{\mathcal{S}}$, then $[y_0]_{\tilde{\mathbb{G}}} = \mathbf{Y}$. Therefore, in view of (III), $\mathcal{S}$ is a PO.

To demonstrate our outcomes, we provide the following examples.

Example 4.1. *Let $\mathbf{Y} = [1, 3]$ with standard metric σ, then $(\mathbf{Y}, \sigma)$ is a CMS. Endow a directed graph $\mathbb{G}$ on $\mathbf{Y}$ by $\mathcal{E}(\mathbb{G}) = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$, then $\mathbb{G}$ is transitive. Assume that $\{y_n\} \subset \mathbf{Y}$ is sequence verifying $(y_n, y_{n+1}) \in \mathcal{E}(\mathbb{G})$, for all $n \in \mathbb{N}$ and $y_n \xrightarrow{\sigma} y$. As $(y_n, y_{n+1}) \notin \{(1, 3), (2, 3)\}$, we have $(y_n, y_{n+1}) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, for every $n \in \mathbb{N}$ and, hence, $\{y_n\} \subset \{1, 2\}$. By closedness of $\{1, 2\}$, we get $(y_n, y) \in \mathcal{E}(\mathbb{G})$. Consequently, $\mathbb{G}$ is a (C)-graph.*

Let $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ be a map by

$$\mathcal{S}(y) = \begin{cases} 1 & \text{if } 1 \leq y \leq 2, \\ 2 & \text{if } 2 < y \leq 3. \end{cases}$$

Define the function $\psi(s) = s/3$, then $\psi \in \Psi$. It is unambiguously accessible that $\mathcal{S}$ is a $(\mathbb{G}, \psi)$-contraction and that $\mathbb{G}$ is weakly connected. Hence by Theorem 4.1, $\mathcal{S}$ is a PO so that $y^* = 1$ is a unique fixed point.

Example 4.2. *Let $\mathbf{Y} = [0, +\infty)$ with standard metric σ, then $(\mathbf{Y}, \sigma)$ is a CMS. Endow a directed graph $\mathbb{G}$ on $\mathbf{Y}$ by $\mathcal{E}(\mathbb{G}) := \{(y, z) \in \mathbf{Y}^2 : y > z\}$, then $\mathbb{G}$ is transitive. Let $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ be a map defined by $\mathcal{S}(y) = y/(y + 1)$, then $\mathcal{S}$ is orbitally $\mathbb{G}$-continuous.*

Define the function $\psi(s) = s/(1 + s)$, then $\psi \in \Psi$. Now, for all $(y, z) \in \mathcal{E}(\mathbb{G})$, we have $(\mathcal{S}y, \mathcal{S}z) \in \mathcal{E}(\mathbb{G})$ and

$$\begin{aligned}
\sigma(\mathcal{S}y, \mathcal{S}z) &= \left|\frac{y}{y+1} - \frac{z}{z+1}\right| = \left|\frac{y-z}{1+y+z+yz}\right| \\
&\leq \frac{y-z}{1+(y-z)} = \frac{\sigma(y,z)}{1+\sigma(y,z)} \\
&\leq \psi(\sigma(y,z)).
\end{aligned}$$

Thus, $\mathcal{S}$ is a $(\mathbb{G}, \psi)$-contraction. It is unambiguously accessible that $\mathbb{G}$ is weakly connected. Hence, by Theorem 4.2, $\mathcal{S}$ is a PO so that $y^* = 0$ is a unique fixed point.

5. An application to BVP

This section deals with the following BVP:

$$\begin{cases} \vartheta'(s) = \zeta(s, \vartheta(s)), & s \in [a, b], \\ \vartheta(a) = \vartheta(b), \end{cases} \tag{5.1}$$

where $\zeta : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Φ will indicate the class of increasing continuous functions $\psi : [0, +\infty) \to [0, +\infty)$, which verify $\psi(s) < s$, for every $s > 0$. Observe that $\Phi \subset \Psi$.

