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Research article

Fixed point theorems for b-generalized contractive mappings with weak continuity conditions

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Abstract: The purpose of this paper was to introduce several b-generalized contractive mappings in the framework of cone b-metric spaces over Banach algebras. The obtained contractions generalized and extended the counterparts in metric spaces, cone metric spaces, and b-metric spaces. Moreover, via weakening the completeness of the spaces, we gave some fixed point theorems for asymptotically regular mappings without considering the orbital continuity and k-continuity of the mappings. Those who need a specification is our results do not rely on the continuity of b-metric and the normality of cones. In addition, some nontrivial examples were presented to illustrate the superiority of our fixed point theorems.

Keywords: fixed point; Kannan type contraction; asymptotic regularity; weak orbital continuity; k-continuity
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1. Introduction and preliminaries

Since the well-known Banach contractive principle [1] was created by Polish mathematician Banach in 1922, the fixed point theory has sprung up like mushrooms, since it is widely used in solving differential equations [2, 3]. A large number of fixed point results for all kinds of contractions have been investigated in the past years. As an example, we recall the following Kannan type fixed point theorem [4]:

Theorem 1. Let (M, d) be a complete metric space, Γ a self-mapping on M, and $K \in [0, \frac{1}{2})$ a constant. If

$$d(\Gamma\varsigma, \Gamma\upsilon) \leq K[d(\varsigma, \Gamma\varsigma) + d(\upsilon, \Gamma\upsilon)], \tag{1.1}$$

holds for all $\varsigma, \upsilon \in M$, then Γ possess a unique fixed point $\varpi \in M$, and for any $\varsigma \in M$, $\Gamma^n\varsigma$ converges to ϖ as $n \to \infty$.

The mapping Γ therein is known as a Kannan type contraction. There are some other important contractive mappings such as the Reich [5,6] type contraction, Ćirić [7] type contraction, Chatterjea [8] type contraction, Hardy and Rogers [9] type contraction, Bianchini [10] type contraction, Λ-contraction [11], and so on [12, 13]. Among them, the Kannan type contraction attracts extensive attentions, since its corresponding fixed point theorem describes the relationship between the contraction and the completeness of metric spaces [14].

In 2007, Huang and Zhang [15] introduced the concept of cone metric space and gave several fixed point theorems under the conditions of normal cones. Immediately afterward, Rezapour and Hamlbarani [16] improved fixed point theorems from [15] by deleting the conditions of normalities of cones. In 2011, Hussian and Shah [17] initiated the notion of the cone b-metric space, which is a sharp generalization of b-metric space and cone metric space. Moreover, they offered several topological properties in cone b-metric spaces. Whereafter, Huang and Xu [18] considered fixed point work for some contractive mappings in cone b-metric spaces. Subsequently, Liu and Xu [19] presented the definition of cone metric space over Banach algebra, which greatly generalizes cone metric space since it replaces Banach space for Banach algebra. The significance of this paper is that they proved the nonequivalence for fixed point results between metric spaces and cone metric spaces over Banach algebras. Since then, a large number of scholars have focused on establishing fixed point results in such spaces; the reader may refer to [20–22] and the references therein. Based on the previous work, Huang and Radenović [23] reintroduced cone b-metric spaces over Banach algebras and exhibited a lot of fixed point theorems in their spaces.

For those who need a specification, among a great deal of fixed point results, asymptotic regularity acts as a basically important role in metric spaces; see ([24], Chapter IX), ([25], Chapter 9), and [26], which is defined as follows.

Definition 1. The mapping $\Gamma : M \to M$ in a metric space (M, d) is called asymptotically regular at $\varsigma \in M$ if $\lim_{n\to\infty} d(\Gamma^n\varsigma, \Gamma^{n+1}\varsigma) = 0$. Γ is said to be asymptotically regular if it is asymptotically regular at each $\varsigma \in M$.

Asymptotic regularity is usually used to obtain fixed point theorems by many researchers; see [27–32]. Recently, Górnicki [33] claimed that there is no necessary connection between asymptotically regular self-mapping and Cauchy sequence. They demonstrated that the asymptotic regularity and continuity are independent conditions. In recent years, numerous interesting results for asymptotically regular mappings in complete metric spaces were given under the assumptions of continuities. The following theorem generalizes the contraction constant from $K \in [0, \frac{1}{2})$ to $K \in [0, \infty)$ for Kannan type contraction.

Theorem 2. ([33]) Let (M, d) be a complete metric space, and let $\alpha \in [0, 1)$ and $K \in [0, \infty)$ be constants. Suppose that $\Gamma : M \to M$ is a continuous and asymptotically regular mapping satisfying

$$d(\Gamma\varsigma, \Gamma\upsilon) \le \alpha d(\varsigma, \upsilon) + K[d(\varsigma, \Gamma\varsigma) + d(\upsilon, \Gamma\upsilon)], \tag{1.2}$$

for all $\varsigma, \upsilon \in M$, then Γ possess a unique fixed point $\varpi \in M$, and for any $\varsigma \in M$, $\Gamma^n\varsigma$ converges to ϖ as $n \to \infty$.

Later on, Bisht [34] made an improvement on the above result by virtue of relaxing the continuity condition for the asymptotically regular mapping in complete metric spaces. As a matter of fact, the continuity of a mapping can lead to orbital continuity and k-continuity, but the converse is not true; see [35, 36].

Theorem 3. Let (M, d) be a complete metric space, and let $\alpha \in [0, 1)$ and $K \in [0, \infty)$ be constants. Suppose that $\Gamma : M \to M$ is an asymptotically regular mapping satisfying

$$d(\Gamma\varsigma, \Gamma\upsilon) \leq \alpha d(\varsigma, \upsilon) + K[d(\varsigma, \Gamma\varsigma) + d(\upsilon, \Gamma\upsilon)], \tag{1.3}$$

for all $\varsigma, \upsilon \in M$, then Γ possess a unique fixed point $\varpi \in M$, and for any $\varsigma \in M$, $\Gamma^n\varsigma$ converges to ϖ as $n \to \infty$, provided that either Γ is orbitally continuous or k-continuous for some $k \geq 1$.

From the three typical theorems given above, we can see that the existence of fixed points on metric spaces is usually inseparable from the completeness of the spaces and often requires the continuity of mappings and asymptotic regularity; see [29,33,34,37]. In most of the abstract spaces involving cones, such as the cone metric spaces, cone b-metric spaces, cone metric spaces over Banach algebras, cone b-metric spaces over Banach algebras, and so on, the fixed points in these spaces generally require the continuity of the cone metric or cone b-metric and the normality of the cone, such as [15, 19, 38], [39, Theorem 5, 6]. The requirements of these conditions are very strong, so how do we find the existence and uniqueness of fixed points for abstract metric spaces that do not satisfy these conditions?

In order to answer this question, throughout this paper, we give some fixed point theorems for asymptotically regular mappings in the setting of cone b-metric spaces over Banach algebras, which contain cone metric spaces, cone b-metric spaces, cone metric spaces over Banach algebras, and so on. As compared to the past results from the literature [29,33,37,39], our results weaken the completeness of the spaces, the orbital continuity, and the k-continuity of the mappings. Furthermore, we brush aside the continuity of the b-metric or the normality of the cone. In addition, we expand some notions from metric spaces, b-metric spaces, cone metric spaces, and cone b-metrics to the abstract cone b-metric spaces over Banach algebras. Besides these progresses, we also give several examples to illustrate that our new notions and theorems are genuine improvements and generalizations to the previous works in the existing literature.

