Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 15172–15189.
DOI: 10.3934/math.2024736
Received: 20 February 2024
Revised: 17 April 2024
Accepted: 22 April 2024
Published: 26 April 2024

Research article

On fixed point and an application of C^*-algebra valued (α, β)-Bianchini-Grandolfi gauge contractions

Moirangthem Pradeep Singh[1], **Yumnam Rohen**[1,2], **Khairul Habib Alam**[1,*], **Junaid Ahmad**[3] and **Walid Emam**[4]

[1] Department of Mathematics, National Institute of Technology Manipur, Imphal 795004, India
[2] Department of Mathematics, Manipur University, Imphal 795003, India
[3] Department of Mathematics and Statistics, International Islamic University, Islamabad 44000, Pakistan
[4] Department of Statistics and Operations Research, Faculty of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: almkrlhb@gmail.com; Tel: +919563451673.

Abstract: It is the purpose of the present paper to obtain certain fixed point outcomes in the sense of C^*-algebra valued metric spaces. Here, we present the definitions of the gauge function, the Bianchini-Grandolfi gauge function, α-admissibility, and (α, β)-admissible Geraghty contractive mapping in the sense of C^*-algebra. Using these definitions, we define (α, β)-Bianchini-Grandolfi gauge contraction of type I and type II. Next, we prove our primary results that the function satisfying our contraction condition has to have a unique fixed point. We also explain our results using examples. Additionally, we discuss some consequent results that can be easily obtained from our primary outcomes. Finally, there is a useful application to integral calculus.

Keywords: Bianchini-Grandolfi gauge function; gauge function; C^*-algebra valued; (α, β)-Bianchini-Grandolfi gauge contraction
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

Several areas of mathematics as well as other fields of study use fixed point theory as the primary motivation for both new findings and the generalization of fixed point theorems that have already been established. In recent decades, the theory of fixed points has been dynamically developing. The Banach theorem is the first significant finding in the theory of fixed points regarding contractive mapping (the principle of contractive mapping). The aforementioned theorem was introduced in

Banach's dissertation [1] in 1922. This particular theorem is a crucial tool for research in many different areas of mathematics. Authors generalized the Banach theorem in different ways (see, [2–7]). One of the generalizations includes proving the theorem in generalized metric spaces (see, [8–11]). Again, to generalize metric spaces, one of the best ways used by authors is to change the co-domain of the metric function. Brzdek et al. [12] generalized the Banach theorem across a variety of spaces (e.g., in quasi-Banach spaces, p-Banach spaces, ultrametric spaces, and dq-metric spaces). Recently, over Banach algebras, Bakr and Hussein [13] studied vector quasi-cone metric spaces.

Ma et al. [8] developed the concept of the C^*-algebra valued metric space (C^*-$avMS$) by substituting the set of real numbers with the collection of all positive elements in a unitary C^*-algebra. In a C^*-algebra, by restricting the codomain of the metric to positive elements, you ensure that the metric reflects this positivity and maintains compatibility with the algebraic operations on positive elements. By focusing on positive elements in the codomain of the metric, you align the metric more closely with the operator norm and its properties. Again, C^*-algebras have a natural order structure defined by positive elements. By considering the set of positive elements in the codomain of the metric, you emphasize this ordered structure and make it more prominent in the metric space. This can be beneficial for studying properties related to order, such as lattice properties or the structure of positive cones. Positive elements have well-defined square roots, which can be useful in various contexts, such as functional calculus or spectral theory. Restricting the codomain of the metric to positive elements can lead to more straightforward calculations and proofs in many cases.

Non-commutative spaces and quantum spaces are commonly exemplified by C^*-algebras. They are crucial to the non-commutative geometry. Since bounded linear operators on Hilbert spaces and matrices are common quantum space problems, the theory should apply to many of these issues. As a result, an alternative to regular metric spaces, C^*-algebras offer non-commutative spaces. The standard metric's symmetry characteristic can be eliminated to create one of the first generalizations of metric spaces. Semi-metricity is another idea that was introduced early on. The self-distance and symmetry properties which are distinctive to the Euclidean metric are satisfied by this kind of metric. The symmetry property is satisfied by the metric in our instance of C^*-$avMS$.

Bianchini and Grandolfi [14] in 1968 defined the Bianchini-Grandolfi gauge function (BGGF) on $[0, +\infty)$, which is nowadays called as the function of the rate of convergence, named by Ptak [15]. In 2006, using the idea of Bianchini-Grandolfi, Proinov [16] introduced a function called the gauge function (GF) with an immediate lemma and proved the higher order of convergence of subsequent approximations as a generalization of the Banach contraction mapping principle.

Samet et al. [3] first proposed α-admissible and α-contractive mappings in 2012, and also produced several numbers of fixed point problems for such contractions, and these notions are now widely accepted. Triangular α-admissible mapping was a concept that Karapinar et al. [4] presented later in 2013, expanding the definition of α-admissible mappings. Geraghty [2] created a fascinating contraction in 1973. In light of this, he looked at a few contractions via auxiliary functions that force the presence and uniqueness of fixed points in arbitrary complete metric spaces. The concept of α-Geragthy contractive mappings, which generalizes the concept of α-admissible mappings, was first described by Cho et al. in [17]. For (α,β)-admissible Geraghty contraction mappings, Chandok [18] demonstrated some intriguing fixed point results in 2015. Later on, authors generalized the theorem in this direction by introducing different auxiliary functions and different rational contractions (see, [5, 11, 19–22]).

Motivated by the preceding discussion, we redefine the gauge function, the BGGF, α-admissibility, Geraghty contractive mapping, and (α,β)-admissible mapping in the C^*-algebra sense. Utilizing these definitions, we present two types of (α,β)-Bianchini-Grandolfi gauge contractions. We then demonstrate our primary findings, which require functions meeting our contraction criterion to have a single fixed point. We also provide some illustrative examples to clarify our outcomes. Finally, using our primary finding we investigate if a solution to the integral equation exists and is unique. The findings in this paper expand a few C^*-*avMS* fixed-point theorems of [2–4, 8, 14, 16].

2. Preliminaries

Throughout the paper, we use C to represent a unitary C^*-algebra [23, 24]. A Banach $*$-algebra with $\|c^*c\| = \|c\|^2$, $\forall c \in C$, is known as a C^*-algebra, where $*$ is involution with $(c_1c_2)^* = c_2^*c_1^*$ and $c_1^{**} = c_1$, $\forall c_1, c_2 \in C$. With respect to the zero element 0_C in C an element $c \in C$ is positive if $0_C \preceq c$. Let

$$C_+ = \{c \in C : 0_C \preceq c\}.$$

Suppose

$$c_1 \preceq c_2 \Leftrightarrow 0_C \preceq c_2 - c_1$$

is the partial ordering defined in it.

