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Research article

Finite-time stability and uniqueness theorem of solutions of nabla fractional (q, h)-difference equations with non-Lipschitz and nonlinear conditions

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Abstract: In this paper, the discrete (q, h)-fractional Bihari inequality is generalized. On the grounds of inequality, the finite-time stability and uniqueness theorem of solutions of (q, h)-fractional difference equations with non-Lipschitz and nonlinear conditions is concluded. In addition, the validity of our conclusion is illustrated by a nonlinear example with a non-Lipschitz condition.

Keywords: fractional (q, h)-difference equation; finite-time stability; uniqueness; non-Lipschitz condition; discrete (q, h)-fractional Bihari inequality
Mathematics Subject Classification: 39A12, 39A70

1. Introduction

In the past few decades, fractional calculus has been widely used in fields such as science and engineering. Many natural systems can be properly modeled by a nonlinear differential equation in the real world, such as neural networks [1], control systems [2, 3], disease models [4], blood production in leukemia patients (Mackey- Glass model) [5], and population dynamics [6]. Compared with classical integer-order systems, fractional systems can more accurately describe the memory characteristics of various materials and processes.

In recent years, q-calculus has been widely used in various areas, such as, in the approximation theory [7], number theory [8], quantum theory [9], and physics [10]. In 2010, the authors (see [11, 12]) introduced (q, h)-calculus as an extension of the basic notions of discrete fractional calculus. Since then, new results involving (q, h)-calculus have continued to emerge [13–15].

The existence and uniqueness of solutions are the basis for studying the stability problem, but they are easily neglected. We can mention [2, 3, 16–23], references therein, etc. However, as far as we know, most nonlinear functions are already mentioned in the existing literature. Fractional difference systems are Lipschitz continuous. The results of the uniqueness theorem for solutions to non-Lipschitz, nonlinear fractional difference equations are rare. In [24], applying the fractional Bihari inequality, the authors investigated the uniqueness theorem of the Caputo difference equation. Motivated by the above discussion, the uniqueness theorem of solutions of fractional (q, h)-difference equations with non-Lipschitz nonlinearities is given in this paper.

Owing to Lazarevic's remarkable and seminal works [25–27], more and more scientists have become increasingly interested in the finite-time stability analysis of fractional delay systems. Since stability analysis is critical in fractional systems, many experts and scholars are dedicated to studying methods of stability analysis, such as Lyapunov's method [28], the Gronwall inequality [29–33], the fractional Fourier transform [34], and Mittag-Leffler matrix functions [35].

In our paper, with the aid of the discrete (q, h)-fractional Bihari inequality, we apply a new method to study the finite-time stability of the (q, h)-fractional difference equation (1.1)

$$\begin{aligned} {}_a^C\nabla^{\mu}_{(q,h)}x(t) &= g(x(t)), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}^{T}_{\sigma(a)}, \\ x(a) &= x_0, \end{aligned} \tag{1.1}$$

where $0 < \mu < 1$, $g : \mathbb{R} \to [0, +\infty)$, $x(t) : \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)} \to \mathbb{R}$, $T \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$.

Remark 1.1. *In [36], fractional Bihari difference inequalities were studied. In our paper, a discrete fractional Bihari inequality with a (q, h) time scale is generalized and developed. Unlike the case in [36], where the existence of the inverse function of the function Φ was not clarified, the proof of discrete (q, h)-fractional Bihari inequality in our paper is more complete and precise.*

The rest of this paper is organized as follows: In Section 2, fundamental concepts of discrete fractional calculus on (q, h)-time scales are illuminated. In Section 3, the discrete (q, h)-fractional Bihari inequality is generalized. In Section 4, the uniqueness and finite-time stability of solutions of fractional (q, h)-difference equations with non-Lipschitz and nonlinear conditions are obtained. In Section 5, an example of a non-Lipschitz condition is presented to illustrate the validity of our conclusion numerically.

2. Preliminaries

In this section, fundamental concepts and conclusions about fractional (q, h)-difference are illuminated.

For any $s, t \in \mathbb{R}$, one has

$$\mathbb{N}_s := \{s, s + 1, s + 2, \dots\},$$
$$\mathbb{N}_s^t := \{s, s + 1, s + 2, \dots, t\}, \quad \text{if } t - s \in \mathbb{N}_1,$$

otherwise, $\mathbb{N}_s^t := \emptyset$.

Definition 2.1. *[12] Let $t_0 > 0$ and $h > 0$. For $q > 1$, the (q, h)-time scale is given by*

$$\mathbb{T}^{t_0}_{(q,h)} := \{t_0 q^m + [m]_q h, m \in \mathbb{Z}\} \cup \Big\{\frac{h}{1 - q}\Big\}, \tag{2.1}$$

where $[m]_q := \frac{q^m-1}{q-1}$, $[m]_1 := m$. Particularly, the $(1,h)$-time scale is given by $\mathbb{T}^{t_0}_{(1,h)} := \{t_0 + mh, m \in \mathbb{Z}\}$.

Let $t_0 > 0$, $q > 1$, $h > 0$, and $a \in \mathbb{T}^{t_0}_{(q,h)}$ with $a > \frac{h}{1-q}$. Restrictions of the time scale $\mathbb{T}^{t_0}_{(q,h)}$ are given as

$$\tilde{\mathbb{T}}^{\sigma^i(a)}_{(q,h)} := \{t \in \mathbb{T}^{t_0}_{(q,h)}, t \geq \sigma^i(a)\}, \qquad \text{for } i = 0, 1, 2, \ldots \tag{2.2}$$

$$_{(q,h)}\tilde{\mathbb{T}}^{M}_{\sigma(a)} := \{t \in \mathbb{T}^{t_0}_{(q,h)}, \sigma(a) \leq t \leq M\}, \qquad \text{for } M \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}. \tag{2.3}$$

Here, $\sigma^0(a) = a$ and $\sigma^i(a) = \sigma(\sigma^{i-1}(a))$ for $i \in \mathbb{N}_+$(analogously for ρ^i).