In this paper, unless special explanations, we always assume that $\mathbb{N}$ is the set of all nonnegative integrals, $\mathcal{A}$ is a real Banach algebra with a unit e, which is said to be a unital Banach algebra, and P is a solid cone of $\mathcal{A}$. Let θ be the null element of $\mathcal{A}$ and $\leq$ as the partial order with respect to P. Since they are some basic notions in [40], we do not repeat them here.

In what follows, we give some necessary definitions and lemmas.

Definition 2. ([41,42]) Let M be a nonempty set and $s \geq 1$ a constant. The mapping $d : M \times M \to \mathbb{R}^+$ is called a b-metric if it satisfies:

(d1) $0 \leq d(\varsigma, \upsilon)$ for all $\varsigma, \upsilon \in M$ and $d(\varsigma, \upsilon) = \theta$ if and only if $\varsigma = \upsilon$;
(d2) $d(\varsigma, \upsilon) = d(\upsilon, \varsigma)$ for all $\varsigma, \upsilon \in M$;
(d3) $d(\varsigma, \upsilon) \leq s[d(\varsigma, \varpi) + d(\varpi, \upsilon)]$ for all $\varsigma, \upsilon, \varpi \in M$.

In this case, (M, d, s) is said to be a b-metric space.

Definition 3. ([17,23]) Let M be a nonempty set and $s \geq 1$ a constant. The mapping $d : M \times M \to \mathcal{A}$ is called a cone b-metric if it satisfies:

(d1) $\theta \leq d(\varsigma, \upsilon)$ for all $\varsigma, \upsilon \in M$ and $d(\varsigma, \upsilon) = \theta$ if and only if $\varsigma = \upsilon$;
(d2) $d(\varsigma, \upsilon) = d(\upsilon, \varsigma)$ for all $\varsigma, \upsilon \in M$;
(d3) $d(\varsigma, \upsilon) \leq s[d(\varsigma, \varpi) + d(\varpi, \upsilon)]$ for all $\varsigma, \upsilon, \varpi \in M$.

In this case, (M, d, s) is said to be a cone b-metric space over Banach algebra (CMSBA).

For some details for the notions of Cauchy sequence, convergent sequence, and completeness of (M, d, s), the reader may refer to [23] and the references therein.

Definition 3. ([43]) Let $\{u_n\}$ be a sequence in P. $\{u_n\}$ is called a c-sequence if for each $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.

Lemma 1. ([23]) Let $\{\varsigma_n\}$ and $\{\upsilon_n\}$ be c-sequences in P and $\alpha, \beta \in P$, then $\{\alpha\varsigma_n + \beta\upsilon_n\}$ is a c-sequence in P.

Lemma 2. ([40]) Let $\varsigma \in \mathcal{A}$ and $r(\varsigma)$ be the spectral radius of ς, i.e.,

$$r(\varsigma) = \lim_{n\to\infty} \|\varsigma^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|\varsigma^n\|^{\frac{1}{n}}.$$

If $r(\varsigma) < 1$, then $e - \varsigma$ is invertible. Further, one has

$$(e - \varsigma)^{-1} = \sum_{i=0}^{\infty} \varsigma^i.$$

Lemma 3. ([20]) Let $\varsigma, \upsilon \in \mathcal{A}$.
(i) If ς and υ commute, then $r(\varsigma\upsilon) \leq r(\varsigma)r(\upsilon)$.
(ii) If $0 \leq r(\varsigma) < 1$, then $r((e - \varsigma)^{-1}) \leq (1 - r(\varsigma))^{-1}$.

2. Main results

In [39], the authors proved that cone b-metric is not necessarily continuous in general even if the cone is normal, which is different from the usual metric or cone metric with a normal cone. In this section, we obtain some fixed point theorems in orbitally complete CMSBA. We omit the usual conditions such as the regularity or normality of cone, the continuity of cone b-metric, and the continuity of the mapping. Inspired by the notions of x_0-orbital continuity [44], Γ-orbital completeness, [7] and weak orbital continuity [45, 46] in metric spaces, we also give the relevant notions in the framework of CMSBA. Moreover, we provide some valuable examples to emphasize the relationships among these notions.

First of all, we give the following concept.

Definition 4. Let (M, d, s) be a CMSBA and Γ a self-mapping on M. Denote $O(\Gamma, \varsigma) = \{\varsigma, \Gamma\varsigma, \Gamma^2\varsigma, \Gamma^3\varsigma, \ldots\}$ as the orbit of $\varsigma \in M$, then:

(i) The mapping Γ is called x_0-orbitally continuous for some $x_0 \in M$ if its restriction to the set $\overline{O(\Gamma, x_0)}$ is continuous, i.e., $\Gamma : \overline{O(\Gamma, x_0)} \to M$ is continuous, where $\overline{O(\Gamma, x_0)}$ represents the closure of $O(\Gamma, x_0)$. Moreover, Γ is said to be orbitally continuous if it is x_0-orbitally continuous at each $x_0 \in M$.

(ii) The mapping Γ is called weakly orbitally continuous if the set $\{y \in M : u = \lim_i \Gamma^{n_i} y$ implies $\Gamma u = \lim_i \Gamma\Gamma^{n_i} y\}$ is nonempty, whenever the set $\{x \in M : u = \lim_i \Gamma^{n_i} x\}$ is nonempty for some $u \in M$.

(iii) The space (M, d, s) is called Γ-orbitally complete if for some $\varsigma \in M$, any Cauchy sequence in $O(\Gamma, \varsigma)$ converges in M.

It is valid that for any self-mapping Γ, every complete space (M, d, s) is Γ-orbitally complete, but a Γ-orbitally complete space (M, d, s) need not be complete. In addition, orbital continuity and k-continuity of Γ imply weak orbital continuity and x_0-orbital continuity, but the converse need not be true. Kindly see the following example.

Example 1. Let $\mathcal{A} = \mathbb{R}^2$ with the norm $\|(\varsigma_1, \varsigma_2)\| = |\varsigma_1| + |\varsigma_2|$ and the multiplication defined by

$$\varsigma\upsilon = (\varsigma_1, \varsigma_2)(\upsilon_1, \upsilon_2) = (\varsigma_1\upsilon_1, \varsigma_1\upsilon_2 + \varsigma_2\upsilon_1),$$

where $\varsigma = (\varsigma_1, \varsigma_2), \upsilon = (\upsilon_1, \upsilon_2) \in \mathcal{A}$. It is easy to see that $\mathcal{A}$ is a unital Banach algebra with its unital element $e = (1, 0)$. Define a cone $P = \{(\varsigma_1, \varsigma_2) \in \mathbb{R}^2 : \varsigma_1, \varsigma_2 \geq 0\}$. Put $M = [0, 1] \times [0, 1]$ and construct a mapping $d : M \times M \to \mathcal{A}$ by

$$d((\varsigma_1, \varsigma_2), (\upsilon_1, \upsilon_2)) = (|\varsigma_1 - \upsilon_1|^2, |\varsigma_2 - \upsilon_2|^2),$$

where $\varsigma = (\varsigma_1, \varsigma_2), \upsilon = (\upsilon_1, \upsilon_2) \in M$. We can easily prove that (M, d, s) is a CMSBA with $s = 2$. Define a self-mapping Γ on M as

$$\Gamma\varsigma = \Gamma(\varsigma_1, \varsigma_2) = \begin{cases} (1, 1), & (\varsigma_1, \varsigma_2) = (0, 0); \\ (1, 1), & (\varsigma_1, \varsigma_2) = (1, 1); \\ (\frac{4}{7}\varsigma_1, \frac{4}{7}\varsigma_2), & \textit{otherwise}. \end{cases}$$

