Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 15190–15201.
DOI:10.3934/math.2024737
Received: 18 February 2024
Revised: 03 April 2024
Accepted: 08 April 2024
Published: 26 April 2024

Research article

On oscillating radial solutions for non-autonomous semilinear elliptic equations

H. Al Jebawy[1,*], **H. Ibrahim**[2] **and Z. Salloum**[2]

[1] American University in Dubai (AUD), Dubai, UAE
[2] Faculty of Science I, Mathematics Department, Lebanese University, Hadath, Lebanon

* **Correspondence:** Email: haljebawy@aud.edu.

Abstract: We consider semilinear elliptic equations of the form $\Delta u + f(|x|, u) = 0$ on $\mathbb{R}^N$ with $f(|x|, u) = q(|x|)g(u)$. These type of equations arise in various problems in applied mathematics, and particularly in the study of population dynamics, solitary waves, diffusion processes, and phase transitions. We show that under suitable assumptions on the nonlinearity f, there exists an oscillating radial solution converging to a zero of the function g. We also study the oscillating and limiting behavior of this solution.

Keywords: semilnear elliptic equation; oscillation theory; positive radial solution
Mathematics Subject Classification: 34C10, 35J61

1. Introduction

The existence and behavior of positive radial solutions of the semilinear elliptic equation

$$\Delta u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{1.1}$$

has been studied by many authors [5–7, 9–12]. The unknown u being radial and smooth, the study of existence shifts to the ordinary differential equation

$$\begin{cases} u'' + \dfrac{N-1}{r}u' + f(u) = 0 \text{ on } \mathbb{R}_+, \\ u(0) = \alpha > 0 \quad \text{and} \quad u'(0) = 0, \end{cases} \tag{1.2}$$

where $f : \mathbb{R}_+ \to \mathbb{R}$ is a locally Lipschitz function satisfying, among other conditions,

$$f(\xi) = 0 \text{ for some } \xi > 0.$$

It was proved (see [11]) that there exists a positive oscillating solution of (1.2) satisfying $\lim_{r\to\infty} u(r) = \xi$. The proof is based on ODE methods and makes an important use of the following identity, which is derived by multiplying the first equation of (1.2) by u' and then integrating by parts:

$$\frac{(u'(b))^2}{2} - \frac{(u'(a))^2}{2} + \int_a^b \frac{N-1}{r}(u'(r))^2 dr + F(u(b)) - F(u(a)) = 0, \tag{1.3}$$

where $F(t) = \int_0^t f(s)ds$ and $0 \le a \le b$. The main advantage, remarkably and frequently taken in [11], of (1.3) is a simple observation that

$$F(u(b)) \le F(u(a)) \quad \text{for} \quad 0 \le a \le b \text{ and } u'(a) = 0. \tag{1.4}$$

To our knowledge, this result of the existence of oscillating, radial, and convergent solutions of (1.1) has not been generalized to non-autonomous equations of the form

$$\Delta u + f(|x|, u) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.5}$$

that appear in various problems in applied mathematics related to, for example, solitary waves for Klein-Gordon equations and the reaction-diffusion equations. Such a generalization is then worth investigating. Let us mention that the existence of radial solutions for semilinear elliptic equations that converges at infinity has attracted the attention of different authors (see for instance [1–3, 5–7, 10, 12]). Smooth radial solutions of (1.5) satisfy the following identity, analogous to (1.3),

$$\frac{(u'(b))^2}{2} - \frac{(u'(a))^2}{2} + \int_a^b \frac{N-1}{r}(u'(r))^2 dr - \int_a^b F_r(r, u(r))dr + F(b, u(b)) - F(a, u(a)) = 0, \tag{1.6}$$

where

$$F(r, t) = \int_0^t f(r, s)ds.$$

The difficulties here are in fact twofold: the determination of the exact limit $\xi = \lim_{r\to\infty} u(r)$ strongly depends on the behavior of $f(r, t)$ when $r \to \infty$, so we may directly get into a limiting problem of u due to wild limiting behavior of f. The second difficulty is to obtain a practical inequality (useful in various technical situations) like (1.4) due to the presence of the term $\int_a^b F_r(r, u(r))dr$ in (1.6). Indeed, maintaining a negative sign of this term is mainly subjected to the radial variation of f, and to the location of the unknown function u. Since our aim is to understand how to generalize the existence result of [11], we see that the consideration of all these conditions for the general nonlinearity f is not our best starting point. For this reason, we hereby consider functions $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ of the form

