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Research article

Efficient spectral collocation method for nonlinear systems of fractional pantograph delay differential equations

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Abstract: Caputo-Hadamard-type fractional calculus involves the logarithmic function of an arbitrary exponent as its convolutional kernel, which causes challenges in numerical approximations. In this paper, we construct and analyze a spectral collocation approach using mapped Jacobi functions as basis functions and construct an efficient algorithm to solve systems of fractional pantograph delay differential equations involving Caputo-Hadamard fractional derivatives. What we study is the error estimates of the derived method. In addition, we tabulate numerical results to support our theoretical analysis.

Keywords: mapped Jacobi functions; spectral methods; convergence analysis; pantograph delay differential equations
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1. Introduction

Fractional differential equations have gained growing attention in recent years and many monographs have appeared [1, 2]. The most common definitions of fractional calculus (differentiation and integration) are for Caputo, Riemann-Liouville, and Grunwald-Letnikov derivatives [3–7]. Compared with these two types of definitions, the Hadamard fractional calculus, which was first introduced in 1892 by Hadamard [8], did not receive much attention. The kernel of the integrand in the definition of the fractional Hadamard derivative contains a logarithmic function with an arbitrary exponent different from the Riemann-Liouville fractional derivatives. Recently, the Hadamard derivative and Hadamard-type fractional differential equations have been useful in practical problems related to mechanics and engineering, such as fracture analysis or both planar and three-dimensional elasticities [9]. Kilbas discussed Hadamard-type fractional differential equations in different spaces [10]. Recently, Ma and Li described the properties of Hadamard calculus [11] and they also proposed the definite conditions for Hadamard-type fractional differential equations.

The Caputo-Hadamard (C-H) fractional derivative is a kind of fractional derivative that is useful in describing abnormal diffusion processes, especially ultra-slow diffusion. Gohar et al. [12] studied the existence and uniqueness of the solution to Caputo-Hadamard fractional differential equations and the corresponding continuation theorem. Wang et al. [13] investigated the stability of the zero solution of a class of nonlinear Hadamard-type fractional differential systems by utilizing a new fractional comparison principle. Belbali et al. [14] discussed the existence, uniqueness, and stability of solutions for a nonlinear fractional differential system consisting of a nonlinear Caputo-Hadamard fractional initial value problem. Aljoudi et al. [15] studied a coupled system of Caputo-Hadamard-type sequential fractional differential equations supplemented with nonlocal boundary conditions involving Hadamard fractional integrals. Dhaniya et al. [16] established the existence, uniqueness, and Hyers-Ulam stability of the solution to the nonlinear Langevin fractional differential equation that involves the C-H and Caputo fractional operators, with nonperiodic and nonlocal integral boundary conditions. Beyenea et al. [17] established sufficient conditions for the existence and uniqueness of solutions to nonlinear Caputo-Hadamard fractional differential equations involving Hadamard integrals and unbounded delays. He et al. [18] considered the Hadamard and the Caputo-Hadamard fractional derivatives and the stability of related systems without and with delay.

Due to the complex form of C-H fractional operators, one often needs to find a suitable numerical scheme to approximate it, which greatly improves the efficiency of the actual calculation process. The studies on numerical methods for nonlinear C-H fractional differential equations are still in their early stages. Gohar [19] studied finite difference methods for fractional differential equations with C-H derivatives and investigated the smoothness properties of the solution. Li et al. [20] obtained the analytical solution to a certain linear fractional partial differential equation with the C-H fractional derivative by introducing a new modified Laplace transform, and derived a numerical algorithm for such kinds of equations. Fan et al. [21] proposed three kinds of numerical formulas for approximating the C-H fractional derivatives, which are called $L1-2$ formula, $L2-1_{\sigma}$formula, and H2N2 formula.

Most numerical methods for solving fractional differential equations are based on local difference schemes. Compared with the previous works, the main contribution of this paper is to extend the results in [22, 23] by constructing and analyzing a nonlocal spectral collocation method for the following system of fractional pantograph delay differential equations:

$$\begin{cases} {}_{\ell}^{CH}D_t^{\rho}X_1(t) = g_1(t, X_1(t), \dots, X_M(t), X_1(qt), \dots, X_M(qt)),\ t \in I, \\ {}_{\ell}^{CH}D_t^{\rho}X_2(t) = g_2(t, X_1(t), \dots, X_M(t), X_1(qt), \dots, X_M(qt)),\ t \in I, \\ \vdots \\ {}_{\ell}^{CH}D_t^{\rho}X_M(t) = g_M(t, X_1(t), \dots, X_M(t), X_1(qt), \dots, X_M(qt)),\ t \in I, \\ X_i(t) = \bar{X}_i(t),\ for\ qt \le \ell\ i = 1, 2, \dots, M, \qquad \ell \in (0, t),\ \rho \in (0, 1), q \in (0, 1), \end{cases} \tag{1.1}$$

where $g_i : I \times \mathbb{R}^M \to \mathbb{R}$ are given continuous functions, $I = (\ell, e\ell)$, and the C-H derivative ${}_{\ell}^{CH}D_t^{\rho}$ of order $0 < \rho < 1$ is given by (2.2).

The outline of this paper is as follows: In Section 2, we introduce some necessary definitions and preliminaries. In Section 3, we construct the spectral collocation scheme. In Section 4, we provide some auxiliary lemmas. The convergence analysis is discussed in Section 5. The numerical results are provided in Section 6.

2. Preliminaries

In this section, some relevant properties of the C-H fractional calculus and the logarithmic Jacobi (*log J*) approximation are presented.

Definition 2.1. The C-H fractional integral with order $\rho > 0$ is defined as [24]

$$ {}_{\ell}J_z^{\rho}X(z) = \frac{1}{\Gamma(\rho)} \int_{\ell}^{z} \kappa^{\rho-1}(z,w)X(w)\frac{dw}{w}, \; z > \ell > 0, \tag{2.1} $$

where $\kappa(z, w) = \log(\frac{z}{w})$.

Definition 2.2. The C-H fractional differential operator of order $0 < \rho < 1$ is given as [1]

$$ {}_{\ell}^{CH}D_z^{\rho}X(z) = \frac{1}{\Gamma(1-\rho)} \int_{\ell}^{z} \kappa^{-\rho}(z,w)\, X'(w)dw. \tag{2.2} $$

Definition 2.3. Let $\rho, \eta > -1$, $I := [\ell, \ell\, e]$, and $\ell > 0$. The *log J* functions of order p are given by [23]

$$ \begin{aligned} \mathscr{P}_p^{\rho,\eta,\ell}(z) &= \mathscr{P}_p^{\rho,\eta}\left(\kappa^2(z,\ell) - 1\right) \quad (\eta, \rho > -1,\; \ell > 0,\; \forall z \in I) \\ &= \frac{\Gamma(p+\rho+1)}{p!\Gamma(p+\rho+\eta+1)} \sum_{k=0}^{p} \binom{p}{k} \frac{\Gamma(p+k+\rho+\eta+1)}{\Gamma(k+\rho+1)} \left(\kappa(z,\ell) - 1\right)^k, \end{aligned} \tag{2.3} $$

where $\mathscr{P}_p^{\rho,\eta}(z)$ is the Jacobi polynomial and it is defined as

$$ \mathscr{P}_p^{\rho,\eta}(z) = \frac{\Gamma(p+\rho+1)}{\Gamma(p+1+\rho+\eta)p!} \sum_{k=0}^{p} \binom{p}{k} \frac{\Gamma(p+k+\rho+\eta+1)}{\Gamma(k+\rho+1)} \left(\frac{z-1}{2}\right)^k. $$

