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Research article

Stochastic prey-predator model with small random immigration

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Abstract: In this paper, we introduce a novel stochastic prey-predator model under random small immigration. Mainly, we prove boundedness for the solution of the model using probabilistic and analytic types of inequalities. Furthermore, possible conditions on the immigration for achieving stochastic square stability are obtained. The immigration of both prey and predator is assumed to be either constant and stochastically perturbed or proportional to the population and stochastically perturbed. In all cases, we arrived at the fact that stability can only be achieved if the immigration is small enough. We also show that as random immigration increases, the dynamic becomes destabilized and could lead to chaos. Lastly, we perform a computational analysis in order to verify the obtained theoretical results.

Keywords: stochastic stability; prey-predator model; boundedness; small random immigration
Mathematics Subject Classification: 92-11, 60H10, 34D20

1. Introduction

Mathematicians have been very motivated to contribute to the study of theoretical ecology, from both theoretical and applied points of view. Theoretical ecology problems are first mathematically modeled, and the resulting models are then solved and described from an ecological standpoint. One of the main topics in the theoretical ecology is the prey-predator interactions, which were first developed independently in the 1920s by Lotka [1] and Voltera [2]. Since then, seeking more realistic results, researchers have been highly involved in studying and developing the so-called Lotka-Voltera model [3–7].

Modifications that took place in the literature are mainly in the model construction, such as the logistic and exponential forms that describe the growth in a better way. Further consideration in this direction are functional and numerical responses, see [8–10]. Various external effects have been investigated to simulate real-life effects on prey predator models [11–17]. However, many unexpected factors affect ecosystems, which makes the use of random noise in the mathematical model necessary, see [18–20] for general account. In [21], the authors address the asymptotic behavior of stochastic version of Lotka-Voltera model. The existence, uniqueness and positivity of global solution are obtained for different stochastic models with slight variations on the models' types [22, 23]. The work [24] investigates numerical results on two-predator one-prey stochastic model.

Recently, immigration have been considered as one of the most important external effects. The importance of examining the immigration element in prey-predator systems arises from the fact that immigration may aid in the survival of species that are on the verge of extinction, as well as in achieving ecological stability. Results obtained by this means are only focused on investigating immigration with deterministic models such as [25, 26]. In 2018,Tahara et al. [27] studied the Lotka Voltera model when affected by small deterministic immigration.

As far as we know, random immigration on prey-predator models has not been considered. The aim of this paper is to initiate such an investigation, namely we study boundedness and stochastic mean square stability for a prey-predator model with small stochastic immigration. Our results are numerically verified.

2. Problem setting

In this section, we are going to present a new modification to intraspecific competition in prey predator systems that has been thoroughly examined in the work of the first author [28]. He studied persistence, extinction, local and global asymptotic stability and established that the system is unconditionally stable. More recently, in [29], a novel mechanism for measuring predator inference for this model was obtained. Our modification, which accounts for small stochastic fluctuations in immigration using a dimensionless model, is as follows:

$$\begin{cases} dN_1 = rN_1\left(1 - \dfrac{N_1}{k}\right)dt - \alpha N_1 N_2 dt + \sigma_1(t, N_1)N_1 dW_1 & (2.1)\\ dN_2 = -mN_2 dt + bN_1N_2 dt - cN_2^2 dt + \sigma_2(t, N_2)N_2 dW_2 & (2.2)\\ N_1(0) = N_{1_0}, \quad N_2(0) = N_{2_0}, \end{cases}$$

where

- N_1 is the density of the prey.
- N_2 is the density of the predator.
- r is the growth rate of the prey.
- k is the systems carrying capacity.
- α is the catching rate of the prey by a predator.
- m is the rate of natural death of a predator.
- b is the efficiency of converting consumed prey into predatory birth.
- c is the intraspesific competition inside the predator.

- W_1 and W_2 represent a one-dimensional real-valued Wiener processes.

The immigration terms for the prey and predator $\sigma_1(t, N_1)$ and $\sigma_2(t, N_2)$ take the following possibilities:

$$\sigma_1(t, N_1) = \epsilon \text{ or } \sigma_1(t, N_1) = \frac{\epsilon}{N_1}, \tag{2.3}$$

here $\epsilon \geq 0$ is a natural number representing immigrants in the prey and $\frac{\epsilon}{N_1}$ is a ratio that representing immigrants proportion of the prey. In similar way, we have

$$\sigma_2(t, N_2) = \delta \text{ or } \sigma_2(t, N_2) = \frac{\delta}{N_2}, \tag{2.4}$$

where $\delta \geq 0$ is a natural number representing immigrants in the predator and $\frac{\delta}{N_2}$ is a ratio representing immigrants proportion of the predator.

Stochastic immigration (constant or proportional to population) is known to be one of the ecological management for population. As a real life example, ship-ballast water could represent a stochastic immigration, see [29] for more details.