Definition 2.2. *[37] Assume x: $\mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$. Then the nabla (q,h)-derivative of the function x can be defined by*

$$\nabla_{(q,h)}x(t) := \frac{x(t) - x(\rho(t))}{\nu(t)} = \frac{x(t) - x(\tilde{q}(t-h))}{(1-\tilde{q})t + \tilde{q}h}, \tag{2.4}$$

where $\tilde{q} = \frac{1}{q}$, $\sigma(t) = qt + h$, $\rho(t) = \tilde{q}(t-h)$, $\nu(t) = t - \rho(t) = (1-\tilde{q})t + \tilde{q}h$.

$$\sigma^{m+1}(t) := \sigma^m(\sigma(t)), \qquad \rho^{m+1}(t) := \rho^m(\rho(t)), \quad m = 0, 1, \cdots$$

The two standard notations above can help us obtain the following equalities:

$$\sigma^m(t) = q^m t + [m]_q h, \quad \rho^m(t) = q^{-m}(t - [m]_q h).$$

Remark 2.3. *For $t \in \mathbb{T}^{t_0}_{(q,h)}$, the following equalities hold:*

1) $\nu(\sigma^k(t)) = q^{k-1}((q-1)t + h)$;
2) $\nu(\rho^i(\sigma^n(t))) = \tilde{q}^i \nu(\sigma^n(t))$.

Definition 2.4. *[12] $\Gamma_{\tilde{q}}$ is the q-Gamma function, which is introduced as*

$$\Gamma_{\tilde{q}}(x) := \frac{(\tilde{q},\tilde{q})_\infty (1-\tilde{q})^{1-x}}{(\tilde{q}^x,\tilde{q})_\infty}, \quad x \in \mathbb{R}\backslash\{0,-1,-2,\cdots\}, \tag{2.5}$$

where $(a,\tilde{q})_\infty = \prod_{k=0}^{\infty}(1 - a\tilde{q}^k)$.

For $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, the q-binomial coefficient is defined by

$$\begin{bmatrix} x \\ k \end{bmatrix}_{\tilde{q}} := \frac{\Gamma_{\tilde{q}}(x+1)}{\Gamma_{\tilde{q}}(k+1)\Gamma_{\tilde{q}}(x-k+1)}. \tag{2.6}$$

Definition 2.5. *[12] The μ-th power function on $\mathbb{T}^{t_0}_{(q,h)}$ is defined as*

$$(t-\tau)^{(\mu)}_{(\tilde{q},h)} := ([t] - [\tau])^{(\mu)}_{\tilde{q}}, \quad t, \tau \in \mathbb{T}^{t_0}_{(q,h)}, \tag{2.7}$$

where $[s]$ is given by $[s] := s + h\tilde{q}/(1-\tilde{q}) = \frac{\nu(s)}{1-\tilde{q}}$ and $\mu \in \mathbb{R}$.

The μ-th nabla fractional (q,h)-Taylor monomial on $\mathbb{T}^{t_0}_{(q,h)}$ is given by

$$\hat{h}_\mu(t,s) := \frac{(t-s)^{(\mu)}_{(\tilde{q},h)}}{\Gamma_{\tilde{q}}(\mu+1)}, \quad t, s \in \mathbb{T}^{t_0}_{(q,h)}, \tag{2.8}$$

where $\mu \in \mathbb{R}, \tilde{q} = \frac{1}{q}$.

Lemma 2.6. *Let $\mu \neq -1, -2, -3, \cdots$. For $s \in \mathbb{T}_{(q,h)}^{\sigma(a)}$, the following fractional equalities can be derived: (1) $\hat{h}_\mu(s, s) = 0$; (2) ${}_\tau\nabla_{(q,h)}\hat{h}_\mu(s, \tau) = -\hat{h}_{\mu-1}(s, \rho(\tau))$; (3) $\int_a^s \hat{h}_\mu(s, \rho(\tau))\nabla_{(q,h)}\tau = \hat{h}_{\mu+1}(s, a)$.*

Lemma 2.7. *([12] q-Pascal rules) Property of the q-binomial coefficients:*

$$\begin{bmatrix} m \\ l \end{bmatrix}_{\tilde{q}} = \begin{bmatrix} m-1 \\ l-1 \end{bmatrix}_{\tilde{q}} + \tilde{q}^l \begin{bmatrix} m-1 \\ l \end{bmatrix}_{\tilde{q}}. \tag{2.9}$$

Lemma 2.7 is valuable in proving the monotony of the function Φ in Eq (3.1).

Definition 2.8. *[12] Assume $f : \mathbb{T}_{(q,h)}^{\sigma(a)} \to \mathbb{R}$ and $s = \sigma^m(a)$, $m \geq 1$. Then, the nabla (q, h)-integral of f from a to s is defined by*

$$\int_a^s f(\tau)\nabla\tau := \sum_{j=1}^{m} f(\sigma^j(a))\nu(\sigma^j(a)) \tag{2.10}$$

with the standard convention that $\int_a^a f(\tau)\nabla\tau = 0$.

Definition 2.9. *[12] The definition of μ-th ($\mu \in \mathbb{R}^+$) nabla (q, h)-fractional integral is given by*

$${}_a\nabla_{(q,h)}^{-\mu} f(t) := \int_a^t \hat{h}_{\mu-1}(t, \rho(\tau))f(\tau)\nabla\tau = \sum_{j=1}^{n} \hat{h}_{\mu-1}(\sigma^n(a), \sigma^{j-1}(a))f(\sigma^j(a))\nu(\sigma^j(a)). \tag{2.11}$$

Definition 2.10. *[12] The μ-th ($m - 1 < \mu \leq m, m \in \mathbb{Z}^+$) Riemann-Liouville (R-L) nabla (q, h)-fractional difference of function f can be defined by*

$${}_a\nabla_{(q,h)}^{\mu} f(t) := \nabla_{(q,h)}^{m}\, {}_a\nabla_{(q,h)}^{-(m-\mu)} f(t). \tag{2.12}$$

Particularly, for $0 < \mu < 1$, one has

$${}_a\nabla_{(q,h)}^{\mu} f(t) := \nabla_{(q,h)}\, {}_a\nabla_{(q,h)}^{-(1-\mu)} f(t). \tag{2.13}$$

Definition 2.11. *[12] The μ-th ($m - 1 < \mu \leq m, m \in \mathbb{Z}^+$) Caputo nabla (q, h)-fractional difference of function f can be introduced as*

$${}_a^C\nabla_{(q,h)}^{\mu} f(t) := {}_a\nabla_{(q,h)}^{-(m-\mu)} \nabla_{(q,h)}^{m} f(t). \tag{2.14}$$