We say that $\tilde{\vartheta} \in C'[a, b]$ is a lower solution of (5.1) if

$$\begin{cases} \tilde{\vartheta}'(s) \leq \zeta(s, \tilde{\vartheta}(s)), & s \in [a, b], \\ \tilde{\vartheta}(a) \leq \tilde{\vartheta}(b). \end{cases}$$

Theorem 5.1. *In addition to the Problem* (5.1)*, assume that there exists $l > 0$ and $\psi \in \Phi$ satisfying*

$$0 \leq [\zeta(s,\beta) + l\beta] - [\zeta(s,\alpha) + l\alpha] \leq l\psi(\beta - \alpha), \quad \textit{for any } \alpha, \beta \in \mathbb{R} \textit{ with } \alpha \leq \beta. \tag{5.2}$$

If the Problem (5.1) *has a lower solution, then it enjoys a unique solution.*

Proof. We can re-express the Eq (5.1) in the following form

$$\begin{cases} \vartheta'(s) + l\vartheta(s) = \zeta(s, \vartheta(s)) + l\vartheta(s), & \text{for every } s \in [a, b], \\ \vartheta(a) = \vartheta(b), \end{cases}$$

which is equivalent to

$$\vartheta(s) = \int_a^b \Lambda(s,\tau)[\zeta(\tau,\vartheta(\tau)) + l\vartheta(\tau)]d\tau, \tag{5.3}$$

where $\Lambda(s,\tau)$ is the Green function so that

$$\Lambda(s,\tau) = \begin{cases} \frac{e^{l(b+\tau-s)}}{e^{lb}-1}, & 0 \le \tau < s \le b, \\ \frac{e^{l(\tau-s)}}{e^{lb}-1}, & 0 \le s < \tau \le b. \end{cases}$$

Let $\mathbf{Y} := C[a,b]$. Define the map $\mathcal{S} : \mathbf{Y} \to \mathbf{Y}$ by

$$(\mathcal{S}\vartheta)(s) = \int_a^b \Lambda(s,\tau)[\zeta(\tau,\vartheta(\tau)) + l\vartheta(\tau)]d\tau, \quad \text{for all } s \in [a,b]. \tag{5.4}$$

On $\mathbf{Y}$, equip directed graph $\mathbb{G}$ defined by

$$\mathcal{E}(\mathbb{G}) = \{(\vartheta,\omega) \in \mathbf{Y} \times \mathbf{Y} : \vartheta(s) \le \omega(s), \text{ for every } s \in [a,b]\}. \tag{5.5}$$

If $\tilde{\vartheta} \in C'[a,b]$ is a lower solution of (5.1), then we have

$$\tilde{\vartheta}'(s) + l\tilde{\vartheta}(s) \le \zeta(s,\tilde{\vartheta}(s)) + l\tilde{\vartheta}(s), \quad \text{for all } s \in [a,b].$$

Multiplying the above inequality by e^{ls}, we obtain

$$(\tilde{\vartheta}(s)e^{ls})' \le [\zeta(s,\tilde{\vartheta}(s)) + l\tilde{\vartheta}(s)]e^{ls}, \quad \text{for all } s \in [a,b],$$

which yields

$$\tilde{\vartheta}(s)e^{ls} \le \tilde{\vartheta}(a) + \int_a^s [\zeta(\tau,\tilde{\vartheta}(\tau)) + l\tilde{\vartheta}(\tau)]e^{l\tau}d\tau, \quad \text{for each } s \in [a,b]. \tag{5.6}$$

Employing $\tilde{\vartheta}(a) \le \tilde{\vartheta}(b)$, we get

$$\tilde{\vartheta}(a)e^{lb} \le \tilde{\vartheta}(b)e^{lb} \le \tilde{\vartheta}(a) + \int_a^b [\zeta(\tau,\tilde{\vartheta}(\tau)) + l\tilde{\vartheta}(\tau)]e^{l\tau}d\tau,$$

i.e.,

$$\tilde{\vartheta}(a) \le \int_a^b \frac{e^{l\tau}}{e^{lb}-1}[\zeta(\tau,\tilde{\vartheta}(\tau)) + l\tilde{\vartheta}(\tau)]d\tau. \tag{5.7}$$