3. Boundedness of solution

This section is concerned with the boundedness of the solutions to systems (2.1) and (2.2). For this, we write the system as follows:

$$\begin{aligned} dN_t &= \mu(t, N_t)dt + \sigma(t, N_t)dW \\ N_t(0) &= N_0, \end{aligned} \tag{3.1}$$

where $\mu : [(0, T) \times \mathbb{R}^+]^2 \to \mathbb{R}$ and $\sigma : [(0, T) \times \mathbb{R}^+]^2 \to \mathbb{R}$ such that

$$\mu(t, N_t) = \begin{bmatrix} rN_1\left(1 - \frac{N_1}{k}\right) - \alpha N_1 N_2 \\ -mN_2 + bN_1N_2 - cN_2^2 \end{bmatrix}, \quad \sigma(t, N_t) = \begin{bmatrix} \sigma_1(t, N_1).N_1 \\ \sigma_2(t, N_2).N_2 \end{bmatrix},$$

$$N_t = \begin{bmatrix} N_{1t} \\ N_{2t} \end{bmatrix}, \quad and \quad dW = \begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix}.$$

The following lemma will be helpful in proving the boundedness of the solution to problem (3.1).

Lemma 3.1. *μ and σ in problem (3.1) satisfy the linear growth condition*

$$\mathbb{E}\|\sigma(t, N_t)\|^2 + \mathbb{E}\|\mu(t, N_t)\|^2 \leq K(1 + \mathbb{E}\|N_t\|^2),$$

where K is some positive constant.

Proof. We begin with (2.1), in which we take the sup and expectation to get

$$\mathbb{E} \sup_{s\in[0,t]} |N_1(s)| \leq N_{1_0} + r\mathbb{E} \sup_{s\in[0,t]} \int_0^s |N_1|ds + \mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \sigma_1(s, N_1)N_1 dW_1 \right|. \tag{3.2}$$

As for the stochastic integral, we use Burkholder-Davis-Gundy's (BDG) inequality and (2.3), then we get

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}|N_1(s)| &\leq N_{1_0} + r\mathbb{E}\sup_{s\in[0,t]}\int_0^s |N_1|ds + \sigma_1 C_1 \mathbb{E}\left(\int_0^t |N_1|^2 ds\right)^{\frac{1}{2}} \\ &\leq N_{1_0} + r\mathbb{E}\sup_{s\in[0,t]}\int_0^s |N_1|ds + \sigma_1 C_1\sqrt{T}\mathbb{E}\sup_{s\in[0,t]}|N_1(s)|, \qquad (3.3)\end{aligned}$$

where C_1 is BDG's constant. Now for σ_1 small enough, we obtain

$$\mathbb{E}\sup_{s\in[0,t]}|N_1(s)| \leq C_2 + C_3\mathbb{E}\sup_{s\in[0,t]}\int_0^s |N_1|ds. \qquad (3.4)$$

Applying Gronwall's inequality to (3.4), we get

$$\mathbb{E}|N_1(t)| \leq C_4. \qquad (3.5)$$

Now,

$$\begin{aligned}\mathbb{E}\|\sigma(t,N)\|^2 &= \mathbb{E}\|\sigma_1(t,N_1)\|^2 + \mathbb{E}\|\sigma_2(t,N_2)\|^2 \\ &\leq \epsilon\mathbb{E}\|N_1\|^2 + \delta\mathbb{E}\|N_2\|^2, \qquad (3.6)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\|\mu(t,N)\|^2 &\leq r^2\mathbb{E}\|N_1\|^2 + b^2\mathbb{E}\|N_1\|^2\|N_2\|^2 \\ &\leq C_5\mathbb{E}\|N_1\|^2(1+\|N_2\|^2). \qquad (3.7)\end{aligned}$$

The proof is concluded based on (3.5)–(3.7). □

Theorem 3.2. *Suppose that N_t is any solution of (3.1), then for $p \geq 2$ and some positive constant C, we have*

$$\mathbb{E}\sup_{s\in[0,t]}|N_s|^p \leq C.$$

Proof. The solution of (3.1) satisfies

$$N_t = N_0 + \int_0^t \mu(s,N_s)ds + \int_0^t \sigma(s,N_s)dW_s.$$

We use the following inequality, see [30]

$$\left(\sum_{k=1}^m a_k\right)^n \leq m^{n-1}\sum_{k=1}^m a_k^n, \qquad (3.8)$$

followed by taking the supermom and the expectation, we obtain

$$\mathbb{E}\sup_{s\in[0,t]}|N_s|^p \leq 3^{p-1}\left(|N_0|^p + \mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \mu(s,N_s)ds\right|^p + \mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \sigma(s,N_s)dW_s\right|^p\right). \qquad (3.9)$$