We define the space of logarithmic functions of order s by

$$ P_s^{\log}(\Omega) := span\left\{1, \kappa(z,\ell), \kappa(z,\ell)^2, \ldots, \kappa(z,\ell)^s\right\}, $$

where $\Omega = [\ell, +\infty),\; \ell > 0$. Let

$$ \chi^{\rho,\eta,\ell}(z) := z^{-1}\kappa(z,\ell)^{\eta}\left(1 - \kappa(z,\ell)\right)^{\rho}. \tag{2.4} $$

We denote by $L^2_{\chi^{\rho,\eta,\ell}}(I)$ the weighted L^2 space with the following inner product and norm:

$$ (X, \phi)_{\chi^{\rho,\eta,\ell}} = \int_I X(z)\phi(z)\chi^{\rho,\eta,\ell}(z)dz, \qquad ||X||_{\chi^{\rho,\eta,\ell}} = (X, X)^{1/2}_{\chi^{\rho,\eta,\ell}}. \tag{2.5} $$

One of the most important properties of the *log J* polynomials is that they are mutually orthogonal in $L^2_{\chi^{\rho,\eta,\ell}}(I)$, i.e.,

$$\begin{aligned} &\left(\mathscr{P}_m^{\rho,\eta,\ell}(z), \mathscr{P}_j^{\rho,\eta,\ell}(z)\right)_{\chi^{\rho,\eta,\ell}} = 0, \quad \forall j \neq m, \\ &\left\|\mathscr{P}_j^{\rho,\eta,\ell}(z)\right\|_{\chi^{\rho,\eta,\ell}} = \widehat{\theta}_j^{\rho,\eta} = \frac{\Gamma(j+\rho+1)\Gamma(j+\eta+1)}{(2j+\rho+\eta+1)j!\Gamma(j+\rho+\eta+1)}. \end{aligned} \tag{2.6}$$

We define the following first-order differential operator:

$$D^1_{\log}\phi(z) = \frac{d}{d\kappa(z,\ell)}\phi(z) = z\phi'(z), \tag{2.7}$$

and an induction leads to

$$D^k_{\log}\phi(z) = \overbrace{D^1_{\log}\cdot D^1_{\log}\cdots D^1_{\log}}^{k}\phi(z). \tag{2.8}$$

We also define the non-uniformly weighted *log J* Sobolev space as

$$B^{i,\ell}_{\rho,\eta}(I) := \{\phi : D^j_{\log}\phi \in L^2_{\chi^{\rho+j,\eta+j,\ell}}(I),\ 0 \leq j \leq i\}, \quad i \in \mathbb{N},$$

with

$$\begin{aligned} &(\psi,\phi)_{B^{i,\ell}_{\rho,\eta}} = \sum_{k=0}^{i}(D^k_{\log}\psi, D^k_{\log}\phi)_{\chi^{\rho+k,\eta+k,\ell}}, \quad \|\phi\|_{B^{i,\ell}_{\rho,\eta}} = (\phi,\phi)^{1/2}_{B^{i,\ell}_{\rho,\eta}}, \\ &|\phi|_{B^{i,\ell}_{\rho,\eta}} = \|D^i_{\log}\phi\|_{\chi^{\rho+i,\eta+i,\ell}}. \end{aligned}$$

For the usual shifted-weighted Jacobi Sobolev space, we define

$$B^i_{\rho,\eta}(\Lambda) := \{\phi : \partial^j_z\phi \in L^2_{\chi^{\rho+j,\eta+j}}(\Lambda), 0 \leq j \leq i\}, \quad i \in \mathbb{N},$$

where $\chi^{\rho,\eta} = (-z+1)^\rho z^\eta$ with $z \in \Lambda = [0,1]$ is the classical Jacobi weight function.

Assume that $x_0 < x_1 < \cdots < x_{M-1} < x_M$ in I are the roots of $\mathscr{P}^{\rho,\eta,\ell}_{M+1}(x)$. Let $z(x) = \log\frac{x}{\ell}$. Then $z_j := z(x_j) = \log\frac{x_j}{\ell}$, $0 \leq j \leq M$, are zeros of $\mathscr{P}^{\rho,\eta}_{M+1}(x)$, and $\{\chi_i\}_{i=0}^M$ are the corresponding weights.

The *log J*-Gauss quadrature enjoys the exactness

$$\int_I X(z)\chi^{\rho,\eta,\ell}(z)dz = \sum_{i=0}^{M} X(z_i)\chi_i, \quad \forall X(z) \in P^{\log}_{2M+1}. \tag{2.9}$$

Hence,

$$\sum_{k=0}^{M}\mathscr{P}^{\rho,\eta,\ell}_q(z_k)\mathscr{P}^{\rho,\eta,\ell}_j(z_k)\chi_k = \widehat{\theta}^{\rho,\eta}_q\delta_{q,j}, \quad \forall\, 0 \leq q+j \leq 2M+1. \tag{2.10}$$

For any $X(\ell e^z) \in C(I)$, the *log J*-Gauss interpolation operator $I^{\rho,\eta,\ell}_{z,M} : C(I) \longrightarrow P^{\log}_M$ is determined uniquely by

$$I^{\rho,\eta,\ell}_{z,M}X(z_q) = X(z_q), \quad 0 \leq q \leq M. \tag{2.11}$$

From the above condition, we have $I^{\rho,\eta,\ell}_{z,M} X = X$ for all $X \in P^{\log}_M$. On the other hand, since $I^{\rho,\eta,\ell}_{z,M} X \in P^{\log}_M$, we can write

$$I^{\rho,\eta,\ell}_{z,M} X(x) = \sum_{i=0}^{M} \widehat{X}^{\rho,\eta,\ell}_i \mathscr{P}^{\rho,\eta,\ell}_i(x), \quad \widehat{X}^{\rho,\eta,\ell}_i = \frac{1}{\widehat{\theta}^{\rho,\eta}_i} \sum_{j=0}^{M} X(x_j)\mathscr{P}^{\rho,\eta,\ell}_i(x_j)\chi_j, \ \forall X \in P^{\log}_M(I). \tag{2.12}$$

The L^∞(I) space is the set of all measurable functions that are essentially bounded. That is, functions g that are bounded almost everywhere on a set of finite measures. The essential supermom norm is used to define the norm of this space and is given as

$$\|g\|_\infty = ess \sup_{x\in I} |g(x)| .$$

Definition 2.4. Let $\mathbf{A}(z) = \left(a_{ij}(z)\right)_{m\times n}$ be an $(m \times n)$ matrix function with $z \in I$. We consider the non-negative real-valued function

$$|\mathbf{A}(z)| = \sum_{i=1}^{m} \sum_{j=1}^{n} \left|a_{ij}(z)\right| , \tag{2.13}$$

and the norms

$$\begin{aligned} \|\mathbf{A}\|_{\chi^{\rho,\eta,\ell}} &:= \left(\int_I |\mathbf{A}(z)|^2 \chi^{\rho,\eta,\ell} dz \right)^{1/2} , \\ \|\mathbf{A}\|_\infty &:= ess \sup_{z\in I} |\mathbf{A}(z)| . \end{aligned} \tag{2.14}$$