Particularly, for $0 < \mu < 1$, one has

$${}_a^C\nabla_{(q,h)}^{\mu} f(t) := {}_a\nabla_{(q,h)}^{-(1-\mu)} \nabla_{(q,h)} f(t).$$

Lemma 2.12. *[38, Theorem 3.39] Given two functions $u, v : \mathbb{T}_{(q,h)}^{\sigma(a)} \to \mathbb{R}$. Then, the integration by parts formula in fractional calculus is obtained as*

$$\int_a^t u(\rho(\tau))\nabla_{(q,h)}v(\tau)\nabla_{(q,h)}\tau = u(\tau)v(\tau)\Big|_{\tau=a}^{t} - \int_a^t v(\tau)\nabla_{(q,h)}u(\tau)\nabla_{(q,h)}\tau, \quad t \in \mathbb{T}_{(q,h)}^{\sigma(a)}. \tag{2.15}$$

Lemma 2.13. *Assume* $0 < \mu < 1$*, and* $y : \tilde{\mathbb{T}}^{a}_{(q,h)} \to \mathbb{R}$*. Then,*

$$ {}^{C}_{a}\nabla^{\mu}_{(q,h)} y(t) = -\hat{h}_{-\mu}(t,a)y(a) + \int_a^t \hat{h}_{-\mu-1}(t,\rho(s))y(s)\nabla_{(q,h)}s. \tag{2.16} $$

Proof. By Definition 2.11, one has

$$ \begin{aligned} {}^{C}_{a}\nabla^{\mu}_{(q,h)} y(t) &= {}_{a}\nabla^{-(1-\mu)}_{(q,h)}\nabla_{(q,h)}y(t) \\ &= \int_a^t \hat{h}_{-\mu}(t,\rho(s))\nabla_{(q,h)}y(s)\nabla_{(q,h)}s \\ &\overset{(2.15)}{=} \int_a^t \hat{h}_{-\mu-1}(t,\rho(s))y(s)\nabla_{(q,h)}s + \hat{h}_{-\mu}(t,s)y(s)\Big|_{s=a}^{t} \\ &= \int_a^t \hat{h}_{-\mu-1}(t,\rho(s))y(s)\nabla_{(q,h)}s - \hat{h}_{-\mu}(t,a)y(a), \end{aligned} $$

where, by convention, $\hat{h}_{-\mu}(t,t) = 0$. □

Lemma 2.14. *[12] If* $\mu \in \mathbb{R}, t, s \in \mathbb{T}^{t_0}_{(q,h)}$*,* $t = \sigma^n(s)$ $(n \in \mathbb{N})$*, then,*

$$ \hat{h}_{\mu}(t,s) = (\nu(t))^{\mu} \begin{bmatrix} \mu + n - 1 \\ n - 1 \end{bmatrix}_{\tilde{q}}. \tag{2.17} $$

Lemma 2.15. *[38] Assume* $x : \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)} \to \mathbb{R}$*, and* $\nu, \mu > 0$*. Then,*

$$ {}_{a}\nabla^{-\nu}_{(q,h)}{}_{a}\nabla^{-\mu}_{(q,h)}x(t) = {}_{a}\nabla^{-\nu-\mu}_{(q,h)}x(t). \tag{2.18} $$

Lemma 2.16. *[38] For* μ*-th* $(m-1 < \mu < m, m \in \mathbb{Z}^+)$*, and* $x : \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)} \to \mathbb{R}$*, it follows:*

$$ {}_{a}\nabla^{-(m-\mu)}_{(q,h)}\,{}_{a}\nabla^{m-\mu}_{(q,h)}x(t) = x(t). \tag{2.19} $$

Lemma 2.17. *[37] For* μ*-th* $(m-1 < \mu < m, m \in \mathbb{Z}^+)$*, and* $x : \tilde{\mathbb{T}}^{a}_{(q,h)} \to \mathbb{R}$*, one has*

$$ {}_{a}\nabla^{-(m-\mu)}_{(q,h)}\,{}^{C}_{a}\nabla^{m-\mu}_{(q,h)}x(t) = x(t) - x(a). \tag{2.20} $$

From Lemmas 2.13 and 2.14, we have the following lemma:

Lemma 2.18. *Assume* $x : \mathbb{T}^{\sigma(a)}_{(q,h)} \to \mathbb{R}$ *and* $t = \sigma^n(a)$*,* $n \geq 1$*,* $0 < \mu < 1$*. For* $n \geq 1$*, it follows:*

$$ \begin{aligned} {}^{C}_{a}\nabla^{\mu}_{(q,h)}x(t) = {} & (\nu(\sigma^n(a)))^{-\mu}x(\sigma^n(a)) - (\nu(\sigma^n(a)))^{-\mu}\begin{bmatrix} -\mu + n - 1 \\ n - 1 \end{bmatrix}_{\tilde{q}} x(a) \\ & + \sum_{i=1}^{n-1}(\nu(\sigma^n(a)))^{-\mu-1}\begin{bmatrix} -\mu + i - 1 \\ i \end{bmatrix}_{\tilde{q}} x(\sigma^{n-i}(a))\nu(\sigma^{n-i}(a)). \end{aligned} \tag{2.21} $$

Proof. For simplicity, we denote $x(\sigma^n(a)) := x(n)$. Owing to Lemma 2.13, it follows:

$$\begin{aligned}
{}_a^C\nabla^{\mu}_{(q,h)}x(n) &= -\hat{h}_{-\mu}(t,a)y(a) + \int_a^t \hat{h}_{-\mu-1}(t,\rho(s))y(s)\nabla_{(q,h)}s \\
&= -\hat{h}_{-\mu}(\sigma^n(a),a)x(a) + \sum_{i=1}^{n}\hat{h}_{-\mu-1}(\sigma^n(a),\sigma^{i-1}(a))x(\sigma^i(a))\nu(\sigma^i(a)) \\
&= -\hat{h}_{-\mu}(\sigma^n(a),a)x(a) + \hat{h}_{-\mu-1}(\sigma^n(a),\sigma^{n-1}(a)\nu(\sigma^n(a))x(n) \\
&\quad + \sum_{i=1}^{n-1}(\nu(\sigma^n(a)))^{-\mu-1}\begin{bmatrix}-\mu+i-1\\ i\end{bmatrix}_{\tilde{q}} x(\sigma^{n-i}(a))\nu(\sigma^{n-i}(a)). \\
&= (\nu(\sigma^n(a)))^{-\mu}x(\sigma^n(a)) - (\nu(\sigma^n(a)))^{-\mu}\begin{bmatrix}-\mu+n-1\\ n-1\end{bmatrix}_{\tilde{q}} x(a) \\
&\quad + \sum_{i=1}^{n-1}(\nu(\sigma^n(a)))^{-\mu-1}\begin{bmatrix}-\mu+i-1\\ i\end{bmatrix}_{\tilde{q}} x(\sigma^{n-i}(a))\nu(\sigma^{n-i}(a)).
\end{aligned}$$