Now we know the definition of C^*-*avMS* on which the whole paper relies.

Definition 2.1. *[8] Let $V \neq \emptyset$. A mapping d: $V \times V \longrightarrow C$ is known to be a C^*-algebra valued metric (C^*-avM) on V if it satisfies*

(i) $d(v_1, v_2) \succeq 0_C$ *and* $d(v_1, v_2) = 0_C$ *iff* $v_1 = v_2$;

(ii) $d(v_1, v_2) = d(v_2, v_1)$;

(iii) $d(v_1, v_3) \preceq d(v_1, v_2) + d(v_2, v_3)$;

for all $v_1, v_2, v_3 \in V$. The triplet (V, C, d) is known as a C^*-avMS.

The following are examples of C^*-*avMS*, one of which is useful to conclude the presence of a solution of an integral equation in this context.

Example 2.1. *Let $V = \mathbb{R}, C = M_2(\mathbb{R})$. Then, with respect to the metric*

$$d(v_1, v_2) = \begin{bmatrix} |v_1 - v_2| & 0 \\ 0 & \alpha|v_1 - v_2| \end{bmatrix}, \quad \forall v_1, v_2 \in \mathbb{R}, \alpha \geq 0,$$

(V, C, d) is a complete C^*-avMS by the completeness of $\mathbb{R}$.

Example 2.2. *Let M be a Lebesgue measurable set, $V = L^\infty(M)$ and $C = B(L^2(M))$, where $B(L^2(M))$ is the family of all bounded linear operators on $L^2(M)$ and $L^2(M)$ represents the Hilbert space of square-integrable functions over M. Then, with the usual operator norm, C is a C^*-algebra. Let d: $V \times V \longrightarrow C$ be given by*

$$d(\zeta_1, \zeta_2) = \pi_{|\zeta_1 - \zeta_2|}, \quad \forall \zeta_1, \zeta_2 \in V,$$

where π_ϕ: $L^2(M) \longrightarrow L^2(M)$ is the multiplicative self mapping defined by

$$\pi_\phi(\zeta) = \phi \cdot \zeta, \quad \forall \zeta \in L^2(M),$$

and $|\zeta|$ for a $\zeta \in L^2(M)$ represents the absolute value of the function ζ. Then d is a C^*-avM and C^*-avMS (V, C, d) is complete.

The following are some basic definitions and examples required to understand the theory in a better way.

Definition 2.2. *[14] A non-decreasing function function ϑ: $[0, +\infty) \longrightarrow [0, +\infty)$ is called BGGF if $\forall c \in [0, +\infty)$,*

$$\sigma(c) = \sum_{i=0}^{\infty} \vartheta^i(c) \text{ is finite},$$

where $\vartheta^0(c) = c$.

Definition 2.3. *[16] A GF of order $i \geq 1$ is a function ϑ: $[0, +\infty) \longrightarrow [0, +\infty)$ satisfying*

(i) $\vartheta(\lambda c) \leq \lambda^i \vartheta(c)$, $\forall\ c \in [0, +\infty)$ and $\lambda \in (0, 1)$;

(ii) $\vartheta(c) < c$, $\forall c \in [0, +\infty) \setminus \{0\}$.

Example 2.3. *Every convex function ϑ on $[0, +\infty)$ such that $\vartheta(0) = 0$ and $\vartheta(c) < c, \forall c \in (0, \infty)$, is a GF on $[0, +\infty)$ of order 1.*

Lemma 2.1. *[16] Every GF of order $i \geq 1$ on $[0, +\infty)$ is a BGGF on $[0, +\infty)$.*

Definition 2.4. *[2] A self mapping H: $V \longrightarrow V$ is known as a Geraghty contraction in an arbitrary metric space (V, d) if, for some mapping β: $(0, +\infty) \longrightarrow [0, 1)$,*

$$\lim_{n \to +\infty} \beta(r_n) = 1 \Rightarrow \lim_{n \to +\infty} r_n = 0$$

satisfies

$$d(Hv_1, Hv_2) \leq \beta(d(v_1, v_2)) d(v_1, v_2), \quad v_1, v_2 \in V.$$

Definition 2.5. *[3] In an arbitrary metric space (V, d), let H be a self mapping and α: $V \times V \longrightarrow [0, +\infty)$. Then, H is known to be α-admissible if*

$$\alpha(Hv_1, Hv_2) \geq 1, \text{ whenever } \alpha(v_1, v_2) \geq 1, \quad \forall v_1, v_2 \in V.$$

Definition 2.6. *[17] In an arbitrary metric space (V, d), let H be a self mapping. Then, there are mappings α: $V \times V \longrightarrow [0, +\infty)$, $\beta \in \mathcal{B}$ satisfying*

$$\alpha(v_1, v_2) d(Hv_1, Hv_2) \leq \beta(M(v_1, v_2)) M(v_1, v_2), \quad v_1, v_2 \in V,$$

where

$$M(v_1, v_2) = \max\{d(v_1, v_2), d(v_1, Hv_1), d(v_2, Hv_2)\}.$$

Then, H is known to be a α-Geraghty generalized contraction.

3. Auxiliary results

Using the theory of C^*-algebra, we redefine some of the above definitions by replacing the co-domain with a C^*-algebra. Some results are constructed using these definitions, and examples are given. In a C^*-algebra C, let us define C_c, C_I, and C_{I_C} by the subsets

$$\{a \in C : ab = ba, \forall\ b \in C\}, \quad \{a \in C : ab = ba, \forall\ b \in C \text{ and } a \preceq I_C\}$$

and

$$\{a \in C : ab = ba, \forall\ b \in C \text{ and } a \succeq I_C\},$$

respectively, where I_C is the unit element.

Definition 3.1. *A function $\vartheta\colon C_c \longrightarrow C_c$ is called a C^*-algebra valued GF of order $i \geq 1$ if*

(i) $\vartheta(\lambda c) \prec \lambda^i \vartheta(c),\ \forall\ c \in C_c$ *and* $\lambda \in C_I$*;*

(ii) $\vartheta(c) \prec c,\ \forall c \in C_c \setminus \{\theta\}$.

The following is an immediate lemma.

