Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 14949–14981.
DOI: 10.3934/math.2024724
Received: 27 February 2024
Revised: 14 April 2024
Accepted: 17 April 2024
Published: 24 April 2024

Research article

Innovative approaches to fractional modeling: Aboodh transform for the Keller-Segel equation

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Abstract: This study focuses on developing efficient numerical techniques for solving the fractional Keller-Segel (KS) model, which is critical in explaining chemotaxis events. Within the Caputo operator framework, the study applied two unique methodologies: The Aboodh residual power series method (ARPSM) and the Aboodh transform iteration method (ATIM). These approaches were used to find precise solutions to the fractional KS equation, resulting in a better understanding of chemotactic behavior in biological systems. The comparative examination of the ARPSM and ATIM revealed their distinct strengths and applications in solving complicated fractional models. The work advances numerical approaches for fractional differential equations and improves our understanding of chemotaxis dynamics using a precise modeling approach.

Keywords: Keller-Segel (KS) model; Aboodh residual power series method (ARPSM); Aboodh transform iteration method (ATIM); Caputo operator
Mathematics Subject Classification: 33B15, 34A34, 35A20, 35A22, 44A10

1. Introduction

A novel mathematical tool for characterizing non-local structures is fractional calculus (FC). Mathematical explanations of many physical problems using fractional derivatives have proved successful in recent generations when applied to situations close to reality. Many authors, including Hadamard, Riemann-Liouville, Coimbra, Grunwald-Letnikov, Riesz, Weyl, Liouville Caputo, Atangana-Baleanu, and Caputo-Fabrizio, have offered crucial definitions of fractional operators [1–4]. The underlying principle of these traditional differential equations is their reliance on integer-order derivatives, which give the order an integer numerical value indicating the number of times that a function is differentiated. As opposed to fractional partial differential equations (FPDEs), these concepts are expanded by the addition of fractional derivatives. This kind of model is necessary for the description of delayed or dependent responses, non-local interactions, and anomalous diffusion. These relationships not only have the power to explain the complex phenomena of physics, biology, finance, and engineering, but they also can predict special events. The solution of FPDEs is based on a series of special numerical methods and analytical techniques, all adapted to face the scale-free and non-integer properties of fractional derivatives. Often, the systems required for modeling real-world phenomena follow interrelated processes that can be described using systems of partial differential equations [5–9]. Such systems have multiple differential equations, with each one describing how the value of a specific physical quantity or any of the interacting systemic components changes with time as they progress.

Computational models often include partial differential equations (PDEs), which are important in applications such as fluid flow, electromagnetics, population dynamics, and quantized mechanics. By studying the behavior of the components, their interactions, and their relationships with each other, one can understand how patterns and dynamics are formed and how the system stabilizes. Studying PDE systems with coupled terms is extremely complex and requires advanced mathematics, such as numerical simulations, perturbation methods, and symmetry analysis, to discern a solution [10–14]. The Hermite colocation method [15], the optimal homotopy asymptotic technique [16], the Adomian decomposition method [17], the homotopy perturbation transform method [18], the Pade approximation and homotopy-Pade technique [19], the invariant subspace method [20], the q-homotopy analysis transform method [21], the homotopy analysis Sumudu transform method [22], and the Sumudu transform series expansion method [23] are some of the sophisticated approaches developed for finding exact solutions to nonlinear FPDE models [24–26]. If perturbation methods are not used, the homotopy analysis method breaks a problem into an endless series of linear problems. This method employs the concept of homotopy from topology to derive a convergent series solution [27, 28]. An approach to homotopy analysis proposed by Liao [29], and the Laplace transform [30] are combined in the Laplace homotopy perturbation method.

A gradient of chemical molecules guides the movement of cells, a process known as chemotaxis, which is essential for cell population self-organization and developmental biology in general. In 1970, Lee Segel and Evelyn Keller presented the first mathematical model of chemotaxis. To further understand how the mould aggregation process works in the chemical-attraction-based cellular slime, they used parabolic approaches [31]. Here, we take a look at the fractional-order system of a KS model that goes like this:

$$\begin{aligned} &D^p_\varphi \beta_1(\psi,\varphi) - a\frac{\partial^2 \beta_1(\psi,\varphi)}{\partial \psi^2} + \frac{\partial}{\partial \psi}\Big(\beta_1(\psi,\varphi)\frac{\partial \varpi(\beta_2)}{\partial \psi}\Big) = 0, \\ &D^p_\varphi \beta_2(\psi,\varphi) - b\frac{\partial^2 \beta_2(\psi,\varphi)}{\partial \psi^2} - c\beta_1(\psi,\varphi) + d\beta_2(\psi,\varphi) = 0, \quad \text{where} \quad 0 < p \leq 1, \end{aligned} \tag{1.1}$$

having IC's:

$$\begin{aligned} \beta_1(\psi,0) &= \beta_{10}(\psi), \\ \beta_2(\psi,0) &= \beta_{20}(\psi). \end{aligned} \tag{1.2}$$

The concentration of amoebae are indicated by the unknown term $\beta_1(\psi,\varphi)$, while the chemical substance of concentration is expressed by $\beta_2(\psi,\varphi)$; $\frac{\partial}{\partial \psi}\Big(\beta_1(\psi,\varphi)\frac{\partial \varpi(\beta_2)}{\partial \psi}\Big)$; stands for the chemotactic word, indicating that the chemicals are attractive to and sensitive to the cells. The sensitivity function is denoted by $\varpi(\beta_2)$, and a, b, c, and d are positive constants. The parameter $0 < p \leq 1$ represents the order of the fractional derivative. Much recent research has focused on the KS model. For example, to solve the KS model, Atangana used a combination of methods, including a modified homotopy perturbation, the homotopy decomposition, and the Laplace transform approach [32–34]. Zayernouri established a fractional class of implicit Adams-Moulton and explicit Adams-Bashforth methods in [35] and so on [36–38].

According to [39], the residual power series method (RPSM) was developed in 2013 by a Jordanian mathematician named Omar Abu Arqub. The RPSM is a semi-analytical approach that uses Taylor's series to integrate the residual error function. It finds convergence series solutions for differential equations. In 2013, RPSM was first used to resolve fuzzy differential equations. A new RPSM method was created by Arqub et al. [40] to quickly get power series solutions for ordinary differential equations (DEs). A new and attractive RPSM approach for fractional DEs problems was developed by Arqub et al. [41]. A novel iterative technique to estimate fractional KdV-burgers equations was presented by El-Ajou et al. [42] utilizing RPSM. A unique method was developed by Xu et al. [43] for solving Boussinesq DEs with fractional power series. Zhang et al. [44] stated that a trustworthy numerical approach was developed. More readings on RPSM may be found in [45–47].

To resolve fractional-order differential equations (FODEs), the research team used two separate approaches. One approach to solving the updated equation is to project it into the space generated by the Aboodh transform. Next, the original equation may be solved by using the inverse Aboodh transform [48]. This novel methodology combines the Sumudu transform with the homotopy perturbation method. Without discretization, linearization, or perturbation, this novel approach may solve PDEs as power series expansions, irrespective of their linearity or nonlinearity. There is a significant reduction in the computations needed to find the coefficients compared to RPSM, which requires several repetitions of calculating distinct fractional derivatives throughout the solution phases. The proposed approach has the potential to provide an accurate and closed-form approximation solution.

The Aboodh transform iterative technique (ATIM) is a significant mathematical achievement for fractional partial differential equations. Complexity and convergence issues may develop when using traditional techniques to solve partial differential equations with fractional derivatives. Keeping a steady computational economy while continually improving approximations allows our new strategy to improve accuracy continuously, avoiding these limits. Due to this discovery, we can tackle difficult problems in applied mathematics, engineering, and physics, which enhances our capacity to identify and understand complex systems governed by fractional partial differential equations [49–51].

The two most basic approaches to solving fractional differential equations are the Aboodh transform iterative technique (ATIM) and the Aboodh residual power series method (ARPSM) [49–53], respectively. These techniques not only provide numerical solutions to PDEs that do not need discretization or linearization but also make the symbolic terms in analytical solutions instantly and visible. The primary objective of this study is to compare and contrast the performance of ARPSM and ATIM in solving the Keller-Segel (KS) model. It is worth mentioning that several linear and nonlinear fractional differential problems have been solved using these two approaches.

2. Basic concepts

Definition 2.1. [54] Let us assume that the function $\beta_1(\psi,\varphi)$ is piecewise continuous with exponential order. The Aboodh transform (AT) is defined as follows, assuming $\tau \geq 0$ for $\beta_1(\psi,\varphi)$,

$$A[\beta_1(\psi,\varphi)] = \Psi(\psi,\xi) = \frac{1}{\xi}\int_0^\infty \beta_1(\psi,\varphi)e^{-\varphi\xi}d\varphi, \quad r_1 \leq \xi \leq r_2.$$

The Aboodh inverse transform (AIT) is specifically described as follows:

$$A^{-1}[\Psi(\psi,\xi)] = \beta_1(\psi,\varphi) = \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty} \Psi(\psi,\varphi)\xi e^{\varphi\xi}d\varphi,$$

where $\psi = (\psi_1,\psi_2,\cdots,\psi_p) \in \mathbb{R}$ and $p \in \mathbb{N}$.

Lemma 2.2. *Let [55,56] $\beta_{11}(\psi,\varphi)$ and $\beta_{12}(\psi,\varphi)$ are two functions. It is assumed that they are piecewise continuous on $[0,\infty[$ and exponentially ordered. Let $A[\beta_{11}(\psi,\varphi)] = \Psi_1(\psi,\varphi), A[\beta_{12}(\psi,\varphi)] = \Psi_2(\psi,\varphi)$ and χ_1,χ_2 are constants. Thus, the following characteristics are true:*

(1) $A[\chi_1\beta_{11}(\psi,\varphi) + \chi_2\beta_{12}(\psi,\varphi)] = \chi_1\Psi_1(\psi,\xi) + \chi_2\Psi_2(\psi,\varphi)$,
(2) $A^{-1}[\chi_1\Psi_1(\psi,\varphi) + \chi_2\Psi_2(\psi,\varphi)] = \chi_1\beta_{11}(\psi,\xi) + \chi_2\beta_{12}(\psi,\varphi)$,
(3) $A[J_\varphi^p\beta_1(\psi,\varphi)] = \frac{\Psi(\psi,\xi)}{\xi^p}$,
(4) $A[D_\varphi^p\beta_1(\psi,\varphi)] = \xi^p\Psi(\psi,\xi) - \sum_{K=0}^{r-1}\frac{{\beta_1}^K(\psi,0)}{\xi^{K-p+2}}, r-1 < p \leq r,\ r \in \mathbb{N}$.