□

Lemma 2.19. *Let $u(\sigma^n(a))$ and $\nu(\sigma^n(a))$ be nonnegative functions, $\beta > 0$ and $n \in \mathbb{N}_1$. If $u(\sigma^i(a)) \le \nu(\sigma^i(a))$ for $1 \le i \le n$, then,*

$${}_a\nabla^{-\beta}_{(q,h)}u(\sigma^n(a)) \le {}_a\nabla^{-\beta}_{(q,h)}\nu(\sigma^n(a)).$$

Proof. Owing to Definition 2.9, one has

$$\begin{aligned}
{}_a\nabla^{-\beta}_{(q,h)}u(\sigma^n(a)) &= \int_a^{\sigma^n(a)} \hat{h}_{\beta-1}(\sigma^n(a),\rho(s))u(s)\nabla_{(q,h)}s \\
&= \sum_{i=1}^{n}\hat{h}_{\nu-1}(\sigma^n(a),\sigma^{i-1}(a))u(\sigma^i(a))\nu(\sigma^i(a)) \\
&= \sum_{i=1}^{n-1}\hat{h}_{\nu-1}(\sigma^n(a),\sigma^{i-1}(a))u(\sigma^i(a))\nu(\sigma^i(a)) + (\nu(\sigma^n(a)))^{\beta}u(\sigma^n(a)).
\end{aligned}$$

Similarly, we can derive

$${}_a\nabla^{-\beta}_{(q,h)}\nu(\sigma^n(a)) = \sum_{i=1}^{n-1}\hat{h}_{\nu-1}(\sigma^n(a),\sigma^{i-1}(a))\nu(\sigma^i(a))\nu(\sigma^i(a)) + (\nu(\sigma^n(a)))^{\beta}\nu(\sigma^n(a)).$$

For $\beta > 0, 1 \le i \le n-1$, it has

$$\begin{aligned}
\hat{h}_{\beta-1}(\sigma^n(a),\sigma^{i-1}(a)) &\overset{(2.17)}{=} (\nu(\sigma^n(a)))^{\beta-1}\begin{bmatrix}\beta-1+n-i\\ n-i\end{bmatrix}_{\tilde{q}} \\
&\overset{(2.6)}{=} (\nu(\sigma^n(a)))^{\beta-1}\frac{\Gamma_{\tilde{q}}(\nu+n-i)}{\Gamma_{\tilde{q}}(n-i+1)\Gamma_{\tilde{q}}(\beta)} \\
&\overset{(2.5)}{=} (\nu(\sigma^n(a)))^{\beta-1}\frac{(\tilde{q}^{n-i+1},\tilde{q})_\infty(\tilde{q}^{\beta},\tilde{q})_\infty}{(\tilde{q}^{\alpha+n-i},\tilde{q})_\infty}
\end{aligned}$$

$$= (\nu(\sigma^n(a)))^{\beta-1}\frac{\prod_{k=0}^{\infty}(1-\tilde{q}^{n-i+1+k})\prod_{k=0}^{\infty}(1-\tilde{q}^{\beta+k})}{\prod_{k=0}^{\infty}(1-\tilde{q}^{\beta+n-i+k})} > 0.$$

Thus, for $1 \le i \le n-1$, one has

$$\hat{h}_{\beta-1}(\sigma^n(a),\sigma^{i-1}(a))u(\sigma^i(a))\nu(\sigma^i(a)) \le \hat{h}_{\beta-1}(\sigma^n(a),\sigma^{i-1}(a))\nu(\sigma^i(a))\nu(\sigma^i(a)),$$

and

$$(\nu(\sigma^n(a)))^{\beta}u(\sigma^n(a)) \le (\nu(\sigma^n(a)))^{\beta}\nu(\sigma^n(a)).$$

Therefore, ${}_a\nabla_{(q,h)}^{-\beta}u(\sigma^n(a)) \le {}_a\nabla_{(q,h)}^{-\beta}\nu(\sigma^n(a))$. □

3. Discrete (q,h)-fractional Bihari inequality

In this section, a new (q,h)-fractional Bihari inequality is developed.

According to the proof of Lemma 3.1 of [24], the discrete (q,h)-fractional Bihari inequality can be further generalized as follows:

Lemma 3.1. *Assume $x : [0,\infty) \to [0,\infty)$, $0 < \mu < 1$. Let x be a continuous and nondecreasing function with $x(0) = 0$. For any positive sequence $\{V(\sigma^n(a))|n \in \mathbb{N}_0\}$, $\Phi(u)$ is a solution to Eq* (3.1),

$${}_a^C\nabla_{(q,h)}^{\mu}\Phi(V(\sigma^n(a))) = \frac{{}_a^C\nabla_{(q,h)}^{\mu}V(\sigma^n(a))}{x(V(\sigma^n(a)))}. \tag{3.1}$$

If $0 < m_2 < m_1$, then $\Phi(m_2) < \Phi(m_1)$.

Proof. For simplicity, we denote $V(\sigma^i(a)) := V_i$, where $i \in \{0,1,2,3,\cdots,n\}$.