Proposition 2.1. It holds for any $\psi(\ell\, e^x) \in B^m_{\rho,\eta}(\Lambda)$, $m \geq 1$ and $M + 1 \geq m \geq q \geq 0$

$$\|D^q_{\log}(\psi - I^{\rho,\eta,\ell}_M \psi)\|_{\chi^{\rho+q,\eta+q,\ell}} \leq c \sqrt{\frac{(1 + M - m)!}{M!}} M^{q-(1+m)/2} \|\partial^m_x \{\psi(\ell\, e^x)\}\|_{\chi^{\rho+m,\eta+m}}, \tag{2.15}$$

and it takes the form

$$\|D^q_{\log}(\psi - I^{\rho,\eta,\ell}_M \psi)\|_{\chi^{\rho+q,\eta+q,\ell}} \leq cM^{q-m} \|\partial^m_x \{\psi(\ell\, e^x)\}\|_{\chi^{\rho+m,\eta+m}}, \ c \approx 1, \quad \text{for fixed} \quad m \quad \text{and} \quad M \gg 1. \tag{2.16}$$

In the case of $q = 0, 1$, we can write

$$\|\psi - I^{\rho,\eta,\ell}_M \psi\|_{\chi^{\rho,\eta,\ell}} \leq cM^{-m} \|\partial^m_x \{\psi(\ell\, e^x)\}\|_{\chi^{\rho+m,\eta+m}}, \tag{2.17}$$

$$\|\partial_x(\psi - I^{\rho,\eta,\ell}_M \psi)\|_{\tilde{\chi}^{\rho,\eta,\ell}} \leq cM^{1-m} \|\partial^m_x \{\psi(\ell\, e^x)\}\|_{\chi^{\rho+m,\eta+m}}, \tag{2.18}$$

where $\tilde{\chi}^{\rho,\eta,\ell} = x\left(1 - \log\left(\frac{x}{\ell}\right)\right)^{\rho+1} \left(\log\left(\frac{x}{\ell}\right)\right)^{\eta+1}$.

Lemma 2.1. [23] For any $\rho, \eta \in (-1, -\frac{1}{2})$ and for all $\psi(x) \in B^{1,\ell}_{\rho,\eta}(I)$, $\psi(\xi) = 0$ for some $\xi \in I$, it holds

$$\|\psi\|_\infty \leq \sqrt{2} \|\partial_x \psi\|^{1/2}_{\tilde{\chi}^{\rho,\eta,\ell}} \|\psi\|^{1/2}_{\chi^{\rho,\eta,\ell}}. \tag{2.19}$$

Proposition 2.2. [23] For $\rho, \eta \in (-1, -\dfrac{1}{2}]$,

$$\|\psi - I^{\rho,\eta,\ell}_M \psi\|_\infty \leq cM^{1/2-m} \|\partial^m_x \psi(\ell\, e^x)\|_{\chi^{\rho+m,\eta+m}}, \ \ \forall\psi(\ell\, e^x) \in B^m_{\rho,\eta}(\Lambda), \ m \geq 1. \tag{2.20}$$

Lemma 2.2. [23]

$$\|I^{\rho,\eta,\ell}_M\|_\infty := \max_{x\in I} \sum_{j=0}^{M} |h^{\rho,\eta,\ell}_j(x)| = \begin{cases} O(\log M), & -1 < \rho, \eta \leq -\frac{1}{2}, \\ O(M^{\mu+\frac{1}{2}}), & \mu = \max(\rho, \eta), \ \textit{otherwise}, \end{cases} \tag{2.21}$$

where $\{h^{\rho,\eta,\ell}_j(x)\}^M_{j=0}$ are the logarithmic Lagrange interpolation functions that are related to $\mathscr{P}^{\rho,\eta,\ell}_{M+1}(x)$.

3. Non-polynomial spectral collocation scheme

To begin with, we rewrite the differential equation (1.1) in the following equivalent compact integral form:

$$Z(t) = Z_\ell + \frac{1}{\Gamma(\rho)} \int_\ell^t (\kappa(t,s))^{\rho-1}\, \mathbf{Q}\,(s, Z(s), Z(q\,s))\, \frac{ds}{s},\ t \in (\ell, e\ell], \tag{3.1}$$

where

$$\begin{aligned}
&Z(t) = [X_1(t), X_2(t), \dots, X_M(t)]^T\,,\\
&Z_\ell = \left[\bar{X}_1(\ell), \bar{X}_2(\ell), \dots, \bar{X}_M(\ell)\right]^T,\\
&Z(q\,t) = \begin{cases} [X_1(q\,t), X_2(q\,t), \dots, X_M(q\,t)]^T\,, & \textit{if} \quad q\,t > \ell,\\[2ex] \left[\bar{X}_1(q\,t), \bar{X}_2(q\,t), \dots, \bar{X}_M(q\,t)\right]^T, & \textit{if} \quad q\,t \le \ell, \end{cases}\\
&\mathbf{Q}(t) = [g_1, g_2, \dots, g_M]^T\,.
\end{aligned}$$

In the following, we will make some useful transformations, which in turn are the basis for the numerical solution scheme and its numerical analysis. In order to convert the integral interval (ℓ, t) to I, we consider

$$\kappa(s,\ell) = \kappa(t,\ell)\kappa(r,\ell),$$

or

$$s = s(t,r) = \ell\left(\frac{r}{\ell}\right)^{\kappa(t,\ell)}.$$

Hence, the system(3.1) becomes

$$Z(t) = Z_\ell + \frac{(\kappa(t,\ell))^\rho}{\Gamma(\rho)} \int_I (1-\kappa(r,\ell))^{\rho-1}\, \mathbf{G}(s(t,r), Z(s(t,r)), Z(q\,s(t,r)))\frac{dr}{r}. \tag{3.2}$$

The non-polynomial spectral collocation scheme for (3.2) is to find $X_{m,N}(t) \in P_N^{\log}(I),\ m = 1, 2, \dots, M$ such that

$$Z_N(t) = Z_\ell + \frac{1}{\Gamma(\rho)} I_{t,N}^{0,0,\ell}(\kappa(t,\ell))^\rho \int_I r^{-1}\,(1-\kappa(r,\ell))^{\rho-1}\, I_{r,N}^{\rho-1,0,\ell}\mathbf{Q}(s(t,r), Z_N(s(t,r)), Z_N(q\,s(t,r)))dr, \tag{3.3}$$

where

$$Z_N(t) = [X_{1,N}, X_{2,N}, \dots, X_{M,N}]^T\,,$$

and $I_{z,N}^{\rho,\eta,\ell}$ the *log* J-Gauss interpolation operator in the z-direction. For simplicity, we will consider the trial functions as

$$X_{m,N}(t) = \sum_{i=0}^{N} X_{m,i}\mathscr{P}_i^{0,0,\ell}(t), \quad m = 1, \dots, M. \tag{3.4}$$