It is clear that Γ is weakly orbitally continuous. Actually, let $\varsigma = (0, 0)$, then $\Gamma^n\varsigma \to (1, 1)$ and $\Gamma(\Gamma^n\varsigma) \to (1, 1) = \Gamma(1, 1)$ as $n \to \infty$. However, Γ is not orbitally continuous, and this is because $\Gamma^n(\varsigma_1, \varsigma_2) = \left(\left(\frac{4}{7}\right)^n \varsigma_1, \left(\frac{4}{7}\right)^n \varsigma_2\right) \to (0, 0)$ as $n \to \infty$, which follows that, for all $\varsigma_1, \varsigma_2 \in (0, 1)$, $\Gamma(\Gamma^n(\varsigma_1, \varsigma_2)) \to (0, 0) \neq \Gamma(0, 0)$ as $n \to \infty$. In the meanwhile, we can prove that Γ is not k-continuous for all $k \in \mathbb{N}$. That is to say, for any integer $k \in \mathbb{N}$, if n tends to ∞, then

$$\Gamma^{k-1}(\Gamma^n(0, 1)) \to (0, 0), \ \ \Gamma^k(\Gamma^n(0, 1)) \to (0, 0) \neq \Gamma(0, 0).$$

Accordingly, we claim that the conditions of k-continuity and orbital continuity of the mapping Γ are stronger than the weak orbital continuity of Γ.

The concept of asymptotic regularity in CMSBA was introduced in [39], which is a sharp generalization of the counterpart in metric spaces. The notion introduced in metric spaces is valid only under normal cones (see [22, Proposition 2.5]) or usual metric spaces (see [33, 34, 47]), whereas our results are discussed in CMSBA wherein the relevant cone is not necessarily normal. Now, we state it here for convenience.

Definition 5. Let (M, d, s) be a CMSBA. The mapping $\Gamma : M \to M$ is said to be asymptotically regular if for every $c \in \mathcal{A}$ with $c \gg \theta$, there is a $N \in \mathbb{N}$ such that for any $n \geq N$, $d(\Gamma^{n+1}\varsigma, \Gamma^n\varsigma) \ll c$ for all $\varsigma \in M$. In other words, $\{d(\Gamma^{n+1}\varsigma, \Gamma^n\varsigma)\}$ is a c-sequence for all $\varsigma \in M$.

Before displaying our main results, we first start with the subsequent notion of b-generalized Kannan-Górnicki type mapping in CMSBA.

Definition 6. The mapping $\Gamma : M \to M$ is called a b-generalized Kannan-Górnicki type mapping if there exists some $h \in P$ such that, for all $\varsigma, \upsilon \in M$, it holds

$$d(\Gamma\varsigma, \Gamma\upsilon) \preceq h[d(\varsigma, \Gamma\varsigma) + d(\upsilon, \Gamma\upsilon)]. \tag{2.1}$$

It is not hard to verify that every b-generalized Kannan-Górnicki type mapping is a sharp extension of Kannan type contractive mapping. The next example fully illustrates this point.

Example 2. Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1] \times C^1_{\mathbb{R}}[0, 1]$ with the norm

$$\|(\varsigma_1, \varsigma_2)\| = \|\varsigma_1\|_\infty + \|\varsigma_2\|_\infty + \|\varsigma_1'\|_\infty + \|\varsigma_2'\|_\infty.$$

Define the multiplication by

$$\varsigma\upsilon = (\varsigma_1, \varsigma_2)(\upsilon_1, \upsilon_2) = (\varsigma_1\upsilon_1, \varsigma_1\upsilon_2 + \varsigma_2\upsilon_1),$$

where $\varsigma = (\varsigma_1, \varsigma_2), \upsilon = (\upsilon_1, \upsilon_2) \in \mathcal{A}$. It is obvious to show that $\mathcal{A}$ is a unital Banach algebra with $e = (1, 0)$. Put $P = \{(\varsigma_1(t), \varsigma_2(t)) \in \mathcal{A} : \varsigma_1(t) \geq 0, \varsigma_2(t) \geq 0, t \in [0, 1]\}$. Let $M = [0, 1] \times [0, 1]$ and define a mapping $d : M \times M \to \mathcal{A}$ by

$$d((\varsigma_1, \varsigma_2), (\upsilon_1, \upsilon_2))(t) = (|\varsigma_1 - \upsilon_1|^2, |\varsigma_2 - \upsilon_2|^2) \cdot \alpha^t, \quad \varsigma = (\varsigma_1, \varsigma_2),\ \upsilon = (\upsilon_1, \upsilon_2) \in M,$$

where $\alpha > 0$ is a constant. Act a mapping $\Gamma : M \to M$ as $\Gamma\varsigma = \Gamma(\varsigma_1, \varsigma_2) = \left(\frac{3}{5}\varsigma_1, \frac{3}{5}\varsigma_2\right)$, and it is obvious that Γ is not a usual Kannan type contractive mapping. Indeed, choose $\varsigma = (0, 0)$, $\upsilon = (1, 1)$, then $\Gamma\varsigma = (0, 0), \Gamma\nu = (\frac{3}{5}, \frac{3}{5})$. There is never $h \in [0, \frac{1}{2})$ satisfying

$$d(\Gamma\varsigma, \Gamma\upsilon)(t) = \left(\frac{9}{25}, \frac{9}{25}\right) \cdot \alpha^t \leq h\left(\frac{4}{25}, \frac{4}{25}\right) \cdot \alpha^t = h[d(\varsigma, \Gamma\varsigma)(t) + d(\upsilon, \Gamma\upsilon)(t)].$$

On the other hand, via simple calculations, one can show that the mapping Γ is a b-generalized Kannan-Górnicki type mapping.

In the sequel, we exhibit a fixed point theorem for asymptotically regular b-generalized Kannan-Górnicki type mapping in orbitally complete CMSBA without depending on the condition of regularity or normality of the cone. Besides that, the mapping and the cone b-metric are not necessarily continuous.

Theorem 4. Let (M, d, s) be a Γ-orbitally complete CMSBA and $\Gamma : M \to M$ an asymptotically regular b-generalized Kannan-Górnicki type mapping. If the mapping Γ is ς_0-orbitally continuous at some $\varsigma_0 \in M$ or weakly orbitally continuous, then Γ possess a unique fixed point $\varpi \in M$, and for any $\varsigma \in M$, $\Gamma^n\varsigma$ converges to ϖ as $n \to \infty$.

Proof. Choose $\varsigma_0 \in M$, and define a Picard iterative sequence $\{\varsigma_n\}$ as $\varsigma_n = \Gamma\varsigma_{n-1} = \Gamma^n\varsigma_0$, $n = 1, 2, \cdots$. If $\varsigma_{n-1} = \varsigma_n$ holds for some $n \in \mathbb{N}$, then the desired result is valid. Without loss of generality, we assume that $\varsigma_{n-1} \neq \varsigma_n$ for any $n \in \mathbb{N}$. By the asymptotical regularity of Γ, the sequence $\{d(\varsigma_n, \varsigma_{n+1})\}$ is a c-sequence. For any $m, n \in \mathbb{N}$ and $m > n$, we have

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &= d(\Gamma\varsigma_{n-1}, \Gamma\varsigma_{m-1}) \\ &\leq h[d(\varsigma_{n-1}, \Gamma\varsigma_{n-1}) + d(\varsigma_{m-1}, \Gamma\varsigma_{m-1})] \\ &= h[d(\varsigma_{n-1}, \varsigma_n) + d(\varsigma_{m-1}, \varsigma_m)]. \end{aligned}$$

Since Γ is asymptotically regular, by Lemma 1 we know $\{d(\varsigma_n, \varsigma_m)\}$ is a c-sequence, which implies that $\{\varsigma_n\}$ is a Cauchy sequence in M. As (M, d, s) is Γ-orbitally complete, there exists some $\varpi \in M$ such that $\varsigma_n \to \varpi$ as $n \to \infty$.