Now, we estimate the last two terms in the R-H-S of (3.9). We start with the stochastic term, in which we apply BDG's inequality, followed by the Hölder inequality for $\frac{1}{p/2}+\frac{1}{p/(p-2)}=1$ to get

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \sigma(s,N_s)dW_s\right|^p &\leq C_6\mathbb{E}\left(\int_0^t |\sigma(s,N_s)|^2\,ds\right)^{\frac{p}{2}}\\ &\leq C_6\mathbb{E}\left(\int_0^t \left(|\sigma(s,N_s)|^2\right)^{\frac{p}{2}}ds\right)^{\frac{p}{2}\times\frac{2}{p}}.\left(\int_0^t (1)^{\frac{p}{p-2}}ds\right)^{\frac{p}{2}\times\frac{p-2}{p}}\\ &\leq C_6T^{\frac{P-2}{2}}\mathbb{E}\int_0^t |\sigma(s,N_s)|^p\,ds. \end{aligned}\tag{3.10}$$

We now use the linear growth condition proved in Lemma 3.1, and (3.8) for power $n=\frac{p}{2}$ to obtain

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \sigma(s,N_s)dW_s\right|^p &\leq C_6T^{\frac{P-2}{2}}K^{\frac{p}{2}}.2^{\frac{p}{2}-1}\mathbb{E}\int_0^t (1+|N_s|^p)\,ds\\ &\leq C_6T^{\frac{P-2}{2}}K^{\frac{p}{2}}.2^{\frac{p}{2}-1}\left[T+\mathbb{E}\int_0^t ||N_s|^p ds\right]\\ &= C_6T^{\frac{P}{2}}K^{\frac{p}{2}}.2^{\frac{p}{2}-1}+C_6T^{\frac{P-2}{2}}K^{\frac{p}{2}}.2^{\frac{p}{2}-1}\mathbb{E}\int_0^t |N_s|^p ds.\end{aligned}\tag{3.11}$$

In order to estimate the second term on the R-H-S of (3.9), we use the Hölder inequality for $\frac{1}{p}+\frac{1}{p/(p-1)}=1$, we have

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \mu(s,N_s)ds\right|^p &\leq \mathbb{E}\sup_{s\in[0,t]}\left(\int_0^s |\mu(s,N_s)|ds\right)^p\\ &\leq \mathbb{E}\left(\int_0^t |\mu(s,N_s)|^p ds\right)^{p\times\frac{1}{p}}.\left(\int_0^t (1)^{\frac{p}{p-1}}ds\right)^{p\times\frac{p-1}{p}}\\ &\leq T^{p-1}\mathbb{E}\left(\int_0^t |\mu(s,N_s)|^p ds\right).\end{aligned}\tag{3.12}$$

We now use Lemma 3.1, and (3.8) for power $n=\frac{p}{2}$, we have

$$\begin{aligned}\mathbb{E}\int_0^t \left(|\mu(s,N_s)|^2\right)^{\frac{p}{2}}ds &\leq K^{\frac{p}{2}}\int_0^t\left(1+|N_s|^2\right)^{\frac{p}{2}}ds\\ &\leq K^{\frac{p}{2}}2^{\frac{p}{2}-1}\mathbb{E}\int_0^t (1+|N_s|^p)\,ds\\ &\leq K^{\frac{p}{2}}2^{\frac{p}{2}-1}\left[T+\mathbb{E}\int_0^t |N_s|^p ds\right]\\ &= K^{\frac{p}{2}}2^{\frac{p}{2}-1}T+K^{\frac{p}{2}}2^{\frac{p}{2}-1}\mathbb{E}\int_0^t |N_s|^p ds.\end{aligned}\tag{3.13}$$

Using (3.13) into (3.12), we have

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \mu(s,N_s)ds\right|^p \leq K^{\frac{p}{2}}2^{\frac{p}{2}-1}T^p+K^{\frac{p}{2}}2^{\frac{p}{2}-1}T^{p-1}\mathbb{E}\int_0^t |N_s|^p ds.\tag{3.14}$$

From (3.9), (3.11) and (3.14), we obtain

$$\mathbb{E}\sup_{s\in[0,t]}|N_s|^p \leq C_7 + C_8\mathbb{E}\int_0^t |N_s|^p ds. \tag{3.15}$$

This and Gronwall's inequality complete the proof. □

4. Stochastic mean square stability

We are now interested in studying stochastic stability in our system around its deterministic equilibrium. Let us emphasize that our system has non-negative equilibrium points:

(1) $P_0 = (0,0)$.
(2) $P_1 = (k,0)$.
(3) $P^* = (N_1^*, N_2^*)$, where $N_1^* = \dfrac{k(br+m)}{b(r+k\alpha)}$ and $N_2^* = \dfrac{r(kb-m)}{b(r+k\alpha)}$.

The following theorem states the feasibility of the equilibrium point $P^* = (N_1^*, N_2^*)$.