Lemma 3.1. *From Definition 3.1, we can conclude that*

(i) $\vartheta(0_C) = 0_C$*;*

(ii) $\frac{\vartheta(c)}{c^i}$ *is non-decreasing on* $C_c \setminus \{0_C\}$.

Definition 3.2. *A non-decreasing C^*-algebra valued GF $\vartheta\colon C_c \longrightarrow C_c$ is called a C^*-algebra valued BGGF if, $\forall c \in C_c$,*

$$\sigma(c) = \sum_{i=0}^{\infty} \vartheta^i(c) \text{ is finite},$$

where $\vartheta^0(c) = c$.

Now we present two examples of GFs.

Example 3.1. *Let $C = M_2(\mathbb{R})$ then C_c is the collection of all diagonal matrices in $M_2(\mathbb{R})$. Suppose ϑ: $C_c \longrightarrow C_c$ is given by*

$$\vartheta\left(\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} \lambda c & 0 \\ 0 & \lambda c \end{bmatrix}\ (0 < \lambda < 1), \quad \forall c \in \mathbb{R}.$$

Then, ϑ is a C^*-algebra valued GF on C_c.

Example 3.2. *Letting*

$$C = M_2\left(\left[0, \left(\frac{1}{k}\right)^{\frac{1}{r-1}}\right)\right)(k > 0, r > 1),$$

then C_c is the collection of all diagonal matrices in $M_2\left(\left[0, \left(\frac{1}{k}\right)^{\frac{1}{r-1}}\right)\right)$. Suppose $\vartheta\colon C_c \longrightarrow C_c$ is given by

$$\vartheta\left(\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} kc^r & 0 \\ 0 & kc^r \end{bmatrix}, \quad \forall c \in \left[0, \left(\frac{1}{k}\right)^{\frac{1}{r-1}}\right).$$

Then, ϑ is a C^*-algebra valued GF on C_c.

The following are analogous lemmas we can derive from the theory discussed in the introduction section.

Lemma 3.2. *Every C^*-algebra valued GF of order $i \geq 1$ on C_c is a C^*-algebra valued BGGF on C_c.*

Proof. The proof can be done in a similar way to Proinov [16]. □

Lemma 3.3. *From Definition 3.2, we can conclude that*

$$\sigma(c) = \sigma(\vartheta(c)) + c.$$

Proof. Since

$$\sigma(c) = \sum_{i=0}^{\infty} \vartheta^i(c) = \vartheta^0(c) + \sum_{i=0}^{\infty} \vartheta^i(\vartheta(c)),$$

the proof is obvious. □

Lemma 3.4. *From Definition 3.2, we can conclude that for every C^*-algebra valued GF function ϑ: $C_c \longrightarrow C_c$,*

$$\sigma(c) = \sum_{i=0}^{\infty} \lambda^i \vartheta^i(c) \text{ is finite for every } \lambda \in C_I,$$

where $\vartheta^0(c) = c$.

Proof. Since $\lambda \in C_I$ implies $\lambda \leq I_C$ and $\sum_{i=0}^{\infty} \vartheta^i(c)$ is finite, the rest of the proof is obvious from the theory of convergent series. □

Now we define C^*-algebra valued Geraghty contraction.

Definition 3.3. *Let H be a self mapping in a C^*-avMS (V, C, d) and $\beta : C_+ \setminus \{0_C\} \longrightarrow C_I$, where*

$$\lim_{n \to +\infty} \beta(r_n) = I_C \Rightarrow \lim_{n \to +\infty} r_n = 0_C$$

satisfies

$$d(Hv_1, Hv_2) \leq \beta(d(v_1, v_2))d(v_1, v_2), \quad \forall v_1, v_2 \in V.$$

Then, H is said to be a C^*-algebra valued Geraghty contraction.

From now on, we denote $\mathfrak{B}$ as the set of maps β, defined in Definition 3.3. The following is an example of a member of $\mathfrak{B}$.

Example 3.3. *Let $C = M_2(\mathbb{R})$. Then, C_I is the collection of all diagonal matrices with elements from $[0, 1)$ in $M_2(\mathbb{R})$. So, the β: $C_+ \setminus \{0_C\} \longrightarrow C_I$ functions given by*

$$\beta\left(\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{1+c} & 0 \\ 0 & \frac{1}{1+c} \end{bmatrix}, \quad \forall c \in (0, +\infty),$$
$$\beta\left(\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{e^c} & 0 \\ 0 & \frac{1}{e^c} \end{bmatrix}, \quad \forall c \in (0, +\infty)$$

are members of $\mathfrak{B}$.

Now we have the significant meaning of C^*-algebra valued α-admissibility.

Definition 3.4. *Let H be a self mapping in a C^*-avMS and α: $V \times V \longrightarrow C$. Then H is known to be C^*-algebra valued α-admissible if, for all $v_1, v_2 \in V$,*

$$\alpha(Hv_1, Hv_2) \succeq I_C, \ \ \textit{whenever} \ \ \alpha(v_1, v_2) \succeq I_C.$$

For example, below H is a C^*-algebra valued α-admissible mapping.

Example 3.4. *Let $C = M_2(\mathbb{R})$. Then, C_{I_C} is the collection of all diagonal matrices with elements from $[1, +\infty)$ in $M_2(\mathbb{R})$. Also, let (V, d) be an arbitrary metric space, and for all the following pairs of mappings H: $V \longrightarrow V$, α: $V \times V \longrightarrow C$ given by*

$$\left(v^3 + \sqrt[7]{v}\ ; \begin{bmatrix} v_1^5 - v_2^5 & 0 \\ 0 & v_1^5 - v_2^5 \end{bmatrix}\right), \left(\sqrt[3]{v}\ ; \begin{bmatrix} e^{v_1 - v_2} & 0 \\ 0 & e^{v_1 - v_2} \end{bmatrix}\right)$$

and

$$\left(v^4 + \ln(v^2 + 1)\ ; \begin{bmatrix} \frac{v_1^3}{1+v_1^3} - \frac{v_2^3}{1+v_2^3} + 1 & 0 \\ 0 & \frac{v_1^3}{1+v_1^3} - \frac{v_2^3}{1+v_2^3} + 1 \end{bmatrix}\right).$$

H is C^*-algebra valued α-admissible. Indeed, whenever

$$\alpha(v_1, v_2) \succeq I_C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we get

$$v_1^5 - v_2^5,\ v_1 - v_2,\ \frac{v_1^3}{1 + v_1^3} - \frac{v_2^3}{1 + v_2^3} + 1 \geq 1.$$

Now, if we apply the corresponding H and calculate $\alpha(Hv_1, Hv_2)$, we see $\alpha(Hv_1, Hv_2) \succeq I_C$.