Definition 2.3. [57] In terms of order p, the Caputo defines the fractional derivative of the function $\beta_1(\psi,\varphi)$ as:

$$D_\varphi^p\beta_1(\psi,\varphi) = J_\varphi^{m-p}{\beta_1}^{(m)}(\psi,\varphi),\ r \geq 0,\ m-1 < p \leq m,$$

where $\psi = (\psi_1,\psi_2,\cdots,\psi_p) \in \mathbb{R}^p$ and $m, p \in R$, J_φ^{m-p} is the R-L integral of $\beta_1(\psi,\varphi)$.

Definition 2.4. [58] Following is the structure of the power series notation:

$$\sum_{r=0}^{\infty}\hbar_r(\psi)(\varphi-\varphi_0)^{rp} = \hbar_0(\varphi-\varphi_0)^0 + \hbar_1(\varphi-\varphi_0)^p + \hbar_2(\varphi-\varphi_0)^{2p} + \cdots,$$

where $\psi = (\psi_1,\psi_2,\cdots,\psi_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. The series concerning φ_0 is referred to as a multiple fractional power series (MFPS), where the series coefficients are $\hbar_r(\psi)'s$ and φ is variable.

Lemma 2.5. *Let us suppose that the exponential order function is $\beta_1(\psi,\varphi)$. In this case, the AT is defined as: $A[\beta_1(\psi,\varphi)] = \Psi(\psi,\xi)$. Hence,*

$$A[D_\varphi^{rp}\beta_1(\psi,\varphi)] = \xi^{rp}\Psi(\psi,\xi) - \sum_{j=0}^{r-1}\xi^{p(r-j)-2}D_\varphi^{jp}\beta_1(\psi,0), \quad 0 < p \leq 1, \tag{2.1}$$

where $\psi = (\psi_1,\psi_2,\cdots,\psi_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$ and $D_\varphi^{rp} = D_\varphi^p.D_\varphi^p.\cdots.D_\varphi^p(r-times)$.

Proof. By induction, we are able to illustrate Eq (2.5). When $r = 1$ is used in Eq (2.5), the following results are obtained:

$$A[D_\varphi^{2p}\beta_1(\psi,\varphi)] = \xi^{2p}\Psi(\psi,\xi) - \xi^{2p-2}\beta_1(\psi,0) - \xi^{p-2}D_\varphi^p\beta_1(\psi,0).$$

Equation (2.5) is true for $r = 1$, according to Lemma 2.2, part (4). After substituting $r = 2$ in Eq (2.5), we get:

$$A[D_r^{2p}\beta_1(\psi,\varphi)] = \xi^{2p}\Psi(\psi,\xi) - \xi^{2p-2}\beta_1(\psi,0) - \xi^{p-2}D_\varphi^p\beta_1(\psi,0). \tag{2.2}$$

Equation (2.2) L.H.S. enables us to determine

$$L.H.S = A[D_\varphi^{2p}\beta_1(\psi,\varphi)]. \tag{2.3}$$

The following way is used to express Eq (2.3):

$$L.H.S = A[D_\varphi^{p}\beta_1(\psi,\varphi)]. \tag{2.4}$$

Assume

$$z(\psi,\varphi) = D_\varphi^{p}\beta_1(\psi,\varphi). \tag{2.5}$$

Thus, Eq (2.4) becomes

$$L.H.S = A[D_\varphi^{p}z(\psi,\varphi)]. \tag{2.6}$$

Implementing the Caputo derivative led to a modification in Eq (2.6).

$$L.H.S = A[J^{1-p}z^{'}(\psi,\varphi)]. \tag{2.7}$$

Equation (2.7) provides the R-L integral for AT, which allows us to deduce the following:

$$L.H.S = \frac{A[z^{'}(\psi,\varphi)]}{\xi^{1-p}}. \tag{2.8}$$

Equation (2.8) is changed into the following form by using the differential characteristic of the AT:

$$L.H.S = \xi^{p}Z(\psi,\xi) - \frac{z(\psi,0)}{\xi^{2-p}}, \tag{2.9}$$

from Eq (2.5), we obtain:

$$Z(\psi,\xi) = \xi^{p}\Psi(\psi,\xi) - \frac{\beta_1(\psi,0)}{\xi^{2-p}},$$

where $A[z(\psi,\varphi)] = Z(\psi,\xi)$. Therefore, Eq (2.9) is transformed to

$$L.H.S = \xi^{2p}\Psi(\psi,\xi) - \frac{\beta_1(\psi,0)}{\xi^{2-2p}} - \frac{D_\varphi^p\beta_1(\psi,0)}{\xi^{2-p}}, \tag{2.10}$$

when $r = K$. Equations (2.5) and (2.10) are compatible. For $r = K$, let's assume that Eq (2.5) holds. Therefore, we substitute $r = K$ into Eq (2.5):

$$A[D_\varphi^{Kp}\beta_1(\psi,\varphi)] = \xi^{Kp}\Psi(\psi,\xi) - \sum_{j=0}^{K-1}\xi^{p(K-j)-2}D_\varphi^{jp}D_\varphi^{jp}\beta_1(\psi,0),\ 0 < p \leq 1. \tag{2.11}$$

Next, we will show how to solve Eq (2.5) for $r = K + 1$. Based on Eq (2.5), we may express

$$A[D_\varphi^{(K+1)p}\beta_1(\psi,\varphi)] = \xi^{(K+1)p}\Psi(\psi,\xi) - \sum_{j=0}^{K}\xi^{p((K+1)-j)-2}D_\varphi^{jp}\beta_1(\psi,0). \tag{2.12}$$

After examining the left side of Eq (2.12), we get

$$L.H.S = A[D_{\varphi}^{Kp}(D_{\varphi}^{Kp})], \tag{2.13}$$

let

$$D_{\varphi}^{Kp} = g(\psi, \varphi),$$

by Eq (2.13), we drive

$$L.H.S = A[D_{\varphi}^{p} g(\psi, \varphi)]. \tag{2.14}$$

Equation (2.14) is modified to provide the following result by using the R-L integral and Caputo derivative:

$$L.H.S = \xi^{p} A[D_{\varphi}^{Kp} \beta_1(\psi, \varphi)] - \frac{g(\psi, 0)}{\xi^{2-p}}. \tag{2.15}$$

Equation (2.15) is derived from Eq (2.11),

$$L.H.S = \xi^{rp} \Psi(\psi, \xi) - \sum_{j=0}^{r-1} \xi^{p(r-j)-2} D_{\varphi}^{jp} \beta_1(\psi, 0). \tag{2.16}$$

In addition, the outcome that follows is obtained from Eq (2.16):

$$L.H.S = A[D_{\varphi}^{rp} \beta_1(\psi, 0)].$$

Thus, for $r = K + 1$, the Eq (2.5) is valid. Equation (2.5) is valid for all positive integers according to the mathematical induction method. □

Here, we find another novel way of looking to MFTS, or multiple fractional Taylor's series. The ARPSM, which will be discussed in more depth later on, will benefit from this formula.

Lemma 2.6. *Assume that $\beta_1(\psi, \varphi)$ represents the exponential order function. The expression $A[\beta_1(\psi, \varphi)] = \Psi(\psi, \xi)$ is the AT of $\beta_1(\psi, \varphi)$. The AT MFTS notation looks like this:*

$$\Psi(\psi, \xi) = \sum_{r=0}^{\infty} \frac{\hbar_r(\psi)}{\xi^{rp+2}}, \xi > 0, \tag{2.17}$$

where, $\psi = (s_1, \psi_2, \cdots, \psi_p) \in \mathbb{R}^p,\ p \in \mathbb{N}$.

Proof. Let us investigate Taylor's series' fractional order expression:

$$\beta_1(\psi, \varphi) = \hbar_0(\psi) + \hbar_1(\psi) \frac{\varphi^p}{\Gamma[p+1]} + +\hbar_2(\psi) \frac{\varphi^{2p}}{\Gamma[2p+1]} + \cdots. \tag{2.18}$$

The following equality is obtained by applying the AT to Eq (2.18):

$$A\left[\beta_1(\psi, \varphi)\right] = A\left[\hbar_0(\psi)\right] + A\left[\hbar_1(\psi) \frac{\varphi^p}{\Gamma[p+1]}\right] + A\left[\hbar_1(\psi) \frac{\varphi^{2p}}{\Gamma[2p+1]}\right] + \cdots$$

This is accomplished by using the AT's characteristics.

$$A\left[\beta_1(\psi, \varphi)\right] = \hbar_0(\psi) \frac{1}{\xi^2} + \hbar_1(\psi) \frac{\Gamma[p+1]}{\Gamma[p+1]} \frac{1}{\xi^{p+2}} + \hbar_2(\psi) \frac{\Gamma[2p+1]}{\Gamma[2p+1]} \frac{1}{\xi^{2p+2}} \cdots$$

A distinct variant of Taylor's series in the AT is therefore obtained. □

Lemma 2.7. *As stated in the new form of Taylor's series 2.17, the MFPS may be represented as $A[\beta_1(\psi,\varphi)] = \Psi(\psi,\xi)$.*

$$\hbar_0(\psi) = \lim_{\xi\to\infty} \xi^2\Psi(\psi,\xi) = \beta_1(\psi,0). \tag{2.19}$$

Proof. This can be determined from the revised version of Taylor's series:

$$\hbar_0(\psi) = \xi^2\Psi(\psi,\xi) - \frac{\hbar_1(\psi)}{\xi^p} - \frac{\hbar_2(\psi)}{\xi^{2p}} - \cdots \tag{2.20}$$

As shown in Eq (2.20), the necessary solution may be obtained by evaluating $\lim_{\xi\to\infty}$ into Eq (2.19) and doing a quick computation. □

Theorem 2.8. *The function $A[\beta_1(\psi,\varphi)] = \Psi(\psi,\xi)$ may be expressed in MFPS form as follows:*

$$\Psi(\psi,\xi) = \sum_{0}^{\infty} \frac{\hbar_r(\psi)}{\xi^{rp+2}},\ \xi > 0,$$

where $\psi = (\psi_1,\psi_2,\cdots,\psi_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. Then we have

$$\hbar_r(\psi) = D_r^{rp}\beta_1(\psi,0),$$

where, $D_\varphi^{rp} = D_\varphi^p.D_\varphi^p.\cdots.D_\varphi^p(r - times)$.