If Γ is ς_0-orbitally continuous, then $\Gamma^n\varsigma_0 \to \varpi$ leads to $\Gamma^{n+1}\varsigma_0 \to \Gamma\varpi$. By the uniqueness of limit, one has $\varpi = \Gamma\varpi$. If Γ is weakly orbitally continuous, then $\Gamma^n\varsigma_0 \to \varpi$ for each $\varsigma_0 \in M$. Hence, we have $\Gamma^n\varrho_0 \to \varpi$ and $\Gamma^{n+1}\varrho_0 \to \Gamma\varpi$ for some $\varrho_0 \in M$, which means that $\varpi = \Gamma\varpi$. Thus, ϖ is a fixed point of Γ.

Now, it remains to prove the uniqueness of ϖ. Assume that there exists another $\upsilon \in M$ such that $\Gamma\upsilon = \upsilon$, then

$$d(\upsilon, \varpi) = d(\Gamma\upsilon, \Gamma\varpi) \leq h[d(\upsilon, \Gamma\upsilon) + d(\varpi, \Gamma\varpi)] = \theta,$$

which yields that $\upsilon = \varpi$. Therefore, ϖ is the unique fixed point of Γ.

Remark 1. We can prove that if Γ is a Kannan type contractive mapping in (M, d, s) and $h \in P$ with $r(h) \in [0, \frac{1}{2})$, then by Lemmas 2 and 3, Γ^m is also a Kannan type contractive mapping with vector $(e-h)^{-(m-1)}h^m$, and

$$r((e-h)^{-(m-1)}h^m) \leq \left(\frac{r(h)}{1-r(h)}\right)^{m-1} r(h)$$

holds for all $m \in \mathbb{N}$ and $m \geq 2$. As a matter of fact, we obtain

$$d(\Gamma^m\varsigma, \Gamma^m\upsilon) = d(\Gamma(\Gamma^{m-1}\varsigma), \Gamma(\Gamma^{m-1}\upsilon)) \leq h[d(\Gamma^{m-1}\varsigma, \Gamma^m\varsigma) + d(\Gamma^{m-1}\upsilon, \Gamma^m\upsilon)],$$

where

$$\begin{aligned} d(\Gamma^{m-1}\varsigma, \Gamma^m\varsigma) + d(\Gamma^{m-1}\upsilon, \Gamma^m\upsilon) \leq\ & h[d(\Gamma^{m-2}\varsigma, \Gamma^{m-1}\varsigma) + d(\Gamma^{m-1}\varsigma, \Gamma^m\varsigma) \\ & + d(\Gamma^{m-2}\upsilon, \Gamma^{m-1}\upsilon) + d(\Gamma^{m-1}\upsilon, \Gamma^m\upsilon)]. \end{aligned}$$

Due to the fact that $r(h) \in [0, \frac{1}{2})$ and Lemma 3, we speculate that $e-h$ is invertible, so

$$\begin{aligned} d(\Gamma^{m-1}\varsigma, \Gamma^m\varsigma) + d(\Gamma^{m-1}\upsilon, \Gamma^m\upsilon) &\leq (e-h)^{-1}h[d(\Gamma^{m-2}\varsigma, \Gamma^{m-1}\varsigma) + d(\Gamma^{m-2}\upsilon, \Gamma^{m-1}\upsilon)] \\ &\leq (e-h)^{-2}h^2[d(\Gamma^{m-3}\varsigma, \Gamma^{m-2}\varsigma) + d(\Gamma^{m-3}\upsilon, \Gamma^{m-2}\upsilon)]. \end{aligned}$$

By the iteration, we have

$$d(\Gamma^m\varsigma, \Gamma^m\upsilon) \leq (e-h)^{-(m-1)}h^m[d(\varsigma, \Gamma\varsigma) + d(\upsilon, \Gamma\upsilon)].$$

On the other hand, on account of Lemma 3, it is palpable that

$$r((e-h)^{-(m-1)}h^m) \leq (r(e-h)^{-1}h)^{m-1}r(h) \leq \left(\frac{r(h)}{1-r(h)}\right)^{m-1} r(h).$$

Thus, Γ^m is a Kannan type mapping, but it is not true if Γ is a generalized Kannan-Górnicki type contractive mapping. See the following example.

Example 3. Let $\mathcal{A}$ and P be defined as the same as in Example 2. Put $M = [\frac{1}{4}, 4] \times [\frac{1}{4}, 4]$ and $\Gamma : M \to M$ as $\Gamma(\varsigma_1, \varsigma_2) = (\frac{1}{\varsigma_1}, \frac{1}{\varsigma_2})$, $\varsigma = (\varsigma_1, \varsigma_2) \in M$. Define a mapping as

$$d(\varsigma, \upsilon)(t) = \begin{cases} (exp(t),\ exp(t)), & \text{if } \varsigma \neq \upsilon; \\ (0,\ 0), & \text{if } \varsigma = \upsilon, \end{cases}$$

where $\varsigma = (\varsigma_1, \varsigma_2), \upsilon = (\upsilon_1, \upsilon_2)$, and $\varsigma = \upsilon$ if $\varsigma_1 = \upsilon_1, \varsigma_2 = \upsilon_2$.

It is not hard to verify that Γ is a b-generalized Kannan-Górnicki type mapping with $h = (1, 1)$. In fact, there are two cases as follows.

Case 1. If $\varsigma = \upsilon$, then $\Gamma\varsigma = \Gamma\upsilon$ and $d(\Gamma\varsigma, \Gamma\upsilon)(t) = (0, 0)$. The inequality (2.1) holds for all $h \in P$.

Case 2. If $\varsigma \neq \upsilon$, then $\Gamma\varsigma \neq \Gamma\upsilon$ and $d(\Gamma\varsigma, \Gamma\upsilon)(t) = (exp(t), exp(t))$. If $\varsigma = (1, 1)$, $\upsilon \neq (1, 1)$, then $\Gamma\varsigma = (1, 1)$, $d(\varsigma, \Gamma\varsigma)(t) = (0, 0)$, and $d(\upsilon, \Gamma\upsilon)(t) = (exp(t), exp(t))$. Thus, the inequality (2.1) holds for $h = (a^t, a^t) \in P$, where $a > 1$ is a constant. If $\varsigma \neq (1, 1)$, $\upsilon = (1, 1)$, we obtain the same result. If $\varsigma \neq (1, 1), \upsilon \neq (1, 1)$, then

$$d(\Gamma\varsigma, \Gamma\upsilon)(t) = (exp(t), exp(t))$$

$$\preceq h\{(exp(t), exp(t)) + (exp(t), exp(t))\}$$
$$= h\{d(\varsigma, \Gamma\varsigma)(t) + d(\upsilon, \Gamma\upsilon)(t)\},$$

for $h = (\frac{1}{2}a^t, \frac{1}{2}a^t) \in P$, where $a > 1$ is a constant.

Notice that Γ^2 is not a b-generalized Kannan-Górnicki type mapping for all $h \in P$. For any $\varsigma, \upsilon \in M$ with $\varsigma \neq \upsilon$, we have $\Gamma^2\varsigma = \varsigma$, $\Gamma^2\upsilon = \upsilon$. Thus, one has

$$d(\Gamma^2\varsigma, \Gamma^2\upsilon)(t) = d(\varsigma, \upsilon)(t) = (exp(t), exp(t)),$$
$$d(\varsigma, \Gamma^2\varsigma)(t) = d(\varsigma, \varsigma)(t) = (0, 0) = d(\upsilon, \Gamma^2\upsilon)(t),$$

which is a contradiction with the fact that $(exp(t), exp(t)) \preceq h(0, 0)$ does not hold for all $h \in P$.