4. Main results

The following definition of (α,β)-Bianchini-Grandolfi gauge contraction of type I is one of our primary definitions.

Definition 4.1. *Let H be a self mapping in a C^*-avMS (V, C, d) and α: $V \times V \longrightarrow C$. Then H is said to be a C^*-algebra valued (α,β)-Bianchini-Grandolfi gauge contraction of type I if for all $v_1, v_2 \in V$ and for some $\beta \in \mathfrak{B}$,*

$$d(Hv_1, Hv_2) \succeq \theta \Rightarrow \alpha(v_1, v_2)d(Hv_1, Hv_2) \preceq \vartheta(\beta(d(v_1, v_2))d(v_1, v_2)). \tag{4.1}$$

Here, ϑ: $C_c \longrightarrow C_c$ is a C^*-algebra valued BGGF.

Now we prove one of our primary results.

Theorem 4.1. *In a complete C^*-avMS (V, C, d), let H: $V \longrightarrow V$ be a C^*-algebra valued (α,β)-Bianchini-Grandolfi gauge contraction of type I. Then, H must have unique $v^* \in V$ such that $Hv^* = v^*$.*

Proof. Let $v_0 \in V$ be arbitrary and $\{v_n\}$ be the sequence defined by

$$v_n = Hv_{n-1}, \forall n \in \mathbb{N}.$$

Suppose there is a $m \in \mathbb{N}$ so that $v_{m+1} = v_m$. Then $Hv_m = v_m$ and v_m is a fixed point. Now we can assume the sequence $\{v_n\}$ is a sequence of distinct terms. Since H is a C^*-algebra valued (α,β)-Bianchini-Grandolfi gauge contraction of type I, we have

$$\begin{aligned} d(v_n, v_{n+1}) &\preceq \alpha(v_n, v_{n+1}) d(v_n, v_{n+1}) \\ &= \alpha(v_n, v_{n+1}) d(Hv_{n-1}, Hv_n) \\ &\preceq \vartheta(\beta(d(v_{n-1}, v_n)) d(v_{n-1}, v_n)) \\ &= \vartheta(\lambda(v_{n-1}, v_n)) \ \ \lambda = \max_{\{v_n\}} \beta \\ &\preceq \lambda \vartheta(d(v_{n-1}, v_n)), \end{aligned}$$

for any $n \in \mathbb{N}$. For $n = 1$,

$$d(v_1, v_2) \preceq \lambda \vartheta(d(v_0, v_1)).$$

For $n = 2$,

$$d(v_2, v_3) \preceq \lambda \vartheta(d(v_1, v_2)) \preceq \lambda \vartheta(\lambda \vartheta(d(v_0, v_1))) \preceq \lambda^2 \vartheta^2(d(v_0, v_1)).$$

Continuing the process, we have

$$d(v_n, v_{n+1}) \preceq \lambda^n \vartheta^n(d(v_0, v_1)), \forall n \in \mathbb{N}.$$

Now,

$$\begin{aligned} d(v_n, v_{n+m}) \preceq & d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+m}) \\ \preceq & d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+2}) + d(v_{n+2}, v_{n+m}) \\ & \vdots \\ \preceq & d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+2}) + \cdots + d(v_{n+m-2}, v_{n+m-1}) + d(v_{n+m-1}, v_{n+m}) \\ \preceq & \lambda^n \vartheta^n(d(v_0, v_1)) + \lambda^{n+1} \vartheta^{n+1}(d(v_0, v_1)) + \cdots + \lambda^{n+m-2} \vartheta^{n+m-2}(d(v_0, v_1)) \\ & + \lambda^{n+m-1} \vartheta^{n+m-1}(d(v_0, v_1)) \\ = & \sum_{j=n}^{n+m-1} \lambda^j \vartheta^j(d(v_0, v_1)) \\ \preceq & \sum_{j=0}^{\infty} \lambda^j \vartheta^j(d(v_0, v_1)) \\ \prec & \infty, \end{aligned}$$

for any $n, m \in \mathbb{N}, m \geq 2$. Which shows that the sequence $\{v_n\}$ is Cauchy. Being (V, C, d) complete C^*-*avMS*, there exist $v^* \in V$ so that $\{v_n\}$ converges to v^*, i.e.,

$$\lim_{n \to \infty} d(v^*, v_n) = 0_C.$$

Again,

$$\begin{aligned} d(Hv^*, v_{n+1}) &\preceq \alpha(Hv^*, v_{n+1}) d(Hv^*, v_{n+1}) \\ &\preceq \vartheta(\beta(d(v^*, v_n)) d(v^*, v_n)) \\ &\preceq \beta(d(v^*, v_n)) \vartheta(d(v^*, v_n)) \\ &\preceq \beta(d(v^*, v_n)) d(v^*, v_n). \end{aligned}$$

Limiting both sides, we have

$$\lim_{n\to\infty} d(Hv^*, v^*) = 0_C \Rightarrow Hv^* = v^*,$$

i.e., v^* is a fixed point of H. Let v^* and v^{**} be two distinct fixed points of H. Then

$$\begin{aligned} d(Hv^*, Hv^{**}) &\preceq \alpha(Hv^*, Hv^{**}) d(Hv^*, Hv^{**}) \\ &\preceq \vartheta(\beta(d(v^*, v^{**})) d(v^*, v^{**})) \\ &\preceq \beta(d(v^*, v^{**})) \vartheta(d(v^*, v^{**})) \\ \Rightarrow d(v^*, v^{**}) &\preceq \beta(d(v^*, v^{**})) d(v^*, v^{**}). \end{aligned}$$

This implies

$$\beta(d(v^*, v^{**})) \succeq I_C,$$

a contradiction. So, v^* is unique in V so that $Hv^* = v^*$. □

The example below demonstrates Theorem 4.1.