Proof. The new Taylor's series is as follows:

$$\hbar_1(\psi) = \xi^{p+2}\Psi(\psi,\xi) - \xi^p\hbar_0(\psi) - \frac{\hbar_2(\psi)}{\xi^p} - \frac{\hbar_3(\psi)}{\xi^{2p}} - \cdots \tag{2.21}$$

$\lim_{\xi\to\infty}$, is applied to (2.21), we get

$$\hbar_1(\psi) = \lim_{\xi\to\infty}(\xi^{p+2}\Psi(\psi,\xi) - \xi^p\hbar_0(\psi)) - \lim_{\xi\to\infty}\frac{\hbar_2(\psi)}{\xi^p} - \lim_{\xi\to\infty}\frac{\hbar_3(\psi)}{\xi^{2p}} - \cdots$$

After calculating the limit, we have the following equality:

$$\hbar_1(\psi) = \lim_{\xi\to\infty}(\xi^{p+2}\Psi(\psi,\xi) - \xi^p\hbar_0(\psi)). \tag{2.22}$$

The result of inserting Lemma 2.5 into Eq (2.22) is as follows:

$$\hbar_1(\psi) = \lim_{\xi\to\infty}(\xi^2 A[D_\varphi^p\beta_1(\psi,\varphi)](\xi)). \tag{2.23}$$

Furthermore, it is transformed into by using Lemma 2.6 to Eq (2.23),

$$\hbar_1(\psi) = D_\varphi^p\beta_1(\psi,0).$$

Again, applying limit $\xi \to \infty$ and using the new form of Taylor's series, we obtain:

$$\hbar_2(\psi) = \xi^{2p+2}\Psi(\psi,\xi) - \xi^{2p}\hbar_0(\psi) - \xi^p\hbar_1(\psi) - \frac{\hbar_3(\psi)}{\xi^p} - \cdots$$

We get the result from Lemma 2.6.

$$\hbar_2(\psi) = \lim_{\xi\to\infty} \xi^2(\xi^{2p}\Psi(\psi,\xi) - \xi^{2p-2}\hbar_0(\psi) - \xi^{p-2}\hbar_1(\psi)). \quad (2.24)$$

Using Lemmas 2.5 and 2.7, we convert Eq (2.24) into

$$\hbar_2(\psi) = D_\varphi^{2p}\beta_1(\psi,0),$$

when the new Taylor's series is put through the same process, the following results are obtained:

$$\hbar_3(\psi) = \lim_{\xi\to\infty} \xi^2(A[D_\varphi^{2p}\beta_1(\psi,p)](\xi)),$$

Lemma 2.7 is used to derive the final equation:

$$\hbar_3(\psi) = D_\varphi^{3p}\beta_1(\psi,0),$$

in general

$$\hbar_r(\psi) = D_\varphi^{rp}\beta_1(\psi,0).$$

Consequently, proof ends here. □

The principles regulating the convergence of Taylor's series in its new form are explained and proven in the following theorem.

Theorem 2.9. *Presented in Lemma 2.6, the formula for multiple fractional Taylor's series may be represented in the following new form:* $A[\beta_1(\psi,\varphi)] = \Psi(\psi,\xi)$. *When* $|\xi^a A[D_\varphi^{(K+1)p}\beta_1(\psi,\varphi)]| \le T$, *for all* $0 < \xi \le s$ *and* $0 < p \le 1$, *the following inequality satisfies the residual* $R_K(\psi,\xi)$ *of the new MFTS:*

$$|R_K(\psi,\xi)| \le \frac{T}{\xi^{(K=1)p+2}},\ 0 < \xi \le s.$$

Proof. Let $A[D_\varphi^{rp}\beta_1(\psi,\varphi)](\xi)$ is defined on $0 < \xi \le s$ for $r = 0,1,2,\cdots,K+1$. Let us assume that $|\xi^2 A[D_{psi},\varphi]^{K+1}{\beta_1}^1]| \le T,$ *on* $0 < \xi \le s$. Determine the following relation using the new Taylor's series:

$$R_K(\psi,\xi) = \Psi(\psi,\xi) - \sum_{r=0}^{K} \frac{\hbar_r(\psi)}{\xi^{rp+2}}. \quad (2.25)$$

Equation (2.25) is converted using Theorem 2.8,

$$R_K(\psi,\xi) = \Psi(\psi,\xi) - \sum_{r=0}^{K} \frac{D_\varphi^{rp}\beta_1(\psi,0)}{\xi^{rp+2}}. \quad (2.26)$$

To solve Eq (2.26), multiply $\xi^{(K+1)a+2}$ on both sides,

$$\xi^{(K+1)p+2}R_K(\psi,\xi) = \xi^2(\xi^{(K+1)p}\Psi(\psi,\xi) - \sum_{r=0}^{K} \xi^{(K+1-r)p-2}D_\varphi^{rp}\beta_1(\psi,0)). \quad (2.27)$$

Lemma 2.5 applied to Eq (2.27) yields

$$\xi^{(K+1)p+2} R_K(\psi, \xi) = \xi^2 A[D_{\varphi}^{(K+1)p} \beta_1(\psi, \varphi)]. \tag{2.28}$$

Taking absolute of Eq (2.28), we get

$$|\xi^{(K+1)p+2} R_K(\psi, \xi)| = |\xi^2 A[D_{\varphi}^{(K+1)p} \beta_1(\psi, \varphi)]|. \tag{2.29}$$

Applying the criteria listed in Eq (2.29) yields the following result:

$$\frac{-T}{\xi^{(K+1)p+2}} \leq R_K(\psi, \xi) \leq \frac{T}{\xi^{(K+1)p+2}}. \tag{2.30}$$

We use Eq (2.30) to get the necessary result,

$$|R_K(\psi, \xi)| \leq \frac{T}{\xi^{(K+1)p+2}}.$$

Thus, a new series convergence criteria is developed. □

3. Proposed methodologies

3.1. Aboodh residual power series method (ARPSM)

The ARPSM rules served as the foundation for our general model solution, which we describe below.

Step 1. The general equation may be simplified to obtain:

$$D_{\varphi}^{p} \beta_1(\psi, \varphi) + \vartheta(\psi) N(\beta_1) - \delta(\psi, \beta_1) = 0. \tag{3.1}$$

Step 2. The two sides of Eq (3.1) are evaluated using the AT in order to get

$$A[D_{\varphi}^{p} \beta_1(\psi, \varphi) + \vartheta(\psi) N(\beta_1) - \delta(\psi, \beta_1)] = 0, \tag{3.2}$$

transformation of Eq (3.2) by using Lemma 2.5. Thus,

$$\Psi(\psi, s) = \sum_{j=0}^{q-1} \frac{D_{\varphi}^{j} \beta_1(\psi, 0)}{s^{qp+2}} - \frac{\vartheta(\psi) Y(s)}{s^{qp}} + \frac{F(\psi, s)}{s^{qp}}, \tag{3.3}$$

where, $A[\delta(\psi, \beta_1)] = F(\psi, s), A[N(\beta_1)] = Y(s)$.

Step 3. Examine the form that the solution to Eq (3.3) takes:

$$\Psi(\psi, s) = \sum_{r=0}^{\infty} \frac{\hbar_r(\psi)}{s^{rp+2}}, \ s > 0,$$

Step 4. To proceed, follow these steps:

$$\hbar_0(\psi) = \lim_{s \to \infty} s^2 \Psi(\psi, s) = \beta_1(\psi, 0).$$

The following outcome by using Theorem 2.9:

$$\hbar_1(\psi) = D_\varphi^p \beta_1(\psi, 0),$$
$$\hbar_2(\psi) = D_\varphi^{2p} \beta_1(\psi, 0),$$
$$\vdots$$
$$\hbar_w(\psi) = D_\varphi^{wp} \beta_1(\psi, 0).$$

Step 5. The $\Psi(\psi, s)$ series that has been K^{th} truncated may be found using the formula below:

$$\Psi_K(\psi, s) = \sum_{r=0}^{K} \frac{\hbar_r(\psi)}{s^{rp+2}}, \ s > 0,$$

$$\Psi_K(\psi, s) = \frac{\hbar_0(\psi)}{s^2} + \frac{\hbar_1(\psi)}{s^{p+2}} + \cdots + \frac{\hbar_w(\psi)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\psi)}{s^{rp+2}}.$$

Step 6. Remember that, in order to derive the following, you must take into consideration both the K^{th}-truncated Aboodh residual function and the Aboodh residual function (ARF) from (3.3) separately:

$$ARes(\psi, s) = \Psi(\psi, s) - \sum_{j=0}^{q-1} \frac{D_\varphi^j \beta_1(\psi, 0)}{s^{jp+2}} + \frac{\vartheta(\psi)Y(s)}{s^{jp}} - \frac{F(\psi, s)}{s^{jp}},$$

and

$$ARes_K(\psi, s) = \Psi_K(\psi, s) - \sum_{j=0}^{q-1} \frac{D_\varphi^j \beta_1(\psi, 0)}{s^{jp+2}} + \frac{\vartheta(\psi)Y(s)}{s^{jp}} - \frac{F(\psi, s)}{s^{jp}}. \tag{3.4}$$

Step 7. Put $\Psi_K(\psi, s)$ into Eq (3.4) rather than use its expansion form,

$$\begin{aligned} ARes_K(\psi, s) =& \Big(\frac{\hbar_0(\psi)}{s^2} + \frac{\hbar_1(\psi)}{s^{p+2}} + \cdots + \frac{\hbar_w(\psi)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\psi)}{s^{rp+2}}\Big) \\ & - \sum_{j=0}^{q-1} \frac{D_\varphi^j \beta_1(\psi, 0)}{s^{jp+2}} + \frac{\vartheta(\psi)Y(s)}{s^{jp}} - \frac{F(\psi, s)}{s^{jp}}. \end{aligned} \tag{3.5}$$

Step 8. Equation (3.5) may be solved by multiplying both sides by s^{Kp+2},

$$\begin{aligned} s^{Kp+2} ARes_K(\psi, s) =& s^{Kp+2}\Big(\frac{\hbar_0(\psi)}{s^2} + \frac{\hbar_1(\psi)}{s^{p+2}} + \cdots + \frac{\hbar_w(\psi)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\psi)}{s^{rp+2}} \\ & - \sum_{j=0}^{q-1} \frac{D_\varphi^j \beta_1(\psi, 0)}{s^{jp+2}} + \frac{\vartheta(\psi)Y(s)}{s^{jp}} - \frac{F(\psi, s)}{s^{jp}}\Big). \end{aligned} \tag{3.6}$$

Step 9. After taking $\lim_{s\to\infty}$, we calculate the solution to Eq (3.6), which is:

$$\begin{aligned} \lim_{s\to\infty} s^{Kp+2} ARes_K(\psi, s) = & \lim_{s\to\infty} s^{Kp+2}\Big(\frac{\hbar_0(\psi)}{s^2} + \frac{\hbar_1(\psi)}{s^{p+2}} + \cdots + \frac{\hbar_w(\psi)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\psi)}{s^{rp+2}} \\ & - \sum_{j=0}^{q-1} \frac{D_\varphi^j \beta_1(\psi, 0)}{s^{jp+2}} + \frac{\vartheta(\psi)Y(s)}{s^{jp}} - \frac{F(\psi, s)}{s^{jp}}\Big). \end{aligned}$$

Step 10. Solving the above equation will provide the value of $\hbar_K(\psi)$,

$$\lim_{s\to\infty}(s^{Kp+2}ARes_K(\psi,s)) = 0,$$

where $K = w+1, w+2, \cdots$.
Step 11. Using a K-truncated series of $\Psi(\psi,s)$, replace the values of $\hbar_K(\psi)$ to get the K-approximate solution of Eq (3.3).
Step 12. Solve $\Psi_K(\psi,s)$ using the AIT to get the K-approximate solution $\beta_{1_K}(\psi,\varphi)$.