In the following, we will extend Kannan's and Górnicki's work to more general theorems. For this purpose, denote Ψ as the set of all the functions $\psi : P \times P \to P$, satisfying the following conditions:

(i) $\psi(\theta, \theta) = \theta$;

(ii) ψ is continuous at (θ, θ).

Definition 7. Let (M, d, s) be a CMSBA and $\Gamma : M \to M$ be a mapping, then

(i) Γ is called a b-generalized Ćirić-Proinov-Górnicki type mapping if there is some $h \in P$ with $r(h) < \frac{1}{s^2}$, such that
$$d(\Gamma\varsigma, \Gamma\upsilon) \preceq hu(\varsigma, \upsilon) + \psi(d(\varsigma, \Gamma\varsigma), d(\upsilon, \Gamma\upsilon)), \tag{2.2}$$
for all $\varsigma, \upsilon \in M$, where $u(\varsigma, \upsilon) \in \{d(\varsigma, \upsilon), d(\varsigma, \Gamma\upsilon), d(\upsilon, \Gamma\varsigma)\}$;

(ii) Γ is called a b-generalized Hardy-Rogers-Proinov-Górnicki type mapping if there are $k, l, j \in P$ with $r(sk + l + j) < \frac{1}{s}$, such that
$$d(\Gamma\varsigma, \Gamma\upsilon) \preceq kd(\varsigma, \upsilon) + ld(\varsigma, \Gamma\upsilon) + jd(\upsilon, \Gamma\varsigma) + \psi(d(\varsigma, \Gamma\varsigma), d(\upsilon, \Gamma\upsilon)), \tag{2.3}$$
for all $\varsigma, \upsilon \in M$;

(iii) Γ is called a b-generalized Reich-Proinov-Górnicki type mapping if there is some $h \in P$ with $r(h) < \frac{1}{s^2}$, such that
$$d(\Gamma\varsigma, \Gamma\upsilon) \preceq hd(\varsigma, \upsilon) + \psi(d(\varsigma, \Gamma\varsigma), d(\upsilon, \Gamma\upsilon)), \tag{2.4}$$
for all $\varsigma, \upsilon \in M$ and for some $\psi \in \Psi$.

By these definitions, we would like to give the following results.

Theorem 5. Let (M, d, s) be a Γ-orbitally complete CMSBA and $\Gamma : M \to M$ an asymptotically regular b-generalized Ćirić-Proinov-Górnicki type mapping. Suppose that Γ is ς_0-orbitally continuous at some $\varsigma_0 \in M$ or weakly orbitally continuous, then Γ possess a unique fixed point $\varpi \in M$, and for any $\varsigma \in M$, $\Gamma^n\varsigma$ converges to ϖ as $n \to \infty$.

Proof. Choose $\varsigma_0 \in M$, and construct a Picard iterative sequence as $\varsigma_n = \Gamma\varsigma_{n-1} = \Gamma^n\varsigma_0$, $n = 1, 2, \cdots$. If $\varsigma_{n-1} = \varsigma_n$ holds for some $n \in \mathbb{N}$, then the desired result is valid. Without loss of generality, we assume that $\varsigma_{n-1} \neq \varsigma_n$ for any $n \in \mathbb{N}$. Since Γ is asymptotically regular, then $\{d(\varsigma_n, \varsigma_{n+1})\}$ is a c-sequence. For any $m, n \in \mathbb{N}$ and $m > n$, we have

$$d(\varsigma_n, \varsigma_m) = d(\Gamma\varsigma_{n-1}, \Gamma\varsigma_{m-1}) \preceq hu(\varsigma_{n-1}, \varsigma_{m-1}) + \psi[d(\varsigma_{n-1}, \Gamma\varsigma_{n-1}), d(\varsigma_{m-1}, \Gamma\varsigma_{m-1})],$$

where

$$u(\varsigma_{n-1}, \varsigma_{m-1}) \in \{d(\varsigma_{n-1}, \varsigma_{m-1}), d(\varsigma_{n-1}, \Gamma\varsigma_{m-1}), d(\varsigma_{m-1}, \Gamma\varsigma_{n-1})\}.$$

Put $\delta_n = d(\varsigma_{n-1}, \Gamma\varsigma_{n-1})$, then $\delta_m = d(\varsigma_{m-1}, \Gamma\varsigma_{m-1})$. Consequently, it establishes that

$$\psi[d(\varsigma_{n-1}, \Gamma\varsigma_{n-1}), d(\varsigma_{m-1}, \Gamma\varsigma_{m-1})] = \psi(\delta_n, \delta_m).$$

Now, we will consider the proof as three cases.
Case 1. If $u(\varsigma_{n-1}, \varsigma_{m-1}) = d(\varsigma_{n-1}, \varsigma_{m-1})$, then

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &\leq hd(\varsigma_{n-1}, \varsigma_{m-1}) + \psi(\delta_n, \delta_m) \\ &\leq h[s(d(\varsigma_{n-1}, \varsigma_n) + d(\varsigma_n, \varsigma_{m-1}))] + \psi(\delta_n, \delta_m) \\ &\leq shd(\varsigma_{n-1}, \varsigma_n) + s^2h[d(\varsigma_n, \varsigma_m) + d(\varsigma_m, \varsigma_{m-1})] + \psi(\delta_n, \delta_m). \end{aligned}$$

In view of $r(h) < \frac{1}{s^2}$, then $e - s^2h$ is invertible and

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &\leq (e - s^2h)^{-1}[shd(\varsigma_{n-1}, \varsigma_n) + s^2hd(\varsigma_m, \varsigma_{m-1}) + \psi(\delta_n, \delta_m)] \\ &= (e - s^2h)^{-1}[sh\delta_n + s^2h\delta_m + \psi(\delta_n, \delta_m)]. \end{aligned}$$

Case 2. If $u(\varsigma_{n-1}, \varsigma_{m-1}) = d(\varsigma_{n-1}, \Gamma\varsigma_{m-1}) = d(\varsigma_{n-1}, \varsigma_m)$, then

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &\leq hd(\varsigma_{n-1}, \varsigma_m) + \psi(\delta_n, \delta_m) \\ &\leq sh[d(\varsigma_{n-1}, \varsigma_n) + d(\varsigma_n, \varsigma_m)] + \psi(\delta_n, \delta_m), \end{aligned}$$

which yields that

$$d(\varsigma_n, \varsigma_m) \leq (e - sh)^{-1}[shd(\varsigma_{n-1}, \varsigma_n) + \psi(\delta_n, \delta_m)] = (e - sh)^{-1}[sh\delta_n + \psi(\delta_n, \delta_m)].$$

Case 3. If $u(\varsigma_{n-1}, \varsigma_{m-1}) = d(\varsigma_{m-1}, \Gamma\varsigma_{n-1}) = d(\varsigma_{m-1}, \varsigma_n)$, then

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &\leq hd(\varsigma_{m-1}, \varsigma_n) + \psi(\delta_n, \delta_m) \\ &\leq sh[d(\varsigma_{m-1}, \varsigma_m) + d(\varsigma_m, \varsigma_n)] + \psi(\delta_n, \delta_m), \end{aligned}$$

which implies that

$$d(\varsigma_n, \varsigma_m) \leq (e - sh)^{-1}[shd(\varsigma_{m-1}, \varsigma_m) + \psi(\delta_n, \delta_m)] = (e - sh)^{-1}[sh\delta_m + \psi(\delta_n, \delta_m)].$$

Since $\{\delta_n\}$ is a c-sequence, as well as $\{\delta_m\}$ for all $m > n$, by the condition that ψ is continuous at (θ, θ), we claim that for all cases, $\{\varsigma_n\}$ is a Cauchy sequence in M. By virtue of the fact that (M, d, s) is Γ-orbitally complete, there exists some $\varpi \in M$ such that $\varsigma_n = \Gamma^n\varsigma_0 \to \varpi$ as $n \to \infty$.