Also, we can use the following approximation:

$$\begin{aligned}
&I_{t,N}^{0,0,\ell} I_{r,N}^{\rho-1,0,\ell}\,(\kappa(t,\ell))^\rho\, g_m\,(s(t,r), Z_N(s(t,r)), Z_N(q\,s(t,r)))\\
&= \sum_{i=0}^{N}\sum_{j=0}^{N} v_{m,i,j}\mathscr{P}_i^{0,0,\ell}(t)\mathscr{P}_j^{\rho-1,0,\ell}(r), \quad m = 1, \dots, M.
\end{aligned} \tag{3.5}$$

A straightforward calculation by using (3.5) and (2.6) gives

$$\begin{aligned}
&\frac{1}{\Gamma(\rho)} I_{t,N}^{0,0,\ell}\left[(\kappa(t,\ell))^{\rho}\int_I r^{-1}\left(1-\kappa(r,\ell)\right)^{\rho-1} I_{r,N}^{\rho-1,0,\ell} g_m(s(t,r),Z_N(s(t,r)),Z_N(qs(t,r)))dr\right]\\
=&\frac{1}{\Gamma(\rho)}\sum_{i=0}^{N}\sum_{j=0}^{N} v_{m,i,j}\mathscr{P}_i^{0,0,\ell}(t)\int_I r^{-1}\left(1-\kappa(r,\ell)\right)^{\rho-1}\mathscr{P}_j^{\rho-1,0,\ell}(r)dr \qquad (3.6)\\
=&\frac{1}{\Gamma(\rho+1)}\sum_{i=0}^{N} v_{m,i,0}\mathscr{P}_i^{0,0,\ell}(t), \quad m=1,\dots,M.
\end{aligned}$$

Let $\left\{\chi_p^{\rho,\eta,\ell}, x_p^{\rho,\eta,\ell}\right\}_{p=0}^{N}$ be the weights and the nodes of Gauss-type logarithmic Jacobi interpolation. A direct application of (3.5) and (2.12) yields

$$\begin{aligned}
v_{m,i,0}=&\rho\,(2i+1)\sum_{p=0}^{N}\sum_{q=0}^{N}\left(\kappa(t_p^{0,0,\ell},\ell)\right)^{\rho}\mathscr{P}_i^{0,0,\ell}\left(t_p^{0,0,\ell}\right)\\
&\times g_m\left(s\left(t_p^{0,0,\ell},r_q^{\rho-1,0,\ell}\right),Z_N\left(s\left(t_p^{0,0,\ell},r_q^{\rho-1,0,\ell}\right)\right),Z_N\left(qs\left(t_p^{0,0,\ell},r_q^{\rho-1,0,\ell}\right)\right)\right)\chi_p^{0,0,\ell}\chi_q^{\rho-1,0,\ell}.
\end{aligned} \qquad (3.7)$$

Hence, we deduce that

$$\sum_{i=0}^{N} X_{m,i}\mathscr{P}_i^{0,0,\ell}(t)=X_\ell\mathscr{P}_0^{0,0,\ell}(t)+\frac{1}{\Gamma(\rho+1)}\sum_{i=0}^{N} v_{m,i,0}\mathscr{P}_i^{0,0,\ell}(t). \qquad (3.8)$$

We compared the coefficients of (3.8) to get

$$\begin{aligned}
X_{m,0}&=Z_\ell+\frac{v_{m,0,0}}{\Gamma(\rho+1)},\\
X_{m,i}&=\frac{v_{m,i,0}}{\Gamma(\rho+1)}, \quad 1\le i\le N, \quad m=1,\dots,M,
\end{aligned} \qquad (3.9)$$

where Z_ℓ is the vector of initial values defined in (3.1).

4. Auxiliary lemmas

Here, we derive the rate of convergence of the scheme (3.3) in the $L^2_{\chi^{0,0,\ell}}$-norm. Accordingly, we introduce some lemmas.

Let $r_i^{\rho,\eta,\ell}$ be the *log* J-Gauss nodes in I, and $s_i^{\rho,\eta,\ell}=s\left(x,r_i^{\rho,\eta,\ell}\right)$. The mapped *log* J-Gauss interpolation operator ${}_x\widetilde{\mathcal{I}}_{s,N}^{\rho,\eta,\ell}: C(\ell,x)\longrightarrow P_N^{\log}(\ell,x)$ is defined by

$${}_x\widetilde{\mathcal{I}}_{s,N}^{\rho,\eta,\ell}u\left(s_i^{\rho,\eta,\ell}\right)=u\left(s_i^{\rho,\eta,\ell}\right), \quad 0\le i\le N. \qquad (4.1)$$

Hence,

$${}_x\widetilde{\mathcal{I}}_{s,N}^{\rho,\eta,\ell}u\left(s_i^{\rho,\eta,\ell}\right)=u\left(s_i^{\rho,\eta,\ell}\right)=u\left(s(x,r_i^{\rho,\eta,\ell})\right)=\mathcal{I}_{r,N}^{\rho,\eta,\ell}u\left(s(x,r_i^{\rho,\eta,\ell})\right), \qquad (4.2)$$

and

$${}_x\widetilde{\mathcal{I}}_{s,N}^{\rho,\eta,\ell}u\,(s)=\mathcal{I}_{r,N}^{\rho,\eta,\ell}u\,(s(x,r))\Big|_{\kappa(r,\ell)=\frac{\kappa(s,\ell)}{\kappa(x,\ell)}}. \qquad (4.3)$$

Moreover, the following results can be easily derived:

$$\begin{aligned}\int_{\ell}^{x} s^{-1}(\kappa(x,s))^{\rho-1}\, {}_{x}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell} X(s)ds &= (\kappa(x,\ell))^{\rho}\int_{I} r^{-1}(1-\kappa(r,\ell))^{\rho-1}\, \mathcal{I}_{r,N}^{\rho-1,0,\ell} X(s(x,r))\,dr \\ &= (\kappa(x,\ell))^{\rho}\sum_{j=0}^{N} X\left(s(x,r_j^{\rho-1,0,\ell})\right)\chi_j^{\rho-1,0,\ell} \\ &= (\kappa(x,\ell))^{\rho}\sum_{j=0}^{N} X\left(s_j^{\rho-1,0,\ell}\right)\chi_j^{\rho-1,0,\ell}.\end{aligned} \tag{4.4}$$

Similarly,

$$\int_{\ell}^{x} s^{-1}(\kappa(x,s))^{\rho-1}\left({}_{x}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell} X(s)\right)^{2} ds = (\kappa(x,\ell))^{\rho}\sum_{j=0}^{N} X^{2}\left(s_j^{\rho-1,0,\ell}\right)\chi_j^{\rho-1,0,\ell}. \tag{4.5}$$

Then, for any $1 \le s \le N+1$, we have

$$\begin{aligned}&\int_{\ell}^{x} s^{-1}(\kappa(x,s))^{\rho-1}\left|\left(\mathcal{I} - {}_{x}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\right)X(s)\right|^{2} ds \\ =&(\kappa(x,\ell))^{\rho}\int_{I} r^{-1}(1-\kappa(r,\ell))^{\rho-1}\left|\left(\mathcal{I}-\mathcal{I}_{r,N}^{\rho-1,0,\ell}\right)X(s(x,r))\right|^{2} dr \\ \le& cN^{-2m}(\kappa(x,\ell))^{\rho}\int_{I} r^{-1}(1-\kappa(r,\ell))^{\rho+m-1}(\kappa(r,\ell))^{m}\left|D_{\log,r}^{m} X(s(x,r))\right|^{2} dr \\ =& cN^{-2m}\int_{\ell}^{x} s^{-1}(\kappa(x,s))^{\rho+m-1}(\kappa(s,\ell))^{m}\left|D_{\log,s}^{m} X(s)\right|^{2} ds,\end{aligned} \tag{4.6}$$

where $\mathcal{I}$ is the identity operator.