3.2. Aboodh transform iteration method (ATIM)

Consider the following PDE of space and time fractional order:

$$D_\varphi^p\beta_1(\psi,\varphi) = \Phi\Big(\beta_1(\psi,\varphi), D_\psi^\varphi\beta_1(\psi,\varphi), D_\psi^{2\varphi}\beta_1(\psi,\varphi), D_\psi^{3\varphi}\beta_1(\psi,\varphi)\Big),\ 0 < p,\varphi \le 1, \tag{3.7}$$

having the IC's

$$\beta_1{}^{(k)}(\psi,0) = h_k,\ k = 0,1,2,\cdots,m-1, \tag{3.8}$$

the function $\beta_1(\psi,\varphi)$ is unknown, while $\Phi\Big(\beta_1(\psi,\varphi), D_\psi^\varphi\beta_1(\psi,\varphi), D_\psi^{2\varphi}\beta_1(\psi,\varphi), D_\psi^{3\varphi}\beta_1(\psi,\varphi)\Big)$ may be a nonlinear operator or linear of $\beta_1(\psi,\varphi), D_\psi^\varphi\beta_1(\psi,\varphi), D_\psi^{2\varphi}\beta_1(\psi,\varphi)$ and $D_\psi^{3\varphi}\beta_1(\psi,\varphi)$. Applying the AT to both sides of Eq (3.7) yields the following equation; for convenience, we will denote $\beta_1(\psi,\varphi)$ using the symbol β_1,

$$A[\beta_1(\psi,\varphi)] = \frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\beta_1{}^{(k)}(\psi,0)}{s^{2-p+k}} + A\Big[\Phi\Big(\beta_1(\psi,\varphi), D_\psi^\varphi\beta_1(\psi,\varphi), D_\psi^{2\varphi}\beta_1(\psi,\varphi), D_\psi^{3\varphi}\beta_1(\psi,\varphi)\Big)\Big]\Big), \tag{3.9}$$

as a result of using the AIT to solve this problem,

$$\beta_1(\psi,\varphi) = A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\beta_1{}^{(k)}(\psi,0)}{s^{2-p+k}} + A\Big[\Phi\Big(\beta_1(\psi,\varphi), D_\psi^\varphi\beta_1(\psi,\varphi), D_\psi^{2\varphi}\beta_1(\psi,\varphi), D_\psi^{3\varphi}\beta_1(\psi,\varphi)\Big)\Big]\Big)\Big]. \tag{3.10}$$

The solution obtained by using the iterative Aboodh transform method is represented as an infinite series,

$$\beta_1(\psi,\varphi) = \sum_{i=0}^{\infty}\beta_{1_i}. \tag{3.11}$$

Since $\Phi\Big(\beta_1, D_\psi^\varphi\beta_1, D_\psi^{2\varphi}\beta_1, D_\psi^{3\varphi}\beta_1\Big)$ is either a nonlinear or linear operator, which can be decomposed as follows:

$$\begin{aligned}
&\Phi\Big(\beta_1, D_\psi^\varphi\beta_1, D_\psi^{2\varphi}\beta_1, D_\psi^{3\varphi}\beta_1\Big)\\
=&\Phi\Big(\beta_{1_0}, D_\psi^\varphi\beta_{1_0}, D_\psi^{2\varphi}\beta_{1_0}, D_\psi^{3\varphi}\beta_{1_0}\Big)\\
&+\sum_{i=0}^{\infty}\left(\Phi\Big(\sum_{k=0}^{i}\Big(\beta_{1_k}, D_\psi^\varphi\beta_{1_k}, D_\psi^{2\varphi}\beta_{1_k}, D_\psi^{3\varphi}\beta_{1_k}\Big)\Big) - \Phi\Big(\sum_{k=1}^{i-1}\Big(\beta_{1_k}, D_\psi^\varphi\beta_{1_k}, D_\psi^{2\varphi}\beta_{1_k}, D_\psi^{3\varphi}\beta_{1_k}\Big)\Big)\right).
\end{aligned} \tag{3.12}$$

Equation (3.10) is changed to the following equation by substituting the values of (3.12) and (3.11).

$$\begin{aligned}\sum_{i=0}^{\infty}\beta_{1_i}(\psi,\varphi) =&A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\beta_1{}^{(k)}(\psi,0)}{s^{2-p+k}} + A[\Phi(\beta_{1_0}, D_\psi^{\varphi}\beta_{1_0}, D_\psi^{2\varphi}\beta_{1_0}, D_\psi^{3\varphi}\beta_{1_0})]\Big)\Big]\\ &+ A^{-1}\Big[\frac{1}{s^p}\Big(A\Big[\sum_{i=0}^{\infty}\Big(\Phi\sum_{k=0}^{i}(\beta_{1_k}, D_\psi^{\varphi}\beta_{1_k}, D_\psi^{2\varphi}\beta_{1_k}, D_\psi^{3\varphi}\beta_{1_k})\Big)\Big]\Big)\Big]\\ &- A^{-1}\Big[\frac{1}{s^p}\Big(A\Big[\Big(\Phi\sum_{k=1}^{i-1}(\beta_{1_k}, D_\psi^{\varphi}\beta_{1_k}, D_\psi^{2\varphi}\beta_{1_k}, D_\psi^{3\varphi}\beta_{1_k})\Big)\Big]\Big)\Big],\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\beta_{1_0}(\psi,\varphi) = A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\beta_1{}^{(k)}(\psi,0)}{s^{2-p+k}}\Big)\Big],\\ &\beta_{1_1}(\psi,\varphi) = A^{-1}\Big[\frac{1}{s^p}\Big(A[\Phi(\beta_{1_0}, D_\psi^{\varphi}\beta_{1_0}, D_\psi^{2\varphi}\beta_{1_0}, D_\psi^{3\varphi}\beta_{1_0})]\Big)\Big],\\ &\vdots\\ \beta_{1_{m+1}}(\psi,\varphi) =&A^{-1}\Big[\frac{1}{s^p}\Big(A\Big[\sum_{i=0}^{\infty}\Big(\Phi\sum_{k=0}^{i}(\beta_{1_k}, D_\psi^{\varphi}\beta_{1_k}, D_\psi^{2\varphi}\beta_{1_k}, D_\psi^{3\varphi}\beta_{1_k})\Big)\Big]\Big)\Big]\\ &- A^{-1}\Big[\frac{1}{s^p}\Big(A\Big[\Big(\Phi\sum_{k=1}^{i-1}(\beta_{1_k}, D_\psi^{\varphi}\beta_{1_k}, D_\psi^{2\varphi}\beta_{1_k}, D_\psi^{3\varphi}\beta_{1_k})\Big)\Big]\Big)\Big],\ m = 1, 2, \cdots.\end{aligned} \tag{3.14}$$

The m-term of Eq (3.7) may be analytically approximated using the following expression:

$$\beta_1(\psi,\varphi) = \sum_{i=0}^{m-1}\beta_{1_i}. \tag{3.15}$$

4. Problem 1

4.1. Problem 1 with ARPSM

Examine the time-fractional KS model with sensitivity term $\varpi(\beta_2) = 1$, as shown in [23]. Then, $\frac{\partial}{\partial\psi}\Big(\beta_1(\psi,\varphi)\frac{\partial\varpi(\beta_2)}{\partial\psi}\Big) = 0$,

$$\begin{aligned}&D_\varphi^p\beta_1(\psi,\varphi) - a\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2} = 0,\\ &D_\varphi^p\beta_2(\psi,\varphi) - b\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} - c\beta_1(\psi,\varphi) + d\beta_2(\psi,\varphi) = 0, \quad \text{where} \quad 0 < p \leq 1,\end{aligned} \tag{4.1}$$

having IC's:

$$\begin{aligned}\beta_1(\psi,0) &= l_1 e^{-\psi^2},\\ \beta_2(\psi,0) &= l_2 e^{-\psi^2}.\end{aligned} \tag{4.2}$$

Using Eq (4.2), AT is applied to Eq (4.1) in order to get

$$\begin{aligned}
&\beta_1(\psi, s) - \frac{l_1 e^{-\psi^2}}{s^2} - \frac{a}{s^p}\Big[\frac{\partial^2 \beta_1(\psi, \varphi)}{\partial \psi^2}\Big] = 0,\\
&\beta_2(\psi, s) - \frac{l_2 e^{-\psi^2}}{s^2} - \frac{b}{s^p}\Big[\frac{\partial^2 \beta_2(\psi, \varphi)}{\partial \psi^2}\Big] - \frac{c}{s^p}\Big[\beta_1(\psi, \varphi)\Big] + \frac{d}{s^p}\Big[\beta_2(\psi, \varphi)\Big] = 0.
\end{aligned} \tag{4.3}$$

The k^{th}-truncated term series are

$$\begin{aligned}
\beta_1(\psi, s) &= \frac{l_1 e^{-\psi^2}}{s^2} + \sum_{r=1}^{k} \frac{f_r(\psi, s)}{s^{rp+1}},\\
\beta_2(\psi, s) &= \frac{l_2 e^{-\psi^2}}{s^2} + \sum_{r=1}^{k} \frac{j_r(\psi, s)}{s^{rp+1}}, \quad r = 1, 2, 3, 4\cdots
\end{aligned} \tag{4.4}$$