By (5.6) and (5.7), we get

$$\begin{aligned} \tilde{\vartheta}(s)e^{ls} &\le \int_a^b \frac{e^{l\tau}}{e^{lb}-1}[\zeta(\tau,\tilde{\vartheta}(\tau)) + l\tilde{\vartheta}(\tau)]d\tau + \int_a^s e^{l\tau}[\zeta(\tau,\tilde{\vartheta}(\tau)) + l\tilde{\vartheta}(\tau)]d\tau \\ &= \int_a^s \frac{e^{l(b+\tau)}}{e^{lb}-1}[\zeta(\tau,\tilde{\vartheta}(\tau)) + l\tilde{\vartheta}(\tau)]d\tau + \int_s^b \frac{e^{l\tau}}{e^{lb}-1}[\zeta(\tau,\tilde{\vartheta}(\tau)) + l\tilde{\vartheta}(\tau)]d\tau, \end{aligned}$$

which yields

$$\tilde{\vartheta}(s) \le \int_a^s \frac{e^{l(b+\tau-s)}}{e^{lb}-1}[\zeta(\tau,\tilde{\vartheta}(\tau)) + l\tilde{\vartheta}(\tau)]d\tau + \int_s^b \frac{e^{l(\tau-s)}}{e^{lb}-1}[\zeta(\tau,\tilde{\vartheta}(\tau)) + l\tilde{\vartheta}(\tau)]d\tau$$

$$
\begin{aligned}
&= \int_a^b \Lambda(s,\tau)[\zeta(\tau,\tilde{\vartheta}(\tau)) + l\tilde{\vartheta}(\tau)]d\tau \\
&= (\mathcal{S}\tilde{\vartheta})(s), \qquad \text{for all } \ s \in [a,b],
\end{aligned}
$$

which implies that $(\tilde{\vartheta}, \mathcal{S}\tilde{\vartheta}) \in \mathcal{E}(\mathbb{G})$. Thus, $\tilde{\vartheta} \in \mathbf{Y}_{\mathcal{S}}$, i.e., $\mathbf{Y}_{\mathcal{S}} \neq \emptyset$.

Now, let $\vartheta, \omega \in \mathbf{Y}$ be chosen arbitrarily. Let $u := \max\{\vartheta, v\}$, then one has $(\vartheta, u) \in \mathcal{E}(\mathbb{G})$ and $(v, u) \in \mathcal{E}(\mathbb{G})$. This yields that $\tilde{\mathbb{G}}$ is connected, i.e., $\mathbb{G}$ is weakly connected.

Define the following metric on $\mathbf{Y}$:

$$
\sigma(\vartheta, \omega) = \sup_{s \in [a,b]} |\vartheta(s) - \omega(s)|, \qquad \text{for all } \ \vartheta, \omega \in \mathbf{Y}. \tag{5.8}
$$

Clearly $(\mathbf{Y}, \sigma)$ is complete. To substantiate that $\mathbb{G}$ is a (C)-graph, let $\{\vartheta_{\mathsf{n}}\} \subset \mathbf{Y}$ be a sequence converging to $\overline{\vartheta} \in \mathbf{Y}$ and verifying $(y_{\mathsf{n}}, y_{\mathsf{n}+1}) \in \mathcal{E}(\mathbb{G})$, for all $\mathsf{n} \in \mathbb{N}$. This implies that $\vartheta_{\mathsf{n}}(s) \leq \overline{\vartheta}(s)$, for all $\mathsf{n} \in \mathbb{N}$ and for all $s \in [a,b]$. By (5.5), we have $(\vartheta_{\mathsf{n}}, \overline{\vartheta}) \in \mathcal{E}(\mathbb{G})$, for all $\mathsf{n} \in \mathbb{N}$. This shows that $\mathbb{G}$ is a (C)-graph.

