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Research article

Existence results for a coupled system of nonlinear fractional functional differential equations with infinite delay and nonlocal integral boundary conditions

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Abstract: This article is devoted to studying a new class of nonlinear coupled systems of fractional differential equations supplemented with nonlocal integro-coupled boundary conditions and affected by infinite delay. We first transform the boundary value problem into a fixed-point problem, and, with the aid of the theory of infinite delay, we assume an appropriate phase space to deal with the obtained problem. Then, the existence result of solutions to the given system is investigated by employing Schaefer's fixed-point theorem, while the uniqueness result is established in view of the Banach contraction mapping principle. The illustrative examples are constructed to ensure the availability of the main results.

Keywords: coupled system; infinite delay; Caputo fractional derivative; integral boundary conditions; existence
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1. Introduction

Differential equations of fractional order have constituted in recent years a significant field of applied research due to their ability to describe many real-world phenomena more accurately than those of integer order. As many processes and changes in nature are affected by delays when they occur, differential equations with finite and infinite delay attracted the scholars to model many applications as delayed differential equations, for example, co-infection of malaria and HIV/AIDS [1, 2], predator and prey models [3, 4], output feedback controller systems [5], BAM neural networks [6], neural networks with Lévy noise network-based control systems [7], HFMD model [8], etc.

The coupled systems of different types of fractional differential equations that are associated with different types of initial and boundary conditions play an important role in mathematical modeling, as such systems occur in various problems of applied sciences such as engineering and physical applications. Therefore, there are many applications and studies dealing with coupled systems; for instance, Hu and Zhang [9] investigated the existence of solutions for a coupled system of p-Laplacian fractional differential equations with infinite-point boundary conditions by applying coincidence degree theory. In [10], the authors derive the existence and uniqueness results for a class of coupled systems involving implicit fractional differential equations with periodic boundary conditions. The existence and uniqueness of integrable solutions for a nonlinear coupled system of fractional differintegral equations with weighted initial conditions were established in [11]. Ahmad et al. [12] obtained, with the aid of Schaefer's and Banach fixed point theorems, the existence results for a nonlinear coupled system involving different orders of both Riemann-Liouville and Caputo generalized fractional derivatives and equipped with Riemann Stieltjes type integral boundary conditions. Aljodi [13] studied a new class of coupled systems of Caputo-Fabrizio differential equations equipped with nonlocal coupled boundary conditions and established the existence and uniqueness results based on Banach and Krasnoselskii fixed point theorems. Zhao [14] discussed the existence criteria for the solutions of a class of coupled systems involving Atangana-Baleanu fractional order differential equations with (p_1, p_1)-Laplacian operators and proved the stability of the obtained system by means of generalized Ulam-Hyers stability. In [15], the authors investigated a new class of coupled implicit systems involving ϱ-fractional derivatives of different orders and anti-periodic boundary conditions. Zhao [16] established important results related to the stability, existence, and uniqueness of the solutions of a coupled system involving Atangana-Baleanu-Caputo fractional differential equations with a Laplacian operator and impulses using generalized Ulam-Hyers for the stability and by an F-contractive operator and a fixed-point theorem on metric space for the uniqueness.

The theory of differential equations with infinite delay has gained much attention since the early 1970s, and has developed rapidly since that time. General axioms and theorems were established to deal with this kind of equation. The appropriate selection of the phase space $\mathfrak{F}$ was very important in the study of differential equations with infinite delay, which was identified by specific axioms, see [17–19]. For more details on the theoretical aspects of differential equations with unbounded delay, we refer the reader to the works [17, 20–23]. Although there has recently been considerable work on fractional differential equations with infinite delay; see, for instance [24–31], the studies on coupled systems of delayed differential equations, especially with infinite delay, are limited; for instance, see [3, 32–34].

Recently, Liu and Zhao [33] investigated the existence results in a Banach space for a coupled system involving neutral integro-functional differential equations of fractional order between 0 and 1, with infinite delay, by applying standard fixed-point theorems.

Our goal in this work is to extend the previous studies on coupled systems with infinite delay by introducing a new class of the form:

$$\begin{cases} {}^{C}D^{\varsigma}_{0^+}u_1(t) = f(t, u_{1t}, u_{2t}), \quad t \in \Omega := [0,1], \\ u_1(t) = \eta_1(t), \quad t \in (-\infty, 0], \\ {}^{C}D^{\gamma}_{0^+}u_2(t) = g(t, u_{1t}, u_{2t}), \quad t \in \Omega := [0,1], \\ u_2 = \eta_2(t), \quad t \in (-\infty, 0], \\ u_1(1) = \lambda_1 \displaystyle\int_0^{\sigma_1} u_2(s)ds, \quad u_2(1) = \lambda_2 \displaystyle\int_0^{\sigma_2} u_1(s)ds, \quad \sigma_1, \sigma_2 \in (0,1), \end{cases} \tag{1.1}$$

where ${}^{C}D^{\varsigma}_{0^+}$, ${}^{C}D^{\gamma}_{0^+}$ are the Caputo derivatives of fractional order $\varsigma, \gamma \in (1,2]$, respectively, and λ_1, λ_2 are constants. $f, g : \Omega \times \mathfrak{T} \to \mathbb{R}$ are continuous functions and $\eta_i \in \mathfrak{T}$ such that $\eta_i(0) = 0$, for $i = 1, 2$, where $\mathfrak{T}$ is denoted as a phase space that is specified in Section 2. The functions $u_{it} : (-\infty, 0] \to \mathbb{R}$, which are elements in $\mathfrak{T}$, are defined as $u_{it}(\tau) = u_i(t+\tau)$, $\tau \le 0$, for $t \in \Omega$ and $u_i : (-\infty, 1] \to \mathbb{R}$, $i = 1, 2$.

The remaining content of this article is presented as follows: Some basic materials related to our work are presented, and the integral equation that is equivalent to the solution for the linear variant of problem (1.1) is derived in Section 2. In Section 3, we establish our main results with the aid of Schaefer's theorem and the contraction mapping principle, and we provide some examples to illustrate the obtained results. Finally, the conclusion is presented in Section 4.

2. Preliminaries

For this paper, we define the phase space $(\mathfrak{T}, \|.\|_{\mathfrak{T}})$ as a seminormed linear space of functions that map $(-\infty, 0]$ into $\mathbb{R}$ and satisfy the following fundamental axioms, see [22]:

(N_1) For a function x that maps $(-\infty, 1]$ into $\mathbb{R}$, such that $x_0 \in \mathfrak{T}$, and for each $t \in [0,1]$, the following conditions hold:

(1) x_t is an element in $\mathfrak{T}$,

(2) $\|x_t\|_{\mathfrak{T}} \le q(t)\sup\{|x(s)| : 0 \le s \le t\} + p(t)\|x_0\|_{\mathfrak{T}}$, where $q, p : [0, \infty) \to [0, \infty)$ are two functions independent of $x(.)$ such that q is a continuous, p is a locally bounded, and

$$q^* = \sup\{|q(t)| : t \in [0,1]\}, \quad p^* = \sup\{|p(t)| : t \in [0,1]\},$$

(3) $|x(t)| \le L\|x_t\|_{\mathfrak{T}}$, where $L \ge 0$ be a constant.

(N_2) For $x(.)$ satisfies (N_1), x_t is a continuous $\mathfrak{T}-$ valued function on $[0,1]$,

(N_3) $\mathfrak{T}$ is a complete space.

Next, we define the space $\mathfrak{T}_a = \{x : (-\infty, 1] \to \mathbb{R} : x|_{(-\infty,0]} \in \mathfrak{T}$ and $x|_{[0,1]} \in C(\Omega, \mathbb{R})\}$ with a seminorm $\|.\|_{\mathfrak{T}_a}$ defined by $\|x\|_{\mathfrak{T}_a} = \|\eta\|_{\mathfrak{T}} + \sup_{s\in\Omega}|x(s)|$, $x \in \mathfrak{T}_a$ and $x(t) = \eta(t)$ for $t \in (-\infty, 0]$.

Definition 2.1. *[35] The Riemann-Liouville fractional integral for a function $h : [0, \infty) \to \mathbb{R}$, of order $\beta > 0$ is defined by*

$$I^{\beta}_{0^+}h(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds, \quad t > 0.$$

Definition 2.2. *[35] The Caputo fractional derivative of order β for a function $h : [0,\infty] \to \mathbb{R}$ with $h(t) \in AC^n[0,\infty)$ is defined by*

$$ {}^C D^{\beta}_{0^+} h(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{\beta-n+1}} ds = I^{n-\beta}_{0^+} h^{(n)}(t),\ t > 0,\ n-1 < \beta < n, $$

where $n = [\beta] + 1$.

Lemma 2.1. *[35] Let $\beta > 0$ and $h(t) \in AC^n[0,\infty)$ or $C^n[0,\infty)$. Then*

$$ (I^{\beta}_{0^+}\ {}^C D^{\beta}_{0^+} h)(t) = h(t) - \sum_{k=0}^{n-1} \frac{h^{(k)}(0)}{k!} t^k,\ t > 0,\ n-1 < \beta < n. \tag{2.1} $$

In the following, we prove an auxiliary lemma that is associated with the linear variant of problem (1.1).