Lemma 4.1. The following estimate holds for the error function $e_N(x) = Z(x) - Z_N(x)$:

$$\|e_N\|_{\chi^{0,0,\ell}} \le \sum_{j=1}^{3}\left\|\Xi_j\right\|_{\chi^{0,0,\ell}}, \tag{4.7}$$

where

$$\begin{aligned}\Xi_1 &= Z(x) - \mathcal{I}_{x,N}^{0,0,\ell} Z(x), \\ \Xi_2 &= \mathcal{I}_{x,N}^{0,0,\ell}\int_{\ell}^{x} \boldsymbol{R}(x,s)(\mathcal{I} - {}_{x}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell})\quad \mathbf{Q}(s,Z(s),Z(q\,s))ds, \\ \Xi_3 &= \mathcal{I}_{x,N}^{0,0,\ell}\int_{\ell}^{x} \boldsymbol{R}(x,s)\quad {}_{x}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}(\mathbf{Q}(s,Z(s),Z(qs)) - \mathbf{Q}(s,Z_N(s),Z_N(qs)))ds,\end{aligned}$$

and $\boldsymbol{R}(x,s) = (R_{ij})$ with $R_{ij} = \frac{s^{-1}(\kappa(x,s))^{\rho-1}}{\Gamma(\rho)}\delta_{ij},\ i,\ j = 1,\dots,M.$

Proof.

$$\|e_N\|_{\chi^{0,0,\ell}} \le \left\|Z - \mathcal{I}_{x,N}^{0,0,\ell} Z\right\|_{\chi^{0,0,\ell}} + \left\|\mathcal{I}_{x,N}^{0,0,\ell} Z - Z_N\right\|_{\chi^{0,0,\ell}}. \tag{4.8}$$

It is clear from (3.1) that

$$\mathcal{I}_{x,N}^{0,0,\ell} Z(x) = Z_\ell + \frac{1}{\Gamma(\rho)}\mathcal{I}_{x,N}^{0,0,\ell}\int_{\ell}^{x} s^{-1}(\kappa(x,s))^{\rho-1}\,\mathbf{Q}(s,Z(s),Z(qs))\,ds, \tag{4.9}$$

and

$$Z_N(x) = Z_\ell + \frac{1}{\Gamma(\rho)}\mathcal{I}^{0,0,\ell}_{x,N}\int_\ell^x s^{-1}\left(\kappa(x,s)\right)^{\rho-1}\,{}_x\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}\mathbf{Q}\left(s, Z_N(s), Z_N(qs)\right)ds. \tag{4.10}$$

Subtracting (4.9) from (4.11) yields

$$\begin{aligned}&\mathcal{I}^{0,0,\ell}_{x,N}Z(x) - Z_N(x)\\ =&\frac{1}{\Gamma(\rho)}\mathcal{I}^{0,0,\ell}_{x,N}\int_\ell^x s^{-1}\left(\kappa(x,s)\right)^{\rho-1}\left(\mathbf{Q}\left(s, Z(s), Z(q\,s)\right) - {}_x\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}\mathbf{Q}\left(s, Z_N(s), Z_N(qs)\right)\right)ds,\end{aligned} \tag{4.11}$$

which has the form:

$$\begin{aligned}&\mathcal{I}^{0,0,\ell}_{x,N}Z(x) - Z_N(x)\\ =&\frac{1}{\Gamma(\rho)}\mathcal{I}^{0,0,\ell}_{x,N}\int_\ell^x s^{-1}\left(\kappa(x,s)\right)^{\rho-1}\left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}\right)\mathbf{Q}\left(s, Z(s), Z(qs)\right)ds\\ &+\frac{1}{\Gamma(\rho)}\mathcal{I}^{0,0,\ell}_{x,N}\int_\ell^x s^{-1}\left(\kappa(x,s)\right)^{\rho-1}\,{}_x\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}\left(\mathbf{Q}\left(s, Z(s), Z(qs)\right) - \mathbf{Q}\left(s, Z_N(s), Z_N(qs)\right)\right)ds.\end{aligned} \tag{4.12}$$

□

5. Convergence analysis

5.1. Convergence analysis in $L^2_{\chi^{0,0,\ell}}$-norm

Theorem 5.1. Let $Z(x)$ be the solutions of the systems (3.1) and (3.3), respectively. Then we have the following estimate:

$$\|Z - Z_N\|_{\chi^{0,0,\ell}} \le cN^{-m}\left(\|D^m_{\log}Z\|^2_{\chi^{m,m,\ell}} + \|D^m_{\log}\mathbf{Q}\left(x, Z(x), Z(q\,x)\right)\|^2_{\chi^{\rho+m-1,m,\ell}}\right), \tag{5.1}$$

where $1 \le m \le N+1$ and $N \ge 1$.

Proof. Using Proposition 2.1, we get

$$\|\Xi_1\|_{\chi^{0,0,\ell}} = \left\|Z - \mathcal{I}^{0,0,\ell}_{x,N}Z\right\|_{\chi^{0,0,\ell}} \le cN^{-m}\|D^m_{\log}Z\|^2_{\chi^{m,m,\ell}} \le cN^{-m}\left\|\partial^m_x Z(\ell\,e^x)\right\|_{\chi^{m,m}}. \tag{5.2}$$

Using the *log* J-Gauss integration formula, gives

$$\begin{aligned}\|\Xi_2\|_{\chi^{0,0,\ell}} &= \left\|\mathcal{I}^{0,0,\ell}_{x,N}\int_\ell^x \mathbf{R}(x,s)\left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}\right)\mathbf{Q}\left(s, Z(s), Z(qs)\right)ds\right\|_{\chi^{0,0,\ell}}\\ &= \left\|\sum_{k=1}^{M}\mathcal{I}^{0,0,\ell}_{x,N}\int_\ell^x R_{kk}(x,s)\left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}\right)g_k\left(s, Z(s), Z(q\,s)\right)ds\right\|_{\chi^{0,0,\ell}}\\ &= \left[\int_I \chi^{0,0,\ell}\left(\sum_{k=1}^{M}\mathcal{I}^{0,0,\ell}_{x,N}\int_\ell^x R_{kk}(x,s)\left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}\right)g_k\left(s, Z(s), Z(q\,s)\right)ds\right)dx\right]^{1/2}\\ &= \left[\sum_{j=0}^{N}\chi^{0,0,\ell}_j\left(\sum_{k=1}^{M}\int_\ell^{x^{0,0,\ell}_j} R_{kk}(x^{0,0,\ell}_j, s)\left(\mathcal{I} - {}_{x^{0,0,\ell}_j}\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}\right)g_k\left(s, Z(s), Z(q\,s)\right)ds\right)^2\right]^{1/2}\\ &\le \left[\sum_{j=0}^{N}\chi^{0,0,\ell}_j\sum_{k=1}^{M}\left(\int_\ell^{x^{0,0,\ell}_j} R_{kk}(x^{0,0,\ell}_j, s)\left(\mathcal{I} - {}_{x^{0,0,\ell}_j}\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}\right)g_k\left(s, Z(s), Z(q\,s)\right)ds\right)^2\sum_{k=1}^{M}(1)^2\right]^{1/2}.\end{aligned}$$