By Lemma 2.18, it follows:

$$\begin{aligned}{}_a^C\nabla_{(q,h)}^{\mu}\Phi(V_n) &= (\nu(\sigma^n(a)))^{-\mu}\Phi(V_n) - (\nu(\sigma^n(a)))^{-\mu}\begin{bmatrix}n-\mu-1\\ n-1\end{bmatrix}_{\tilde{q}}\Phi(V_0)\\ &\quad+\sum_{i=1}^{n-1}(\nu(\sigma^n(a)))^{-\mu-1}\begin{bmatrix}i-\mu-1\\ i\end{bmatrix}_{\tilde{q}}\nu(\sigma^{n-i}(a))\Phi(V_{n-i}),\\ {}_a^C\nabla_{(q,h)}^{\mu}V_n &= (\nu(\sigma^n(a)))^{-\mu}V_n - (\nu(\sigma^n(a)))^{-\mu}\begin{bmatrix}n-\mu-1\\ n-1\end{bmatrix}_{\tilde{q}}V_0\\ &\quad+\sum_{i=1}^{n-1}(\nu(\sigma^n(a)))^{-\mu-1}\begin{bmatrix}i-\mu-1\\ i\end{bmatrix}_{\tilde{q}}\nu(\sigma^{n-i}(a))V_{n-i}.\end{aligned}$$

According to Eq (3.1), one has

$$\begin{aligned}&(\nu(\sigma^n(a)))^{-\mu}\Phi(V_n) - (\nu(\sigma^n(a)))^{-\mu}\begin{bmatrix}n-\mu-1\\ n-1\end{bmatrix}_{\tilde{q}}\Phi(V_0)\\ &\quad+\sum_{i=1}^{n-1}(\nu(\sigma^n(a)))^{-\mu-1}\begin{bmatrix}i-\mu-1\\ i\end{bmatrix}_{\tilde{q}}\nu(\sigma^{n-i}(a))\Phi(V_{n-i})\end{aligned}$$

$$
\begin{aligned}
&= (\nu(\sigma^n(a)))^{-\mu} V_n - (\nu(\sigma^n(a)))^{-\mu} \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}} V_0 \\
&\quad + \sum_{i=1}^{n-1} (\nu(\sigma^n(a)))^{-\mu-1} \begin{bmatrix} i-\mu-1 \\ i \end{bmatrix}_{\tilde{q}} \nu(\sigma^{n-i}(a)) V_{n-i},
\end{aligned}
$$

which leads easily to

$$
\begin{aligned}
\Phi(V_n) &= \frac{V_n}{x(V_n)} + \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \left(\Phi(V_0) - \frac{V_0}{x(V_n)}\right) \\
&\quad + \sum_{i=1}^{n-1} \begin{bmatrix} i-\mu-1 \\ i \end{bmatrix}_{\tilde{q}} (\nu(\sigma^n(a)))^{-1} \nu(\sigma^{n-i}(a)) \Big(\frac{V_{n-i}}{x(V_n)} - \Phi(V_{n-i})\Big).
\end{aligned} \tag{3.2}
$$

In particular, we take a sequence $\{V_j\}$ that satisfies the following conditions:

$$
V_j = \begin{cases} m_1, & \text{if } 0 \le j \le n-1, \\ m_2, & \text{if } j \ge n, \end{cases}
$$

where $0 < m_2 < m_1$.

For $0 < m_2 < m_1$, attending to Eq (3.2), it follows:

$$
\begin{aligned}
\Phi(m_2) &= \Phi(V_n) \\
&= \frac{m_2}{x(m_2)} + \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \left(\Phi(m_2) - \frac{m_1}{x(m_2)}\right) \\
&\quad + \sum_{i=1}^{n-1} \begin{bmatrix} i-\mu-1 \\ i \end{bmatrix}_{\tilde{q}} (\nu(\sigma^n(a)))^{-1} \nu(\sigma^{n-i}(a)) \Big(\frac{m_1}{x(m_2)} - \Phi(m_1)\Big) \\
&= \frac{m_2}{x(m_2)} + \frac{m_1}{x(m_2)} \Big(\sum_{i=1}^{n-1} \begin{bmatrix} i-\mu-1 \\ i \end{bmatrix}_{\tilde{q}} (\nu(\sigma^n(a)))^{-1} \nu(\sigma^{n-i}(a)) - \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \Big) \\
&\quad + \Phi(m_1) \Big(-\sum_{i=1}^{n-1} \begin{bmatrix} i-\mu-1 \\ i \end{bmatrix}_{\tilde{q}} (\nu(\sigma^n(a)))^{-1} \nu(\sigma^{n-i}(m_1)) + \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \Big) \\
&= \frac{m_2}{x(m_2)} + \frac{m_1}{x(m_2)} \Big(\sum_{i=1}^{n-1} \begin{bmatrix} i-\mu-1 \\ i \end{bmatrix}_{\tilde{q}} (\nu(\sigma^n(a)))^{-1} \nu(\rho^i(\sigma^n(a))) - \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \Big) \\
&\quad + \Phi(m_1) \Big(-\sum_{i=1}^{n-1} \begin{bmatrix} i-\mu-1 \\ i \end{bmatrix}_{\tilde{q}} (\nu(\sigma^n(a)))^{-1} \nu(\rho^i(\sigma^n(a))) + \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \Big) \\
&\overset{Remark 2.3(2)}{=} \frac{m_2}{x(m_2)} + \frac{m_1}{x(m_2)} \Big(\sum_{i=1}^{n-1} \begin{bmatrix} i-\mu-1 \\ i \end{bmatrix}_{\tilde{q}} \tilde{q}^i - \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \Big) \\
&\quad + \Phi(m_1) \Big(-\sum_{i=1}^{n-1} \begin{bmatrix} i-\mu-1 \\ i \end{bmatrix}_{\tilde{q}} \tilde{q}^i + \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \Big) \\
&\overset{(2.9)}{=} \frac{m_2}{x(m_2)} + \frac{m_1}{x(m_2)} \Big(\sum_{i=1}^{n-1} \begin{bmatrix} i-\mu \\ i \end{bmatrix}_{\tilde{q}} - \sum_{i=1}^{n-1} \begin{bmatrix} i-\mu-1 \\ i-1 \end{bmatrix}_{\tilde{q}} - \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \Big)
\end{aligned}
$$

$$+\Phi(m_1)\Big(-\sum_{i=1}^{n-1}\begin{bmatrix} i-\mu \\ i \end{bmatrix}_{\tilde{q}} + \sum_{i=1}^{n-1}\begin{bmatrix} i-\mu-1 \\ i-1 \end{bmatrix}_{\tilde{q}} + \begin{bmatrix} n-\mu-1 \\ n-1 \end{bmatrix}_{\tilde{q}}\Big)$$

$$= \frac{m_2}{x(m_2)} - \frac{m_1}{x(m_2)} + \Phi(m_1). \tag{3.3}$$

Thus, for $0 < m_2 < m_1$, one has $\Phi(m_2) < \Phi(m_1)$. The proof is completed. □

Remark 3.2. *If* $0 < m_2 < m_1$, *and* $\Phi(m_2)$ *is defined as* (3.3), *one has*

$$\lim_{m_2 \to 0^+} \Phi(m_2) = -\infty,$$

which is critical in the proof of the uniqueness theorem.