Lemma 2.2. *Let $h_1, h_2 \in C(0,1)$ and $u_1, u_2 \in AC(\Omega, \mathbb{R}) \cap \mathfrak{T}_a$ and*

$$ \Lambda_1 = 1 - \frac{\lambda_1 \lambda_2 \sigma_1^2 \sigma_2^2}{4} \neq 0. \tag{2.2} $$

Then, the unique solution to the problem is:

$$ \begin{cases} {}^C D^{\varsigma}_{0^+} u_1(t) = h_1(t),\ \ {}^C D^{\gamma}_{0^+} u_2(t) = h_2(t), \quad t \in \Omega := [0,1], \\ u_1(t) = \eta_1(t),\ \ u_2(t) = \eta_2(t), \qquad\qquad t \in (-\infty, 0], \\ u_1(1) = \lambda_1 \displaystyle\int_0^{\sigma_1} u_2(s) ds, \quad u_2(1) = \lambda_2 \displaystyle\int_0^{\sigma_2} u_1(s) ds, \ \ \sigma_1, \sigma_2 \in (0,1), \end{cases} \tag{2.3} $$

is given by:

$$ u_1(t) = \begin{cases} \eta_1(t), \ \ t \in (-\infty, 0], \\ \dfrac{1}{\Gamma(\varsigma)} \displaystyle\int_0^t (t-s)^{\varsigma-1} h_1(s) ds + \dfrac{t}{\Lambda_1} \Bigg\{ \dfrac{\lambda_1}{\Gamma(\gamma)} \displaystyle\int_0^{\sigma_1} \int_0^s (s-\tau)^{\gamma-1} h_2(\tau) d\tau ds \\ -\dfrac{1}{\Gamma(\varsigma)} \displaystyle\int_0^1 (1-s)^{\varsigma-1} h_1(s) ds + \dfrac{\lambda_1 \lambda_2 \sigma_1^2}{2\Gamma(\varsigma)} \displaystyle\int_0^{\sigma_2} \int_0^s (s-\tau)^{\varsigma-1} h_1(\tau) d\tau ds \\ -\dfrac{\lambda_1 \sigma_1^2}{2\Gamma(\gamma)} \displaystyle\int_0^1 (1-s)^{\gamma-1} h_2(s) ds \Bigg\}, \ \ t \in [0,1]. \end{cases} \tag{2.4} $$

$$ u_2(t) = \begin{cases} \eta_2(t), \ \ t \in (-\infty, 0], \\ \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t (t-s)^{\gamma-1} h_2(s) ds + \dfrac{t}{\Lambda_1} \Bigg\{ \dfrac{\lambda_2}{\Gamma(\varsigma)} \displaystyle\int_0^{\sigma_2} \int_0^s (s-\tau)^{\varsigma-1} h_1(\tau) d\tau ds \\ -\dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^1 (1-s)^{\gamma-1} h_2(s) ds + \dfrac{\lambda_1 \lambda_2 \sigma_2^2}{2\Gamma(\gamma)} \displaystyle\int_0^{\sigma_1} \int_0^s (s-\tau)^{\gamma-1} h_2(\tau) d\tau ds \\ -\dfrac{\lambda_2 \sigma_2^2}{2\Gamma(\varsigma)} \displaystyle\int_0^1 (1-s)^{\varsigma-1} h_1(s) ds \Bigg\}, \ \ t \in [0,1]. \end{cases} \tag{2.5} $$

Proof. Applying $I^{\varsigma}_{0^+}$ and $I^{\gamma}_{0^+}$ to the first and second differential equations in (2.3), respectively, and then in view of Lemma 2.1, for $t \in [0, 1]$, we find

$$u_1(t) = \frac{1}{\Gamma(\varsigma)}\int_0^t (t-s)^{\varsigma-1} h_1(s)ds + c_1 + c_2 t, \tag{2.6}$$

$$u_2(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} h_2(s)ds + c_3 + c_4 t, \tag{2.7}$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Using the first boundary conditions: $u_i(0) = \eta_i(0) = 0$, $i = 1, 2$ in (2.6) and (2.7), respectively, we get $c_1 = c_3 = 0$. Consequently, (2.6) and (2.7) have the form:

$$u_1(t) = \frac{1}{\Gamma(\varsigma)}\int_0^t (t-s)^{\varsigma-1} h_1(s)ds + c_2 t, \tag{2.8}$$

$$u_2(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} h_2(s)ds + c_4 t. \tag{2.9}$$

From the second boundary conditions: $u_1(1) = \lambda_1 \int_0^{\sigma_1} u_2(s)ds$, $u_2(1) = \lambda_2 \int_0^{\sigma_2} u_1(s)ds$, together with (2.8) and (2.9), it implies that

$$c_2 - \frac{\lambda_1 \sigma_1^2}{2} c_4 = \frac{\lambda_1}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1} h_2(\tau)d\tau ds - \frac{1}{\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1} h_1(s)ds,$$

$$c_4 - \frac{\lambda_2 \sigma_2^2}{2} c_2 = \frac{\lambda_2}{\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1} h_1(\tau)d\tau ds - \frac{1}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} h_2(s)ds.$$

By solving these two equations together, we get

$$\begin{aligned} c_2 = \frac{1}{1 - \frac{\lambda_1\lambda_2\sigma_1^2\sigma_2^2}{4}} \Bigg\{ & \frac{\lambda_1}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1} h_2(\tau)d\tau ds - \frac{1}{\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1} h_1(s)ds \\ & + \frac{\lambda_1\lambda_2\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1} h_1(\tau)d\tau ds - \frac{\lambda_1\sigma_1^2}{2\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} h_2(s)ds \Bigg\}, \end{aligned}$$

$$\begin{aligned} c_4 = \frac{1}{1 - \frac{\lambda_1\lambda_2\sigma_1^2\sigma_2^2}{4}} \Bigg\{ & \frac{\lambda_2}{\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1} h_1(\tau)d\tau ds - \frac{1}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} h_2(s)ds \\ & + \frac{\lambda_1\lambda_2\sigma_2^2}{2\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1} h_2(\tau)d\tau ds - \frac{\lambda_2\sigma_2^2}{2\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1} h_1(s)ds \Bigg\}. \end{aligned}$$

By replacing the values of c_2 and c_4 in (2.8) and (2.9), respectively, we obtain solutions (2.4) and (2.5). The converse of the lemma can be proved by direct computation. □

3. Existence and uniqueness results

To establish our main results for problem (1.1), in view of Lemma 2.2, let us transform problem (1.1) into a fixed-point problem by introducing an operator $\mathfrak{J} := (\mathfrak{J}_1, \mathfrak{J}_2) : \Pi \to \Pi$ as

$$\mathfrak{J}_1(u_1,u_2)(t) = \begin{cases} \eta_1(t), \quad t \in (-\infty, 0], \\ \displaystyle\int_0^t \frac{(t-s)^{\varsigma-1}}{\Gamma(\varsigma)} f(s,u_{1s},u_{2s})ds + \frac{t}{\Lambda_1}\Bigg\{ \frac{\lambda_1}{\Gamma(\gamma)} \int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1} g(\tau,u_{1\tau},u_{2\tau})d\tau ds \\ \displaystyle -\int_0^1 \frac{(1-s)^{\varsigma-1}}{\Gamma(\varsigma)} f(s,u_{1s},u_{2s})ds + \frac{\lambda_1\lambda_2\sigma_1^2}{2\Gamma(\varsigma)} \int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1} f(\tau,u_{1\tau},u_{2\tau})d\tau ds \\ \displaystyle -\frac{\lambda_1\sigma_1^2}{2\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} g(s,u_{1s},u_{2s})ds \Bigg\}, \quad t \in [0,1], \end{cases}$$

and

$$\mathfrak{J}_2(u_1,u_2)(t) = \begin{cases} \eta_2(t), \quad t \in (-\infty, 0], \\ \displaystyle\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} g(s,u_{1s},u_{2s})ds + \frac{t}{\Lambda_1}\Bigg\{ \frac{\lambda_2}{\Gamma(\varsigma)} \int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1} f(\tau,u_{1\tau},u_{2\tau})d\tau ds \\ \displaystyle -\int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} g(s,u_{1s},u_{2s})ds + \frac{\lambda_1\lambda_2\sigma_2^2}{2\Gamma(\gamma)} \int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1} g(\tau,u_{1\tau},u_{2\tau})d\tau ds \\ \displaystyle -\frac{\lambda_2\sigma_2^2}{2\Gamma(\varsigma)} \int_0^1 (1-s)^{\varsigma-1} f(s,u_{1s},u_{2s})ds \Bigg\}, \quad t \in [0,1], \end{cases}$$

where $\Pi = \mathfrak{T}_a \times \mathfrak{T}_a$ is a seminormed space endowed with the seminorm

$$\|(u_1,u_2)\|_\Pi = \|u_1\|_{\mathfrak{T}_a} + \|u_2\|_{\mathfrak{T}_a}, \quad \text{for } (u_1,u_2) \in \Pi.$$

For $t \in [0,1]$, let us assume the solution $(u_1,u_2) \in \Pi$, that satisfies (2.4) and (2.5), to be a decomposition of two functions (v_1,v_2) and $(\bar{w}_1,\bar{w}_2)$, such that $(u_1,u_2)(t) = (v_1,v_2)(t) + (w_1,w_2)(t)$, which implies $(u_{1t},u_{2t}) = (v_{1t},v_{2t}) + (\bar{w}_{1t},\bar{w}_{2t})$.