If Γ is ς_0-orbitally continuous, then $\Gamma^n\varsigma_0 \to \varpi$ implies $\Gamma^{n+1}\varsigma_0 \to \Gamma\varpi$. By the uniqueness of limit, we get $\varpi = \Gamma\varpi$. If Γ is weakly orbitally continuous, then $\Gamma^n\varsigma_0 \to \varpi$ for each $\varsigma_0 \in M$. Based on the weak orbital continuity of Γ, we obtain $\Gamma^n\varrho_0 \to \varpi$ and $\Gamma^{n+1}\varrho_0 \to \Gamma\varpi$ for some $\varrho_0 \in M$, which follows from $\Gamma^{n+1}\varrho_0 \to \varpi$ that $\varpi = \Gamma\varpi$. In other words, ϖ is a fixed point of Γ.

Finally, we will show the uniqueness of the fixed point. Actually, assume that Γ has another fixed point υ, i.e., $\Gamma\upsilon = \upsilon$, then

$$d(\upsilon, \varpi) = d(\Gamma\upsilon, \Gamma\varpi) \leq hu(\upsilon, \varpi) + \psi(d(\upsilon, \Gamma\upsilon), d(\varpi, \Gamma\varpi)) \leq hd(\upsilon, \varpi) + \psi(\theta, \theta) = hd(\upsilon, \varpi),$$

where

$$u(\upsilon, \varpi) \in \{d(\upsilon, \varpi), d(\upsilon, \Gamma\varpi), d(\varpi, \Gamma\upsilon)\} = d(\upsilon, \varpi).$$

Making full use of $r(h) < \frac{1}{s^2} < 1$, we acquire $\upsilon = \varpi$. Therefore, we finish the proof.

Similar to Theorem 5, we have the following results on the Eqs (2.3) and (2.4). We omit their proofs.

Theorem 6. Let (M, d, s) be a Γ-orbitally complete CMSBA and $\Gamma : M \to M$ an asymptotically regular b-generalized Hardy-Rogers-Proinov-Górnicki type mapping. Suppose that Γ is ς_0-orbitally continuous at some $\varsigma_0 \in M$ or weakly orbitally continuous, then Γ possess a unique fixed point $\varpi \in M$, and for any $\varsigma \in M$, $\Gamma^n\varsigma$ converges to ϖ as $n \to \infty$.

Theorem 7. Let (M, d, s) be a Γ-orbitally complete CMSBA and $\Gamma : M \to M$ an asymptotically regular b-generalized Reich-Proinov-Górnicki type mapping. Suppose that Γ is ς_0-orbitally continuous at some $\varsigma_0 \in M$ or weakly orbitally continuous, then Γ possess a unique fixed point $\varpi \in M$, and for any $\varsigma \in M$, $\Gamma^n\varsigma$ converges to ϖ as $n \to \infty$.

Remark 2. Our theorems greatly generalize and improve the results from [29, 33, 34, 37]. The results in the existing literature always rely strongly on the completeness of the spaces. Moreover, some classic conclusions such as [29, Theorem 2.3], [33, Theorem 2.6], and [37, Theorems 4.1 and 4.2] always depend on the continuity of the mapping or need the orbital continuity or k-continuity (see [34, Theorem 2.1]). Throughout this paper, we use the weak orbital continuity instead of the usual continuity, and utilize the orbital completeness instead of the general completeness for the spaces. The improvements weaken the conditions. As a consequence, we have more conveniences for applications in the future. Furthermore, these results are obtained in CMSBA with a non-normal cone and discontinuous cone b-metric, which are not equivalent to the theorems in cone b-metric spaces or b-metric spaces. They may offer us more applications since there are lots of non-normal cones (see [16]).

We give a nontrivial example in which the mapping Γ has a fixed point since it satisfies our conditions (2.2)–(2.4). However, it does not satisfies the condition (1.3), which means that our results have their superiorities.

Example 4. Let $\mathcal{A}$, P, and d be defined as the same as in Example 1. Choose a set $M = [0, +\infty) \times [0, +\infty)$ and a mapping $\Gamma : M \to M$ as

$$\Gamma(\varsigma_1, \varsigma_2) = \left(\frac{\varsigma_1}{\varsigma_1 + 1}, \frac{\varsigma_2}{\varsigma_2 + 1}\right),$$

for all $\varsigma = (\varsigma_1, \varsigma_2) \in M$. Set $\psi(\varsigma, \upsilon) = \sqrt{\varsigma} + \sqrt{\upsilon}$, where $\varsigma = (\varsigma_1, \varsigma_2), \upsilon = (\upsilon_1, \upsilon_2) \in M$. For any $\varsigma, \upsilon \in M$, we get

$$d(\varsigma, \Gamma\varsigma) = \left(\left|\varsigma_1 - \frac{\varsigma_1}{\varsigma_1 + 1}\right|^2, \left|\varsigma_2 - \frac{\varsigma_2}{\varsigma_2 + 1}\right|^2\right) = \left(\frac{\varsigma_1^4}{(\varsigma_1 + 1)^2}, \frac{\varsigma_2^4}{(\varsigma_2 + 1)^2}\right).$$

Similarly, we have

$$d(\upsilon, \Gamma\upsilon) = \left(\frac{\upsilon_1^4}{(\upsilon_1 + 1)^2}, \frac{\upsilon_2^4}{(\upsilon_2 + 1)^2}\right).$$

Thus, we obtain

$$\psi(d(\varsigma, \Gamma\varsigma), d(\upsilon, \Gamma\upsilon)) = \left(\frac{\varsigma_1^2}{\varsigma_1 + 1} + \frac{\upsilon_1^2}{\upsilon_1 + 1}, \frac{\varsigma_2^2}{\varsigma_2 + 1} + \frac{\upsilon_2^2}{\upsilon_2 + 1}\right),$$

and

$$d(\Gamma\varsigma, \Gamma\upsilon) = \left(\left|\frac{\varsigma_1 - \upsilon_1}{(\varsigma_1 + 1)(\upsilon_1 + 1)}\right|^2, \left|\frac{\varsigma_2 - \upsilon_2}{(\varsigma_2 + 1)(\upsilon_2 + 1)}\right|^2\right).$$

After simple calculations, we can see that Γ satisfies the conditions (2.2)–(2.4) for the corresponding $h \in P$. The other conditions of Theorem 5 are also satisfied. Hence, the mapping Γ has a unique fixed point $\varsigma = (0, 0)$ in M.

However, by taking $\varsigma = (0, 0)$, $\upsilon = (\frac{1}{n}, \frac{1}{m})$, we can prove that Γ is not a usual Kannan type contractive mapping since it does not satisfy (1.3). In view of $\Gamma\varsigma = (0, 0)$, $\Gamma\upsilon = (\frac{1}{n+1}, \frac{1}{m+1})$, it means that

$$\begin{aligned} d(\Gamma\varsigma, \Gamma\upsilon) &= \left(\frac{1}{(n+1)^2}, \frac{1}{(m+1)^2}\right) \\ &\leq \alpha\left(\frac{1}{n^2}, \frac{1}{m^2}\right) + K\left[(0, 0) + \left(\frac{1}{n^2(n+1)^2}, \frac{1}{m^2(m+1)^2}\right)\right]. \end{aligned}$$

Hence, one has

$$\frac{1}{(n+1)^2} \leq \frac{\alpha}{n^2} + \frac{K}{n^2(n+1)^2},$$

where $\alpha \in [0, 1)$ and $K \geq 0$ are constants. This leads to a contradiction for large enough n because

$$1 \leq \alpha(1 + \frac{1}{n})^2 + \frac{K}{n^2},$$

implies $\alpha \geq 1$. Therefore, based on Theorem 5, we can infer that the mapping Γ has a unique fixed point in M, but we cannot make the same conclusion by Theorem 3.