$$f(r, t) = q(r)g(t), \tag{1.7}$$

where $q : \mathbb{R}_+ \longrightarrow]0, \infty[$ is a positive, increasing C^1 function with $\lim_{r\to\infty} q(r) = q_\infty < \infty$, and $g : \mathbb{R}_+ \to \mathbb{R}$ is a locally Lipschitz function satisfying the following conditions:

$$g < 0 \text{ in } (0, \xi) \text{ with } g(0) = g(\xi) = 0 \text{ for some } \xi > 0, \tag{1.8}$$

$$\exists\, \eta > \xi \text{ such that } \int_0^{\eta} g(t)dt = 0 \text{ and } g > 0 \text{ in } (\xi, \eta), \tag{1.9}$$

$$g'(\xi) > 0. \tag{1.10}$$

As an immediate consequence, we deduce that $f(r,t)$ is decreasing in r for $0 < t < \xi$, increasing in r for $\xi < t < \eta$, and $\lim_{r\to\infty} f(r,t) = f_\infty(t) = q_\infty g(t)$. Moreover, for every $r \geq 0$, we have

$$\int_0^{\eta} f(r,t)dt = \int_0^{\eta} q(r)g(t)dt = q(r)\int_0^{\eta} g(t)dt = 0.$$

2. Main result

Since we are interested in radial solutions of (1.5) with f given by (1.7), we consider the following initial value problem on $[0,\infty[$:

$$\begin{cases} u'' + \dfrac{N-1}{r}u' + q(r)g(u) = 0, \\ u(0) = \alpha > 0 \quad \text{and} \quad u'(0) = 0, \end{cases} \tag{2.1}$$

where g satisfies (1.8)–(1.10). Then, for every $\alpha \in (0,\eta)$ with $\alpha \neq \xi$, (2.1) admits a solution u that remains positive for all $r > 0$ (see for instance [8]). Furthermore, we prove the following result:

Theorem 2.1. *If f satisfies (1.7)–(1.10), then for every $\alpha \in (0,\eta)$ with $\alpha \neq \xi$, the solution u of (2.1) oscillates (has infinitely many local maxima and local minima) with $\lim_{r\to\infty} u(r) = \xi$ in such a way that the local maxima of u are strictly decreasing to ξ at ∞ and the local minima are strictly increasing to ξ at ∞, and the distance between two consecutive zeros of $u - \xi$ tends to $\frac{\pi}{\sqrt{q_\infty g'(\xi)}}$.*

We adopt the shooting method used in [4], which consists of varying α in $(0,\eta)$ to obtain a radial oscillating solution of (2.1). The main ingredient of our proof is the energy Eq (1.6) that now reads

$$\frac{(u'(b))^2}{2} - \frac{(u'(a))^2}{2} + \int_a^b \frac{N-1}{r}(u'(r))^2 dr$$
$$-\int_a^b q'(r)\Big(\int_0^{u(r)} g(s)ds\Big)dr + q(b)\int_0^{u(b)} g(s)ds - q(a)\int_0^{u(a)} g(s)ds = 0. \tag{2.2}$$

Also, multiplying (2.1) by u' and integrating between $0 \leq a \leq b$ with $u'(a) = u'(b) = 0$ gives

$$\int_a^b q(r)g(u(r))u'(r)dr \leq 0. \tag{2.3}$$

This inequality plays a crucial role in regards to the monotonicity of the local extrema of u. Finally, a direct integration of (2.1) between $0 \leq a \leq b$ leads to

$$u'(b) - u'(a) + \int_a^b \frac{N-1}{r}u'(r)dr + \int_a^b q(r)g(u(r))dr = 0. \tag{2.4}$$

Finally, if $v(r) = r^{\frac{N-1}{2}}(u(r) - \xi)$, then

$$v'' + \left\{ \frac{q(r)g(u(r))}{u(r) - \xi} - \frac{(N-1)(N-3)}{4r^2} \right\} v = 0, \tag{2.5}$$

where we use the convention that $\frac{g(u)}{u-\xi} = g'(\xi)$ when $u = \xi$.