The function $(v_1,v_2)(.) : (-\infty,1] \times (-\infty,1] \to \mathbb{R}^2$ is defined by

$$(v_1,v_2)(t) = \begin{cases} (\eta_1,\eta_2)(t), \quad t \in (-\infty,0], \\ (0,0), \quad t \in (0,1], \end{cases} \tag{3.1}$$

which yield $(v_{10},v_{20}) = (\eta_1,\eta_2)$. Also, the function $(\bar{w}_1,\bar{w}_2)(.) : (-\infty,1] \times (-\infty,1] \to \mathbb{R}^2$ is defined by

$$(\bar{w}_1,\bar{w}_2)(t) = \begin{cases} (0,0), \quad t \in (-\infty,0], \\ (w_1,w_2)(t), \quad t \in (0,1], \end{cases} \tag{3.2}$$

where the function $(w_1,w_2)(.)$ satisfies

$$\begin{aligned} w_1(t) &= \frac{1}{\Gamma(\varsigma)} \int_0^t (t-s)^{\varsigma-1} f(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds \\ &\quad + \frac{t}{\Lambda_1}\Bigg\{ \frac{\lambda_1}{\Gamma(\gamma)} \int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1} g(\tau, v_{1\tau}+\bar{w}_{1\tau}, v_{2\tau}+\bar{w}_{2\tau})d\tau ds \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1} f(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds \\
&+\frac{\lambda_1\lambda_2\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1} f(\tau, v_{1\tau}+\bar{w}_{1\tau}, v_{2\tau}+\bar{w}_{2\tau})d\tau ds \\
&-\frac{\lambda_1\sigma_1^2}{2\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} g(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds\Bigg\},
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
w_2(t) \;=\; &\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} g(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds \\
&+\frac{t}{\Lambda_1}\Bigg\{\frac{\lambda_2}{\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1} f(\tau, v_{1\tau}+\bar{w}_{1\tau}, v_{2\tau}+\bar{w}_{2\tau})d\tau ds \\
&-\frac{1}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} g(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds \\
&+\frac{\lambda_1\lambda_2\sigma_2^2}{2\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1} g(\tau, v_{1\tau}+\bar{w}_{1\tau}, v_{2\tau}+\bar{w}_{2\tau})d\tau ds \\
&-\frac{\lambda_2\sigma_2^2}{2\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1} f(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds\Bigg\}.
\end{aligned} \tag{3.4}
$$

Thus, for every $(w_1, w_2) \in \Pi$, $(w_1, w_2)(0) = (0, 0)$.

Now, set $\mathfrak{T}'_a = \{w \in \mathfrak{T}_a$ such that $w_0 = 0\}$ and introduce a seminorm $\|.\|_{\mathfrak{T}'_a}$ on $\mathfrak{T}'_a$ by

$$\|w\|_{\mathfrak{T}'_a} = \|w_0\|_{\mathfrak{T}} + \sup_{t\in[0,1]} |w(t)| = \sup_{t\in[0,1]} |w(t)|,\;\; w \in \mathfrak{T}'_a,$$

which means that $\|.\|_{\mathfrak{T}'_a}$ is indeed a norm in $\mathfrak{T}'_a$ and consequently, the space $(\mathfrak{T}'_a, \|.\|_{\mathfrak{T}'_a})$ is a Banach space. Furthermore, consider the Banach space $\Pi' = \mathfrak{T}'_a \times \mathfrak{T}'_a$ with the norm

$$\|(w_1, w_2)\|_{\Pi'} = \|w_1\|_{\mathfrak{T}'_a} + \|w_2\|_{\mathfrak{T}'_a},$$

for $(w_1, w_2) \in \Pi'$, and define the operator $\mathfrak{S} = (\mathfrak{S}_1, \mathfrak{S}_2) : \Pi' \to \Pi'$ by

$$\mathfrak{S}(w_1(t), w_2(t)) := (\mathfrak{S}_1(w_1(t), w_2(t)), \mathfrak{S}_2(w_1(t), w_2(t))), \tag{3.5}$$

where

$$
\begin{aligned}
\mathfrak{S}_1(w_1(t), w_2(t)) \;=\; &\frac{1}{\Gamma(\varsigma)}\int_0^t (t-s)^{\varsigma-1} f(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds \\
&+\frac{t}{\Lambda_1}\Bigg\{\frac{\lambda_1}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1} g(\tau, v_{1\tau}+\bar{w}_{1\tau}, v_{2\tau}+\bar{w}_{2\tau})d\tau ds \\
&-\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1} f(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds \\
&+\frac{\lambda_1\lambda_2\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1} f(\tau, v_{1\tau}+\bar{w}_{1\tau}, v_{2\tau}+\bar{w}_{2\tau})d\tau ds \\
&-\frac{\lambda_1\sigma_1^2}{2\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} g(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds\Bigg\},\;\; t \in (0, 1],
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
\mathfrak{S}_2(w_1(t), w_2(t)) &= \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} g(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds \\
&+\frac{t}{\Lambda_1}\Big\{\frac{\lambda_2}{\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1} f(\tau, v_{1\tau}+\bar{w}_{1\tau}, v_{2\tau}+\bar{w}_{2\tau})d\tau ds \\
&-\frac{1}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} g(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds \\
&+\frac{\lambda_1\lambda_2\sigma_2^2}{2\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1} g(\tau, v_{1\tau}+\bar{w}_{1\tau}, v_{2\tau}+\bar{w}_{2\tau})d\tau ds \\
&-\frac{\lambda_2\sigma_2^2}{2\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1} f(s, v_{1s}+\bar{w}_{1s}, v_{2s}+\bar{w}_{2s})ds\Big\}, \quad t\in(0,1].
\end{aligned} \tag{3.7}
$$

Clearly, if the operator $\mathfrak{S}$ has a fixed point, then $\mathfrak{J}$ has a fixed point, and vice versa.

Further, to establish our main results, we introduce the following hypotheses:

(C_1) There exist continuous nonnegative functions $\alpha_i, \varphi_i \in C([0,1], \mathbb{R}^+)$, $i = 1,2,3$, such that

$$|f(t,u_1,u_2)| \le \alpha_1(t) + \alpha_2(t)\|u_1\|_{\mathfrak{T}} + \alpha_3(t)\|u_2\|_{\mathfrak{T}}, \quad \forall (u_1,u_2)\in\mathfrak{T}, \forall t\in\Omega,$$

$$|g(t,u_1,u_2)| \le \varphi_1(t) + \varphi_2(t)\|u_1\|_{\mathfrak{T}} + \varphi_3(t)\|u_2\|_{\mathfrak{T}}, \quad \forall (u_1,u_2)\in\mathfrak{T}, \forall t\in\Omega.$$

(C_2) There exist constants $\ell_i > 0, \chi_i > 0$, $i = 1,2$, such that

$$|f(t,u_1,u_2) - f(t,u_1^*,u_2^*)| \le \ell_1\|u_1-u_1^*\|_{\mathfrak{T}} + \ell_2\|u_2-u_2^*\|_{\mathfrak{T}}, \ \forall (u_1,u_2),(u_1^*,u_2^*)\in\mathfrak{T}, \forall t\in\Omega,$$

$$|g(t,u_1,u_2) - g(t,u_1^*,u_2^*)| \le \chi_1\|u_1-u_1^*\|_{\mathfrak{T}} + \chi_2\|u_2-u_2^*\|_{\mathfrak{T}}, \ \forall (u_1,u_2),(u_1^*,u_2^*)\in\mathfrak{T}, \forall t\in\Omega.$$

In the following, for brevity, we use the notations:

$$\Lambda_2 = \frac{1}{\Gamma(\varsigma+1)} + \frac{1}{|\Lambda_1|\Gamma(\varsigma+1)} + \frac{|\lambda_2|\sigma_2^2}{2|\Lambda_1|\Gamma(\varsigma+1)} + \frac{|\lambda_2|\sigma_2^{\varsigma+1}}{|\Lambda_1|\Gamma(\varsigma+2)}\Big[1+\frac{|\lambda_1|\sigma_1^2}{2}\Big], \tag{3.8}$$

$$\Lambda_3 = \frac{1}{\Gamma(\gamma+1)} + \frac{1}{|\Lambda_1|\Gamma(\gamma+1)} + \frac{|\lambda_1|\sigma_1^2}{2|\Lambda_1|\Gamma(\gamma+1)} + \frac{|\lambda_1|\sigma_1^{\gamma+1}}{|\Lambda_1|\Gamma(\gamma+2)}\Big[1+\frac{|\lambda_2|\sigma_2^2}{2}\Big], \tag{3.9}$$

and

$$\Phi = \min\Big\{1 - q^*\Big(\bar{\alpha}_2\Lambda_2 + \bar{\varphi}_2\Lambda_3\Big), 1 - q^*\Big(\bar{\alpha}_3\Lambda_2 + \bar{\varphi}_3\Lambda_3\Big)\Big\},$$

where $\bar{\alpha}_i = \sup\{|\alpha_i(t)| : t\in\Omega\}$ and $\bar{\varphi}_i = \sup\{|\varphi_i(t)| : t\in\Omega\}$, $i = 1,2,3$.

The aim of our first result is to provide sufficient criteria that ensure the existence of solutions for problem (1.1) in view of Schaefer's theorem [36].

Lemma 3.1. *(Schaefer) [36] . For a Banach space B, assume that $\mathcal{P} : B \to B$ is a continuous and compact mapping on B. Then $\mathcal{P}$ has a fixed point $\bar{u}\in B$, if the set of all solutions of the equation $u = \rho\mathcal{P}u$, for $0 < \rho < 1$, is bounded.*

Theorem 3.1. *Let $f, g : \Omega \times \mathfrak{T} \to \mathbb{R}$ be continuous functions, and condition (C_1) holds true. Then problem (1.1) has at least one solution on $(-\infty, 1]$, if the following inequalities are satisfied:*

$$q^*(\bar{\alpha}_2\Lambda_2 + \bar{\varphi}_2\Lambda_3) < 1 \quad and \quad q^*(\bar{\alpha}_3\Lambda_2 + \bar{\varphi}_3\Lambda_3) < 1, \tag{3.10}$$

where Λ_2, Λ_3 are respectively introduced by (3.8) and (3.9).