Example 4.1. *Let $V = [1, +\infty)$, $C = M_2(\mathbb{R})$, and $d\colon V \times V \longrightarrow C$ be described as*

$$d(v_1, v_2) = \begin{bmatrix} |v_1 - v_2| & 0 \\ 0 & |v_1 - v_2| \end{bmatrix}.$$

Now, the mapping $H\colon V \longrightarrow V$ given by

$$Hv = \frac{1}{2}\left(\frac{v+1}{v} + 1\right)$$

is such that

$$\begin{aligned} d(Hv_1, Hv_2) &= \begin{bmatrix} \frac{1}{2}\left|\frac{v_1+1}{v_1} - \frac{v_2+1}{v_2}\right| & 0 \\ 0 & \frac{1}{2}\left|\frac{v_1+1}{v_1} - \frac{v_2+1}{v_2}\right| \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\left|\frac{v_2-v_1}{v_1 v_2}\right| & 0 \\ 0 & \frac{1}{2}\left|\frac{v_2-v_1}{v_1 v_2}\right| \end{bmatrix} \\ &\preceq \begin{bmatrix} \frac{1}{2}\frac{|v_1-v_2|}{1+|v_1-v_2|} & 0 \\ 0 & \frac{1}{2}\frac{|v_1-v_2|}{1+|v_1-v_2|} \end{bmatrix}, \quad \forall v_1, v_2 \in V. \end{aligned}$$

That is, for

$$\alpha(v_1, v_2) = \begin{cases} I_C, & v_1 \in V, \\ 0_C, & \textit{otherwise}, \end{cases}$$

$$\vartheta\left(\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} \frac{c}{2} & 0 \\ 0 & \frac{c}{2} \end{bmatrix}, \quad \forall \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \in C$$

and

$$\beta\left(\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{1+c} & 0 \\ 0 & \frac{1}{1+c} \end{bmatrix}, \quad \forall \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \in C_+ \setminus \{0_C\},$$

H satisfies all conditions of Theorem 4.1, having a unique fixed point.

We now present the definition of C^*-algebra valued (α, β)-Bianchini-Grandolfi gauge contraction of type II.

Definition 4.2. *Let H be a self mapping in a C^*-avMS (V, C, d) and $\alpha\colon V \times V \longrightarrow C$. Then H is said to be a C^*-algebra valued (α, β)-Bianchini-Grandolfi gauge contraction of type II if for all $v_1, v_2 \in V$ and for some $\beta \in \mathfrak{B}$, $d(Hv_1, Hv_2) \geq \theta$ implies*

$$\alpha(v_1, v_2) d(Hv_1, Hv_2) \preceq \vartheta(\beta(m(v_1, v_2)) m(v_1, v_2)), \tag{4.2}$$

where $\vartheta\colon C_c \longrightarrow C_c$ is a C^*-algebra valued BGGF and

$$m(v_1, v_2) = \max\{d(v_1, v_2), d(v_1, Hv_1), d(v_2, Hv_2)\}.$$

The result below utilizes (α, β)-Bianchini-Grandolfi gauge contraction of type II.

Theorem 4.2. *In a complete C^*-avMS (V, C, d), let $H\colon V \longrightarrow V$ be a C^*-algebra valued (α, β)-Bianchini-Grandolfi gauge contraction of type II. Then H must have unique $v^* \in V$ such that $Hv^* = v^*$.*

Proof. Let $v_0 \in V$ be arbitrary and $\{v_n\}$ be the sequence defined by

$$v_n = Hv_{n-1}, \quad \forall n \in \mathbb{N}.$$

Suppose there is some $m \in \mathbb{N}$ such that $v_{m+1} = v_m$. Then $Hv_m = v_m$ and v_m is a fixed point. Now we can assume the sequence $\{v_n\}$ is a sequence of distinct terms. Since H is a C^*-algebra valued (α, β)-Bianchini-Grandolfi gauge contraction of type II, we have

$$\begin{aligned} d(v_n, v_{n+1}) &\preceq \alpha(v_n, v_{n+1}) d(v_n, v_{n+1}) \\ &= \alpha(v_n, v_{n+1}) d(Hv_{n-1}, Hv_n) \\ &\preceq \vartheta\left(\beta(v_{n-1}, v_n) \max\{d(v_{n-1}, v_n), d(v_{n-1}, Hv_{n-1}), d(v_n, Hv_n)\}\right) \\ &= \vartheta(\lambda \max\{d(v_{n-1}, v_n), d(v_n, v_{n+1})\})) \quad \lambda = \max_{\{v_n\}} \beta \preceq g \\ &\preceq \lambda \vartheta(\max\{d(v_{n-1}, v_n), d(v_n, v_{n+1})\})), \end{aligned}$$

for any $n \in \mathbb{N}$. Letting

$$\max\{d(v_{n-1}, v_n), d(v_n, v_{n+1})\} = d(v_n, v_{n+1}),$$

then

$$d(v_n, v_{n+1}) \preceq \lambda \vartheta(d(v_n, v_{n+1})) \preceq \lambda d(v_n, v_{n+1}).$$

This implies $\lambda \succeq I_C$, a contradiction. So,

$$\max\{d(v_{n-1}, v_n), d(v_n, v_{n+1})\} = d(v_{n-1}, v_n)$$

and

$$d(v_n, v_{n+1}) \preceq \lambda\vartheta(d(v_{n-1}, v_n)),$$

for any $n \in \mathbb{N}$. For $n = 1$,

$$d(v_1, v_2) \preceq \lambda\vartheta(d(v_0, v_1)).$$

For $n = 2$,

$$d(v_2, v_3) \preceq \lambda\vartheta(d(v_1, v_2)) \preceq \lambda\vartheta(\lambda\vartheta(d(v_0, v_1))) \preceq \lambda^2\vartheta^2(d(v_0, v_1)).$$

Continuing the process, we get

$$d(v_n, v_{n+1}) \preceq \lambda^n\vartheta^n(d(v_0, v_1)), \quad \forall n \in \mathbb{N}.$$

Now,

$$\begin{aligned}
d(v_n, v_{n+m}) \preceq & d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+m}) \\
\preceq & d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+2}) + d(v_{n+2}, v_{n+m}) \\
& \vdots \\
\preceq & d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+2}) + \cdots + d(v_{n+m-2}, v_{n+m-1}) + d(v_{n+m-1}, v_{n+m}) \\
\preceq & \lambda^n\vartheta^n(d(v_0, v_1)) + \lambda^{n+1}\vartheta^{n+1}(d(v_0, v_1)) + \cdots + \lambda^{n+m-2}\vartheta^{n+m-2}(d(v_0, v_1)) \\
& + \lambda^{n+m-1}\vartheta^{n+m-1}(d(v_0, v_1)) \\
= & \sum_{j=n}^{n+m-1} \lambda^j\vartheta^j(d(v_0, v_1)) \\
\preceq & \sum_{j=0}^{\infty} \lambda^j\vartheta^j(d(v_0, v_1)) \\
\prec & \infty,
\end{aligned}$$