Remark 3.3. *According to equality* (3.2), *we can clarify the existence of a solution to Eq* (3.1).

Theorem 3.4. (Discrete (q,h)-fractional Bihari inequality) *Suppose* $0 < \mu < 1$, $u(t) : \tilde{\mathbb{T}}^a_{(q,h)} \to [0,+\infty)$, *and* $c > 0$ *is a constant. Assume* $[0,+\infty) \to [0,+\infty)$, *and* ϕ *is a continuous and nondecreasing function with* $\phi(0) = 0$. *If*

$$u(t) \leq c + {}_a\nabla^{-\mu}_{(q,h)}\phi(u(t)), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}^T_{\sigma(a)}, \tag{3.4}$$

where $T \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$, then,

$$u(t) \leq \Phi^{-1}\Big(\Phi(c) + \hat{h}_\mu(t,a)\Big), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}^T_{\sigma(a)}, \tag{3.5}$$

where $\Phi(u)$ is a solution to

$${}^C_a\nabla^{\mu}_{(q,h)}\Phi(V(\sigma^n(a))) = \frac{{}^C_a\nabla^{\mu}_{(q,h)}V(\sigma^n(a))}{\phi(V(\sigma^n(a)))}$$

for any positive sequence $\{V(\sigma^n(a)) | n \in \mathbb{N}_0\}$.

Proof. Let $v(t)$ be the right-hand side of the inequality (3.4), namely

$$v(t) = c + {}_a\nabla^{-\mu}_{(q,h)}\phi(u(t)), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}^T_{\sigma(a)}.$$

It follows that

$$u(t) \leq v(t), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}^T_{\sigma(a)}. \tag{3.6}$$

For $t \in {}_{(q,h)}\tilde{\mathbb{T}}^T_{\sigma(a)}$, one has

$$\begin{aligned}
{}^C_a\nabla^{\mu}_{(q,h)}v(t) &= {}_a\nabla^{-(1-\mu)}_{(q,h)}\nabla_{(q,h)}v(t) \\
&= {}_a\nabla^{-(1-\mu)}_{(q,h)}\nabla_{(q,h)}\Big(c + {}_a\nabla^{-\mu}_{(q,h)}\phi(u(t))\Big) \\
&= {}_a\nabla^{-(1-\mu)}_{(q,h)}\nabla_{(q,h)a}\nabla^{-\mu}_{(q,h)}\phi(u(t)) \\
&\overset{(2.13)}{=} {}_a\nabla^{-(1-\mu)}_{(q,h)}\,{}_a\nabla^{1-\mu}_{(q,h)}\phi(u(t)) \\
&\overset{(2.19)}{=} \phi(u(t)).
\end{aligned}$$

Combining the monotonicity of ϕ and the inequality (3.6) yields

$$ {}_a^C\nabla_{(q,h)}^{\mu} v(t) \leq \phi(v(t)), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}_{\sigma(a)}^{T}. $$

Therefore,

$$ {}_a^C\nabla_{(q,h)}^{\mu} \Phi(v(t)) = \frac{{}_a^C\nabla_{(q,h)}^{\mu} v(t)}{\phi(v(t))} \leq \frac{\phi(v(t))}{\phi(v(t))} = 1, \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}_{\sigma(a)}^{T}. \tag{3.7} $$

Using Lemma 2.19 on the inequality (3.7), one has

$$ {}_a\nabla_{(q,h)}^{-\mu} {}_a^C\nabla_{(q,h)}^{\mu} \Phi(v(t)) \leq \hat{h}_{\mu}(t,a), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}_{\sigma(a)}^{T}, \tag{3.8} $$

where we use

$$ {}_a\nabla_{(q,h)}^{-\mu} 1 = \int_0^t \hat{h}_{\mu-1}(t,\rho(\tau))\nabla_{(q,h)}\tau \overset{\text{Lem.2.6}}{=} \hat{h}_{\mu}(t,a). $$

By Lemma 2.17, we get

$$ \Phi(v(t)) \leq \Phi(v(a)) + \hat{h}_{\mu}(t,a), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}_{\sigma(a)}^{T}. \tag{3.9} $$

Using the monotonicity of Φ and the inequality (3.9) obtains

$$ v(t) \leq \Phi^{-1}\Big(\Phi(v(a)) + \hat{h}_{\mu}(t,a)\Big), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}_{\sigma(a)}^{T}. $$

Consequently,

$$ u(t) \leq v(t) \leq \Phi^{-1}\Big(\Phi(c) + \hat{h}_{\mu}(t,a)\Big), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}_{\sigma(a)}^{T}. $$

This completes the proof. □

4. Applications of discrete (q, h)-fractional Bihari inequality

In this section, by using the discrete (q, h)-fractional Bihari inequality, the uniqueness and finite-time stability of solutions to the fractional (q, h)-difference equation are derived.

Consider the fractional (q, h)-difference initial value problem

$$ \begin{aligned} {}_a^C\nabla_{(q,h)}^{\mu} y(t) &= g(y(t)), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}_{\sigma(a)}^{T}, \\ y(a) &= y_0, \end{aligned} \tag{4.1} $$

where $0 < \mu < 1$, $g : \mathbb{R} \to [0, +\infty)$, $y(t) : \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)} \to \mathbb{R}$, $T \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$.

Theorem 4.1. **(Uniqueness theorem)** *If a solution of the (q, h)-fractional difference initial value problem (FDIVP)* (4.1) *exists, then the Eq* (4.1) *has a unique solution on ${}_{(q,h)}\tilde{\mathbb{T}}_{\sigma(a)}^{T}$ if the function f satisfies:*

$$ |g(y(t)) - g(\tilde{y}(t))| \leq \phi(|y(t) - \tilde{y}(t)|), \tag{4.2} $$

where $\phi(u) : [0, \infty) \to [0, \infty)$, ϕ is a continuous and nondecreasing function with $\phi(0) = 0$.