Using Cauchy-Schwarz inequality leads to the following estimate:

$$
\begin{aligned}
\|\Xi_2\|_{\chi^{0,0,\ell}} &\leq C\left[\sum_{j=0}^{N}\sum_{k=1}^{M}\chi_j^{0,0,\ell}\int_{\ell}^{x_j^{0,0,\ell}} R_{kk}(x_j^{0,0,\ell},s)\,ds\int_{\ell}^{x_j^{0,0,\ell}} R_{kk}(x_j^{0,0,\ell},s)\left|\left(\mathcal{I}-{}_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\right)g_k\left(s,Z(s),Z(q\,s)\right)\right|^2 ds\right]^{1/2}\\
&\leq C\left[\sum_{j=0}^{N}\sum_{k=1}^{M}\chi_j^{0,0,\ell}\left(\kappa(x_j^{0,0,\ell},\ell)\right)^{\rho}\int_{\ell}^{x_j^{0,0,\ell}} s^{-1}\left(\kappa(x_j^{0,0,\ell},s)\right)^{\rho-1}\left|\left(\mathcal{I}-{}_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\right)g_k\left(s,Z(s),Z(q\,s)\right)\right|^2 ds\right]^{1/2}\\
&\leq C\left(\sum_{j=0}^{N}\chi_j^{0,0,\ell}\left(\kappa(x_j^{0,0,\ell},\ell)\right)^{\rho}\right)^{1/2}\left(\sum_{k=1}^{M}\int_{\ell}^{x_j^{0,0,\ell}} s^{-1}\left(\kappa(x_j^{0,0,\ell},s)\right)^{\rho-1}\left|\left(\mathcal{I}-{}_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\right)g_k\left(s,Z(s),Z(q\,s)\right)\right|^2 ds\right)^{1/2}\\
&\leq C\left(\sum_{j=0}^{N}\chi_j^{0,0,\ell}\left(\kappa(x_j^{0,0,\ell},\ell)\right)^{\rho}\right)^{1/2}\left(\int_{\ell}^{x_j^{0,0,\ell}} s^{-1}\left(\kappa(x_j^{0,0,\ell},s)\right)^{\rho-1}\left|\left(\mathcal{I}-{}_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\right)\mathbf{Q}\left(s,Z(s),Z(qs)\right)\right|^2 ds\right)^{1/2}\\
&\leq cN^{-m}\left[\sum_{j=0}^{N}\chi_j^{0,0,\ell}\left(\kappa(x_j^{0,0,\ell},\ell)\right)^{\rho}\int_{\ell}^{x_j^{0,0,\ell}} s^{-1}\left(\kappa(x_j^{0,0,\ell},s)\right)^{\rho+m-1}\left(\kappa(s,\ell)\right)^{m}\left|D_{\log,s}^{m}\mathbf{Q}\left(s,Z(s),Z(qs)\right)\right|^2 ds\right]^{1/2}\\
&\leq cN^{-m}\|D_{\log}^{m}\mathbf{Q}\left(\cdot,Z(\cdot),Z(q\,\cdot)\right)\|^2_{\chi^{\rho+m-1,m,\ell}}.
\end{aligned}
\tag{5.3}
$$

An estimate for the term $\|E_3\|_{\chi^{0,0,\ell}}$ can be obtained by using the *log J*-Gauss integration formula, to give

$$
\begin{aligned}
\|\Xi_3\|^2_{\chi^{0,0,\ell}} &= \left\|\mathbf{R}(x,s)\mathcal{I}_{x,N}^{0,0,\ell}\int_{\ell}^{x}{}_{x}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}(\mathbf{Q}(s,Z(s),Z(q\,s))-\mathbf{Q}(s,Z_N(s),Z_N(q\,s)))ds\right\|^2_{\chi^{0,0,\ell}}\\
&=\frac{1}{\Gamma^2(\rho)}\int_I\chi^{0,0,\ell}\times\left(\sum_{k=1}^{M}\mathcal{I}_{x,N}^{0,0,\ell}\int_{\ell}^{x}s^{-1}\left(\kappa(x,s)\right)^{\rho-1}{}_{x}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\left(g_k\left(s,Z(s),Z(q\,s)\right)-g_k\left(s,Z_N(s),Z_N(q\,s)\right)\right)ds\right)^2 dx\\
&=\frac{1}{\Gamma^2(\rho)}\sum_{j=0}^{N}\chi_j^{0,0,\ell}\times\left(\int_{\ell}^{x_j^{0,0,\ell}}s^{-1}\left(\kappa(x,s)\right)^{\rho-1}\sum_{k=1}^{M}{}_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\left(g_k\left(s,Z(s),Z(q\,s)\right)-g_k\left(s,Z_N(s),Z_N(q\,s)\right)\right)ds\right)^2.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&\|\Xi_3\|^2_{\chi^{0,0,\ell}}\\
\leq&\frac{1}{\Gamma^2(\rho)}\sum_{j=0}^{N}\chi_j^{0,0,\ell}\int_{\ell}^{x_j^{0,0,\ell}}s^{-1}\left(\kappa(x,s)\right)^{\rho-1}ds\\
&\times\int_{\ell}^{x_j^{0,0,\ell}}s^{-1}\left(\kappa(x,s)\right)^{\rho-1}\left(\left|\sum_{k=1}^{M}{}_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\left(g_k\left(s,Z(s),Z(q\,s)\right)-g_k\left(s,Z_N(s),Z_N(q\,s)\right)\right)\right|\right)^2 ds\\
\leq&\frac{1}{\Gamma^2(\rho)}\sum_{j=0}^{N}\chi_j^{0,0,\ell}\left(\log\frac{x}{\ell}\right)^{\rho}\int_{\ell}^{x_j^{0,0,\ell}}s^{-1}\left(\kappa(x,s)\right)^{\rho-1}\\
&\times\left(\sum_{k=1}^{M}\left|{}_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\left(g_k\left(s,Z(s),Z(q\,s)\right)-g_k\left(s,Z_N(s),Z_N(q\,s)\right)\right)\right|\right)^2 ds,
\end{aligned}
\tag{5.4}
$$

and using the logarithmic Jacobi-Gauss quadrature formula (4.4), we obtain

$$\begin{aligned}\|\Xi_3\|_{\chi^{0,0,\ell}} \leq &\frac{1}{\Gamma(\rho+1)}\left[\sum_{j=0}^{N}\rho\chi_j^{0,0,\ell}\left(\kappa(x_j^{0,0,\ell},\ell)\right)^{2\rho}\right.\\ &\times\sum_{q=0}^{N}\chi_q^{\rho-1,0,\ell}\left(\sum_{k=1}^{M}\left|g_k\left(s\left(x_j^{0,0,\ell},r_q^{\rho-1,0,\ell}\right),Z\left(s\left(x_j^{0,0,\ell},r_q^{\rho-1,0,\ell}\right)\right),Z\left(q\,s\left(x_j^{0,0,\ell},r_q^{\rho-1,0,\ell}\right)\right)\right)\right.\right.\\ &-\left.\left.\left. g_k\left(s\left(x_j^{0,0,\ell},r_q^{\rho-1,0,\ell}\right),Z_N\left(s\left(x_j^{0,0,\ell},r_q^{\rho-1,0,\ell}\right)\right),Z_N\left(q\,s\left(x_j^{0,0,\ell},r_q^{\rho-1,0,\ell}\right)\right)\right)\right|\right)^2\right]^{1/2}.\end{aligned} \tag{5.5}$$