In the end, we will provide an example to illustrate the genuine amelioration between our results and those in the existing literature. It satisfies all the conditions of Theorem 7, then there exists a unique fixed point. However, the results in the references do not get the corresponding conclusions since they require some additional conditions.

Example 5. Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 3]$ with the norm $\|\varsigma\| = \|\varsigma\|_\infty + \|\varsigma'\|_\infty$. The multiplication of $\mathcal{A}$ is defined by its usual pointwise multiplication, then $\mathcal{A}$ is a unital Banach algebra with $e = 1$. Set $M = [0, 3)$ and define $d(\varsigma, \upsilon)(t) = |\varsigma - \upsilon|^2\phi$ for all $\varsigma, \upsilon \in M$ and $\phi \in P = \{f(t) \in \mathcal{A} : f(t) \geq 0, t \in [0, 1]\}$, then P is a non-normal cone in CMSBA (M, d, s) with the coefficient $s = 2$. Choose $h(t) = \frac{t}{20} + \frac{1}{6}$ and $\psi(\varsigma(t), \upsilon(t)) = \varsigma(t) + \upsilon(t)$ for all $\varsigma(t), \upsilon(t) \in P$. We have

$$h^n(t) = \left(\frac{t}{20} + \frac{1}{6}\right)^n, \quad (h^n(t))' = \frac{n}{20}\left(\frac{t}{20} + \frac{1}{6}\right)^{n-1},$$

which follows that

$$\|h^n\| = \|h^n\|_\infty + \|(h^n)'\|_\infty = \frac{n}{20}\left(\frac{13}{60}\right)^{n-1}\left(1 + \frac{13}{3n}\right).$$

Consequently, we obtain

$$r(h) = \lim_{n\to\infty} \|h^n\|^{\frac{1}{n}} = \frac{13}{60} < \frac{1}{4} = \frac{1}{s^2}.$$

Define the mapping $\Gamma : M \to M$ by

$$\Gamma\varsigma = \begin{cases} \frac{6}{5}, & \varsigma \in [0, 2); \\ \frac{3+2\varsigma}{5}, & \varsigma \in [2, 3). \end{cases}$$

Thus, Γ is asymptotically regular and weakly orbitally continuous, but not orbitally continuous or k-continuous. In fact, for any $\varsigma \in [2, 3)$, letting n tend to ∞, we have

$$\Gamma^n \varsigma \to 1, \ \ \Gamma(\Gamma^n \varsigma) \to 1 \neq \Gamma 1 = \frac{6}{5},$$

which yields that Γ is not orbitally continuous. By using the same method, we can prove that Γ is not k-continuous. Moreover, the space (M, d, s) is Γ-orbitally complete but not complete. Now, we will utilize three cases to show that the inequality (2.4) is satisfied.

(1) For all $\varsigma, \upsilon \in [0, 2)$, it is clear that

$$d(\Gamma\varsigma, \Gamma\upsilon)(t) = \left|\frac{6}{5} - \frac{6}{5}\right|^2 \phi \leq hd(\varsigma, \upsilon)(t) + \psi(d(\varsigma, \Gamma\varsigma)(t), d(\upsilon, \Gamma\upsilon)(t)).$$

(2) For all $\varsigma, \upsilon \in [2, 3)$, one gets

$$\begin{aligned} d(\Gamma\varsigma, \Gamma\upsilon)(t) &= \left|\frac{3 + 2\varsigma}{5} - \frac{3 + 2\upsilon}{5}\right|^2 \phi \\ &\leq \left(\frac{t}{20} + \frac{1}{6}\right)|\varsigma - \upsilon|^2 \phi \\ &= hd(\varsigma, \upsilon)(t) + \psi(d(\varsigma, \Gamma\varsigma)(t), d(\upsilon, \Gamma\upsilon)(t)). \end{aligned}$$

(3) For all $\varsigma \in [0, 2)$, $\upsilon \in [2, 3)$, we arrive at

$$\begin{aligned} d(\Gamma\varsigma, \Gamma\upsilon)(t) &= \left|\frac{3 + 2\upsilon}{5} - \frac{6}{5}\right|^2 \phi \\ &\leq \left(\frac{t}{20} + \frac{1}{6}\right)|\varsigma - \upsilon|^2 \phi + \left|\varsigma - \frac{6}{5}\right|^2 \phi + \left|\upsilon - \frac{3 + 2\upsilon}{5}\right|^2 \phi \\ &= hd(\varsigma, \upsilon)(t) + \psi(d(\varsigma, \Gamma\varsigma)(t), d(\upsilon, \Gamma\upsilon)(t)). \end{aligned}$$

Since $\left|\upsilon - \frac{3+2\upsilon}{5}\right|^2 = \frac{(3\upsilon-3)^2}{25}$ and the fact that

$$\frac{(3\upsilon - 3)^2}{25} - \frac{(2\upsilon - 3)^2}{25} = \frac{5\upsilon^2 - 6\upsilon}{25} > 0,$$

for all $\upsilon \in [2, 3)$, we have

$$d(\Gamma\varsigma, \Gamma\upsilon)(t) \leq hd(\varsigma, \upsilon)(t) + \psi(d(\varsigma, \Gamma\varsigma)(t), d(\upsilon, \Gamma\upsilon)(t)),$$

for all $\varsigma \in [2, 3)$, $\upsilon \in [0, 2)$. Therefore, by making the most of Theorem 7, we claim that Γ has a unique fixed point in M.

3. Conclusions

In this paper, we give some fixed point results for b-generalized contractive mappings with some weak continuity conditions. First, we introduce the concepts of x_0-orbital continuity, weak orbital continuity, and Γ-orbital completeness in CMSBA. A fixed point theorem is obtained under these conditions for the asymptotically regular b-generalized Kannan-Górnicki type mapping, which is very different from the classic Kannan-type fixed point theorem. Second, we study the existence and uniqueness of the fixed points for b-generalized Ćirić-Proinov-Górnicki type mapping, b-generalized Hardy-Rogers-Proinov-Górnicki type mapping, and b-generalized Reich-Proinov-Górnicki type mapping, respectively. At last, we provide several examples to illustrate the genuine improvement between our results and the theorems in the literature. In these theorems, we utilize the weak orbital continuity and orbital completeness instead of the general continuity and the completeness. Furthermore, these results are obtained in CMSBA with a non-normal cone and discontinuous cone b-metric, which are not equivalent to the theorems in cone b-metric spaces or b-metric spaces. It means we have more applications in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits at leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181.
2. Z. H. Li, X. B. Shu, F. Xu, The existence of upper and lower solutions to second order random impulsive differential equation with boundary value problem, *AIMS Math.*, **5** (2020), 6189–6210. https://doi.org/10.3934/math.2020398
3. S. Li, L. X. Shu, X. B. Shu, F. Xu, Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays, *Stochastics*, **91** (2019), 857–872. https://doi.org/10.1080/17442508.2018.1551400
4. R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60** (1968), 71–76.
5. S. Reich, Some remarks concerning contraction mappings, *Can. Math. Bull.*, **14** (1971), 121–124. https://doi.org/10.4153/CMB-1971-024-9
6. S. Reich, Fixed points of contractive functions, *Boll. Un. Mat. Ital.*, **5** (1972), 26–42.
7. L. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **45** (1974), 267–273.