Proof. We start the proof by showing that the operator $\mathfrak{S} : \Pi' \to \Pi'$ defined by (3.5) is continuous and maps any bounded subset of Π' into a relatively compact subset of Π'; that is, $\mathfrak{S}$ is completely continuous. Clearly, the continuity of the operator $\mathfrak{S} : \Pi' \to \Pi'$ follows the continuity of the functions f and g. Now, let us consider the bounded set $\mathfrak{B}_{\bar{r}} = \{(w_1, w_2) : \|(w_1, w_2)\| < \bar{r}\} \subset \Pi'$. Then, positive constants $\mathcal{M}_f$ and $\mathcal{M}_g$ can be found such that

$$\begin{aligned}
\left|f(s, v_{1s} + \bar{w}_{1s}, v_{2s} + \bar{w}_{2s})\right| &\leq \bar{\alpha}_1 + \bar{\alpha}_2 \|v_{1s} + \bar{w}_{1s}\|_{\mathfrak{T}} + \bar{\alpha}_3 \|v_{2s} + \bar{w}_{2s}\|_{\mathfrak{T}} \\
&\leq \bar{\alpha}_1 + \bar{\alpha}_2 \,(q^* \sup_{s\in\Omega} |w_1(s)| + p^*\|\eta_1\|_{\mathfrak{T}}) + \bar{\alpha}_3 \,(q^* \sup_{s\in\Omega} |w_2(s)| + p^*\|\eta_2\|_{\mathfrak{T}}) \\
&\leq \bar{\alpha}_1 + p^*(\bar{\alpha}_2 \|\eta_1\|_{\mathfrak{T}} + \bar{\alpha}_3 \|\eta_2\|_{\mathfrak{T}}) + q^*(\bar{\alpha}_2 \|w_1\|_{\mathfrak{T}'_a} + \bar{\alpha}_3 \|w_2\|_{\mathfrak{T}'_a}) \\
&\leq \bar{\alpha}_1 + p^*(\bar{\alpha}_2 \|\eta_1\|_{\mathfrak{T}} + \bar{\alpha}_3 \|\eta_2\|_{\mathfrak{T}}) + \alpha^* q^* \bar{r} = \mathcal{M}_f.
\end{aligned}$$

Simmilarly

$$\left|g(s, v_{1s} + \bar{w}_{1s}, v_{2s} + \bar{w}_{2s})\right| \leq \bar{\varphi}_1 + p^*(\bar{\varphi}_2 \|\eta_1\|_{\mathfrak{T}} + \bar{\varphi}_3 \|\eta_2\|_{\mathfrak{T}}) + \varphi^* q^* \bar{r} = \mathcal{M}_g,$$

where
$\|v_{1s} + \bar{w}_{1s}\|_{\mathfrak{T}} \leq \|v_{1s}\|_{\mathfrak{T}} + \|\bar{w}_{1s}\|_{\mathfrak{T}} \leq q^* \sup\{|w_1(s)| : s \in \Omega\} + p^*\|\eta_1\|_{\mathfrak{T}} = q^*\|w_1\|_{\mathfrak{T}'_a} + p^*\|\eta_1\|_{\mathfrak{T}}$,
$\|v_{2s} + \bar{w}_{2s}\|_{\mathfrak{T}} \leq \|v_{2s}\|_{\mathfrak{T}} + \|\bar{w}_{2s}\|_{\mathfrak{T}} \leq q^* \sup\{|w_2(s)| : s \in \Omega\} + p^*\|\eta_1\|_{\mathfrak{T}} = q^*\|w_2\|_{\mathfrak{T}'_a} + p^*\|\eta_2\|_{\mathfrak{T}}$,
and $\alpha^* = \max\{\bar{\alpha}_2, \bar{\alpha}_3\}$, $\varphi^* = \max\{\bar{\varphi}_2, \bar{\varphi}_3\}$.

Then, for any $(w_1, w_2) \in \mathfrak{B}_{\bar{r}}$, $t \in \Omega$, we have

$$\begin{aligned}
|\mathfrak{S}_1(w_1(t), w_2(t))| &\leq \frac{1}{\Gamma(\varsigma)} \int_0^t (t-s)^{\varsigma-1} |f(s, v_{1s} + \bar{w}_{1s}, v_{2s} + \bar{w}_{2s})| ds \\
&\quad + \frac{t}{|\Lambda_1|} \left\{ \frac{|\lambda_1|}{\Gamma(\gamma)} \int_0^{\sigma_1} \int_0^s (s-\tau)^{\gamma-1} |g(\tau, v_{1\tau} + \bar{w}_{1\tau}, v_{2\tau} + \bar{w}_{2\tau})| d\tau ds \right. \\
&\quad + \frac{1}{\Gamma(\varsigma)} \int_0^1 (1-s)^{\varsigma-1} |f(s, v_{1s} + \bar{w}_{1s}, v_{2s} + \bar{w}_{2s})| ds \\
&\quad + \frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma)} \int_0^{\sigma_2} \int_0^s (s-\tau)^{\varsigma-1} |f(\tau, v_{1\tau} + \bar{w}_{1\tau}, v_{2\tau} + \bar{w}_{2\tau})| d\tau ds \\
&\quad \left. + \frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} |g(s, v_{1s} + \bar{w}_{1s}, v_{2s} + \bar{w}_{2s})| ds \right\} \\
&\leq \frac{1}{\Gamma(\varsigma)} \int_0^t (t-s)^{\varsigma-1} \mathcal{M}_f ds + \frac{t}{|\Lambda_1|} \left\{ \frac{|\lambda_1|}{\Gamma(\gamma)} \int_0^{\sigma_1} \int_0^s (s-\tau)^{\gamma-1} \mathcal{M}_g d\tau ds \right. \\
&\quad + \frac{1}{\Gamma(\varsigma)} \int_0^1 (1-s)^{\varsigma-1} \mathcal{M}_f ds + \frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma)} \int_0^{\sigma_2} \int_0^s (s-\tau)^{\varsigma-1} \mathcal{M}_f d\tau ds \\
&\quad \left. + \frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} \mathcal{M}_g ds \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{M}_f\left\{\int_0^t \frac{(t-s)^{\varsigma-1}}{\Gamma(\varsigma)}ds+\frac{t}{|\Lambda_1|\Gamma(\varsigma)}\int_0^1(1-s)^{\varsigma-1}ds\right.\\
&\quad\left.+\frac{t|\lambda_1||\lambda_2|\sigma_1^2}{2|\Lambda_1|\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s(s-\tau)^{\varsigma-1}d\tau ds\right\}\\
&\quad+\mathcal{M}_g\left\{\frac{t|\lambda_1|}{|\Lambda_1|\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s(s-\tau)^{\gamma-1}d\tau ds+\frac{t|\lambda_1|\sigma_1^2}{2|\Lambda_1|\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}ds\right\}\\
&\leq \mathcal{M}_f\left\{\frac{1}{\Gamma(\varsigma+1)}+\frac{1}{|\Lambda_1|\Gamma(\varsigma+1)}+\frac{|\lambda_1||\lambda_2|\sigma_1^2\sigma_2^{\varsigma+1}}{2|\Lambda_1|\Gamma(\varsigma+2)}\right\}\\
&\quad+\mathcal{M}_g\left\{\frac{|\lambda_1|\sigma_1^{\gamma+1}}{|\Lambda_1|\Gamma(\gamma+2)}+\frac{|\lambda_1|\sigma_1^2}{2|\Lambda_1|\Gamma(\gamma+1)}\right\}.
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
|\mathfrak{S}_2(w_1(t),w_2(t))| &\leq \mathcal{M}_f\left\{\frac{|\lambda_2|\sigma_2^{\varsigma+1}}{|\Lambda_1|\Gamma(\varsigma+2)}+\frac{|\lambda_2|\sigma_2^2}{2|\Lambda_1|\Gamma(\varsigma+1)}\right\}\\
&\quad+\mathcal{M}_g\left\{\frac{1}{\Gamma(\gamma+1)}+\frac{1}{|\Lambda_1|\Gamma(\gamma+1)}+\frac{|\lambda_1||\lambda_2|\sigma_2^2\sigma_1^{\gamma+1}}{2|\Lambda_1|\Gamma(\gamma+2)}\right\}.
\end{aligned}
$$

Therefore, for any $(w_1,w_2)\in\mathfrak{B}_{\bar{r}}$, we get

$$
\begin{aligned}
\|\mathfrak{S}(w_1,w_2)\|_{\Pi'} &= \|\mathfrak{S}_1(w_1,w_2)\|_{\Pi'}+\|\mathfrak{S}_2(w_1,w_2)\|_{\Pi'}\\
&\leq \mathcal{M}_f\Lambda_2+\mathcal{M}_g\Lambda_3,
\end{aligned}
$$

which yields that the operator $\mathfrak{S}$ is uniformly bounded.