Theorem 4.1. *If $kb > m$, then $P^* = (N_1^*, N_2^*)$ is feasible.*

We shall study the mean square stochastic stability of systems (2.1) and (2.2) around their positive equilibrium.

$$[left = \{]dN_1 = rN_1\left(1 - \frac{N_1}{k}\right)dt - \alpha N_1N_2dt + \sigma_1(t,N_1)(N_1 - N_1^*)dW_1 \tag{4.1}$$
$$dN_2 = -mN_2dt + bN_1N_2dt - cN_2^2dt + \sigma_2(t,N_2)(N_2 - N_2^*)dW_2 \tag{4.2}$$
$$N_1(0) = N_{1_0}, \quad N_2(0) = N_{2_0}.$$

For this, we use the change of variables

$$U_1 = N_1 - N_1^*, \quad and \quad U_2 = N_2 - N_2^*.$$

With this, we obtain the linearized stochastic differential equation around the point $P^* = (N_1^*, N_2^*)$ as follows:

$$dU_t = F(U_t)dt + G(U_t)dW, \tag{4.3}$$

where

$$U_t = \begin{bmatrix} U_{1t} \\ U_{2t} \end{bmatrix},$$

$$F(U_t) = \begin{bmatrix} \frac{N_1^*\partial H_1(N_1^*,N_2^*)}{\partial N_1} + H_1 & \frac{N_1^*\partial H_1(N_1^*,N_2^*)}{\partial N_2} \\ \frac{N_2^*\partial H_2(N_1^*,N_2^*)}{\partial N_1} & \frac{N_2^*\partial H_2(N_1^*,N_2^*)}{\partial N_2} + H_2 \end{bmatrix}\begin{bmatrix} U_{1t} \\ U_{2t} \end{bmatrix},$$

where

$$H_1(N_1^*, N_2^*) = r\left(1 - \frac{N_1}{k}\right) - aN_2.$$
$$H_2(N_1^*, N_2^*) = -m + bN_1 - cN_2.$$

Thus,

$$F(U_t) = \begin{bmatrix} \frac{-rN_1^*}{k} & -aN_1^* \\ bN_2^* & -cN_2^* \end{bmatrix} \begin{bmatrix} U_{1t} \\ U_{2t} \end{bmatrix},$$

and

$$G(U_t) = \begin{bmatrix} \sigma_1(U_{1t})U_{1t} & 0 \\ 0 & \sigma_2(U_{2t})U_{2t} \end{bmatrix} \begin{bmatrix} U_{1t} \\ U_{2t} \end{bmatrix}.$$

Let $Q_T = [t_0, \infty] \times \mathbb{R}^2$, where $t_0 \geq 0$ and let $C_0^2(Q_T)$ be the set of all functions $Z : (t, v) \in Q_T \mapsto Z(t, v) \in \mathbb{R}^+$ that are continuously differentiable in time and twice continuously differentiable with respect to the second variable. Now, following [31] the Markov process generator of (4.3) is given by

$$LZ(t, U) = \frac{\partial Z(t, U_t)}{\partial t} + F^T(U_t)\frac{\partial Z(t, U_t)}{\partial U_t} + \frac{1}{2}Tr\left[G^T(U_t)\frac{\partial^2 Z(t, U_t)}{\partial U_t^2}G(U_t)\right], \tag{4.4}$$

where $Tr[A]$ is the trace of matrix A and A^T is the transposition of matrix A. The following theorem for mean-square stability is collected from [31].

Theorem 4.2. *Assume that U_t is a solution of (4.3) and that $Z(t, U) \in C_0^2(Q_T)$ such that*

$$\begin{aligned} &K_1|U_t|^2 \leq Z(t, U_t) \leq k_2|U_t|^2 \\ &LZ(t, U) \leq -k_3|U_t|^2, \quad k_1, k_2, k_3 > 0. \end{aligned}$$

Then, the trivial solution of (4.3) is asymptotically mean-square stable.

Next, we have

Theorem 4.3. *Suppose that*

$$\sigma_1^2 < \frac{2rN_1^*}{k} \tag{4.5}$$

and

$$\sigma_2^2 < 2cN_2^*. \tag{4.6}$$

Then, the trivial solution of (4.3) is asymptotically mean-square stable.