Proof. Assume that $y(t)$ and $\tilde{y}(t)$ are solutions of Eq (4.1). Applying the operator ${}_a\nabla^{-\mu}_{(q,h)}$ on both sides of Eq (4.1), we obtain:

$$ {}_a\nabla^{-\mu}_{(q,h)}\,{}_a^C\nabla^{\mu}_{(q,h)}y(t) = y(t) - y_0 = {}_a\nabla^{-\mu}_{(q,h)}g(y(t)), $$

which implies

$$ y(t) = y_0 + {}_a\nabla^{-\mu}_{(q,h)}g(y(t)). $$

Similarly, we get

$$ \tilde{y}(t) = y_0 + {}_a\nabla^{-\mu}_{(q,h)}g(\tilde{y}(t)). $$

Thus,

$$ \begin{aligned} |y(t) - \tilde{y}(t)| &\le {}_a\nabla^{-\mu}_{(q,h)}|g(y(t)) - g(\tilde{y}(t))| \\ &\le {}_a\nabla^{-\mu}_{(q,h)}\phi(|y(t) - \tilde{y}(t)|) \\ &< \varepsilon + {}_a\nabla^{-\mu}_{(q,h)}\phi(|y(t) - \tilde{y}(t)|). \end{aligned} $$

Due to Theorem 3.4, it becomes

$$ |y(t) - \tilde{y}(t)| < \Phi^{-1}\Big(\Phi(\varepsilon) + \hat{h}_{\mu}(t,a)\Big), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}^{T}_{\sigma(a)}, $$

where $\Phi(u)$ is a solution of

$$ {}_a^C\nabla^{\mu}_{(q,h)}\Phi(V(\sigma^n(a))) = \frac{{}_a^C\nabla^{\mu}_{(q,h)}V(\sigma^n(a))}{\phi(V(\sigma^n(a)))} $$

for any positive sequence $\{V(\sigma^n(a))|n \in \mathbb{N}_0\}$. According to Remark 3.2, one has

$$ \lim_{\varepsilon\to 0^+} \Phi(\varepsilon) = -\infty. $$

From here, it follows:

$$ \lim_{\xi\to -\infty} \Phi^{-1}(\xi) = 0. $$

Hence,

$$ \lim_{\varepsilon\to 0^+} \Phi^{-1}\Big(\Phi(\varepsilon) + \hat{h}_{\mu}(t,a)\Big) = 0 \tag{4.3} $$

for $t \in {}_{(q,h)}\tilde{\mathbb{T}}^{T}_{\sigma(a)}$. Since ε is arbitrary, Eq (4.3) means that $y(t) = \tilde{y}(t)$ for $t \in {}_{(q,h)}\tilde{\mathbb{T}}^{T}_{\sigma(a)}$. Thus, the uniqueness theorem is proved. □

Definition 4.2. *[27, 32] The fractional difference initial value problem* (4.1) *is finite-time stable w.r.t.* $\{T, \delta, \epsilon\}$ *with* $0 < \delta < \epsilon$ *if and only if* $\|x_0\| < \delta$ *implies* $\|x(t)\| < \epsilon$ *for any* $t \in {}_{(q,h)}\tilde{\mathbb{T}}^{T}_{\sigma(a)}$.

Theorem 4.3. *Let* $0 < \mu < 1$ *and* $g : \mathbb{R} \to [0, +\infty)$*, where* g *is a continuous and nondecreasing function with* $g(0) = 0$*. If*

$$ \delta + \phi(\delta)\hat{h}_{\alpha}(t,a) \le \epsilon, \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}^{T}_{\sigma(a)}, \tag{4.4} $$

for $0 < \delta < \epsilon$ and $T \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$, then the system (4.1) is finite-time stable w.r.t $\{T, \delta, \epsilon\}$, where $\Psi(u)$ is a solution of

$$ {}_a^C\nabla^{\mu}_{(q,h)}\Psi(V(\sigma^n(a))) = \frac{{}_a^C\nabla^{\mu}_{(q,h)}V(\sigma^n(a))}{g(V(\sigma^n(a)))} $$

for any positive sequence $\{V(\sigma^n(a))|n \in \mathbb{N}_0\}$.

Proof. Applying the operator ${}_a\nabla^{-\mu}_{(q,h)}$ on both sides of (4.1) yields

$$ {}_a\nabla^{-\mu}_{(q,h)}{}^C_a\nabla^{\mu}_{(q,h)}y(t) = y(t) - y_0 = {}_a\nabla^{-\mu}_{(q,h)}g(y(t)), $$

which implies

$$ |y(t)| \leq |y_0| + |{}_a\nabla^{-\mu}_{(q,h)}g(y(t))| < \delta + {}_a\nabla^{-\mu}_{(q,h)}g(|y(t)|). $$

According to Theorem 3.4, one has

$$ |y(t)| < \Psi^{-1}\Big(\Psi(\delta) + \hat{h}_\mu(t,a)\Big), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}^T_{\sigma(a)}, $$

where $\Psi(u)$ is a solution of

$$ {}^C_a\nabla^{\mu}_{(q,h)}\Psi(V(\sigma^n(a))) = \frac{{}^C_a\nabla^{\mu}_{(q,h)}V(\sigma^n(a))}{g(V(\sigma^n(a)))} $$

for any positive sequence $\{V(\sigma^n(a))|n \in \mathbb{N}_0\}$.

Formula (3.3) and inequality (4.4) lead to

$$ \Psi(\delta) - \Psi(\epsilon) = \frac{\delta - \epsilon}{g(\delta)} \leq -\hat{h}_\mu(t,a), $$

where $t \in {}_{(q,h)}\tilde{\mathbb{T}}^T_{\sigma(a)}$. Thus,

$$ \Psi(\delta) + \hat{h}_\mu(t,a) \leq \Psi(\epsilon), \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}^T_{\sigma(a)}. $$

By using Lemma 3.4, one has

$$ |x(t)| < \Psi^{-1}\Big(\Psi(\delta) + \hat{h}_\mu(t,a)\Big) \leq \epsilon, \quad t \in {}_{(q,h)}\tilde{\mathbb{T}}^T_{\sigma(a)}. $$

This ends the proof. □

5. An illustrative example

In the last section, a nonlinear example is provided to numerically illustrate the results.