Finally, let $(\vartheta, \omega) \in \mathcal{E}(\mathbb{G})$. By (5.2), we obtain

$$
\zeta(s, \vartheta(s)) + l\vartheta(s) \leq \zeta(s, \omega(s)) + l\omega(s), \qquad \text{for all } \ s \in [a,b]. \tag{5.9}
$$

By (5.4), (5.9), and $\Lambda(s,\tau) > 0$, for all $s, \tau \in [a,b]$, we get

$$
\begin{aligned}
(\mathcal{S}\vartheta)(s) &= \int_a^b \Lambda(s,\tau)[\zeta(\tau,\vartheta(\tau)) + l\vartheta(\tau)]d\tau \\
&\leq \int_a^b \Lambda(s,\tau)[\zeta(\tau,\omega(\tau)) + l\omega(\tau)]d\tau \\
&= (\mathcal{S}\omega)(s), \qquad \text{for all } \ s \in [a,b],
\end{aligned}
$$

which in view of (5.5) yields that $(\mathcal{S}\vartheta, \mathcal{S}\omega) \in \mathcal{E}(\mathbb{G})$. Again, by using (5.2), (5.4), and (5.8), we get

$$
\begin{aligned}
\sigma(\mathcal{S}\vartheta, \mathcal{S}\omega) &= \sup_{s \in [a,b]} |(\mathcal{S}\vartheta)(s) - (\mathcal{S}\omega)(s)| = \sup_{s \in [a,b]} \left((\mathcal{S}\omega)(s) - (\mathcal{S}\vartheta)(s)\right) \\
&\leq \sup_{s \in [a,b]} \int_a^b \Lambda(s,\tau)[\zeta(\tau,\omega(\tau)) + l\omega(\tau) - \zeta(\tau,\vartheta(\tau)) - l\vartheta(\tau)]d\tau \\
&\leq \sup_{s \in [a,b]} \int_a^b \Lambda(s,\tau) l\psi(\omega(\tau) - \vartheta(\tau))d\tau.
\end{aligned} \tag{5.10}
$$

Observe $0 \leq \omega(\tau) - \vartheta(\tau) \leq \sigma(\vartheta, \omega)$. Hence, by monotonicity of ψ, we get

$$
\psi(\omega(\tau) - \vartheta(\tau)) \leq \psi(\sigma(\vartheta, \omega)).
$$

Hence, (5.10) reduces to

$$
\begin{aligned}
\sigma(\mathcal{S}\vartheta, \mathcal{S}\omega) &\leq l\psi(\sigma(\vartheta,\omega)) \sup_{s \in [a,b]} \int_a^b \Lambda(s,\tau)d\tau \\
&= l\psi(\sigma(\vartheta,\omega)) \sup_{s \in [a,b]} \frac{1}{e^{lb} - 1}\left[\frac{1}{l} e^{l(b+\tau-s)}\Big|_0^s + \frac{1}{l} e^{l(\tau-s)}\Big|_s^b\right]
\end{aligned}
$$

$$
\begin{aligned}
&= l\psi(\sigma(\vartheta,\omega))\frac{1}{l(e^{lb}-1)}(e^{lb}-1) \\
&= \psi(\sigma(\vartheta,\omega)),
\end{aligned}
$$

so that

$$
\sigma(\mathcal{S}\vartheta,\mathcal{S}\omega) \le \psi(\sigma(\vartheta,\omega)), \qquad \text{for all } (\vartheta,\omega)\in\mathcal{E}(\mathbb{G}).
$$

Therefore, $\mathcal{S}$ is a $(\mathbb{G},\psi)$-contraction. Consequently, by Theorem 4.1, $\mathcal{S}$ is PO. Thus, the unique fixed point of $\mathcal{S}$ forms the unique solution of (5.1).

6. Conclusions

In 2010, Bojor [5] established the fixed point results under a $(\mathbb{G},\psi)$-contraction due to Matkowski [2]. In this work, we employed a $(\mathbb{G},\psi)$-contraction involving control function of Boyd and Wong [1]. Applying our outcomes, we discussed the existence and uniqueness of solution of BVP (5.1), whereas a lower solution of the BVP exists. Our results generalized and extended the results of Jachymski [3] and Fallahi and Aghanians [7]. In the future, our results can be generalized for $(\mathbb{G},\psi,\phi)$-contraction, employing a pair of control functions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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