Proof. Let us consider a Markov process generator

$$Z(t, U) = \frac{1}{2}\left[\alpha_1 U_1^2 + \alpha_2 U_2^2\right], \tag{4.7}$$

where $\alpha_i, \ i = 1, 2$ to be determined shortly.

$$\frac{\partial Z}{\partial t} = 0, \tag{4.8}$$

$$\frac{\partial Z}{\partial U} = \begin{pmatrix} \alpha_1 U_1 & \alpha_2 U_2, \end{pmatrix}$$

$$F^T(U_t)\frac{\partial Z}{\partial U} = \alpha_1\left(\frac{-rN_1^*}{k}\right)U_1^2 - \alpha_1 aN_1^* U_1U_2 + \alpha_2 bN_2^* U_1U_2 - \alpha_2 cN_2^* U_2^2, \tag{4.9}$$

$$\frac{1}{2}Tr\left[G^T(U_t)\frac{\partial^2 Z(t,U_t)}{\partial U_t^2}G(U_t)\right] = \frac{1}{2}\alpha_1\sigma_1^2U_1^2 + \frac{1}{2}\alpha_2\sigma_2^2U_2^2. \tag{4.10}$$

Using (4.8)–(4.10) into (4.4), we get

$$-\alpha_1\left(\frac{-rN_1^*}{k} - \frac{1}{2}\sigma_1^2\right)U_1^2 - \alpha_2\left(cN_2^* - \frac{1}{2}\sigma_2\right)U_2^2 - \left(\alpha_1 aN_1^* - \alpha_2 bN_2^*\right)U_1U_2. \tag{4.11}$$

Choosing $\alpha_1 aN_1^* = \alpha_2 bN_2^*$ in (4.11), we get

$$LZ(t,U) = -\alpha_1\left(\frac{rN_1^*}{k} - \frac{1}{2}\sigma_1^2\right)U_1^2 - \alpha_2\left(cN_2^* - \frac{1}{2}\sigma_2\right)U_2^2.$$

This, together with the assumptions in the theorems completes the proof. □

Corollary 4.4. *The system is unstable if conditions* (4.5) *or* (4.6) *are not met.*

Remark 4.5. *Let us note that the effect of the stochastic immigration in the prey depends on both growth rate of the prey and carrying capacity of the system, and that of the predator depends on the intraspesific competition inside predator.*

5. Numerical simulations and discussion

In this section, we aim to investigate the random effects of immigration on the stability of prey predator dynamics (2.1) and (2.2) numerically taking into account Theorem 4.3. We use the command "StochasticRungeKuttaScalarNoise" in the MATHEMATICA 11.3 program as a method to execute the numerical simulations, according to the Wolfram website [33]. The parameters and initial conditions are given with the following values:

$$r = 1.5, k = 3, \alpha = 2.5, m = 0.65, c = 0.75, b = 1.25, N_1(0) = 2, N_2(0) = 2.$$

Remark 5.1. *The choice of the parameters and initial conditions values is due to the fact that our model is dimensionless, so we only concern ourselves with values that satisfy the theoretical results, see Theorems 4.3 and Corollary 4.4.*

As regards to the intensity noise, the values of ϵ and δ are taken for three possible situations. These situations are applied to each of the following cases:

Case I	$\sigma_1(t,N_1) = \epsilon$	$\sigma_2(t,N_2) = \delta$
Case II	$\sigma_1(t,N_1) = \epsilon$	$\sigma_2(t,N_2) = \frac{\delta}{N_2}$
Case III	$\sigma_1(t,N_1) = \frac{\epsilon}{N_1}$	$\sigma_2(t,N_2) = \frac{\delta}{N_2}$
Case IV	$\sigma_1(t,N_1) = \frac{\epsilon}{N_1}$	$\sigma_2(t,N_2) = \delta$

where

- Case I represents the number of constant immigrants in the prey as well as in the predator.
- Case II represents the number of constant immigrants in the prey, with immigration proportional to the population in the predator.
- Case III represents immigration that proportional to the population of the prey of immigration that is proportional to the population of the predator.
- Case IV represents immigration that is proportional to the population of the prey with constant immigrants in the predator.

In each of the above cases, we check three different consideration for ϵ and δ. These are:

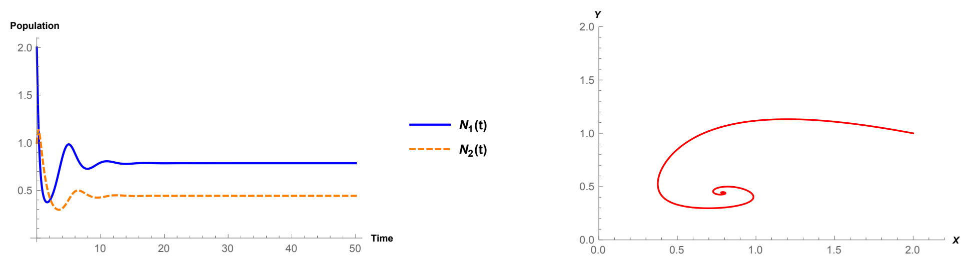
(1) The conditions (4.5) and (4.6) of theorem 4.3 are satisfied.
(2) Condition (4.5) holds true, and condition (4.6) does not.
(3) Both conditions (4.5) and (4.6) are not satisfied.