for any $n, m \in \mathbb{N}, m \geq 2$, which shows that the sequence $\{v_n\}$ is Cauchy. Being (V, C, d) complete C^*-$avMS$, there exist $v^* \in V$ so that $\{v_n\}$ converges to v^*, i.e.,

$$\lim_{n\to\infty} d(v^*, v_n) = \lim_{n\to\infty} d(v^*, v_{n+1}) = 0_C.$$

Again,

$$\begin{aligned}
d(Hv^*, v_{n+1}) &\preceq \alpha(Hv^*, v_{n+1})d(Hv^*, v_{n+1}) \\
&\preceq \vartheta(\beta(d(v^*, v_n)) \max\{d(v^*, v_n), d(v^*, Hv^*), d(v_n, Hv_n)\}) \\
&\preceq \beta(d(v^*, v_n))\vartheta(\max\{d(v^*, v_n), d(v_n, v_{n+1})\}) \\
&\preceq \beta(d(v^*, v_n)) \max\{d(v^*, v_n), d(v_n, v_{n+1})\}.
\end{aligned}$$

Limiting both sides, we have

$$\lim_{n\to\infty} d(Hv^*, v^*) = 0_C \Rightarrow Hv^* = v^*,$$

i.e., v^* is a fixed point of H. Let v^* and v^{**} be two distinct fixed points of H. Then

$$\begin{aligned}
d(Hv^*, Hv^{**}) &\preceq \alpha(Hv^*, Hv^{**})d(Hv^*, Hv^{**})\\
&\preceq \vartheta(\beta(d(v^*, v^{**}))\max\{d(v^*, v^{**}), d(v^*, Hv^*), d(v^{**}, Hv^{**})\})\\
&\preceq \beta(d(v^*, v^{**}))\vartheta(d(v^*, v^{**}))\\
\Rightarrow d(v^*, v^{**}) &\preceq \beta(d(v^*, v^{**}))d(v^*, v^{**}).
\end{aligned}$$

This implies $\beta(d(v^*, v^{**})) \succeq I_C$, a contradiction. So, v^* is unique in V such that $Hv^* = v^*$. □

Now we illustrate an example satisfying Theorem 4.2.

Example 4.2. *Let $V = [0, 1]$, $C = M_2(\mathbb{R})$, and $d\colon V \times V \longrightarrow C$ be given by*

$$d(v_1, v_2) = \begin{bmatrix} |v_1 - v_2| & 0 \\ 0 & |v_1 - v_2| \end{bmatrix}.$$

Now, the mapping $H\colon V \longrightarrow V$, given by

$$Hv = \begin{cases} \frac{v}{3}, & v \in [0, 1), \\ \frac{1}{6}, & v = 1, \end{cases}$$

is such that

$$\begin{aligned}
d(Hv_1, Hv_2) &= \begin{bmatrix} \left|\frac{v_1-v_2}{3}\right| & 0 \\ 0 & \left|\frac{v_1-v_2}{3}\right| \end{bmatrix}\\
&\preceq \begin{bmatrix} \frac{7}{8}\frac{m(v_1,v_2)}{e^{m(v_1,v_2)}} & 0 \\ 0 & \frac{7}{8}\frac{m(v_1,v_2)}{e^{m(v_1,v_2)}} \end{bmatrix},
\end{aligned}$$

$\forall v_1, v_2 \in V$, where

$$m(v_1, v_2) = \max\{|v_1 - v_2|, |v_1 - Hv_1|, |v_2 - Hv_2|\}.$$

That is, for

$$\alpha(v_1, v_2) = \begin{cases} I_C, & v_1 \in V, \\ 0_C, & \textit{otherwise}, \end{cases}$$

$$\vartheta\left(\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} \frac{7c}{8} & 0 \\ 0 & \frac{7c}{8} \end{bmatrix}, \quad \forall \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \in C$$

and

$$\beta\left(\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{e^c} & 0 \\ 0 & \frac{1}{e^c} \end{bmatrix}, \quad \forall \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \in C_+ \setminus \{0_C\}.$$

H satisfies all requirements of Theorem 4.2, having a unique fixed point.

The following are some consequent results that can be easily obtained from Theorems 4.1 and 4.2.

Corollary 4.1. *In a complete C^*-avMS (V, C, d), let H be a self mapping and $\alpha\colon V \times V \longrightarrow C_{I_C}$. Let self-mapping H be such that, for all $v_1, v_2 \in V$ and for some $\beta \in \mathfrak{B}$, $d(Hv_1, Hv_2) \succeq \theta$ implies*

$$\alpha(v_1, v_2)d(Hv_1, Hv_2) \preceq \vartheta(\beta(m'(v_1, v_2))m'(v_1, v_2)), \tag{4.3}$$

here, $\vartheta\colon C_c \longrightarrow C_c$ is a C^*-algebra valued BGGF and

$$m'(v_1, v_2) = \max\left\{d(v_1, v_2), d(v_1, Hv_1), \frac{d(v_1, Hv_1)d(v_2, Hv_2)}{d(v_1, v_2)}\right\}.$$

Then, there is unique $v^* \in V$ such that $Hv^* = v^*$.

Corollary 4.2. *In a complete C^*-avMS (V, C, d), let H be a self mapping and $\alpha\colon V \times V \longrightarrow C_{I_C}$. Let self-mapping H be such that, for all $v_1, v_2 \in V$ and for some $\beta \in \mathfrak{B}$, $d(Hv_1, Hv_2) \succeq \theta$ implies*

$$\alpha(v_1, v_2)d(Hv_1, Hv_2) \preceq \vartheta(\beta(m''(v_1, v_2))m''(v_1, v_2)), \tag{4.4}$$

here, $\vartheta\colon C_c \longrightarrow C_c$ is a C^*-algebra valued BGGF and

$$m''(v_1, v_2) = \max\left\{d(v_1, v_2), d(v_1, Hv_1), d(v_2, Hv_2), \frac{d(v_1, Hv_2)d(v_2, Hv_1)}{d(v_1, v_2)}\right\}.$$

Then, there is unique $v^* \in V$ such that $Hv^* = v^*$.