8. S. K. Chatterjea, Fixed-point theorems, *C. R. Acad. Bulg. Sci.*, **25** (1972), 15–18.

9. G. E. Hardy, T. D. Rogers, A generalization of a fixed point theorem of Reich, *Can. Math. Bull.*, **16** (1973), 201–206. https://doi.org/10.4153/CMB-1973-036-0

10. R. M. T. Bianchini, Su un problema di S. Reich riguardante la teoria dei punti fissi, *Boll. Un. Mat. Ital.*, **5** (1972), 103–108.

11. J. Brzdek, Comments on fixed point results in classes of function with values in a b-metric space, *RACSAM*, **116** (2022), 1–17. https://doi.org/10.1007/s13398-021-01173-6

12. I. A. Rus, Some variants of contraction principle, generalizations and applications, *Stud. U. Babeş-Bol. Mat.*, **61** (2016), 343–358.

13. V. Berinde, A. Petrusel, I. A. Rus, Remarks on the terminology of the mappings in fixed point iterative methods in metric spaces, *Fixed Point Theory*, **24** (2023), 525–540. https://doi.org/10.24193/fpt-ro.2023.2.05

14. P. V. Subrahmanyam, Completeness and fixed points, *Monatsh. Math.*, **80** (1975), 325–330. https://doi.org/10.1007/BF01472580

15. L. G. Huang, X. Zhang, Cone metric space and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087

16. S. Rezapour, R. Hamlbarani, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", *J. Math. Anal. Appl.*, **345** (2008), 719–724. https://doi.org/10.1016/j.jmaa.2008.04.049

17. N. Hussian, M. H. Shah, KKM mappings in cone b-metric spaces, *Comput. Math. Appl.*, **62** (2011), 1677–1684. https://doi.org/10.1016/j.camwa.2011.06.004

18. H. P. Huang, S. Y. Xu, Fixed point theorems of contractive mappings in cone b-metric spaces and applications, *Fixed Point Theory Appl.*, **112** (2013), 1–10. https://doi.org/10.1186/1687-1812-2013-112

19. H. Liu, S. Y. Xu, Fixed point theorems of quasicontractions on cone metric spaces with banach algebras, *Fixed Point Theory Appl.*, **2013** (2013), 1–10. https://doi.org/10.1155/2013/187348

20. S. Y. Xu, S. Radenović, Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, *Fixed Point Theory Appl.*, **102** (2014), 1–12. https://doi.org/10.1186/1687-1812-2014-102

21. Y. Han, S. Y. Xu, Some new theorems on c-distance without continuity in cone metric spaces over Banach algebras, *J. Funct. Space.*, **2018** (2018). https://doi.org/10.1155/2018/7463435

22. H. P. Huang, G. T. Deng, S. Radenović, Some topological properties and fixed point results in cone metric spaces over Banach algebras, *Positivity*, **23** (2019), 21–34. https://doi.org/10.1007/s11117-018-0590-5

23. H. P. Huang, S. Radenović, Some fixed point results of generalized Lipschitz mappings on cone b-metric spaces over Banach algebras, *J. Comput. Anal. Appl.*, **20** (2016), 566–583.

24. J. M. A. Toledano, T. D. Benavides, G. L. Acedo, *Measures of noncompactness in metric fixed point theory*, Basel: Birkhäuser, 1997. https://doi.org/10.1007/978-3-0348-8920-9

25. F. E. Browder, W. V. Peryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, *B. Am. Math. Soc.*, **72** (1966), 571–575.
26. K. Goebel, W. A. Kirk, *Topics in metric fixed point theory*, Cambridge: Cambridge University Press, 1990. https://doi.org/10.1017/CBO9780511526152
27. M. Edelstein, R. O'Brien, Nonexpansive mappings, asymptotic regularity and successive approximation, *J. Lond. Math. Soc.*, **17** (1978), 547–554. https://doi.org/10.1112/jlms/s2-17.3.547
28. J. Górnicki, Geometrical coefficients and the structure of the fixed-point set of asymptotically regular mappings, *Nonlinear Anal.*, **74** (2011), 1190–1199. https://doi.org/10.1016/j.na.2010.09.059
29. S. Panja, K. Roy, M. Saha, R. K. Bisht, Some fixed point theorems via asymptotic regularity, *Filomat*, **34** (2020), 1621–1627. https://doi.org/10.2298/FIL2005621P
30. S. Reich, I. Shafrir, The asymptotic behavior of firmly nonexpansive mappings, *P. Am. Math. Soc.*, **101** (1987), 246–250.
31. A. Wísnicki, On the structure of fixed-point sets of asymptotically regular semigroups, *J. Math. Anal. Appl.*, **393** (2012), 177–184. https://doi.org/10.1016/j.jmaa.2012.03.036
32. A. Jeribi, N. Kaddachi, Z. Laouar, Fixed point theorems for weakly asymptotically regular mappings in banach spaces with an application, *Numer. Func. Anal. Opt.*, **43** (2022), 68–87. https://doi.org/10.1080/01630563.2021.2001820
33. J. Górnicki, Remarks on asymptotic regularity and fixed points, *J. Fixed Point Theory Appl.*, **21** (2019). https://doi.org/10.1007/s11784-019-0668-0
34. R. K. Bisht, A note on the fixed point theorem of Górnicki, *J. Fixed Point Theory Appl.*, **21** (2019). https://doi.org/10.1007/s11784-019-0695-x
35. L. B. Ćirić, On contraction type mappings, *Math. Balkanica*, **1** (1971).
36. A. Pant, R. P. Pant, Fixed points and continuity of contractive maps, *Filomat*, **31**, (2017), 3501–3506. https://doi.org/10.2298/FIL1711501P
37. P. D. Proinov, Fixed point theorems in metric spaces, *Nonlinear Anal.*, **64** (2006), 546–557. https://doi.org/10.1016/j.na.2005.04.044
38. J. Fernandez, N. Malviya, B. Fisher, The asymptotically regularity and sequences in partial cone b-metric spaces with application, *Filomat*, **30** (2016), 2749–2760. https://doi.org/10.2298/FIL1610749F
39. Y. Han, S. Y. Xu, Generalized Reich-Ćirić-Rus-type and Kannan-type contractions in cone b-metric spaces over Banach algebras, *J. Math.*, **2021** (2021), 1–11. https://doi.org/10.1155/2021/7549981
40. W. Rudin, *Functional Analysis*, New York, **45** (1991).
41. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis*, **1** (1993), 5–11. https://api.semanticscholar.org/CorpusID:116827708
42. V. Berinde, M. Păcurar, The early developments in fixed point theory on b-metric spaces, *Carpathian J. Math.*, **38** (2022), 523–538.
43. Z. Kadelburg, S. Radenović, A note on various types of cones and fixed point results in cone metric spaces, *Asian J. Math. Appl.*, **2013** (2013).

44. D. S. Jaggi, Fixed point theorems for orbitally continuous functions, *Mat. Vesnik*, **1** (1977), 129–135.
45. A. Pant, R. P. Pant, M. C. Joshi, Caristi type and Meir-Keeler type fixed point theorems, *Filomat*, **33** (2019), 3711–3721. https://doi.org/10.2298/FIL1912711P
46. R. K. Bisht, An overview of the emergence of weaker continuity notions, various classes of contractive mappings and related fixed point theorems, *J. Fixed Point Theory Appl.*, **25** (2023). https://doi.org/10.1007/s11784-022-01022-y
47. J. Górnicki, Fixed point theorems for Kannan type mappings, *J. Fixed Point Theory Appl.*, **19** (2017), 2145–2152. https://doi.org/10.1007/s11784-017-0402-8