Now, to prove that $\mathfrak{S}$ is equicontinuous on Π', take $0<t_1<t_2<1$, and $(w_1,w_2)\in\mathfrak{B}_{\bar{r}}$. Then

$$
\begin{aligned}
&|\mathfrak{S}_1(w_1(t_2),w_2(t_2))-\mathfrak{S}_1(w_1(t_1),w_2(t_1))|\\
=&\left|\frac{1}{\Gamma(\varsigma)}\int_0^{t_1}\Big((t_2-s)^{\varsigma-1}-(t_1-s)^{\varsigma-1}\Big)f(s,v_{1s}+\bar{w}_{1s},v_{2s}+\bar{w}_{2s})ds\right.\\
&+\frac{1}{\Gamma(\varsigma)}\int_{t_1}^{t_2}(t_2-s)^{\varsigma-1}f(s,v_{1s}+\bar{w}_{1s},v_{2s}+\bar{w}_{2s})ds\\
&+\frac{t_2-t_1}{\Lambda_1}\left\{\frac{\lambda_1}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s(s-\tau)^{\gamma-1}g(\tau,v_{1\tau}+\bar{w}_{1\tau},v_{2\tau}+\bar{w}_{2\tau})d\tau ds\right.\\
&-\frac{1}{\Gamma(\varsigma)}\int_0^1(1-s)^{\varsigma-1}f(s,v_{1s}+\bar{w}_{1s},v_{2s}+\bar{w}_{2s})ds\\
&+\frac{\lambda_1\lambda_2\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s(s-\tau)^{\varsigma-1}f(\tau,v_{1\tau}+\bar{w}_{1\tau},v_{2\tau}+\bar{w}_{2\tau})d\tau ds\\
&\left.\left.-\frac{\lambda_1\sigma_1^2}{2\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}g(s,v_{1s}+\bar{w}_{1s},v_{2s}+\bar{w}_{2s})ds\right\}\right|\\
\leq&\frac{1}{\Gamma(\varsigma)}\int_0^{t_1}\left|(t_2-s)^{\varsigma-1}-(t_1-s)^{\varsigma-1}\right|\mathcal{M}_f\,ds+\frac{1}{\Gamma(\varsigma)}\int_{t_1}^{t_2}\left|(t_2-s)^{\varsigma-1}\right|\mathcal{M}_f\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|t_2-t_1|}{|\Lambda_1|}\Big(\frac{|\lambda_1|}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1}\,\mathcal{M}_g\,d\tau ds+\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1}\,\mathcal{M}_f\,ds\\
&+\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1}\,\mathcal{M}_f\,d\tau ds+\frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}\,\mathcal{M}_g\,ds\Big)\\
\leq\;& \mathcal{M}_f\Big[\frac{2(t_2-t_1)^{\varsigma}+|t_2^{\varsigma}-t_1^{\varsigma}|}{\Gamma(\varsigma+1)}+\frac{(t_2-t_1)}{|\Lambda_1|}\Big(\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma+2)}\sigma_2^{\varsigma+1}+\frac{1}{\Gamma(\varsigma+1)}\Big)\Big]\\
&+\mathcal{M}_g\frac{(t_2-t_1)}{|\Lambda_1|}\Big(\frac{|\lambda_1|}{\Gamma(\gamma+2)}\sigma_1^{\gamma+1}+\frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma+1)}\Big).
\end{aligned}
$$

Likewise, we can find that

$$
\begin{aligned}
&|\mathfrak{S}_2(w_1(t_2),w_2(t_2))-\mathfrak{S}_2(w_1(t_1),w_2(t_1))|\\
\leq\;& \mathcal{M}_g\Big[\frac{2(t_2-t_1)^{\gamma}+|t_2^{\gamma}-t_1^{\gamma}|}{\Gamma(\gamma+1)}+\frac{(t_2-t_1)}{|\Lambda_1|}\Big(\frac{|\lambda_1||\lambda_2|\sigma_2^2}{2\Gamma(\gamma+2)}\sigma_1^{\gamma+1}+\frac{1}{\Gamma(\gamma+1)}\Big)\Big]\\
&+\mathcal{M}_f\frac{(t_2-t_1)}{|\Lambda_1|}\Big(\frac{|\lambda_2|}{\Gamma(\varsigma+2)}\sigma_2^{\varsigma+1}+\frac{|\lambda_2|\sigma_2^2}{2\Gamma(\varsigma+1)}\Big).
\end{aligned}
$$

According to the above inequalities, we show that $|\mathfrak{S}_1(w_1(t_2),w_2(t_2))-\mathfrak{S}_1(w_1(t_1),w_2(t_1))|\to 0$ as $t_1\to t_2$ independently of $(w_1,w_2)\in\mathfrak{B}_{\bar{r}}$. Hence, all the hypotheses of the Arzelá-Ascoli theorem are satisfied, and consequently, we conclude that the operator $\mathfrak{S}:\Pi'\to\Pi'$ is completely continuous.

Finally, let us define the set Ψ by

$$\Psi=\{(w_1,w_2)\in\Pi'|(w_1,w_2)=\xi\mathfrak{S}(w_1,w_2),0<\xi<1\}.$$

Then, we need to prove that Ψ is bounded. Let $(w_1,w_2)\in\Psi$, then $(w_1,w_2)=\xi\mathfrak{S}(w_1,w_2),\ 0<\xi<1$. For any $t\in\Omega$, we get

$$w_1(t)=\xi\mathfrak{S}_1(w_1(t),w_2(t)),\ w_2(t)=\xi\mathfrak{S}_2(w_1(t),w_2(t)),$$

$$
\begin{aligned}
|w_1(t)| &= \xi|\mathfrak{S}_{\mathbf{1}}(w_1(t),w_2(t))|\\
&\leq \frac{1}{\Gamma(\varsigma)}\int_0^t (t-s)^{\varsigma-1}|f(s,v_{1s}+\bar{w}_{1s},v_{2s}+\bar{w}_{2s})|ds\\
&\quad+\frac{t}{|\Lambda_1|}\Big(\frac{|\lambda_1|}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1}|g(\tau,v_{1\tau}+\bar{w}_{1\tau},v_{2\tau}+\bar{w}_{2\tau})|d\tau ds\\
&\quad+\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1}|f(s,v_{1s}+\bar{w}_{1s},v_{2s}+\bar{w}_{2s})|ds\\
&\quad+\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1}|f(\tau,v_{1\tau}+\bar{w}_{1\tau},v_{2\tau}+\bar{w}_{2\tau})|d\tau ds\\
&\quad+\frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}|g(s,v_{1s}+\bar{w}_{1s},v_{2s}+\bar{w}_{2s})|ds\Big)\\
&\leq \frac{1}{\Gamma(\varsigma)}\int_0^t (t-s)^{\varsigma-1}\Big[\bar{\alpha}_1+(q^*\|w_1\|_{\mathfrak{z}'_a}+p^*\|\eta_1\|_{\mathfrak{z}})\bar{\alpha}_2+(q^*\|w_2\|_{\mathfrak{z}'_a}+p^*\|\eta_2\|_{\mathfrak{z}})\bar{\alpha}_3\Big]ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{t}{|\Lambda_1|}\Big\{\frac{|\lambda_1|}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1}\Big[\bar{\varphi}_1+(q^*\|w_1\|_{\mathfrak{T}'_a}+p^*\|\eta_1\|_{\mathfrak{T}})\bar{\varphi}_2 \\
&+(q^*\|w_2\|_{\mathfrak{T}'_a}+p^*\|\eta_2\|_{\mathfrak{T}})\bar{\varphi}_3\Big]d\tau ds \\
&+\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1}\Big[\bar{\alpha}_1+(q^*\|w_1\|_{\mathfrak{T}'_a}+p^*\|\eta_1\|_{\mathfrak{T}})\bar{\alpha}_2+(q^*\|w_2\|_{\mathfrak{T}'_a}+p^*\|\eta_2\|_{\mathfrak{T}})\bar{\alpha}_3\Big]ds \\
&+\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1}\Big[\bar{\alpha}_1+(q^*\|w_1\|_{\mathfrak{T}'_a}+p^*\|\eta_1\|_{\mathfrak{T}})\bar{\alpha}_2 \\
&+(q^*\|w_2\|_{\mathfrak{T}'_a}+p^*\|\eta_2\|_{\mathfrak{T}})\bar{\alpha}_3\Big]d\tau ds \\
&+\frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}\Big[\bar{\varphi}_1+(q^*\|w_1\|_{\mathfrak{T}'_a}+p^*\|\eta_1\|_{\mathfrak{T}})\bar{\varphi}_2+(q^*\|w_2\|_{\mathfrak{T}'_a}+p^*\|\eta_2\|_{\mathfrak{T}})\bar{\varphi}_3\Big]ds\Big\} \\
\le\ &\Big[\bar{\alpha}_1+(q^*\|w_1\|_{\mathfrak{T}'_a}+p^*\|\eta_1\|_{\mathfrak{T}})\bar{\alpha}_2+(q^*\|w_2\|_{\mathfrak{T}'_a}+p^*\|\eta_2\|_{\mathfrak{T}})\bar{\alpha}_3\Big] \\
&\times\Big(\frac{1}{\Gamma(\varsigma+1)}+\frac{1}{|\Lambda_1|\Gamma(\varsigma+1)}+\frac{|\lambda_1||\lambda_2|\sigma_1^2\sigma_2^{\varsigma+1}}{2|\Lambda_1|\Gamma(\varsigma+2)}\Big) \\
&+\Big[\bar{\varphi}_1+(q^*\|w_1\|_{\mathfrak{T}'_a}+p^*\|\eta_1\|_{\mathfrak{T}})\bar{\varphi}_2+(q^*\|w_2\|_{\mathfrak{T}'_a}+p^*\|\eta_2\|_{\mathfrak{T}})\bar{\varphi}_3\Big] \\
&\times\Big(\frac{|\lambda_1|}{|\Lambda_1|\Gamma(\gamma+2)}\sigma_1^{\gamma+1}+\frac{|\lambda_1|\sigma_1^2}{2|\Lambda_1|\Gamma(\gamma+1)}\Big).
\end{aligned}
$$