Using the Lipschitz condition, we obtain

$$\|\Xi_3\|_{\chi^{0,0,\ell}} \leq \frac{L}{\Gamma(\rho+1)}\times\left[\sum_{j=0}^{N}\rho\chi_j^{0,0,\ell}\left(\kappa(x_j^{0,0,\ell},\ell)\right)^{2\rho}\sum_{q=0}^{N}\left(\sum_{i=1}^{M}\chi_q^{\rho-1,0,\ell}\left|X_i\left(s\left(x_j^{0,0,\ell},r_q^{\rho-1,0,\ell}\right)\right)-X_{N,i}\left(s\left(x_j^{0,0,\ell},r_q^{\rho-1,0,\ell}\right)\right)\right|\right)^2\right]^{1/2}, \tag{5.6}$$

using (4.5), we get

$$\begin{aligned}\|\Xi_3\|_{\chi^{0,0,\ell}} \leq &\frac{L}{\Gamma(\rho+1)}\times\left[\sum_{j=0}^{N}\rho\chi_j^{0,0,\ell}\left(\kappa(x_j^{0,0,\ell},\ell)\right)^{\rho}\int_{\ell}^{x_j^{0,0,\ell}}s^{-1}\left(\kappa(x_j^{0,0,\ell},s)\right)^{\rho-1}\left(\sum_{i=1}^{M}\left|_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\left(X_i(s)-X_{N,i}(s)\right)\right|\right)^2ds\right]^{1/2}.\\ \|E_3\|_{\chi^{0,0,\ell}} \leq &\frac{L}{\Gamma(\rho+1)}\left(\sum_{j=0}^{N}\rho\chi_j^{0,0,\ell}\left(\kappa(x_j^{0,0,\ell},\ell)\right)^{\rho}\right)^{1/2}\\ &\times\max_{0\leq j\leq N}\left(\int_{\ell}^{x_j^{0,0,\ell}}s^{-1}\left(\kappa(x_j^{0,0,\ell},s)\right)^{\rho-1}\left(\sum_{i=1}^{M}\left|_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}_{s,N}^{\rho-1,0,\ell}\left(X_i(s)-X_{i,N}(s)\right)\right|\right)^2ds\right)^{1/2}.\end{aligned} \tag{5.7}$$

For any $x_j^{0,0,\ell} \in I$. Let $f(\rho) = \left(\kappa(x_j^{0,0,\ell},\ell)\right)^{\rho}$. We note that $f(\rho)$ is a convex function of ρ. Hence, by Jensen's inequality for all $\rho \in (0,1)$,

$$f(\rho) = (1-\rho)f(0) + \rho f(1).$$

The above inequality yields

$$\begin{aligned}\rho\sum_{j=0}^{N}\chi_j^{0,0,\ell}\left(\kappa(x_j^{0,0,\ell},\ell)\right)^{\rho} &\leq \rho\sum_{j=0}^{N}\chi_j^{0,0,\ell}\left[1-\rho+\rho\left(\kappa(x_j^{0,0,\ell},\ell)\right)\right]\\ &\leq \rho\left[1-\rho+\rho\int_I s^{-1}\left(\log\frac{x}{a}\right)dx\right] \leq \rho\left(1-\frac{\rho}{2}\right) \leq \frac{1}{2}.\end{aligned} \tag{5.8}$$

Hence, by using the above inequality, the triangle inequality, (4.6) and (4.5), we deduce that

$$
\begin{aligned}
\|\Xi_3\|_{\chi^{0,0,\ell}} &\leq \frac{L}{\sqrt{2}\,\Gamma(\rho+1)} \max_{0\leq j\leq N}\left(\int_{\ell}^{x_j^{0,0,\ell}} s^{-1}\left(\kappa(x_j^{0,0,\ell},s)\right)^{\rho-1}\left(\sum_{i=1}^{M}\left|_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}\left(X_i(s)-X_{N,i}(s)\right)\right|\right)^2 ds\right)^{1/2}\\
&\leq \frac{L}{\sqrt{2}\,\Gamma(\rho+1)}\times\max_{0\leq j\leq N}\left[\left(\int_{\ell}^{x_j^{0,0,\ell}} s^{-1}\left(\kappa(x_j^{0,0,\ell},s)\right)^{\rho-1}\left(\sum_{i=1}^{M}\left|_{x_j^{0,0,\ell}}\widetilde{\mathcal{I}}^{\rho-1,0,\ell}_{s,N}X_i(s)-X_i(s)\right|\right)^2 ds\right)^{1/2}\right.\\
&\quad\left.+\left(\int_{\ell}^{x_j^{0,0,\ell}} s^{-1}\left(\kappa(x_j^{0,0,\ell},s)\right)^{\rho-1}\left(\sum_{i=1}^{M}\left|X_i(s)-X_{N,i}(s)\right|\right)^2 ds\right)^{1/2}\right]\\
&\leq cN^{-m}\max_{0\leq j\leq N}\left(\int_{\ell}^{x_j}(\kappa(s,\ell))^m\left(\sum_{i=1}^{M}\left|D^m_{\log,s}X_i(s)\right|\right)^2 ds\right)^{1/2}\\
&\quad+\frac{L}{\sqrt{2}\,\Gamma(\rho+1)}\times\max_{0\leq j\leq N}\left(\int_{\ell}^{x_j^{0,0,\ell}} s^{-1}\left(\kappa(x_j^{0,0,\ell},s)\right)^{\rho-1}\left(\sum_{i=1}^{M}\left|X_i(s)-X_{N,i}(s)\right|\right)^2 ds\right)^{1/2}\\
&\leq cN^{-m}\|D^m_{\log}Z\|^2_{\chi^{m,m,\ell}}+\frac{L}{\sqrt{2}\Gamma(\rho+1)}\|e_N\|^2_{\chi^{m,m,\ell}}.
\end{aligned}
\tag{5.9}
$$

Hence, a combination of (5.2), (5.3), (5.9) and the Lipschitz constant $L < \Gamma(\rho+1)$ leads to the desired result. □

6. Numerical results

In order to illustrate the significance of our key findings, we provide two numerical examples in this section.