Corollary 4.3. *In a complete C^*-avMS (V, C, d), let H be a self mapping and $\alpha\colon V \times V \longrightarrow C_{I_C}$. Let self-mapping H be such that, for all $v_1, v_2 \in V$ and for some $\beta \in \mathfrak{B}$, $d(Hv_1, Hv_2) \succeq \theta$ implies*

$$\alpha(v_1, v_2)d(Hv_1, Hv_2) \preceq \vartheta(\beta(m'''(v_1, v_2))m'''(v_1, v_2)), \tag{4.5}$$

here, $\vartheta\colon C_c \longrightarrow C_c$ is a C^*-algebra valued BGGF and

$$m'''(v_1, v_2) = \max\left\{d(v_1, v_2), d(v_2, Hv_2), \frac{d(v_1, Hv_1)d(v_1, Hv_2) + d(v_2, Hv_2)d(v_2, Hv_1)}{d(v_1, Hv_2) + d(v_2, Hv_1)}\right\}.$$

Then, there is unique $v^* \in V$ such that $Hv^* = v^*$.

Corollary 4.4. *In a complete C^*-avMS (V, C, d), let H be a self mapping and $\alpha\colon V \times V \longrightarrow C_{I_C}$. Let self-mapping H be such that, for all $v_1, v_2 \in V$ and for some $\beta \in \mathfrak{B}$, $d(Hv_1, Hv_2) \succeq \theta$ implies*

$$\alpha(v_1, v_2)d(Hv_1, Hv_2) \preceq \vartheta(\beta(m''''(v_1, v_2))m''''(v_1, v_2)), \tag{4.6}$$

here, $\vartheta\colon C_c \longrightarrow C_c$ is a C^*-algebra valued BGGF and

$$m''''(v_1, v_2) = \max\left\{d(v_1, v_2), d(v_1, Hv_1), \frac{d(v_2, Hv_2)(1 + d(v_1, Hv_1))}{1 + d(v_1, v_2)}\right\}.$$

Then, there is unique $v^* \in V$ such that $Hv^* = v^*$.

Remark 4.1. *In [10], the authors demonstrated that the notion of C^*-avM is a not so great generalization in metric fixed point theory. The authors proved that by utilizing a homeomorphism, the Banach mapping principle in the C^*-avMS can be converted to the same in real metric spaces. In our case, we used three auxiliary functions, namely α, β, and ϑ, to define (α, β)-Bianchini-Grandolfi gauge contraction of type I and type II. Following the method of proving the result in [10], we see that the homeomorphism will not work in our contractions.*

5. Application

In addition to being useful in dynamical systems, engineering, computer science, game theory, physics, neural networks, and many other fields, fixed point theory is also used to solve differential and integral equations, which are utilized to explore the solutions of many mathematical models. We discover a result for a particular type of the following integral in a finite measurable set M as an application of Theorem 4.1:

$$v(t) = \int_M F(t, s, v(s))ds + g(t), t, s \in M, \tag{5.1}$$

where $F\colon M \times M \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g \in L^\infty(M)$.

Now, considering the complete C^*-$avMS$ (V, C, d) as in Example 2.2, if we define a self mapping $H\colon V \longrightarrow V$ by

$$Hv(t) = \int_M F(t, s, v(s))ds + g(t); t, s \in M, \ \forall v \in L^\infty(M),$$

then determining the presence of a solution to (5.1) is analogous to determining the presence of a fixed point in H.

We now state and support our conclusion:

Theorem 5.1. *The integral equation of type (5.1) has a unique solution if* $F\colon M \times M \times \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies*

$$|F(t, s, v_1(s)) - F(t, s, v_2(s))| \leq \frac{\lambda}{m} e^{-\|(v_1 - v_2)\|} \|v_1 - v_2\|,$$

where $|\lambda| < 1$ *and* $m = \int_M ds$.

Proof. For any $\phi \in L^2(M)$, we get

$$\begin{aligned} d(Hv_1(t), Hv_2(t)) &= \pi_{|Hv_1(t) - Hv_2(t)|} \\ &= \pi_{\left|\int_M F(t,s,v_1(s))ds - \int_M F(t,s,v_2(s))ds\right|} \\ &= \pi_{\left|\int_M (F(t,s,v_1(s)) - F(t,s,v_2(s)))ds\right|} \\ &= \phi \cdot \left| \int_M (F(t, s, v_1(s)) - F(t, s, v_2(s)))ds \right| \end{aligned}$$

with

$$\|d(Hv_1(t), Hv_2(t))\| = \left\| \int_M (F(t, s, v_1(s)) - F(t, s, v_2(s)))ds \right\|,$$

and for

$$\vartheta(c) = \lambda c \ (\|\lambda\| < I_C), \quad \beta(c) = e^{-c},$$

we have

$$\begin{aligned}\|\vartheta(\beta(d(v_1,v_2)d(v_1,v_2))\| &= \|\lambda\beta(d(v_1,v_2)d(v_1,v_2)\| \\ &= \|\lambda e^{-d(v_1,v_2)}d(v_1,v_2)\| \\ &= \|\lambda\|\,\|e^{-(v_1-v_2)}\|\,\|v_1-v_2\|.\end{aligned}$$

Hence, for

$$\alpha(v_1,v_2)=\begin{cases} I_C, & v_1\in L^\infty(M),\\ 0_C, & \text{otherwise,}\end{cases}$$

we have

$$\begin{aligned}\|\alpha(v_1,v_2)d(Hv_1(t),Hv_2(t))\| &\le \|d(Hv_1(t),Hv_2(t))\| \\ &= \Big\|\int\limits_M (F(t,s,v_1(s))-F(t,s,v_2(s)))ds\Big\| \\ &\le \int\limits_M \|(F(t,s,v_1(s))-F(t,s,v_2(s)))\|ds \\ &\le \int\limits_M \|\frac{\lambda}{m}\|\; e^{-\|v_1-v_2\|}\;\|v_1-v_2\|ds \\ &\le \|\frac{\lambda}{m}\|\;\|e^{-(v_1-v_2)}\|\;\|v_1-v_2\|\int\limits_M ds \\ &\le \|\vartheta(\beta(d(v_1,v_2)d(v_1,v_2))\|.\end{aligned}$$

Thus, by Theorem 4.1, there is unique $v^*\in L^\infty(M)$ with $Hv^*=v^*$, i.e., there exists a unique solution of (5.1). □

Example 5.1. *Consider a finite measurable set $M=[0,1]$ and the complete C^*-avMS (V,C,d) as in Example 2.2. If*

$$F(t,s,v(s))=(t-s)v(s),\quad g(t)=t^2,$$

for all $s,t\in M$, problem (5.1) *will become nontrivial. Now, considering α,β, and ϑ same as in the proof of Theorem 5.1, we see the problem satisfies Theorem 5.1 for $m=1$ and*

$$\|e^{-(v_1-v_2)}\|<\lambda<1.$$

Clearly, we obtain

$$v(t)=t^2+\frac{3}{13}t-\frac{17}{78}$$

as the unique solution of the nontrivial problem.