Analogously, we can obtain

$$
\begin{aligned}
|w_2(t)| &= \xi|\mathfrak{S}_2(w_1(t),w_2(t))| \\
&\le \Big[\bar{\alpha}_1+(q^*\|w_1\|_{\mathfrak{T}'_a}+p^*\|\eta_1\|_{\mathfrak{T}})\bar{\alpha}_2+(q^*\|w_2\|_{\mathfrak{T}'_a}+p^*\|\eta_2\|_{\mathfrak{T}})\bar{\alpha}_3\Big] \\
&\quad\times\Big(\frac{|\lambda_2|}{|\Lambda_1|\Gamma(\varsigma+2)}\sigma_2^{\varsigma+1}+\frac{|\lambda_2|\sigma_2^2}{2|\Lambda_1|\Gamma(\varsigma+1)}\Big) \\
&\quad+\Big[\bar{\varphi}_1+(q^*\|w_1\|_{\mathfrak{T}'_a}+p^*\|\eta_1\|_{\mathfrak{T}})\bar{\varphi}_2+(q^*\|w_2\|_{\mathfrak{T}'_a}+p^*\|\eta_2\|_{\mathfrak{T}})\bar{\varphi}_3\Big] \\
&\quad\times\Big(\frac{1}{\Gamma(\gamma+1)}+\frac{1}{|\Lambda_1|\Gamma(\gamma+1)}+\frac{|\lambda_1||\lambda_2|\sigma_2^2}{2|\Lambda_1|\Gamma(\gamma+2)}\sigma_1^{\gamma+1}\Big).
\end{aligned}
$$

In consequence, we have

$$
\begin{aligned}
\|w_1\|_{\mathfrak{T}'_a}+\|w_2\|_{\mathfrak{T}'_a} \le\ &\bar{\alpha}_1\Lambda_2+\bar{\varphi}_1\Lambda_3+p^*\|\eta_1\|_{\mathfrak{T}}\Big[\bar{\alpha}_2\Lambda_2+\bar{\varphi}_2\Lambda_3\Big]+p^*\|\eta_2\|_{\mathfrak{T}}\Big[\bar{\alpha}_3\Lambda_2+\bar{\varphi}_3\Lambda_3\Big] \\
&+q^*\|w_1\|_{\mathfrak{T}'_a}\Big[\bar{\alpha}_2\Lambda_2+\bar{\varphi}_2\Lambda_3\Big]+q^*\|w_2\|_{\mathfrak{T}'_a}\Big[\bar{\alpha}_3\Lambda_2+\bar{\varphi}_3\Lambda_3\Big].
\end{aligned}
$$

Hence, by definition of Φ and the conditions (3.10), we get

$$
\|(w_1,w_2)\|_{\Pi'} \le \frac{\bar{\alpha}_1\Lambda_2+\bar{\varphi}_1\Lambda_3+p^*\Big(\|\eta_1\|_{\mathfrak{T}}\Big[\bar{\alpha}_2\Lambda_2+\bar{\varphi}_2\Lambda_3\Big]+\|\eta_2\|_{\mathfrak{T}}\Big[\bar{\alpha}_3\Lambda_2+\bar{\varphi}_3\Lambda_3\Big]\Big)}{\Phi}.
$$

This shows that $\|(w_1,w_2)\|_{\Pi'}$ is bounded for $t\in\Omega$, and, as a result, the set Ψ is bounded. Therefore, in view of the conclusion of Lemma 3.1, we deduce that the operator $\mathfrak{S}$ has at least one fixed point on $(-\infty,1]$, and hence there exists at least one solution to problem (1.1) on $(-\infty,1]$. □

The next result deals with the uniqueness of the solution for problem (1.1) by utilizing Banach's contraction mapping principle.

Theorem 3.2. *Let $f, g : [0,1] \times \mathfrak{T} \to \mathbb{R}$ be continuous functions satisfying the condition (C_2). Then there exists a unique solution to problem (1.1) on $(-\infty, 1]$, if*

$$q^*(\ell\Lambda_2 + \chi\Lambda_3) < 1, \tag{3.11}$$

where $\ell = \max\{\ell_1, \ell_2\}, \chi = \max\{\chi_1, \chi_2\}$ and Λ_2, Λ_3 are respectively given by (3.8) and (3.9).

Proof. Let us fix r to satisfy the following:

$$r > \frac{M_1\Lambda_2 + M_2\Lambda_3 + p^*(\ell\Lambda_2 + \chi\Lambda_3)(\|\eta_1\|_{\mathfrak{T}} + \|\eta_2\|_{\mathfrak{T}})}{1 - q^*(\ell\Lambda_2 + \chi\Lambda_3)},$$

where $M_1 = \sup_{t\in[0,1]} |f(t,0,0)|$, $M_2 = \sup_{t\in[0,1]} |g(t,0,0)|$, and consider the operator $\mathfrak{S} : \Pi' \to \Pi'$, defined by (3.5). Then we show that $\mathfrak{S}\mathfrak{P}_r \subset \mathfrak{P}_r$, where

$$\mathfrak{P}_r = \{w \in \Pi' : \|(w_1, w_2)\|_{\Pi'} \leq r\}.$$

For $(w_1, w_2) \in \mathfrak{P}_r$, we get

$$\begin{aligned}
&|\mathfrak{S}_1(w_1(t), w_2(t))| \\
\leq\ & \frac{1}{\Gamma(\varsigma)}\int_0^t (t-s)^{\varsigma-1}[|f(s, v_{1s} + \bar{w}_{1s}, v_{2s} + \bar{w}_{2s}) - f(s,0,0)| + M_1]ds \\
&+\frac{t}{|\Lambda_1|}\Big(\frac{|\lambda_1|}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1}[|g(\tau, v_{1\tau} + \bar{w}_{1\tau}, v_{2\tau} + \bar{w}_{2\tau}) - g(\tau,0,0)| + M_2]d\tau ds \\
&+\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1}[|f(s, v_{1s} + \bar{w}_{1s}, v_{2s} + \bar{w}_{2s}) - f(s,0,0)| + M_1]ds \\
&+\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1}[|f(\tau, v_{1\tau} + \bar{w}_{1\tau}, v_{2\tau} + \bar{w}_{2\tau}) - f(\tau,0,0)| + M_1]d\tau ds \\
&+\frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}[|g(s, v_{1s} + \bar{w}_{1s}, v_{2s} + \bar{w}_{2s}) - g(s,0,0)| + M_2]ds\Big) \\
\leq\ & \frac{1}{\Gamma(\varsigma)}\int_0^t (t-s)^{\varsigma-1}[\ell_1\|v_{1s} + \bar{w}_{1s}\|_{\mathfrak{T}} + \ell_2\|v_{2s} + \bar{w}_{2s}\|_{\mathfrak{T}} + M_1]ds \\
&+\frac{t}{|\Lambda_1|}\Big(\frac{|\lambda_1|}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s (s-\tau)^{\gamma-1}[\chi_1\|v_{1\tau} + \bar{w}_{1\tau}\|_{\mathfrak{T}} + \chi_2\|v_{2\tau} + \bar{w}_{2\tau}\|_{\mathfrak{T}} + M_2]d\tau ds \\
&+\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-s)^{\varsigma-1}[\ell_1\|v_{1s} + \bar{w}_{1s}\|_{\mathfrak{T}} + \ell_2\|v_{2s} + \bar{w}_{2s}\|_{\mathfrak{T}} + M_1]ds \\
&+\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s (s-\tau)^{\varsigma-1}[\ell_1\|v_{1\tau} + \bar{w}_{1\tau}\|_{\mathfrak{T}} + \ell_2\|v_{2\tau} + \bar{w}_{2\tau}\|_{\mathfrak{T}} + M_1]d\tau ds \\
&+\frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}[\chi_1\|v_{1s} + \bar{w}_{1s}\|_{\mathfrak{T}} + \chi_2\|v_{2s} + \bar{w}_{2s}\|_{\mathfrak{T}} + M_2]ds\Big) \\
\leq\ & \Big[\ell_1(q^*\|w_1\|_{\mathfrak{T}'_a} + p^*\|\eta_1\|_{\mathfrak{T}}) + \ell_2(q^*\|w_2\|_{\mathfrak{T}'_a} + p^*\|\eta_2\|_{\mathfrak{T}}) + M_1\Big] \\
&\times\Big(\frac{1}{\Gamma(\varsigma+1)} + \frac{1}{|\Lambda_1|\Gamma(\varsigma+1)} + \frac{|\lambda_1||\lambda_2|\sigma_1^2\sigma_2^{\varsigma+1}}{2|\Lambda_1|\Gamma(\varsigma+2)}\Big)
\end{aligned}$$

$$+\Big[\chi_1(q^*\|w_1\|_{\mathfrak{X}'_a}+p^*\|\eta_1\|_{\mathfrak{X}})+\chi_2(q^*\|w_2\|_{\mathfrak{X}'_a}+p^*\|\eta_2\|_{\mathfrak{X}})+M_2\Big]$$
$$\times\Big(\frac{|\lambda_1|\sigma_1^{\gamma+1}}{|\Lambda_1|\Gamma(\gamma+2)}+\frac{|\lambda_1|\sigma_1^2}{2|\Lambda_1|\Gamma(\gamma+1)}\Big),$$

which yields for $t\in\Omega$

$$\begin{aligned}\|\mathfrak{S}_1(w_1,w_2)\|_{\Pi'} &\le \Big\{\ell\Big(\frac{1}{\Gamma(\varsigma+1)}+\frac{1}{|\Lambda_1|\Gamma(\varsigma+1)}+\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2|\Lambda_1|\Gamma(\varsigma+2)}\sigma_2^{\gamma+1}\Big)\\ &\quad+\chi\Big(\frac{|\lambda_1|}{|\Lambda_1|\Gamma(\gamma+2)}\sigma_1^{\gamma+1}+\frac{|\lambda_1|\sigma_1^2}{2|\Lambda_1|\Gamma(\gamma+1)}\Big)\Big\}\\ &\quad\times\Big\{q^*(\|w_1\|_{\mathfrak{X}'_a}+\|w_2\|_{\mathfrak{X}'_a})+p^*(\|\eta_1\|_{\mathfrak{X}}+\|\eta_2\|_{\mathfrak{X}})\Big\}\\ &\quad+M_1\Big(\frac{1}{\Gamma(\varsigma+1)}+\frac{1}{|\Lambda_1|\Gamma(\varsigma+1)}+\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2|\Lambda_1|\Gamma(\varsigma+2)}\sigma_2^{\varsigma+1}\Big)\\ &\quad+M_2\Big(\frac{|\lambda_1|}{|\Lambda_1|\Gamma(\gamma+2)}\sigma_1^{\gamma+1}+\frac{|\lambda_1|\sigma_1^2}{2|\Lambda_1|\Gamma(\gamma+1)}\Big).\end{aligned}$$