Proof of Theorem 2.1. We only consider the case $\alpha \in]\xi, \eta[$. The case $\alpha \in]0, \xi[$ is treated similarly. The proof is divided into several steps.

Step 1. ($0 < u(r) < \eta$ for all $r \geq 0$)

Let us show that if $0 < u(0) = \alpha < \eta$, then $0 < u(r) < \eta$ for all $r \geq 0$. This inequality satisfied by u ensures a negative sign for the term $\int_0^{u(r)} g(s)ds$ appearing in (2.2), and thus leads to useful results later on. Let

$$R = \inf\{r > 0 : u(r) = 0 \text{ or } u(r) = \eta\},$$

and assume that $R < \infty$. Since $u(0) = \alpha$ with $\alpha \neq 0$ and $\alpha \neq \eta$, then there exists $\delta > 0$ such that $u(r) \neq 0$ and $u(r) \neq \eta$ for all $0 < r < \delta$. Hence, $R > \delta > 0$. Again, using the continuity of u, we get that

$$u(R) = 0 \quad \text{or} \quad u(R) = \eta.$$

The important point is that $0 < u(r) < \eta$ for $0 \leq r < R$, and so by using (2.2) with $a = 0$ and $b = R$, and owing to the fact that $\int_0^{u(r)} g(s)ds \leq 0$ for $0 \leq r < R$, $q' \geq 0$, $u'(0) = 0$, and $\int_0^{u(R)} g(s)ds = 0$, we obtain

$$q(0) \int_0^{\alpha} g(s)ds \geq 0.$$

But, $q(0) > 0$ and $\int_0^{\alpha} g(s)ds < 0$, and hence there is a contradiction. This proves $R = \infty$.

Step 2. ($\liminf_{r\to\infty} u(r) > 0$ and $\limsup_{r\to\infty} u(r) < \eta$)

Since $u > 0$, then $\liminf_{r\to\infty} u(r) \geq 0$. Assume that $\liminf_{r\to\infty} u(r) = 0$, then there exists a sequence (r_n) of positive numbers such that

$$\lim_{n\to\infty} r_n = \infty \quad \text{and} \quad \lim_{n\to\infty} u(r_n) = 0. \tag{2.6}$$

Applying (2.2) for $a = 0$ and $b = r_n$, we get

$$\frac{(u'(r_n))^2}{2} + \int_0^{r_n} \frac{N-1}{r}(u'(r))^2 dr - \int_0^{r_n} q'(r)(\int_0^{u(r)} g(s)ds)dr + q(r_n)\int_0^{u(r_n)} g(s)ds - q(0)\int_0^{\alpha} g(s)ds = 0.$$

The first three terms of this equation are nonnegative, so

$$q(r_n)\int_0^{u(r_n)} g(s)ds \leq q(0)\int_0^{\alpha} g(s)ds, \tag{2.7}$$

and using (2.6), we get that

$$\lim_{n\to\infty} q(r_n) = q_\infty > 0 \quad \text{and} \quad \lim_{n\to\infty} \int_0^{u(r_n)} g(s)ds = 0,$$

therefore, by taking the limit $n \to \infty$ in (2.7), we finally obtain

$$q(0)\int_0^{\alpha} g(s)ds \geq 0.$$

This is a contradiction since $q(0) > 0$ and $\int_0^{\alpha} g(s)ds < 0$. The proof that $\limsup_{r\to\infty} u(r) < \eta$ is done in a similar manner.

Step 3. (u is an oscillating function)

Let us show that u oscillates on $[0, \infty[$. First, note that

$$u''(0) = -q(0)g(\alpha) < 0,$$

and then, by the regularity of u, there exists $\delta > 0$ such that u decreases on $]0, \delta[$. Let

$$r_1 = \sup\{\delta > 0 : u \text{ is decreasing on }]0, \delta[\},$$

then $r_1 < \infty$. Suppose this is not true, i.e., $r_1 = \infty$, then u decreases to a limit $0 < \ell \leq \alpha$. We observe that $\ell > 0$ since $\liminf_{r\to\infty} u(r) > 0$ by step 2. This is an essential observation to ensure that $g(\ell) \neq 0$ if $\ell \neq \xi$. Two cases can be considered:

• Case $\ell \neq \xi$. Without loss of generality, we assume $\ell > \xi$. Applying the mean value theorem between $n \in \mathbb{N}$ and $n + 1$ we get $u(n + 1) - u(n) = u'(b_n)$, $n < b_n < n + 1$, and hence the existence of a sequence (b_n) such that

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} u'(b_n) = 0.$$

Applying inequality (2.4) between 1 and b_n, we get

$$u'(b_n) - u'(1) + \int_1^{b_n} \frac{N-1}{r} u'(r)dr + \int_1^{b_n} q(r)g(u(r))dr = 0. \tag{2.8}$$

Straightforward computations give

$$0 \geq \int_1^{b_n} \frac{N-1}{r} u'(r)dr \geq (N-1)(u(b_n) - u(1)) \geq (N-1)(\ell - u(1)),$$

and so, as $\lim_{n\to\infty} u'(b_n) = 0$, the first three terms of (2.8) are bounded. On the other hand, $g(u(r)) > 0$ since $\xi < u(r) \leq \alpha$ and then $q(r)g(u(r)) > 0$ with $\lim_{r\to\infty} q(r)g(u(r)) = q_\infty g(\ell) > 0$. Consequently,

$$\lim_{n\to\infty} \int_1^{b_n} q(r)g(u(r))dr = \infty,$$

which is in contradiction with (2.8). The case $\ell < \xi$ is treated similarly, possibly with the application of (2.4) with a large enough to ensure a negative sign of the term $q(r)g(u(r))$ that converges, as $r \to \infty$, to $q_\infty g(\ell) < 0$, leading to the same kind of contradiction as above.

• Case $\ell = \xi$. Since $\lim_{r\to\infty} u(r) = \xi$, then, using (1.10),

$$\lim_{r\to\infty} \frac{q(r)g(u(r))}{u(r) - \xi} = q_\infty g'(\xi) > 0.$$

Hence, for r large enough, say $r > R_0$, we have

$$\frac{q(r)g(u(r))}{u(r)-\xi} - \frac{(N-1)(N-3)}{4r^2} > \epsilon^2 > 0, \tag{2.9}$$

for some $\epsilon > 0$. Therefore, by the Sturm comparison principle applied to ODE (2.5), we deduce that v must vanish infinitely many times in $(R_0, +\infty)$, which leads to a contradiction.

Another approach to see this contradiction is as follows. Since v is a solution of (2.5), then we deduce from (2.9) that $v'' < 0$ for $r > R_0$, which implies that $v'(r) \searrow L \in [-\infty, +\infty[$ as $r \to \infty$. If $L < 0$, then $v(r) \to -\infty$ and this is impossible by the positivity of v. Otherwise, if $L \geq 0$, then $v' > 0$ and v increases on $[R_0, \infty[$, and thus $v(r) \geq v(R_0) > 0$ for $r \geq R_0$. Again, by (2.9) we get $v''(r) \leq -\epsilon^2 v(R_0) < 0$, and consequently $v'(r) \to -\infty$ as $r \to \infty$, and this is also impossible by the positivity of v'.

The oscillation. From all that precedes, we deduce that $r_1 < \infty$, $u'(r_1) = 0$ and u is increasing on $]r_1, r_1 + \delta_1[$ for some $\delta_1 > 0$. This, together with the equation $u''(r_1) = -q(r_1)g(u(r_1))$ and the fact that $q > 0$ and $g(r) > 0$ for $\xi < r \leq \alpha$, show that $u(r_1) \leq \xi$. However, if $u(r_1) = \xi$, then, by the uniqueness of the ODE, we get $u \equiv \xi$, which leads to a contradiction. Finally,

$$u(r_1) < \xi.$$

By essentially repeating the same arguments of this step, we are lead to the existence of $r_2 \in]r_1, \infty[$ such that u is increasing on $]r_1, r_2[$, $u'(r_2) = 0$ and u is decreasing on $]r_2, r_2 + \delta_2[$ for some $\delta_2 > 0$. Here, it is very important to remark that a part of the method of showing $r_2 \neq \infty$ will essentially depend on the fact that $\limsup_{r\to\infty} u(r) < \eta$, as proved in step 2.