Aboodh residual functions (ARFs) are

$$\begin{aligned}
&A_\varphi Res(\psi, s) = \beta_1(\psi, s) - \frac{l_1 e^{-\psi^2}}{s^2} - \frac{a}{s^p}\Big[\frac{\partial^2 \beta_1(\psi, \varphi)}{\partial \psi^2}\Big] = 0,\\
&A_\varphi Res(\psi, s) = \beta_2(\psi, s) - \frac{l_2 e^{-\psi^2}}{s^2} - \frac{b}{s^p}\Big[\frac{\partial^2 \beta_2(\psi, \varphi)}{\partial \psi^2}\Big] - \frac{c}{s^p}\Big[\beta_1(\psi, \varphi)\Big] + \frac{d}{s^p}\Big[\beta_2(\psi, \varphi)\Big] = 0,
\end{aligned} \tag{4.5}$$

and the k^{th}-LRFs as:

$$\begin{aligned}
&A_\varphi Res_k(\psi, s) = \beta_{1k}(\psi, s) - \frac{l_1 e^{-\psi^2}}{s^2} - \frac{a}{s^p}\Big[\frac{\partial^2 \beta_{1k}(\psi, \varphi)}{\partial \psi^2}\Big] = 0,\\
&A_\varphi Res_k(\psi, s) = \beta_{2k}(\psi, s) - \frac{l_2 e^{-\psi^2}}{s^2} - \frac{b}{s^p}\Big[\frac{\partial^2 \beta_{2k}(\psi, \varphi)}{\partial \psi^2}\Big] - \frac{c}{s^p}\Big[\beta_{1k}(\psi, \varphi)\Big] + \frac{d}{s^p}\Big[\beta_{2k}(\psi, \varphi)\Big] = 0,
\end{aligned} \tag{4.6}$$

To determine $f_r(\psi, s)$ and $j_r(\psi, s)$, for $r = 1, 2, 3, ...$ Then, we iteratively solve $\lim_{s\to\infty}(s^{rp+1})$ by multiplying the resulting equation by s^{rp+1}, substituting the r^{th}-Aboodh residual function Eq (4.6) for the r^{th}-truncated series Eq (4.4). $A_\varphi Res_{\beta_1,r}(\psi, s)) = 0$ and $A_\varphi Res_{\beta_2,r}(\psi, s)) = 0$, and $r = 1, 2, 3, \cdots$. Putting $a = 0.5, b = 3, c = 1$ and $d = 0.8$ and taking the values of $l_1 = 160$ and $l_2 = 120$, we find the first few terms as:

$$\begin{aligned}
f_1(\psi, s) &= e^{-\psi^2}\left(320\beta_2{}^2 - 160\right),\\
j_1(\psi, s) &= e^{-\psi^2}\left(1440\psi^2 - 656\right),
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
f_2(\psi, s) &= e^{-\psi^2}\left(640\psi^4 - 1920\psi^2 + 480\right),\\
j_2(\psi, s) &= e^{-\psi^2}\left(17280\psi^4 - 51904\psi^2 + 12941\right),
\end{aligned} \tag{4.8}$$

and so on.

Putting $f_r(\psi, s)$, for $r = 1, 2, 3, \cdots$, in Eq (4.4), we get

$$\begin{aligned}
\beta_1(\psi, s) &= \frac{e^{-\psi^2}\left(320\psi^2 - 160\right)}{s^{p+1}} + \frac{e^{-\psi^2}\left(640\psi^4 - 1920\psi^2 + 480\right)}{s^{2p+1}} + \frac{160e^{-\psi^2}}{s^2} + \cdots,\\
\beta_2(\psi, s) &= \frac{e^{-\psi^2}\left(1440\psi^2 - 656\right)}{s^{p+1}} + \frac{e^{-\psi^2}\left(17280\psi^4 - 51904\psi^2 + 12941\right)}{s^{2p+1}} + \frac{120e^{-\psi^2}}{s^2} + \cdots.
\end{aligned} \tag{4.9}$$

The AIT may be used to get

$$\begin{aligned}
\beta_1(\psi,\varphi) &= \frac{e^{-\psi^2}\varphi^p\left(320\psi^2-160\right)}{\Gamma(p+1)} + \frac{e^{-\psi^2}\varphi^{2p}\left(640\psi^4-1920\psi^2+480\right)}{\Gamma(2p+1)} + 160e^{-\psi^2} + \cdots, \\
\beta_2(\psi,s) &= \frac{e^{-\psi^2}\varphi^p\left(1440\psi^2-656\right)}{\Gamma(p+1)} + \frac{e^{-\psi^2}\varphi^{2p}\left(17280\psi^4-51904\psi^2+12941\right)}{\Gamma(2p+1)} + 120e^{-\psi^2} + \cdots.
\end{aligned} \tag{4.10}$$

4.2. Problem 1 with ATIM

$$\begin{aligned}
D_\varphi^p\beta_1(\psi,\varphi) &= a\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2}, \\
D_\varphi^p\beta_2(\psi,\varphi) &= b\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} + c\beta_1(\psi,\varphi) - d\beta_2(\psi,\varphi), \quad \text{where} \quad 0 < p \le 1,
\end{aligned} \tag{4.11}$$

having IC's:

$$\begin{aligned}
\beta_1(\psi,0) &= l_1e^{-\psi^2}, \\
\beta_2(\psi,0) &= l_2e^{-\psi^2},
\end{aligned} \tag{4.12}$$

By using the AT on each side of Eq (4.11), we are able to get the following result:

$$\begin{aligned}
A[D_\varphi^p\beta_1(\psi,\varphi)] &= \frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\beta_1{}^{(k)}(\psi,0)}{s^{2-p+k}} + A\Big[a\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2}\Big]\Big), \\
A[D_\varphi^p\beta_2(\psi,\varphi)] &= \frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\beta_2{}^{(k)}(\psi,0)}{s^{2-p+k}} + A\Big[b\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} + c\beta_1(\psi,\varphi) - d\beta_2(\psi,\varphi)\Big]\Big),
\end{aligned} \tag{4.13}$$

using the AIT on each side of 4.13, we get the following result:

$$\begin{aligned}
\beta_1(\psi,\varphi) &= A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\beta_1{}^{(k)}(\psi,0)}{s^{2-p+k}} + A\Big[a\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2}\Big]\Big)\Big], \\
\beta_2(\psi,\varphi) &= A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\beta_2{}^{(k)}(\psi,0)}{s^{2-p+k}} + A\Big[b\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} + c\beta_1(\psi,\varphi) - d\beta_2(\psi,\varphi)\Big]\Big)\Big].
\end{aligned} \tag{4.14}$$

The equation that is produced as a consequence of applying the AT in an iterative manner is as follows:

$$\begin{aligned}
\beta_{1_0}(\psi,\varphi) =& A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{\beta_1{}^{(k)}(\psi,0)}{s^{2-p+k}}\Big)\Big] \\
&= A^{-1}\Big[\frac{\beta_1(\psi,0)}{s^2}\Big] \\
&= l_1e^{-\psi^2},
\end{aligned}$$

$$\begin{aligned}\beta_{20}(\psi,\varphi) =& A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{{\beta_2}^{(k)}(\psi,0)}{s^{2-p+k}}\Big)\Big]\\ &= A^{-1}\Big[\frac{\beta_2(\psi,0)}{s^2}\Big]\\ &= l_2 e^{-\psi^2}.\end{aligned}$$

We replaced the RL integral in Eq (4.11) to get the equivalent variant.

$$\begin{aligned}\beta_1(\psi,\varphi) &= l_1 e^{-\psi^2} - A\Big[a\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2}\Big],\\ \beta_2(\psi,\varphi) &= l_2 e^{-\psi^2} - A\Big[b\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} + c\beta_1(\psi,\varphi) - d\beta_2(\psi,\varphi)\Big].\end{aligned} \tag{4.15}$$

Putting $a = 0.5, b = 3, c = 1$, and $d = 0.8$ and taking the values of $l_1 = 160$ and $l_2 = 120$, the following terms are then acquired by using the ATIM procedure:

$$\begin{aligned}\beta_{10}(\psi,\varphi) &= 160e^{-\psi^2},\\ \beta_{20}(\psi,\varphi) &= 120e^{-\psi^2},\\ \beta_{11}(\psi,\varphi) &= \frac{e^{-\psi^2}\left(320\psi^2-160\right)\varphi^p}{\Gamma(p+1)},\\ \beta_{21}(\psi,\varphi) &= \frac{e^{-\psi^2}\left(1440\psi^2-656\right)\varphi^p}{\Gamma(p+1)},\\ \beta_{12}(\psi,\varphi) &= \frac{e^{-\psi^2}\left(640\psi^4-1920\psi^2+478\right)\varphi^{2p}}{\Gamma(2p+1)},\\ \beta_{22}(\psi,\varphi) &= \frac{e^{-\psi^2}zeta^{2p}\left(\left(364.8\ -832\psi^2\right)\Gamma(p+1)+p\left(17280\psi^4-51072\psi^2+12576\right)\Gamma(p)\right)}{\Gamma(p+1)\Gamma(2p+1)}.\end{aligned} \tag{4.16}$$

The following is the final ATIM solution:

$$\begin{aligned}\beta_1(\psi,\varphi) &= \beta_{10}(\psi,\varphi)+\beta_{11}(\psi,\varphi)+\beta_{12}(\psi,\varphi)+\cdots,\\ \beta_2(\psi,\varphi) &= \beta_{20}(\psi,\varphi)+\beta_{21}(\psi,\varphi)+\beta_{22}(\psi,\varphi)+\cdots.\end{aligned} \tag{4.17}$$

$$\begin{aligned}\beta_1(\psi,\varphi) =& 160e^{-\psi^2} + \frac{e^{-\psi^2}\left(320\psi^2-160\right)\varphi^p}{\Gamma(p+1)} + \frac{e^{-\psi^2}\left(640\psi^4-1920\psi^2+478\right)\varphi^{2p}}{\Gamma(2p+1)}+\cdots,\\ \beta_2(\psi,\varphi) =& 120e^{-\psi^2} + \frac{e^{-\psi^2}\left(1440\psi^2-656\right)\varphi^p}{\Gamma(p+1)}\\ &+\frac{e^{-\psi^2}zeta^{2p}\left(\left(364.8\ -832\psi^2\right)\Gamma(p+1)+p\left(17280\psi^4-51072\psi^2+12576\right)\Gamma(p)\right)}{\Gamma(p+1)\Gamma(2p+1)}+\cdots.\end{aligned} \tag{4.18}$$