Note that conditions (4.5) and (4.6) tell us that the stability of the systems (2.1) and (2.2) can only beachieved if the immigration is small enough for both prey and predator.

Figure 1 is taken to represent the dynamic with free immigration, which shows stability as it is known that the corresponding deterministic model is always stable [28]. The rest of the figures are categorized into three sets. The first set (i.e. Figures 2, 5, 8 and 11) represents the small intensities of immigration at $\epsilon = 0.25$ and $\delta = 0.12$ for all cases are mentioned in the table above. Let us note that conditions 4.5 and 4.6 of Theorem 4.3 are always met in this set of consideration, which means that from biological point of view the system is stable. We also note that when the intensities are small enough, stochastic immigration has less effect with full stability of the system.

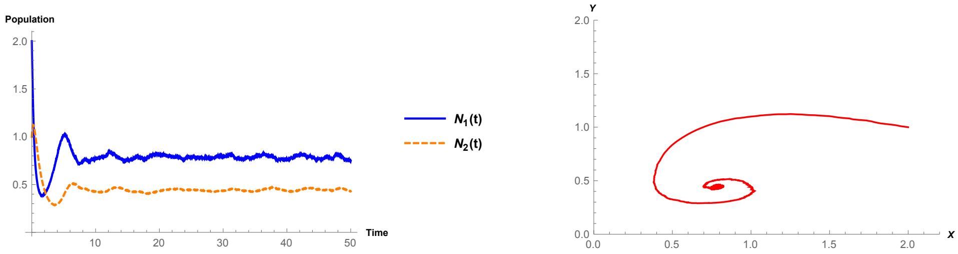
The second set (i.e. Figures 3, 6, 9 and 12) represent the case in which the intensities are taken in such a way that only one of the two conditions of Theorem 4.3 satisfied. Therefore, by Corollary 4.4 the system is unstable. We also note that the increase in stochastic immigration has a clear impact (oscillation) on the system.

The last set (i.e. Figures 4, 7, 10 and 13) represents the case in which the intensities are taken to be large enough so that the two conditions of Theorem 4.3 are not satisfied. Therefore Corollary 4.4 says that the system is not stable, and the increase in stochastic immigration, in addition to the instability, high oscillation.



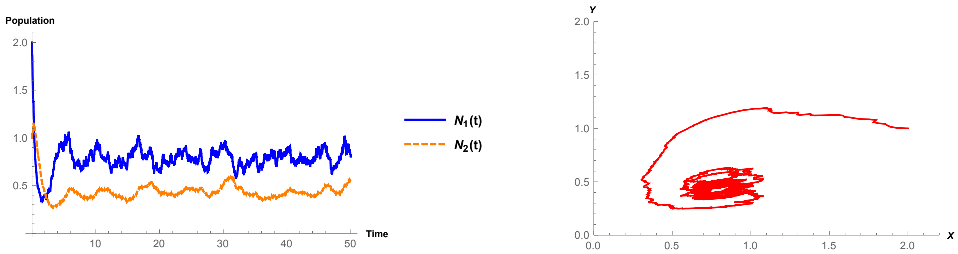
(a) Time series **(b)** Phase plane

Figure 1. Time series and phase plane with free immigration when $\epsilon = \delta = 0$.



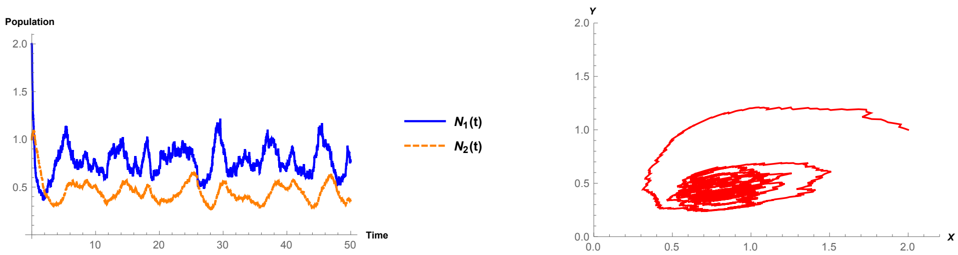
(a) Time series **(b)** Phase plane

Figure 2. Time series and phase plane of case I when $\epsilon = 0.25$ and $\delta = 0.12$.



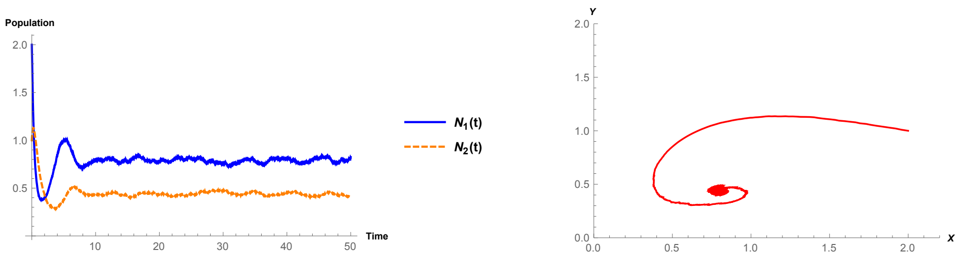
(a) Time series **(b)** Phase plane

Figure 3. Time series and phase plane of case I when $\epsilon = 1.5$ and $\delta = 0.5$.