Again, incidentally,

$$u(r_2) > \xi.$$

We redo the same analysis to conclude that u has infinitely many local maxima and local minima. More precisely, there exists a sequence $(r_n)_{n\geq 1}$ such that

$$r_1 < r_2 < \cdots < r_k < \cdots \to \infty,$$

$$u(r_{2k}), k \geq 1, \text{ are local maxima with } u(r_{2k}) > \xi,$$

and

$$u(r_{2k-1}), k \geq 1, \text{ are local minima with } u(r_{2k-1}) < \xi.$$

For the simplicity of notation we set

$$u_i := u(r_i) \quad \text{for} \quad i \in \mathbb{N}.$$

Step 4. ($\{u_{2k-1}\}_{k\geq 1}$ is increasing and $\{u_{2k}\}_{k\geq 1}$ is decreasing)

We only show that the sequence $\{u_{2k-1}\}_{k\geq 1}$ is increasing. To show that $\{u_{2k}\}_{k\geq 1}$ is decreasing we follow the exact same arguments. First note that, since

$$u_{2k-1} < \xi < u_{2k} \tag{2.10}$$

and u is increasing on $]r_{2k-1}, r_{2k}[$, then there exists a point $\bar{r} \in]r_{2k-1}, r_{2k}[$ such that

$$u(\bar{r}) = \xi, \quad u \leq \xi \text{ on }]r_{2k-1}, \bar{r}[\quad \text{and} \quad u \geq \xi \text{ on }]\bar{r}, r_{2k}[. \tag{2.11}$$

Owing to (2.10) and the regularity of u, we infer that $u' \neq 0$ on some nonempty open subinterval of $]r_{2k-1}, r_{2k}[$. Therefore, using (2.3) with $a = r_{2k-1}$ and $b = r_{2k}$, we get

$$\int_{r_{2k-1}}^{r_{2k}} q(r)g(u(r))u'(r)dr < 0,$$

and so

$$\int_{r_{2k-1}}^{\bar{r}} q(r)g(u(r))u'(r)dr + \int_{\bar{r}}^{r_{2k}} q(r)g(u(r))u'(r)dr < 0. \tag{2.12}$$

Now, using (1.8), (1.9), (2.11), the non-negativity of u', and the monotonicity of q in (2.12), we obtain

$$q(\bar{r}) \int_{r_{2k-1}}^{\bar{r}} g(u(r))u'(r)dr + q(\bar{r}) \int_{\bar{r}}^{r_{2k}} g(u(r))u'(r)dr < 0,$$

and thus

$$q(\bar{r}) \int_{r_{2k-1}}^{r_{2k}} g(u(r))u'(r)dr < 0.$$

But, $q > 0$, and therefore

$$\int_{u_{2k-1}}^{u_{2k}} g(s)ds < 0. \tag{2.13}$$

We reuse (2.3) with $a = r_{2k}$ and $b = r_{2k+1}$ to get

$$\int_{r_{2k}}^{r_{2k+1}} q(r)g(u(r))u'(r)dr < 0.$$

Following a similar approach, we also note that

$$u_{2k+1} < \xi < u_{2k}$$

leading to the existence of $\underline{r} \in]r_{2k}, r_{2k+1}[$ with $u(\underline{r}) = \xi$, and thanks here to the non-positivity of u', the monotonicity of q, and the sign of $g(u)$ on $]r_{2k}, r_{2k+1}[$,

$$q(\underline{r}) \int_{r_{2k}}^{\underline{r}} g(u(r))u'(r)dr + q(\underline{r}) \int_{\underline{r}}^{r_{2k+1}} g(u(r))u'(r)dr < 0.$$

Consequently,

$$\int_{u_{2k}}^{u_{2k+1}} g(s)ds < 0. \tag{2.14}$$

Combining (2.13) and (2.14), we deduce that

$$\int_{u_{2k-1}}^{u_{2k+1}} g(s)ds < 0.$$

Finally, as $u_{2k-1}, u_{2k+1} \in]0, \xi[$ and since $g < 0$ on $]0, \xi[$, the previous inequality asserts that

$$u_{2k-1} < u_{2k+1},$$

and therefore the sequence $\{u_{2k-1}\}_{k\geq 1}$ is increasing. Having $u_{2k-1} \leq \xi$ for all k, we also deduce that

$$\lim_{k\to\infty} u_{2k-1} = \gamma \leq \xi.$$

Similarly, $\{u_{2k}\}_{k\geq 1}$ is decreasing; $u_{2k} \geq \xi$ for all k, and therefore