5. Problem 2

5.1. Problem 2 with ARPSM

Examine the KS model of fractional order as stated in [23] with sensitivity term $\varpi(\beta_2) = \beta_2(\psi,\varphi)$. Then, the function $\frac{\partial}{\partial\psi}\Big(\beta_1(\psi,\varphi)\frac{\partial\varpi(\beta_2)}{\partial\psi}\Big) = \beta_1(\psi,\varphi)\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} + \frac{\partial\beta_1(\psi,\varphi)}{\partial\psi}\frac{\partial\beta_2(\psi,\varphi)}{\partial\psi}$,

$$\begin{aligned}
&D^p_\varphi\beta_1(\psi,\varphi) - a\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2} + \beta_1(\psi,\varphi)\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} + \frac{\partial\beta_1(\psi,\varphi)}{\partial\psi}\frac{\partial\beta_2(\psi,\varphi)}{\partial\psi} = 0,\\
&D^p_\varphi\beta_2(\psi,\varphi) - b\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} - c\beta_1(\psi,\varphi) + d\beta_2(\psi,\varphi) = 0, \quad \text{where} \quad 0 < p \leq 1,
\end{aligned} \tag{5.1}$$

having IC's:

$$\begin{aligned}
&\beta_1(\psi,0) = l_1e^{-\psi^2},\\
&\beta_2(\psi,0) = l_2e^{-\psi^2}.
\end{aligned} \tag{5.2}$$

AT is applied to Eq (5.1), the following results are obtained using Eq (5.2):

$$\begin{aligned}
&\beta_1(\psi,s) - \frac{l_1e^{-\psi^2}}{s^2} - \frac{a}{s^p}\Big[\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2}\Big] + \frac{1}{s^p}A_\varphi\Big[A^{-1}_\varphi\beta_1(\psi,\varphi)\times A^{-1}_\varphi\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2}\Big]\\
&+\frac{1}{s^p}A_\varphi\Big[\frac{\partial A^{-1}_\varphi\beta_1(\psi,\varphi)}{\partial\psi}\times\frac{\partial A^{-1}_\varphi\beta_2(\psi,\varphi)}{\partial\psi}\Big] = 0,\\
&\beta_2(\psi,s) - \frac{l_2e^{-\psi^2}}{s^2} - \frac{b}{s^p}\Big[\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2}\Big] - \frac{c}{s^p}\Big[\beta_1(\psi,\varphi)\Big] + \frac{d}{s^p}\Big[\beta_2(\psi,\varphi)\Big] = 0.
\end{aligned} \tag{5.3}$$

The k^{th} truncated term series are

$$\begin{aligned}
&\beta_1(\psi,s) = \frac{l_1e^{-\psi^2}}{s^2} + \sum_{r=1}^{k}\frac{f_r(\psi,s)}{s^{rp+1}},\\
&\beta_2(\psi,s) = \frac{l_2e^{-\psi^2}}{s^2} + \sum_{r=1}^{k}\frac{j_r(\psi,s)}{s^{rp+1}}, \quad r = 1,2,3,4\cdots.
\end{aligned} \tag{5.4}$$

Aboodh residual functions (ARFs) are

$$\begin{aligned}
A_\varphi Res(\psi,s) =&\beta_1(\psi,s) - \frac{l_1e^{-\psi^2}}{s^2} - \frac{a}{s^p}\Big[\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2}\Big] + \frac{1}{s^p}A_\varphi\Big[A^{-1}_\varphi\beta_1(\psi,\varphi)\times A^{-1}_\varphi\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2}\Big]\\
&+\frac{1}{s^p}A_\varphi\Big[\frac{\partial A^{-1}_\varphi\beta_1(\psi,\varphi)}{\partial\psi}\times\frac{\partial A^{-1}_\varphi\beta_2(\psi,\varphi)}{\partial\psi}\Big] = 0,\\
A_\varphi Res(\psi,s) =&\beta_2(\psi,s) - \frac{l_2e^{-\psi^2}}{s^2} - \frac{b}{s^p}\Big[\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2}\Big] - \frac{c}{s^p}\Big[\beta_1(\psi,\varphi)\Big] + \frac{d}{s^p}\Big[\beta_2(\psi,\varphi)\Big] = 0,
\end{aligned} \tag{5.5}$$

and the k^{th}-LRFs as:

$$
\begin{aligned}
A_{\varphi}Res_k(\psi,s) =&\beta_{1k}(\psi,s) + \frac{l_1 e^{-\psi^2}}{s^2} - \frac{a}{s^p}\Big[\frac{\partial^2\beta_{1k}(\psi,\varphi)}{\partial\psi^2}\Big] + \frac{1}{s^p}A_{\varphi}\Big[A_{\varphi}^{-1}\beta_{1k}(\psi,\varphi)\times A_{\varphi}^{-1}\frac{\partial^2\beta_{2k}(\psi,\varphi)}{\partial\psi^2}\Big] \\
&+ \frac{1}{s^p}A_{\varphi}\Big[\frac{\partial A_{\varphi}^{-1}\beta_{1k}(\psi,\varphi)}{\partial\psi}\times\frac{\partial A_{\varphi}^{-1}\beta_{2k}(\psi,\varphi)}{\partial\psi}\Big] = 0, \\
A_{\varphi}Res_k(\psi,s) =&\beta_{2k}(\psi,s) - \frac{l_2 e^{-\psi^2}}{s^2} - \frac{b}{s^p}\Big[\frac{\partial^2\beta_{2k}(\psi,\varphi)}{\partial\psi^2}\Big] - \frac{c}{s^p}\Big[\beta_{1k}(\psi,\varphi)\Big] + \frac{d}{s^p}\Big[\beta_{2k}(\psi,\varphi)\Big] = 0.
\end{aligned}
\tag{5.6}
$$

To determine $f_r(\psi,s)$ and $j_r(\psi,s)$, for $r = 1,2,3,....$ Then, we iteratively solve $\lim_{s\to\infty}(s^{rp+1})$ by multiplying the resulting equation by s^{rp+1}, substituting the r^{th}-Aboodh residual function Eq (5.6) for the r^{th}-truncated series Eq (5.4). $A_{\varphi}Res_{\beta_1,r}(\psi,s)) = 0$ and $A_{\varphi}Res_{\beta_2,r}(\psi,s)) = 0$, and $r = 1,2,3,\cdots$. Putting $a = 0.5, b = 3, c = 1$, and $d = 0.8$ and taking the values of $l_1 = 160$ and $l_2 = 120$, we find the first few terms as:

$$
\begin{aligned}
f_1(\psi,s) &= e^{-\psi^2}\left(320\psi^2 - 160\right) - 38400e^{-2\psi^2}, \\
j_1(\psi,s) &= e^{-\psi^2}\left(1440\psi^2 - 656\right),
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
f_2(\psi,s) =&e^{-3\psi^2}\left(18432000\psi^2 + 9216000\right) + e^{-2\psi^2}\left(785920 - 1612800\psi^2\right) \\
&+ e^{-\psi^2}\left(640\psi^4 - 1920\psi^2 + 480\right), \\
j_2(\psi,s) =&e^{-\psi^2}\left(17280\psi^4 - 51904\psi^2 + 12941\right) - 38400e^{-2\psi^2},
\end{aligned}
\tag{5.8}
$$

and so on.

Equation (5.4) is used to obtain $f_r(\psi,s)$ for $r = 1,2,3,\cdots$,

$$
\begin{aligned}
&\beta_1(\psi,s) = \frac{160e^{-\psi^2}}{s^2} + \frac{e^{-\psi^2}\left(320\psi^2 - 160\right) - 38400e^{-2\psi^2}}{s^{p+1}} \\
&+ \frac{e^{-3\psi^2}\left(18432000\psi^2 + 9216000\right) + e^{-2\psi^2}\left(785920 - 1612800\psi^2\right) + e^{-\psi^2}\left(640\psi^4 - 1920\psi^2 + 480\right)}{s^{2p+1}} + \cdots, \\
&\beta_2(\psi,s) = \frac{120e^{-\psi^2}}{s^2} + \frac{e^{-\psi^2}\left(1440\psi^2 - 656\right)}{s^{p+1}} + \frac{e^{-\psi^2}\left(17280\psi^4 - 51904\psi^2 + 12941\right) - 38400e^{-2\psi^2}}{s^{2p+1}} + \cdots.
\end{aligned}
\tag{5.9}
$$

AIT is applied to get

$$
\begin{aligned}
&\beta_1(\psi,\varphi) = 160e^{-\psi^2} + \frac{e^{-\psi^2}\varphi^p\left(320\psi^2 - 160\right) - 38400e^{-2\psi^2}}{\Gamma(p+1)} \\
&+ \frac{e^{-3\psi^2}\varphi^{2p}\left(18432000\psi^2 + 9216000\right) + e^{-2\psi^2}\left(785920 - 1612800\psi^2\right) + e^{-\psi^2}\left(640\psi^4 - 1920\psi^2 + 480\right)}{\Gamma(2p+1)} + \cdots, \\
&\beta_2(\psi,\varphi) = \frac{120e^{-\psi^2}}{s^2} + \frac{e^{-\psi^2}\varphi^p\left(1440\psi^2 - 656\right)}{\Gamma(p+1)} + \frac{e^{-\psi^2}\varphi^{2p}\left(17280\psi^4 - 51904\psi^2 + 12941\right) - 38400e^{-2\psi^2}}{\Gamma(2p+1)} + \cdots.
\end{aligned}
\tag{5.10}
$$