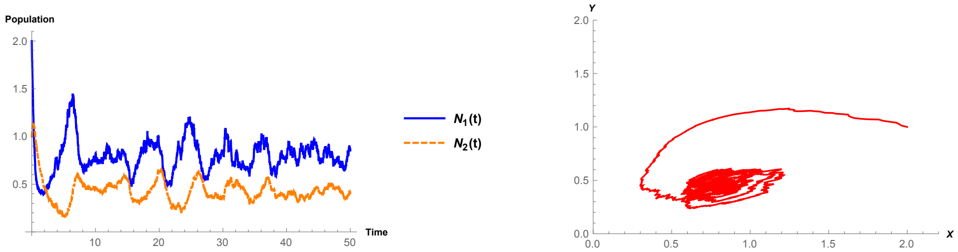
(a) Time series **(b)** Phase plane

Figure 4. Time series and phase plane of case I when $\epsilon = 1.9$ and $\delta = 0.9$.



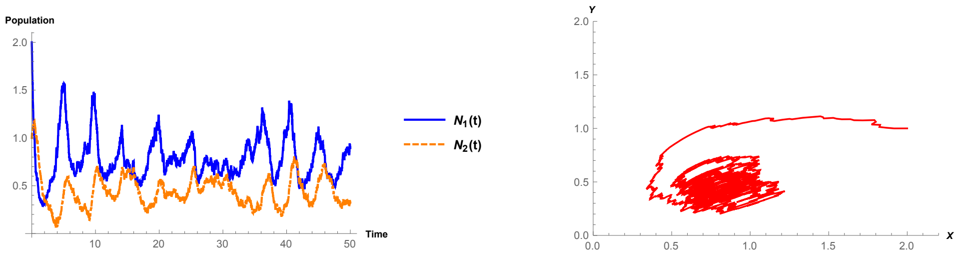
(a) Time series **(b)** Phase plane

Figure 5. Time series and phase plane of case II when $\epsilon = 0.25$ and $\delta = 0.12$.



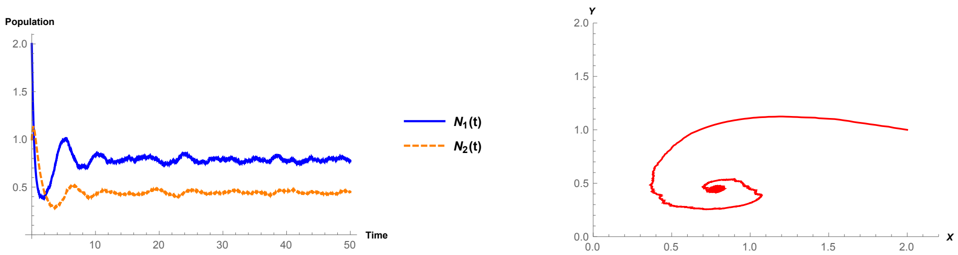
(a) Time series **(b)** Phase plane

Figure 6. Time series and phase plane of case II when $\epsilon = 1.5$ and $\delta = 0.5$.



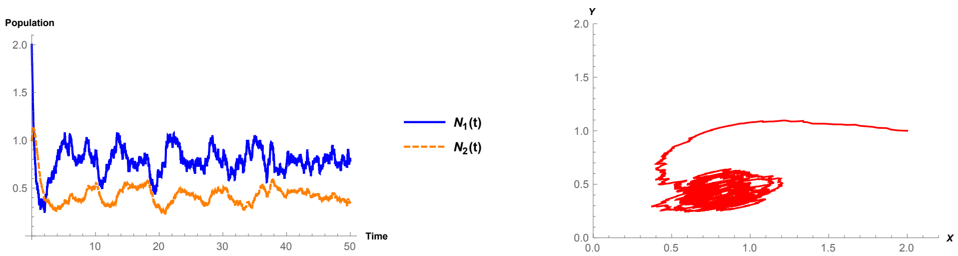
(a) Time series **(b)** Phase plane

Figure 7. Time series and phase plane of case II when $\epsilon = 1.9$ and $\delta = 0.9$.