$$\lim_{k\to\infty} u_{2k} = \beta \geq \xi.$$

A particular case. Assume that $\xi = \frac{\eta}{2}$ and

$$g(s) = s(s-\xi)(\eta - s),$$

then g is antisymmetric with respect to $s = \xi$. In such a situation we may show the monotonicity of $\{u_{2k-1}\}_{k\geq 1}$ and $\{u_{2k}\}_{k\geq 1}$ by a different approach. We only give an idea of the proof by showing

$$u_1 < u_3. \tag{2.15}$$

We first show that $u_1 < \eta - u_2$. Assume to the contrary that $u_1 \geq \eta - u_2$ (see Figure 1).

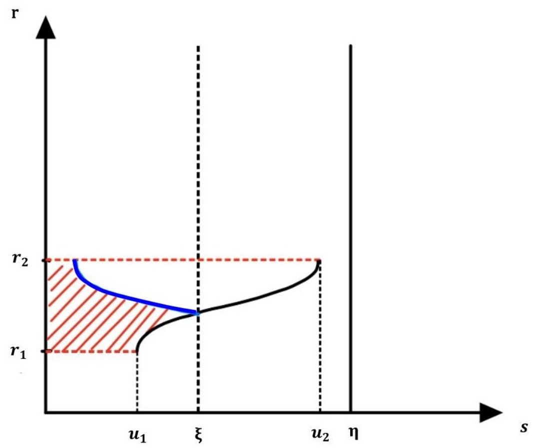

Figure 1. Case $u_1 \geq \eta - u_2$.

By applying (2.2) with $a = r_1$ and $b = r_2$, we get

$$q(r_2)\int_0^{u_2} g(s)ds - q(r_1)\int_0^{u_1} g(s)ds < \int_{r_1}^{r_2}\int_0^{u(r)} q'(r)g(s)dsdr,$$

and as g is antisymmetric with respect to $s = \xi$, then

$$\int_{r_1}^{r_2}\int_0^{u(r)} q'(r)g(s)dsdr = \iint_{\mathcal{R}} q'(r)g(s)dsdr,$$

and therefore

$$q(r_2)\int_0^{u_2} g(s)ds - q(r_1)\int_0^{u_1} g(s)ds < \iint_{\mathcal{R}} q'(r)g(s)dsdr, \tag{2.16}$$

where $\mathcal{R}$ is the shaded area in Figure 1. Notice that, since $u_1 \geq \eta - u_2$ and $q'(r)g(s) \leq 0$ on $\mathcal{R}$, then

$$\iint_{\mathcal{R}} q'(r)g(s)dsdr \leq \int_{r_1}^{r_2}\int_0^{\eta-u_2} q'(r)g(s)dsdr = q(r_2)\int_0^{\eta-u_2} g(s)ds - q(r_1)\int_0^{\eta-u_2} g(s)ds.$$

Using this inequality in (2.16), we finally get

$$q(r_2)\int_0^{u_2} g(s)ds - q(r_1)\int_0^{u_1} g(s)ds < q(r_2)\int_0^{\eta-u_2} g(s)ds - q(r_1)\int_0^{\eta-u_2} g(s)ds.$$

Again, the antisymmetry of g implies

$$\int_0^{\eta-u_2} g(s)ds = \int_0^{u_2} g(s)ds,$$

and then

$$q(r_1)\int_0^{u_1} g(s)ds > q(r_1)\int_0^{u_2} g(s)ds,$$

hence

$$\int_{u_1}^{u_2} g(s)ds < 0,$$

which is in contradiction with the fact that $u_1 \geq \eta - u_2$ and the definition of g. Consequently,

$$u_1 < \eta - u_2. \tag{2.17}$$

We now show that $u_1 < u_3$. Assume to the contrary that $u_1 \geq u_3$. Using this inequality together with (2.17), we draw Figure 2 below.

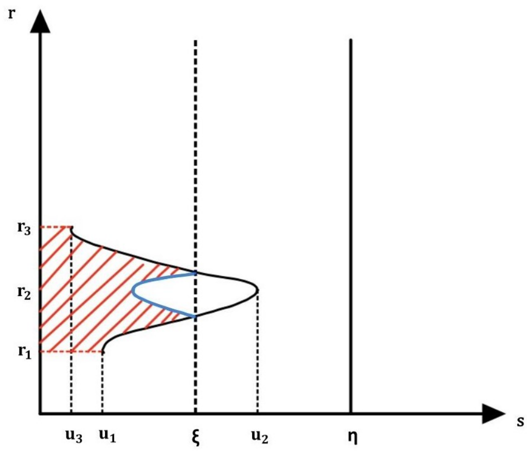

Figure 2. Case $u_1 \geq u_3$.