5.2. Problem 2 with ATIM

$$
\begin{aligned}
D_\varphi^p \beta_1(\psi,\varphi) &= a\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2} - \beta_1(\psi,\varphi)\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi} - \frac{\partial\beta_1(\psi,\varphi)}{\partial\psi}\frac{\partial\beta_2(\psi,\varphi)}{\partial\psi},\\
D_\varphi^p \beta_2(\psi,\varphi) &= b\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} + c\beta_1(\psi,\varphi) - d\beta_2(\psi,\varphi), \quad \text{where} \quad 0 < p \leq 1,
\end{aligned} \tag{5.11}
$$

having IC's:

$$
\begin{aligned}
\beta_1(\psi,0) &= l_1 e^{-\psi^2},\\
\beta_2(\psi,0) &= l_2 e^{-\psi^2},
\end{aligned} \tag{5.12}
$$

when applying the AT to both sides of Eq (5.11), we get the following result:

$$
\begin{aligned}
A[D_\varphi^p \beta_1(\psi,\varphi)] &= \frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{{\beta_1}^{(k)}(\psi,0)}{s^{2-p+k}} + A\Big[a\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2} - \beta_1(\psi,\varphi)\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi} - \frac{\partial\beta_1(\psi,\varphi)}{\partial\psi}\frac{\partial\beta_2(\psi,\varphi)}{\partial\psi}\Big]\Big),\\
A[D_\varphi^p \beta_2(\psi,\varphi)] &= \frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{{\beta_2}^{(k)}(\psi,0)}{s^{2-p+k}} + A\Big[b\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} + c\beta_1(\psi,\varphi) - d\beta_2(\psi,\varphi)\Big]\Big),
\end{aligned} \tag{5.13}
$$

applying the AIT to both sides of Eq (5.13) yields the following result:

$$
\begin{aligned}
\beta_1(\psi,\varphi) &= A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{{\beta_1}^{(k)}(\psi,0)}{s^{2-p+k}} + A\Big[a\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2} - \beta_1(\psi,\varphi)\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi} - \frac{\partial\beta_1(\psi,\varphi)}{\partial\psi}\frac{\partial\beta_2(\psi,\varphi)}{\partial\psi}\Big]\Big)\Big],\\
\beta_2(\psi,\varphi) &= A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{{\beta_2}^{(k)}(\psi,0)}{s^{2-p+k}} + A\Big[b\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} + c\beta_1(\psi,\varphi) - d\beta_2(\psi,\varphi)\Big]\Big)\Big].
\end{aligned} \tag{5.14}
$$

This equation is obtained by using the AT's iterative procedure:

$$
\begin{aligned}
\beta_{1_0}(\psi,\varphi) =& A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{{\beta_1}^{(k)}(\psi,0)}{s^{2-p+k}}\Big)\Big]\\
&= A^{-1}\Big[\frac{\beta_1(\psi,0)}{s^2}\Big]\\
&= l_1 e^{-\psi^2},
\end{aligned}
$$

$$
\begin{aligned}
\beta_{2_0}(\psi,\varphi) =& A^{-1}\Big[\frac{1}{s^p}\Big(\sum_{k=0}^{m-1}\frac{{\beta_2}^{(k)}(\psi,0)}{s^{2-p+k}}\Big)\Big]\\
&= A^{-1}\Big[\frac{\beta_2(\psi,0)}{s^2}\Big]\\
&= l_2 e^{-\psi^2}.
\end{aligned}
$$

The RL integral is applied to Eq (5.1) to yield the equivalent form.

$$\begin{aligned}\beta_1(\psi,\varphi) &= l_1 e^{-\psi^2} - A\Big[a\frac{\partial^2\beta_1(\psi,\varphi)}{\partial\psi^2} - \beta_1(\psi,\varphi)\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi} - \frac{\partial\beta_1(\psi,\varphi)}{\partial\psi}\frac{\partial\beta_2(\psi,\varphi)}{\partial\psi}\Big],\\ \beta_2(\psi,\varphi) &= l_2 e^{-\psi^2} - A\Big[b\frac{\partial^2\beta_2(\psi,\varphi)}{\partial\psi^2} + c\beta_1(\psi,\varphi) - d\beta_2(\psi,\varphi)\Big].\end{aligned} \tag{5.15}$$

Putting $a = 0.5, b = 3, c = 1$, and $d = 0.8$ and taking the values of $l_1 = 160$ and $l_2 = 120$, the following terms are then acquired by using the ATIM procedure: These terms are obtained using the ATIM process,

$$\begin{aligned}
\beta_{10}(\psi,\varphi) =& 160e^{-\psi^2}, \qquad \beta_{20}(\psi,\varphi) = 120e^{-\psi^2},\\
\beta_{11}(\psi,\varphi) =& \frac{e^{-2\beta_2{}^2}\left(e^{\beta_2{}^2}\left(320\beta_2{}^2-160\right)-38400\right)\varphi^p}{\Gamma(p+1)}, \qquad \beta_{21}(\psi,\varphi) = \frac{e^{-x^2}\left(1440x^2-656\right)t^p}{\Gamma(p+1)},\\
\beta_{12}(\psi,\varphi) =& -\Big(225\ 4^{6-p}e^{-2\psi^2}\varphi^p\Big(p\Gamma(p)\Big(4^p\Gamma\Big(p+\frac{1}{2}\Big)\Big(e^{-\psi^2}\Big(-240\psi^4-10.66\psi^2\\
&+e^{\psi^2}\Big(\psi^4-0.044\psi^2+0.727\Big)+174.66\Big)\varphi^{2p}\Gamma(2p+1)+0.0416\Gamma(p+1)\Gamma(3p+1)\Big)\\
&+\varphi^p\Gamma(p+1)\Gamma(3p+1)\Big(3.10179\psi^2+\Big(-0.0012\psi^4\\
&+35.4528\psi^2+17.7236\Big)\sinh\left(\psi^2\right)+\Big(-0.00123\psi^4-35.4453\psi^2-17.7254\Big)\cosh\left(\psi^2\right)\\
&-1.5115\Big)\Big)-0.0738\Gamma(p+1)\Gamma(2p+1)\Gamma(3p+1)\Big)\Big)/\Big(p\Gamma(p)\Gamma\Big(p+\frac{1}{2}\Big)\Gamma(p+1)^2\Gamma(3p+1)\Big),\\
\beta_{22}(\psi,\varphi) =& \Big(e^{-2\psi^2}\varphi^{2p}\Big(e^{\psi^2}\Big(\Big(524.8-1152\psi^2\Big)\Gamma(p+1)\\
&+p\Big(17280\psi^4-50752\psi^2+12416\Big)\Gamma(p)\Big)-38400p\Gamma(p)\Big)\Big)/\Big(\Gamma(p+1)\Gamma(2p+1)\Big).
\end{aligned} \tag{5.16}$$

The following is the ATIM procedure's ultimate solution:

$$\begin{aligned}\beta_1(\psi,\varphi) &= \beta_{10}(\psi,\varphi)+\beta_{11}(\psi,\varphi)+\beta_{12}(\psi,\varphi)+\cdots,\\ \beta_2(\psi,\varphi) &= \beta_{20}(\psi,\varphi)+\beta_{21}(\psi,\varphi)+\beta_{22}(\psi,\varphi)+\cdots.\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\beta_1(\psi,\varphi) =& 160e^{-\psi^2}+\Big(e^{-2\beta_2{}^2}\Big(e^{\beta_2{}^2}\Big(320\beta_2{}^2-160\Big)-38400\Big)\varphi^p\Big)/\Big(\Gamma(p+1)\Big)\\
&-\Big(225\ 4^{6-p}e^{-2\psi^2}\varphi^p\times\Big(p\Gamma(p)\Big(4^p\Gamma\Big(p+\frac{1}{2}\Big)\Big(e^{-\psi^2}\Big(-240\psi^4-10.66\psi^2\\
&+e^{\psi^2}\Big(\psi^4-0.044\psi^2+0.727\Big)+174.66\Big)\varphi^{2p}\Gamma(2p+1)\\
&+0.0416\Gamma(p+1)\Gamma(3p+1)\Big)+\varphi^p\Gamma(p+1)\Gamma(3p+1)\Big(3.10179\psi^2\\
&+\Big(-0.0012\psi^4+35.4528\psi^2+17.7236\Big)\sinh\left(\psi^2\right)\\
&+\Big(-0.00123\psi^4-35.4453\psi^2-17.7254\Big)\cosh\left(\psi^2\right)-1.5115\Big)\Big)\\
&-0.0738\Gamma(p+1)\Gamma(2p+1)\Gamma(3p+1)\Big)\Big)/\Big(p\Gamma(p)\Gamma\Big(p+\frac{1}{2}\Big)\Gamma(p+1)^2\Gamma(3p+1)\Big)+\cdots,
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\beta_2(\psi,\varphi) =& 120e^{-\psi^2}+\Big(e^{-x^2}\left(1440x^2-656\right)t^p\Big)/\Big(\Gamma(p+1)\Big)+\Big(e^{-2\psi^2}\varphi^{2p}\Big(e^{\psi^2}\Big(\Big(524.8-1152\psi^2\Big)\Gamma(p+1)\\
&+p\Big(17280\psi^4-50752\psi^2+12416\Big)\Gamma(p)\Big)-38400p\Gamma(p)\Big)\Big)/\Big(\Gamma(p+1)\Gamma(2p+1)\Big)+\cdots.
\end{aligned} \tag{5.19}$$

6. Graphics and tables discussion

In Problem 1, we embark on a comprehensive exploration of the solutions $\beta_1(\psi, \varphi)$ and $\beta_2(\psi, \varphi)$ through both graphical and numerical analyses employing two distinct methodologies: the Aboodh residual power series method (ARPSM) and the Aboodh transform iteration method (ATIM). Beginning with $\beta_1(\psi, \varphi)$, Figure 1 offers an insightful depiction of the approximate solution obtained via ARPSM for a specific value of $p = 1$. Building upon this foundation, Figure 2 extends the analysis, providing both 3D and 2D representations to elucidate the influence of varying p on the solution when $\varphi = 0.1$. Similarly, Figures 3 and 4 delve into the corresponding analyses for $\beta_2(\psi, \varphi)$. These visualizations offer a nuanced understanding of how changes in the parameter p affect the behavior of the solutions across different dimensions. In conjunction with the graphical exploration, Tables 1 and 2 complement our investigation by presenting detailed fractional order analyses for ARPSM applied to $\beta_1(\psi, \varphi)$ and $\beta_2(\psi, \varphi)$, respectively. These tables provide valuable insights into the fractional characteristics of the solutions and contribute to a comprehensive understanding of their properties.

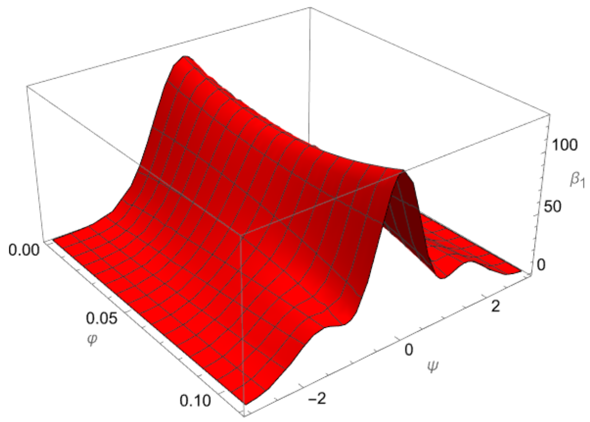

Figure 1. Approximate solution of $\beta_1(\psi, \varphi)$ via ARPSM for $p = 1$.