Similarly, one can find that

$$\begin{aligned}\|\mathfrak{S}_2(w_1,w_2)\|_{\Pi'} &\le \Big\{\ell\Big(\frac{|\lambda_2|}{|\Lambda_1|\Gamma(\varsigma+2)}\sigma_2^{\varsigma+1}+\frac{|\lambda_2|\sigma_2^2}{2|\Lambda_1|\Gamma(\varsigma+1)}\Big)\\ &\quad+\chi\Big(\frac{1}{\Gamma(\gamma+1)}+\frac{1}{|\Lambda_1|\Gamma(\gamma+1)}+\frac{|\lambda_1||\lambda_2|\sigma_2^2}{2|\Lambda_1|\Gamma(\gamma+2)}\sigma_1^{\gamma+1}\Big)\Big\}\\ &\quad\times\Big\{q^*(\|w_1\|_{\mathfrak{X}'_a}+\|w_2\|_{\mathfrak{X}'_a})+p^*(\|\eta_1\|_{\mathfrak{X}}+\|\eta_2\|_{\mathfrak{X}})\Big\}\\ &\quad+M_1\Big(\frac{|\lambda_2|}{|\Lambda_1|\Gamma(\varsigma+2)}\sigma_2^{\varsigma+1}+\frac{|\lambda_2|\sigma_2^2}{2|\Lambda_1|\Gamma(\varsigma+1)}\Big)\\ &\quad+M_2\Big(\frac{1}{\Gamma(\gamma+1)}+\frac{1}{|\Lambda_1|\Gamma(\gamma+1)}+\frac{|\lambda_1||\lambda_2|\sigma_2^2}{2|\Lambda_1|\Gamma(\gamma+2)}\sigma_1^{\gamma+1}\Big).\end{aligned}$$

Consequently, for any $(w_1,w_2)\in\mathfrak{P}_r$, we have

$$\begin{aligned}\|\mathfrak{S}(w_1,w_2)\|_{\Pi'} &= \|\mathfrak{S}_1(w_1,w_2)\|_{\Pi'}+\|\mathfrak{S}_2(w_1,w_2)\|_{\Pi'}\\ &\le (\ell\Lambda_2+\chi\Lambda_3)\Big[q^*\|(w_1,w_2)\|_{\Pi'}+p^*(\|\eta_1\|_{\mathfrak{X}},\|\eta_2\|_{\mathfrak{X}})\Big]+M_1\Lambda_2+M_2\Lambda_3\\ &< r.\end{aligned}$$

So, we conclude that $\mathfrak{S}$ maps $\mathfrak{P}_r$ into itself.

Next, to establish the contraction of the operator $\mathfrak{S}$, let $(w_1,w_2),(w_1^*,w_2^*)\in\Pi', t\in[0,1]$. Then, by (C_2), we get

$$\begin{aligned}&|\mathfrak{S}_1(w_1(t),w_2(t))-\mathfrak{S}_1(w_1^*(t),w_2^*(t))|\\ \le\ &\frac{1}{\Gamma(\varsigma)}\int_0^t(t-s)^{\varsigma-1}\Big[|f(s,v_{1s}+\bar{w}_{1s},v_{2s}+\bar{w}_{2s})-f(s,v_{1s}+\bar{w}^*_{1s},v_{2s}+\bar{w}^*_{2s})|\Big]ds\end{aligned}$$

$$
\begin{aligned}
& +\frac{t}{|\Lambda_1|}\Big\{\frac{|\lambda_1|}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s(s-\tau)^{\gamma-1}\Big[|g(\tau,v_{1\tau}+\bar{w}_{1\tau},v_{2\tau}+\bar{w}_{2\tau})-g(\tau,v_{1\tau}+\bar{w}^*_{1\tau},v_{2\tau}+\bar{w}^*_{2\tau})|\Big]d\tau ds \\
& +\frac{1}{\Gamma(\varsigma)}\int_0^1(1-s)^{\varsigma-1}\Big[|f(s,v_{1s}+\bar{w}_{1s},v_{2s}+\bar{w}_{2s})-f(s,v_{1s}+\bar{w}^*_{1s},v_{2s}+\bar{w}^*_{2s})|\Big]ds \\
& +\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s(s-\tau)^{\varsigma-1}\Big[|f(\tau,v_{1\tau}+\bar{w}_{1\tau},v_{2\tau}+\bar{w}_{2\tau})-f(\tau,v_{1\tau}+\bar{w}^*_{1\tau},v_{2\tau}+\bar{w}^*_{2\tau})|\Big]d\tau ds \\
& +\frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\Big[|g(s,v_{1s}+\bar{w}_{1s},v_{2s}+\bar{w}_{2s})-g(s,v_{1s}+\bar{w}^*_{1s},v_{2s}+\bar{w}^*_{2s})|\Big]ds\Big\} \\
\le\ & \frac{1}{\Gamma(\varsigma)}\int_0^t(t-s)^{\varsigma-1}\Big[\ell_1\|w_{1s}-w^*_{1s}\|_{\mathfrak{T}}+\ell_2\|w_{2s}-w^*_{2s}\|_{\mathfrak{T}}\Big]ds \\
& +\frac{t}{|\Lambda_1|}\Big\{\frac{|\lambda_1|}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s(s-\tau)^{\gamma-1}\Big[\chi_1\|w_{1\tau}-w^*_{1\tau}\|_{\mathfrak{T}}+\chi_2\|w_{2\tau}-w^*_{2\tau}\|_{\mathfrak{T}}\Big]d\tau ds \\
& +\frac{1}{\Gamma(\varsigma)}\int_0^1(1-s)^{\varsigma-1}\Big[\ell_1\|w_{1s}-w^*_{1s}\|_{\mathfrak{T}}+\ell_2\|w_{2s}-w^*_{2s}\|_{\mathfrak{T}}\Big]ds \\
& +\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s(s-\tau)^{\varsigma-1}\Big[\ell_1\|w_{1\tau}-w^*_{1\tau}\|_{\mathfrak{T}}+\ell_2\|w_{2\tau}-w^*_{2\tau}\|_{\mathfrak{T}}\Big]d\tau ds \\
& +\frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\Big[\chi_1\|w_{1s}-w^*_{1s}\|_{\mathfrak{T}}+\chi_2\|w_{2s}-w^*_{2s}\|_{\mathfrak{T}}\Big]ds\Big\} \\
\le\ & \frac{1}{\Gamma(\varsigma)}\int_0^t(t-s)^{\varsigma-1}\Big[\ell_1 q^*\|w_1-w^*_1\|_{\mathfrak{T}'_a}+\ell_2 q^*\|w_2-w^*_2\|_{\mathfrak{T}'_a}\Big]ds \\
& +\frac{t}{|\Lambda_1|}\Big\{\frac{|\lambda_1|}{\Gamma(\gamma)}\int_0^{\sigma_1}\int_0^s(s-\tau)^{\gamma-1}\Big[\chi_1 q^*\|w_1-w^*_1\|_{\mathfrak{T}'_a}+\chi_2 q^*\|w_2-w^*_2\|_{\mathfrak{T}'_a}\Big]d\tau ds \\
& +\frac{1}{\Gamma(\varsigma)}\int_0^1(1-s)^{\varsigma-1}\Big[\ell_1 q^*\|w_1-w^*_1\|_{\mathfrak{T}'_a}+\ell_2 q^*\|w_2-w^*_2\|_{\mathfrak{T}'_a}\Big]ds \\
& +\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2\Gamma(\varsigma)}\int_0^{\sigma_2}\int_0^s(s-\tau)^{\varsigma-1}\Big[\ell_1 q^*\|w_1-w^*_1\|_{\mathfrak{T}'_a}+\ell_2 q^*\|w_2-w^*_2\|_{\mathfrak{T}'_a}\Big]d\tau ds \\
& +\frac{|\lambda_1|\sigma_1^2}{2\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\Big[\chi_1 q^*\|w_1-w^*_1\|_{\mathfrak{T}'_a}+\chi_2 q^*\|w_2-w^*_2\|_{\mathfrak{T}'_a}\Big]ds\Big\} \\
\le\ & q^*\Big\{\ell\Big[\frac{1}{\Gamma(\varsigma+1)}+\frac{1}{|\Lambda_1|\Gamma(\varsigma+1)}+\frac{|\lambda_1||\lambda_2|\sigma_1^2}{2|\Lambda_1|\Gamma(\varsigma+2)}\sigma_2^{\varsigma+1}\Big] \\
& +\chi\Big[\frac{|\lambda_1|}{|\Lambda_1|\Gamma(\gamma+2)}\sigma_1^{\gamma+1}+\frac{|\lambda_1|\sigma_1^2}{2|\Lambda_1|\Gamma(\gamma+1)}\Big]\Big\}(\|w_1-w^*_1\|_{\mathfrak{T}'_a}+\|w_2-w^*_2\|_{\mathfrak{T}'_a}).
\end{aligned}
$$