6. Conclusions and open problems

Since the concept of the C^*-$avMS$ is a relatively recent addition to the body of literature, in this paper, we expanded the redefined definitions of the gauge function, BGGF, α-admissibility, Geraghty contractive mapping, and (α,β)-admissible mapping in a C^*-algebra sense. Utilizing these definitions, we presented two types of (α,β)-Bianchini-Grandolfi gauge contractions. We then demonstrated our primary findings in a C^*-$avMS$, which require functions meeting our contraction criterion to have a single fixed point. These findings generalized a few findings of [2–4, 8, 14, 16] in the C^*-algebra sense. We also provided examples to clarify our results. Finally, using our primary finding, we investigated if a solution to an integral equation exists and is unique.

We have the following open problems:

(i) Can the primary findings of this study be expressed in terms of a lattice in place of the underlying C^*-algebra?
(ii) As in [10], is it possible to derive our contraction results?

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The study was funded by Researchers Supporting Project number (RSPD2024R749), King Saud University, Riyadh, Saudi Arabia. Moirangthem Pradeep Singh (first author) is supported by MHRD, India.

Conflict of interest

The authors declare no conflicts of interest.

References

1. S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, *Fund. Math.*, **3** (1922), 133–181. https://doi.org/10.4064/FM-3-1-133-181
2. M. A. Geraghty, On contractive mappings, *Proc. Amer. Math. Soc.*, **40** (1973), 604–608. https://doi.org/10.1090/S0002-9939-1973-0334176-5
3. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal.*, **75** (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014
4. E. Karapinar, P. Kumam, P. Salimi, On α-ψ-Meir-Keeler contractive mappings, *Fixed Point Theory Appl.*, **2013** (2013), 94. https://doi.org/10.1186/1687-1812-2013-94
5. T. Stephen, Y. Rohen, Fixed points of generalized rational (α,β,Z)-contraction mappings under simulation functions, *J. Math. Comput. Sci.*, **24** (2022), 345–357. https://doi.org/10.22436/jmcs.024.04.07
6. W. Atiponrat, A. Khemphet, W. Chaiwino, T. Suebcharoen, P. Charoensawan, Common best proximity point theorems for generalized dominating with graphs and applications in differential equations, *Mathematics*, **12** (2024), 306. https://doi.org/10.3390/math12020306

7. T. Suebcharoen, W. Atiponrat, K. Chaichana, Fixed point theorems via auxiliary functions with applications to two-term fractional differential equations with nonlocal boundary conditions, *AIMS Math.*, **8** (2023), 7394–7418. https://doi.org/10.3934/math.2023372
8. Z. Ma, L. Jiang, H. Sun, C^*-algebra-valued metric spaces and related fixed point theorems, *Fixed Point Theory Appl.*, **2014** (2014), 206. https://doi.org/10.1186/1687-1812-2014-206
9. K. H. Alam, Y. Rohen, A. Tomar, On fixed point and its application to the spread of infectious diseases model in M_v^b-metric space, *Math. Methods Appl. Sci.*, **47** (2024), 6489–6503. https://doi.org/10.1002/mma.9933
10. H. H. Alsulami, R. P. Agarwal, E. Karapınar, F. Khojasteh, A short note on C^*-valued contraction mappings, *J. Inequal. Appl.*, **2016** (2016), 50. https://doi.org/10.1186/s13660-016-0992-5
11. M. P. Singh, Y. Rohen, N. Saleem, K. H. Alam, K. A. Singh, A. Razzaque, On fixed point equations involving Geraghty type contractions with solution to integral equation, *Mathematics*, **11** (2023), 4882. https://doi.org/10.3390/math11244882
12. J. Brzdek, E. Karapinar, A. Petruşel, A fixed point theorem and the Ulam stability in generalized dq-metric spaces, *J. Math. Anal. Appl.*, **467** (2018), 501–520. https://doi.org/10.1016/j.jmaa.2018.07.022
13. S. M. A. A. Bakr, H. K. Hussein, Some results on coupled coincidence points in vector quasi cone metric spaces over Banach algebras with satisfactory cones, *J. Math. Comput. Sci.*, **31** (2023), 305–317. https://doi.org/10.22436/jmcs.031.03.06
14. R. M. Bianchini, M. Grandolfi, Transformazioni di tipo contracttivo generalizzato in uno spazio metrico, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend.*, **45** (1968), 212–216.
15. V. Ptak, The rate of convergence of Newton's process, *Numer. Math.*, **25** (1975), 279–285. https://doi.org/10.1007/BF01399416
16. P. D. Proinov, A generalization of the Banach contraction principle with high order of convergence of successive approximations, *Nonlinear Anal.*, **67** (2007), 2361–2369. https://doi.org/10.1016/j.na.2006.09.008
17. S. H. Cho, J. S. Bae, E. Karapinar, Fixed point theorems for α-Geraghty contraction type maps in metric spaces, *Fixed Point Theory Appl.*, **2013** (2013), 329. https://doi.org/10.1186/1687-1812-2013-329
18. S. Chandok, Some fixed point theorems for (α,β)-admissible Geraghty type contractive mappings and related results, *Math. Sci.*, **9** (2015), 127–135. https://doi.org/10.1007/s40096-015-0159-4
19. K. H. Alam, Y. Rohen, N. Saleem, Fixed points of (α,β,F^*) and (α,β,F^{**})-weak Geraghty contractions with an application, *Symmetry*, **15** (2023), 243. https://doi.org/10.3390/sym15010243
20. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 94. https://doi.org/10.1186/1687-1812-2012-94
21. D. Singh, V. Chauhan, P. Kumam, V. Joshi, P. Thounthong, Applications of fixed point results for cyclic Boyd-Wong type generalized F-ψ-contractions to dynamic programming, *J. Math. Comput. Sci.*, **17** (2017), 200–215. https://doi.org/10.22436/JMCS.017.02.02

22. B. Khomdram, N. Priyobarta, Y. Rohen, T. Indubala, Remarks on (α,β)-admissible mappings and fixed points under $\mathscr{Z}$-contraction mappings, *J. Math.*, **2021** (2021), 6697739. https://doi.org/10.1155/2021/6697739

23. G. J. Murphy, *C^*-algebras and operator theory*, Academic Press, 1990.

24. Q. H. Xu, T. E. D. Bieke, Z. Q. Chen, *Introduction to operator algebras and noncommutative L_p spaces*, Science Press, 2010.