By applying (2.2) with $a = r_1$ and $b = r_3$, we get

$$q(r_3)\int_0^{u_3} g(s)ds - q(r_1)\int_0^{u_1} g(s)ds < \int_{r_1}^{r_3}\int_0^{u(r)} q'(r)g(s)dsdr.$$

Similar arguments as above yield

$$q(r_3)\int_0^{u_3} g(s)ds - q(r_1)\int_0^{u_1} g(s)ds < \iint_{\mathcal{R}} q'(r)g(s)dsdr,$$

where $\mathcal{R}$ is the shaded area in Figure 2, and then

$$q(r_3)\int_0^{u_3} g(s)ds - q(r_1)\int_0^{u_1} g(s)ds < \int_{r_1}^{r_3}\int_0^{u_3} q'(r)g(s)dsdr = q(r_3)\int_0^{u_3} g(s)ds - q(r_1)\int_0^{u_3} g(s)ds.$$

As a result, we obtain

$$\int_{u_3}^{u_1} g(s)ds > 0,$$

which leads to a contradiction, and inequality (2.15) is therefore valid.

Step 5. $(\gamma = \beta = \xi)$

We know that $\sup_{r\geq 0} u(r) = \alpha < \eta$ and $\inf_{r\geq 0} u(r) = u(r_1) > 0$. Since $g'(\xi) > 0$, and by the boundedness of q, we deduce that there exist $c_1, c_2 > 0$ such that for every $r \geq 0$ we have

$$c_1 < \frac{q(r)g(u(r))}{u(r)-\xi} < c_2.$$

Therefore, for r large enough (say $r \geq R_1$), we deduce that there exist $\epsilon_1, \epsilon_2 > 0$ such that

$$\epsilon_1^2 < \frac{q(r)g(u(r))}{u(r)-\xi} - \frac{(N-1)(N-3)}{4r^2} < \epsilon_2^2.$$

Recall that $v(r) = r^{\frac{N-1}{2}}(u(r)-\xi)$ solves (2.5). Then, using the Sturm comparison theorem, we deduce that for $r \geq R_1$ we have

$$\frac{\pi}{\epsilon_2} < \text{ distance between two consecutive zeros of } u(r)-\xi < \frac{\pi}{\epsilon_1}.$$

Consequently, there exists $c > 0$ such that

$$\sup_{k\geq 0}(r_k - r_{k-1}) \leq c.$$

Then, applying Schwarz's inequality, we get

$$\beta - \gamma < |u(r_k) - u(r_{k-1})| < c^{1/2}\left(\int_{r_{k-1}}^{r_k} |u'(r)|^2 dr\right)^{1/2}.$$

Therefore, for k large enough (say $k \geq k_0$), we have

$$\int_{r_{k-1}}^{r_k} \frac{(u'(r))^2}{r}dr \geq \frac{1}{r_k}\frac{(\beta-\gamma)^2}{c} \geq \frac{c'}{r_{k-1}}\frac{(\beta-\gamma)^2}{c} \geq \frac{c'(\beta-\gamma)^2}{c^2}\int_{r_{k-1}}^{r_k}\frac{dr}{r},$$

where c' is a positive constant that depends only on k_0. Summing over all $k \geq k_0$, we get

$$\int_{r_{k_0}}^{\infty} \frac{(u'(r))^2}{r}dr \geq \frac{c'(\beta-\gamma)^2}{c^2}\int_{r_{k_0}}^{\infty}\frac{dr}{r}. \tag{2.18}$$

Moreover, by taking $a = 0$ and $b = r_k \to \infty$ in (2.2), we note that

$$\int_{r_{k_0}}^{\infty} \frac{(u'(r))^2}{r} dr < \infty.$$

Therefore, we deduce from (2.18) that $\beta = \gamma$. Moreover, since $\gamma < \xi < \beta$, we finally get $\beta = \gamma = \xi$.
Step 6. (conclusion)