In a similar manner, we get

$$
\begin{aligned}
& |\mathfrak{S}_2(w_1(t),w_2(t))-\mathfrak{S}_2(w^*_1(t),w^*_2(t))| \\
\le\ & q^*\Big\{\ell\Big[\frac{|\lambda_2|}{|\Lambda_1|\Gamma(\varsigma+2)}\sigma_2^{\varsigma+1}+\frac{|\lambda_2|\sigma_2^2}{2|\Lambda_1|\Gamma(\varsigma+1)}\Big] \\
& +\chi\Big[\frac{1}{\Gamma(\gamma+1)}+\frac{1}{|\Lambda_1|\Gamma(\gamma+1)}+\frac{|\lambda_1||\lambda_2|\sigma_2^2}{2|\Lambda_1|\Gamma(\gamma+2)}\sigma_1^{\gamma+1}\Big]\Big\}(\|w_1-w^*_1\|_{\mathfrak{T}'_a}+\|w_2-w^*_2\|_{\mathfrak{T}'_a}).
\end{aligned}
$$

Consequently, it follows from the foregoing inequalities that

$$\begin{aligned}\|\mathfrak{S}(w_1,w_2)-\mathfrak{S}(w_1^*,w_2^*)\|_{\Pi'} &= \|\mathfrak{S}_1(w_1,w_2)-\mathfrak{S}_1(w_1^*,w_2^*)\|_{\Pi'}+\|\mathfrak{S}_2(w_1,w_2)-\mathfrak{S}_2(w_1^*,w_2^*)\|_{\Pi'}\\ &\leq q^*(\ell\Lambda_2+\chi\Lambda_3)\|(w_1,w_2)-(w_1^*,w_2^*)\|_{\Pi'},\end{aligned}$$

which, together with the condition (3.11), implies that $\mathfrak{S}$ is a contraction mapping. Therefore, we deduce from the conclusion of the Banach fixed-point theorem that $\mathfrak{S}$ has a unique fixed point. This ensures the existence of a unique solution to problem (1.1) on $(-\infty,1]$. □

3.1. Examples

Consider the following coupled system:

$$\begin{cases} {}^{C}D_{0^+}^{3/2}u_1(t)=f(t,u_{1t},u_{2t}), \quad t\in\Omega:=[0,1],\\ u_1(t)=\eta_1(t), \quad t\in(-\infty,0],\\ {}^{C}D_{0^+}^{5/4}u_2(t)=g(t,u_{1t},u_{2t}), \quad t\in\Omega:=[0,1],\\ u_2=\eta_2(t), \quad t\in(-\infty,0],\\ u_1(1)=1/2\displaystyle\int_0^{2/5}u_2(s)ds, \quad u_2(1)=\displaystyle\int_0^{1/3}u_1(s)ds, \end{cases} \tag{3.12}$$

where $\varsigma=3/2$, $\gamma=5/4$, $\lambda_1=1/2$, $\lambda_2=1$, $\sigma_1=2/5$, $\sigma_2=1/3$, and $f(t,u_{1t},u_{2t})$, $g(t,u_{1t},u_{2t})$, $\eta_1(t)$, and $\eta_2(t)$ will be fixed later.

We find by using the data given in (3.12) that $\Lambda_1=0.9977778$, $\Lambda_2=1.568185488$, $\Lambda_3=1.828971108$, where Λ_1, Λ_2 and Λ_3 are respectively given by (2.2), (3.8), and (3.9).

Let $\delta>0$ and set $\mathfrak{T}_\delta=\{u\in C((-\infty,0],\mathbb{R}) : \lim\limits_{\tau\to-\infty}e^{\delta\tau}u(\tau)$ exists in $\mathbb{R}\}$, with the norm $\|u\|_\delta=\sup\limits_{-\infty<\tau\leq0}e^{\delta\tau}|u(\tau)|$. It is clear that the space $\mathfrak{T}_\delta$ satisfies the axioms of phase space, and $p(t)=q(t)=L=1$, see [24]. Now, let us take the space $\Pi_\delta=\mathfrak{T}_\delta\times\mathfrak{T}_\delta$ with the norm

$$\|(u_1,u_2)\|_{\Pi_\delta}=\|u_1\|_{\mathfrak{T}_\delta}+\|u_2\|_{\mathfrak{T}_\delta}, \text{ for all } (u_1,u_2)\in\Pi_\delta.$$

One can take $\eta_1(t)=\dfrac{e^t-1}{2}$ and $\eta_2(t)=e^{3t}-1$, which are continuous functions such that $\eta_1(0)=\eta_2(0)=0$ and $\lim\limits_{t\to-\infty}e^{\delta t}\eta_1(t)<\infty$, $\lim\limits_{t\to-\infty}e^{\delta t}\eta_2(t)<\infty$. Thus, $\eta_1,\eta_2\in\mathfrak{T}_\delta$. Obviously, $(\eta_1,\eta_2)\in\Pi_\delta$ and $(\eta_1(0),\eta_2(0))=(0,0)$.

In order to illustrate Theorem 3.2, we chose

$$f(t,u_{1t},u_{2t})=\frac{4t^2}{8+t^3}+\frac{e^{-\delta t}\sin u_{1t}}{\sqrt{64+t}}+\frac{e^{-\delta t}u_{2t}|u_{1t}|}{\sqrt{t^2+900}\,(1+|u_{1t}|)}, \tag{3.13}$$

$$g(t,u_{1t},u_{2t})=\frac{2t^4}{5}+\frac{e^{-\delta t}\tan^{-1}u_{1t}}{16(t^2+1)}+\frac{e^{-\delta t}u_{2t}}{\sqrt{t+12}}. \tag{3.14}$$

Clearly,

$$|f(t,u_{1t},u_{2t})|=\frac{4t^2}{8+t^3}+\frac{1}{\sqrt{64+t}}\|u_{1t}\|_{\mathfrak{T}_\delta}+\frac{1}{\sqrt{t^2+900}}\|u_{2t}\|_{\mathfrak{T}_\delta},$$

$$|g(t, u_{1t}, u_{2t})| = \frac{2t^4}{5} + \frac{1}{16(t^2+1)}\|u_{1t}\|_{\mathfrak{T}_\delta} + \frac{1}{\sqrt{t+12}}\|u_{2t}\|_{\mathfrak{T}_\delta},$$

and note that the assumption (C_1) is satisfied with $\alpha_1(t) = \frac{4t^2}{8+t^3}, \alpha_2(t) = \frac{1}{\sqrt{64+t}}, \alpha_3 = \frac{1}{\sqrt{t^2+900}}$ and $\varphi_1 = \frac{2t^4}{5}, \varphi_2 = \frac{1}{16(t^2+1)}, \varphi_3 = \frac{1}{\sqrt{t+12}}$. Moreover, we find

$$q^*(\bar{\alpha}_2\Lambda_2 + \bar{\varphi}_2\Lambda_3) \approx 0.3103338802 < 1,$$
$$q^*(\bar{\alpha}_3\Lambda_2 + \bar{\varphi}_3\Lambda_3) \approx 0.5802513305 < 1.$$

Hence all the conditions of Theorem 3.1 are satisfied, and as a consequence, the problem (3.12) with $f(t, u_{1t}, u_{2t})$ and $g(t, u_{1t}, u_{2t})$ given by (3.13) and (3.14), respectively, has at least one solution on $(-\infty, 1]$.

Next, to demonstrate the applicability of Theorem 3.2, let us assume

$$f(t, u_{1t}, u_{2t}) = \frac{e^{-\delta t}\sin u_{1t}}{25+t} + \frac{e^{-\delta t}|u_{2t}|}{12+|u_{2t}|} + \ln 7, \tag{3.15}$$

$$g(t, u_{1t}, u_{2t}) = \frac{te^{-\delta t}u_{1t}}{24} + \frac{e^{-\delta t}\sin u_{2t}}{\sqrt{t^2+36}}. \tag{3.16}$$

Clearly, f and g satisfy condition (C_2) with $\ell_1 = 1/25, \ell_2 = 1/12, \chi_1 = 1/24, \chi_2 = 1/6$, and so $\ell = 1/12, \chi = 1/6$. Also,

$$q^*(\ell\Lambda_2 + \chi\Lambda_3) \approx 0.4355106420 < 1.$$

So, all the assumptions of Theorem 3.1 hold true, and, according to its conclusion, problem (3.12) with $f(t, u_{1t}, u_{2t})$ and $g(t, u_{1t}, u_{2t})$ given by (3.15) and (3.16), respectively, has a unique solution on $(-\infty, 1]$.

4. Conclusions

As coupled systems have gained intensive interest due to their important applications in real-world phenomena, we have considered in this paper a new class of coupled systems involving nonlinear fractional differential equations that are affected by infinite delay and complemented with nonlocal integral boundary conditions. We have investigated the existence of solutions for problem (1.1) by applying Schaefer's fixed point theorem, while for the uniqueness result, the contraction mapping principle has been employed. To deal with the differential equations with infinite delay, we needed to select an appropriate phase space that satisfies the axioms given in [22]. To guarantee the applicability of our results, illustrative examples have been constructed. The results presented in this paper take on importance as a new contribution to the study of nonlinear coupled systems with infinite delay that extends the literature on this subject. For further studies of the coupled systems with infinite delay, by following the papers [14, 16, 30], we can extend our work in this article by discussing the stability and simulation results for the solutions of the obtained system. Also, the differential equations in the problem at hand can be replaced by implicit differential equations of different types of fractional derivatives with the p-Laplacian operator, and a variety of fixed-point theorems can be applied based on our previous works [12, 15].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict interests.

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