Example 6.1. We consider the following initial value problem:

$$
{}^{CH}_{1}D^{\rho}_{t}X(t) = g(x), \quad X(1) = 0,\ t\in(1,e),\ \rho\in(0,1]. \tag{6.1}
$$

Table 1 shows a comparison of the maximum absolute errors that are obtained from the method that we have presented and those given in [22] and [21]. The numerical results depict that, by using the method proposed in this paper, higher accuracy is achieved.

Table 1. A comparison between the maximum absolute errors of presented method and methods in [22] and [21] with $\rho = 0.5$ for Example 6.1.

N	Error	Error [22]	N [21]	Error [21]
9	5.3874×10^{-9}	9.1283×10^{-6}	20	1.2500×10^{-3}
10	2.7917×10^{-9}	2.2831×10^{-6}	40	2.8647×10^{-4}
11	1.5313×10^{-9}	7.1562×10^{-7}	80	6.6144×10^{-5}
12	8.8111×10^{-10}	2.6679×10^{-7}	160	1.5345×10^{-5}

To investigate numerically the stability of the spectral collocation scheme, we consider the initial value problem (6.1) and the following problems, whose right-hand side, the initial value, and the order of the differential operator suffer perturbations.

$$ {}_{1}^{CH}D_{t}^{\rho}Y(t) = g(x) + \varepsilon_{g},\ Y(1) = 0,\ t \in (1,e),\ \rho = 0.5. \tag{6.2} $$

$$ {}_{1}^{CH}D_{t}^{\rho+\varepsilon_{\rho}}Y(t) = g(x),\ Y(1) = 0,\ t \in (1,e),\ \rho = 0.5,\ \varepsilon_{\rho} \in (-0.5, 0.5). \tag{6.3} $$

$$ {}_{1}^{CH}D_{t}^{\rho}Y(t) = g(x),\ Y(1) = \varepsilon_{Y_0},\ t \in (1,e),\ \rho = 0.5. \tag{6.4} $$

The maximum absolute errors $|X_N - Y_N|$, where X_N is the numerical solution of problem (6.1) and Y_N is the numerical solution of the perturbed problems (6.2), (6.3), and (6.4), are displayed in Tables 2, 3, and 4, respectively. We observe that $\|X_N - Y_N\|_\infty = \mathrm{O}(\varepsilon_g)$, $\|X_N - Y_N\|_\infty = \mathrm{O}(\varepsilon_\rho)$, and $\|X_N - Y_N\|_\infty = \mathrm{O}(\varepsilon_{y_0})$, respectively, independently of N.

Table 2. Maximum of the absolute errors, $|X_N - Y_N|$, where X_N is the numerical solution of problem (6.1) and Y_N is the numerical solution of the perturbed problem (6.2) with several values of ε_g.

N	$\varepsilon_g = 0.1$	$\varepsilon_g = 0.01$	$\varepsilon_g = 0.001$
5	1.130×10^{-1}	1.131×10^{-2}	1.131×10^{-3}
10	1.127×10^{-1}	1.127×10^{-2}	1.127×10^{-3}
15	1.128×10^{-1}	1.128×10^{-2}	1.128×10^{-3}

Table 3. Maximum of the absolute errors, $|X_N - Y_N|$, where X_N is the numerical solution of problem (6.1) and Y_N is the numerical solution of the perturbed problem (6.3) with several values of ε_ρ.

N	$\varepsilon_\rho = 0.1$	$\varepsilon_\rho = 0.01$	$\varepsilon_\rho = 0.001$
5	1.192×10^{-1}	1.249×10^{-2}	1.255×10^{-3}
10	1.192×10^{-1}	1.249×10^{-2}	1.255×10^{-3}
15	1.192×10^{-1}	1.249×10^{-2}	1.255×10^{-3}

Table 4. Maximum of the absolute errors, $|X_N - Y_N|$, where X_N is the numerical solution of problem (6.1) and Y_N is the numerical solution of the perturbed problem (6.4) with several values of ε_{Y_0}.

N	$\varepsilon_{Y_0} = 0.1$	$\varepsilon_{Y_0} = 0.01$	$\varepsilon_{Y_0} = 0.001$
5	1.000×10^{-1}	1.000×10^{-2}	1.000×10^{-3}
10	1.000×10^{-1}	1.000×10^{-2}	1.000×10^{-3}
15	1.000×10^{-1}	1.000×10^{-2}	1.000×10^{-3}

Example 6.2. We consider the following coupled system:

$$ \begin{aligned} {}_{1}^{CH}D_{t}^{\rho}X_1(t) &= X_2^2(qt) + g_1(t), \quad \rho \in (0,1), \\ {}_{1}^{CH}D_{t}^{\rho}X_2(t) &= X_1^2(qt) + g_2(t), \quad \rho \in (0,1). \end{aligned} \tag{6.5} $$

For this problem, the exact solution is given as

$$X_1(t) = (\log t)^5 + 2(\log t)^3,$$

$$X_2(t) = -(\log t)^4 + 2(\log t)^3.$$

We employ the proposed method to solve this problem with various N and ρ values. In Table 5, we list the errors for different values of N and ρ. The numerical results show the convergence of the scheme, which confirms our error analysis.

Table 5. The errors with the fractional orders $\rho = 0.2, 0.4, 0.6, 0.8$, and $q = 3/4$ for Example 6.2.

		The errors for X_1		
N	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$
5	6.8×10^{-8}	4.6×10^{-7}	1.3×10^{-6}	2.1×10^{-6}
10	6.7×10^{-10}	6.0×10^{-9}	2.4×10^{-8}	5.0×10^{-8}
15	3.9×10^{-11}	4.2×10^{-10}	1.9×10^{-9}	4.9×10^{-9}
20	5.0×10^{-12}	6.0×10^{-11}	3.1×10^{-10}	8.8×10^{-10}
25	9.9×10^{-13}	1.3×10^{-11}	7.5×10^{-11}	2.3×10^{-10}
30	2.6×10^{-13}	7.2×10^{-13}	2.3×10^{-11}	7.6×10^{-11}
		The errors for X_2		
N	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$
5	5.7×10^{-8}	4.3×10^{-7}	1.3×10^{-6}	2.1×10^{-6}
10	5.3×10^{-10}	5.3×10^{-9}	2.2×10^{-8}	4.9×10^{-8}
15	3.0×10^{-11}	3.6×10^{-10}	1.8×10^{-9}	4.7×10^{-9}
20	3.8×10^{-12}	5.2×10^{-11}	2.9×10^{-10}	8.4×10^{-10}
25	7.6×10^{-13}	1.1×10^{-11}	7.0×10^{-11}	2.2×10^{-10}
30	2.0×10^{-13}	3.21×10^{-12}	2.1×10^{-11}	7.2×10^{-11}

7. Conclusions

We provided a collocation spectral scheme for nonlinear systems of fractional pantograph delay differential equations. We constructed a mapped Jacobi spectral collocation scheme, described its effective implementation, and derived its convergence analysis. In addition, we provided a numerical example to support our theoretical analysis. The numerical results demonstrate the accuracy and effectiveness of the proposed scheme. We also conclude that the described technique produces very accurate results, even when employing a small number of base functions. Preserving some important mathematical properties and physical structures, such as existence, positivity preservation, the maximum principle, long-time behavior, and singular solutions, may be considered in future work [25, 26].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors assert that they do not have any known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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