Finally, we claim that the distance between two consecutive zeros of $u(r) - \xi$ tends to $\frac{\pi}{\sqrt{q_\infty g'(\xi)}}$ as $r \to \infty$. In fact, since $u(r) \to \xi$ as $r \to \infty$, then

$$h(r) = \frac{q(r)g(r)}{u(r) - \xi} - \frac{(N-1)(N-3)}{4r^2} \underset{r\to\infty}{\longrightarrow} q_\infty g'(\xi).$$

Therefore, for $\epsilon > 0$, one can find R large enough such that for every $r \geq R$ we have

$$q_\infty g'(\xi) - \epsilon < h(r) < q_\infty g'(\xi) + \epsilon.$$

Therefore, applying the Sturm comparison theorem again on (2.5), we deduce that

$$\frac{\pi}{\sqrt{q_\infty g'(\xi) + \epsilon}} < \text{distance between two consecutive zeros of } u(r) - \xi < \frac{\pi}{\sqrt{q_\infty g'(\xi) - \epsilon}}.$$

Taking the limit as $\epsilon \to 0$ we get the desired result. □

3. Conclusions

To summarize, we were finally able to generalize the existence of an oscillating radial solution that converges to a root of f in the non-autonomous case despite the difficulties that rise from the presence of the terms related to $q(r)$ in the energy Eq (2.2). Furthermore, inequality (2.3) allows us to prove the monotonicity of the local extrema. The question that arises now is whether we can generalize these results for f having a singularity at 0; more precisely, for $f(r, u) = q(r)g(u)$ with $g(u) = u^{-\alpha}$ for some $\alpha < 1$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

References

1. H. Berestycki, P. Lions, Existence d'ondes solitaires dans des problèmes non-linéaires du type Klein-Gordon, *C. R. Acad. Sci. A*, **288** (1979), 395–398.

2. H. Berestycki, P. Lions, Existence of a ground state in nonlinear equations of the Klein-Gordon type, *Variational inequalities and complementarity problems (Proc. Internat. School, Erice, 1978)*, 1980, 35–51.

3. M. Berger, On the existence and structure of stationary states for a nonlinear Klein-Gordon equation, *J. Funct. Anal.*, **9** (1972), 249–261. http://dx.doi.org/10.1016/0022-1236(72)90001-8

4. A. Boscaggin, F. Colasuonno, B. Noris, Multiple positive solutions for a class of p-Laplacian Neumann problems without growth conditions, *ESAIM: COCV*, **24** (2018), 1625–1644. http://dx.doi.org/10.1051/cocv/2017074

5. C. Coffman, Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions, *Arch. Rational Mech. Anal.*, **46** (1972), 81–95. http://dx.doi.org/10.1007/BF00250684

6. H. Ibrahim, Radial solutions of semilinear elliptic equations with prescribed asymptotic behavior, *Math. Nachr.*, **293** (2020), 1481–1489. http://dx.doi.org/10.1002/mana.201900003

7. H. Ibrahim, H. Al Jebawy, E. Nasreddine, Radial and asymptotically constant solutions for nonautonomous elliptic equations, *Appl. Anal.*, **100** (2021), 3132–3144. http://dx.doi.org/10.1080/00036811.2020.1712368

8. H. Ibrahim, E. Nasreddine, On the existence of nonautonomous ODE with application to semilinear elliptic equations, *Mediterr. J. Math.*, **15** (2018), 64. http://dx.doi.org/10.1007/s00009-018-1112-1

9. M. Khuddush, K. Prasad, B. Bharathi, Denumerably many positive radial solutions to iterative system of nonlinear elliptic equations on the exterior of a ball, *Nonlinear Dynamics and Systems Theory*, **23** (2023), 95–106.

10. Z. Nehari, On a nonlinear differential equation arising in nuclear physics, *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, **62** (1963), 117–135.

11. W. Ni, On the positive radial solutions of some semilinear elliptic equations on $\mathbb{R}^n$, *Appl. Math. Optim.*, **9** (1982), 373–380. http://dx.doi.org/10.1007/BF01460131

12. G. Ryder, Boundary value problems for a class of nonlinear differential equations, *Pac. J. Math.*, **22** (1967), 477–503. http://dx.doi.org/10.2140/pjm.1967.22.477