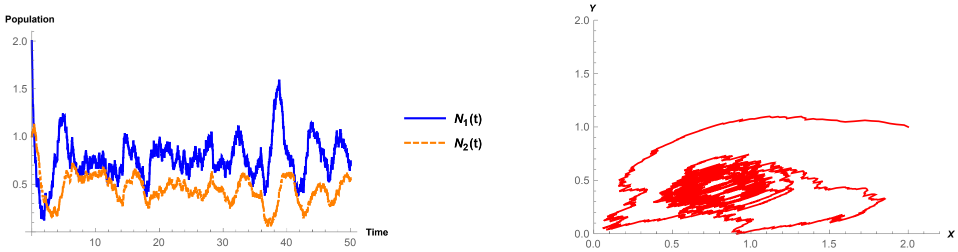
(a) Time series **(b)** Phase plane

Figure 8. Time series and phase plane of case III when $\epsilon = 0.25$ and $\delta = 0.12$.



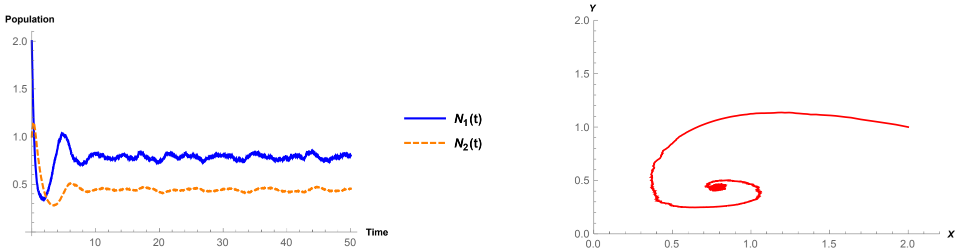
(a) Time series **(b)** Phase plane

Figure 9. Time series and phase plane of case III when $\epsilon = 1.5$ and $\delta = 0.5$.



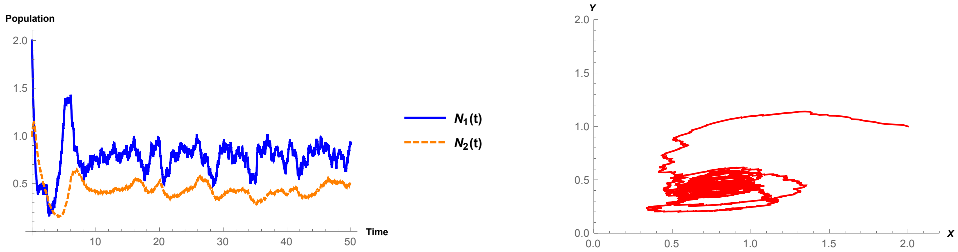
(a) Time series **(b)** Phase plane

Figure 10. Time series and phase plane of case III when $\epsilon = 1.9$ and $\delta = 0.9$.



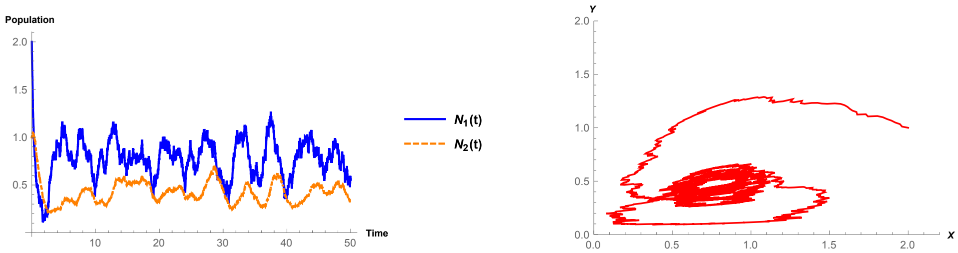
(a) Time series **(b)** Phase plane

Figure 11. Time series and phase plane of case IV when $\epsilon = 0.25$ and $\delta = 0.12$.



(a) Time series **(b)** Phase plane

Figure 12. Time series and phase plane of case IV when $\epsilon = 1.5$ and $\delta = 0.5$.



(a) Time series **(b)** Phase plane

Figure 13. Time series and phase plane of case IV when $\epsilon = 1.9$ and $\delta = 0.9$.

6. Conclusions

- If the random immigration is constant or proportional to the population and small enough, the system is always stable.
- The numerical results are in total agreement with the theoretical results.
- In contrast to the deterministic version of this model, which is always stable, random immigration increases the dynamical behavior of the model by giving stability and instability cases.
- As the random immigration increases, the dynamic is destabilized with higher oscillations, which leads to chaos.
- In the usual sense, immigration makes the prey-predator system more stable, see, for example, [27]. Surprisingly in our study, the random immigration increases caused destabilization of the system, this could be interpreted by the paradox of enrichment phenomena [32].
- As future work, we suggest more studies on the influence of stochastic immigration on more complicated prey-predator models, especially on functional and numerical responses, such as Holloing type II, Beddington–DeAngelis and Crowley-Martin functional responses.We believe that tiny stochastic immigration could stabilize unstable deterministic models, however the outcome remains uncertain.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Authors have no conflicts of interest.

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