Example 5.1. *Consider the Caputo nabla fractional difference equation*

$$ \begin{aligned} {}^C_1\nabla^{0.5}_{(2,1)}x(t) &= x^{\frac{1}{3}}(t), \quad t \in {}_{(2,1)}\tilde{\mathbb{T}}^T_{\sigma(1)} \\ x(1) &= 0.0005, \end{aligned} \tag{5.1} $$

where $T \in \tilde{\mathbb{T}}^{\sigma(1)}_{(2,1)}$. We have $g(x(t)) = x^{\frac{1}{3}}(t)$ and $\delta = 0.001$. To begin, we need to prove the existence and uniqueness of the solution to (5.1). Let $t = \sigma^n(1)$, where $t \in {}_{(2,1)}\tilde{\mathbb{T}}^T_{\sigma(1)}$. According to (2.21), we have

$$ {}^C_1\nabla^{0.5}_{(q,h)}x(t) = (\nu(\sigma^n(1)))^{-0.5}x(\sigma^n(1)) - (\nu(\sigma^n(1)))^{-0.5}\begin{bmatrix} -0.5+n-1 \\ n-1 \end{bmatrix}_{\tilde{q}} x(1) $$

$$
\begin{aligned}
&+\sum_{i=1}^{n-1}(\nu(\sigma^n(1)))^{-0.5-1}\begin{bmatrix}-0.5+i-1\\ i\end{bmatrix}_{\tilde{q}} x(\sigma^{n-i}(1))\nu(\sigma^{n-i}(1))\\
&= x^{\frac{1}{3}}(\sigma^n(1)).
\end{aligned}
$$

For simplicity, we define $x(\sigma^n(1)) := x_n$, then,

$$
\begin{aligned}
&(\nu(\sigma^n(1)))^{-0.5}x_n - x_n^{\frac{1}{3}} - (\nu(\sigma^n(1)))^{-0.5}\begin{bmatrix}-0.5+n-1\\ n-1\end{bmatrix}_{\tilde{q}} x_0\\
&+\sum_{i=1}^{n-1}(\nu(\sigma^n(1)))^{-0.5-1}\begin{bmatrix}-0.5+i-1\\ i\end{bmatrix}_{\tilde{q}} \nu(\sigma^{n-i}(1))x_{n-i} = 0.
\end{aligned}
$$

Let

$$
\begin{aligned}
\varphi_n(x_n) = &(\nu(\sigma^n(1)))^{-0.5}x_n - x_n^{\frac{1}{3}} - (\nu(\sigma^n(1)))^{-0.5}\begin{bmatrix}-0.5+n-1\\ n-1\end{bmatrix}_{\tilde{q}} x_0\\
&+\sum_{i=1}^{n-1}(\nu(\sigma^n(1)))^{-0.5-1}\begin{bmatrix}-0.5+i-1\\ i\end{bmatrix}_{\tilde{q}} \nu(\sigma^{n-i}(1))x_{n-i}.
\end{aligned}
\tag{5.2}
$$

Let

$$
\varphi_n(x_n) = 0(n \in \mathbb{N}_1). \tag{5.3}
$$

Next, we will discuss the existence of a positive solution x_n to Eq (5.3) using the concept of strong induction. Assume $x_i > 0$ for $i \in \mathbb{N}_1^{k-1}$.

Since $\begin{bmatrix}-0.5+k-1\\ k-1\end{bmatrix}_{\tilde{q}} > 0$ and $\begin{bmatrix}-0.5+i-1\\ i\end{bmatrix}_{\tilde{q}} < 0$, $x_i > 0$ for $i \in \mathbb{N}_1^{k-1}$, one has

$$
\begin{aligned}
\varphi_k(0) = &-(\nu(\sigma^k(1)))^{-\alpha}\begin{bmatrix}-0.5+k-1\\ k-1\end{bmatrix}_{\tilde{q}} x_0\\
&+\sum_{i=1}^{k-1}(\nu(\sigma^k(1)))^{-0.5-1}\begin{bmatrix}-0.5+i-1\\ i\end{bmatrix}_{\tilde{q}} \nu(\sigma^{k-i}(1))x_{k-i} < 0.
\end{aligned}
$$

From Eq (5.2), we have $\lim_{x_k\to+\infty}\varphi_k(x_k) = +\infty$. Note that $\varphi_k(x_k)$ is continuous with respect to x_k, thus, the existence of a positive solution x_n to Eq (5.3) is obtained.

Then, applying Theorem 4.1, the uniqueness of the solution to Eq (5.1) is proven.

We assume that there are two positive solutions $z(t)$ and $\tilde{z}(t)$ to Eq (5.1) with $z(1) = \tilde{z}(1) = 0.0005$, hence, we obtain the following inequality:

$$
\begin{aligned}
|g(z(t)) - g(\tilde{z}(t))| &= |z^{\frac{1}{3}}(t) - \tilde{z}^{\frac{1}{3}}(t)|\\
&= \frac{|z(t)-\tilde{z}(t)|}{z^{\frac{2}{3}}(t) + z^{\frac{1}{3}}(t)\tilde{z}^{\frac{1}{3}}(t) + \tilde{z}^{\frac{2}{3}}(t)}\\
&\le \frac{|z(t)-\tilde{z}(t)|}{|z(t)-\tilde{z}(t)|^{\frac{2}{3}}}
\end{aligned}
$$

$$= |z(t) - \tilde{z}(t)|^{\frac{1}{3}}$$
$$= \phi(|z(t) - \tilde{z}(t)|),$$

where $\phi(|z(t)|) = |z(t)|^{\frac{1}{3}}$. By Theorem 4.1, we obtain that the solution to Eq (5.1) is unique.

Finally, using the criterion (4.4) in Theorem 4.3, the largest possible bounds ϵ of the system (5.1) are shown in Table 1.

Table 1. ϵ for $\delta = 0.001$ and T varies in Example 5.1.

T	3	7	15	31	63
ϵ	0.143	0.260	0.403	0.593	0.854

6. Conclusions

In this paper, we have studied the discrete fractional Bihari inequality and applied it to obtain the uniqueness and finite-time stability of solutions of fractional difference equations with non-Lipschitz nonlinearities. In addition, we have provided an example to illustrate the effectiveness and rationality of the uniqueness and finite-time stability numerically.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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