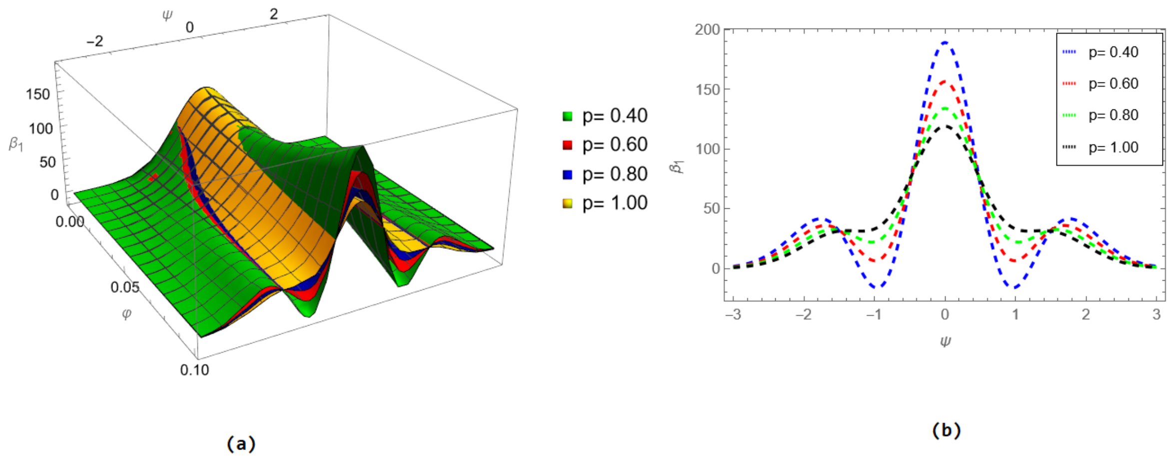

Figure 2. $\beta_1(\psi, \varphi)$, (a) shows three-dimensional analysis of different values of p; (b) shows two-dimensional analysis of different values of p at $\varphi = 0.1$ via ARPSM.

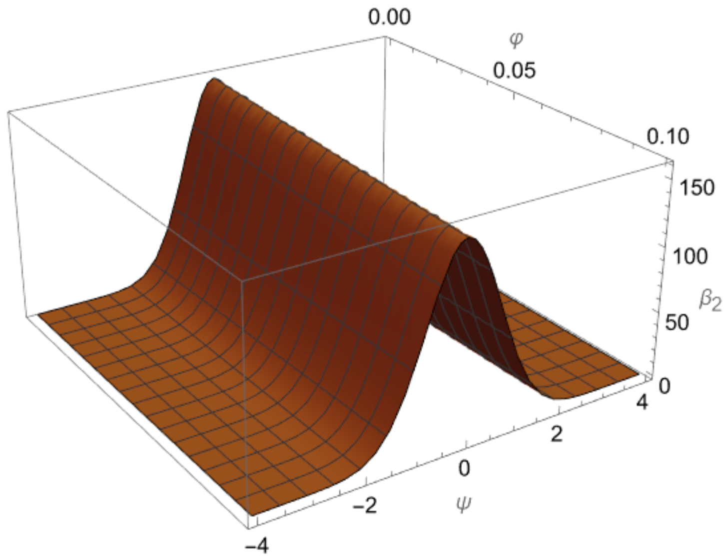

Figure 3. Approximate solution of $\beta_2(\psi, \varphi)$ via ARPSM for $p = 1$.

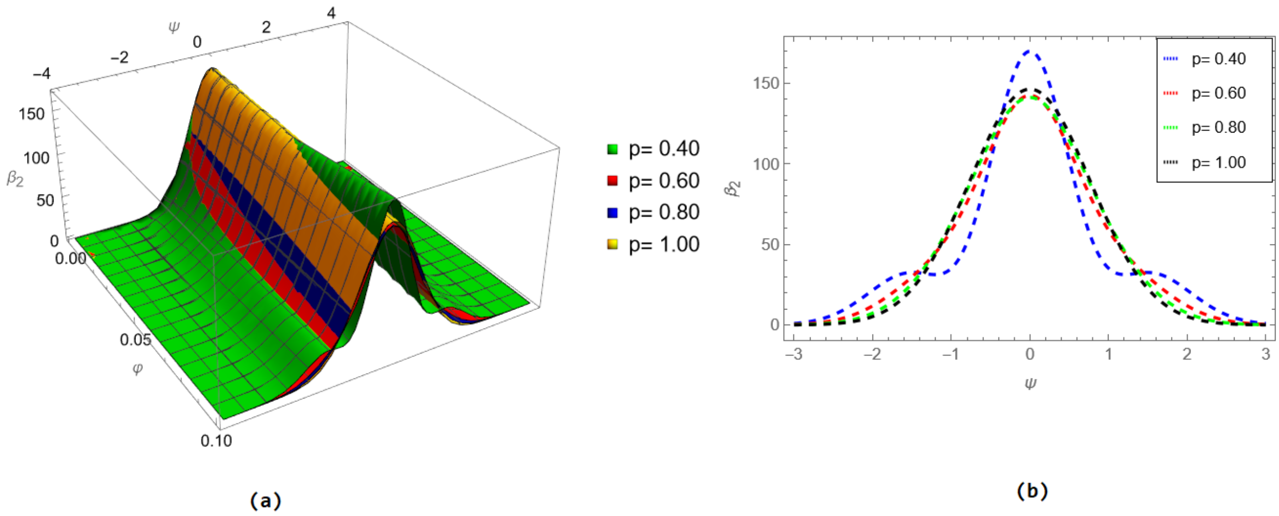

Figure 4. $\beta_2(\psi, \varphi)$: (a) shows three-dimensional analysis of different values of p; (b) shows two-dimensional analysis of different values of p at $\varphi = 0.1$ via ARPSM.

Table 1. Analysis of various values of fractional order of ARPSM of Problem 1 of $\beta_1(\psi, \varphi)$.

φ	$p = 0.40$	$p = 0.60$	$p = 0.80$	$p = 1.00$
0.2	0.0048028	0.0022338	0.00098417	0.00043033
0.4	0.0082096	0.0049227	0.00273349	0.00144405
0.6	0.0112581	0.0078699	0.00506050	0.00305916
0.8	0.0140966	0.0110044	0.00787829	0.00527565
1.0	0.0167894	0.0142882	0.0111338	0.00809353

Table 2. Analysis of various values of fractional order of ARPSM of Problem 1 of $\beta_2(\psi, \varphi)$.

φ	$p = 0.40$	$p = 0.60$	$p = 0.80$	$p = 1.00$
0.2	0.121739	0.054477	0.022375	0.0086336
0.4	0.211321	0.124300	0.066837	0.0334867
0.6	0.291879	0.201620	0.127169	0.0745724
0.8	0.367092	0.284277	0.200911	0.1318910
1.0	0.438569	0.371156	0.286582	0.2054420

Shifting focus to the ATIM method, Figures 5 and 7 display the approximate solutions of $\beta_1(\psi, \varphi)$ and $\beta_2(\psi, \varphi)$ for $p = 1$, respectively. Figures 6 and 8 further extend the analysis, offering insights into the impact of varying p at $\varphi = 0.1$. The fractional order sensitivity is examined through Tables 3 and 4 for $\beta_1(\psi, \varphi)$ and $\beta_2(\psi, \varphi)$ under ATIM.

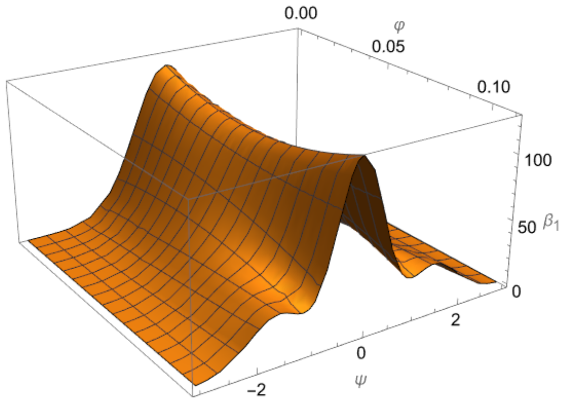

Figure 5. Approximate solution of $\beta_1(\psi, \varphi)$ via ATIM for $p = 1$.

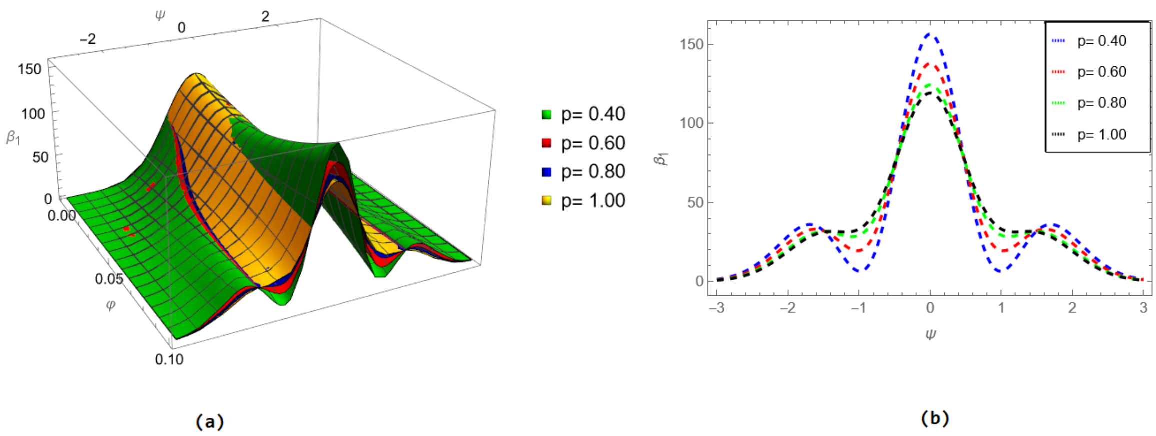

Figure 6. $\beta_1(\psi, \varphi)$: (a) shows three-dimensional analysis of different values of p; (b) shows two dimensional analysis of different values of p at $\varphi = 0.1$ via ATIM.

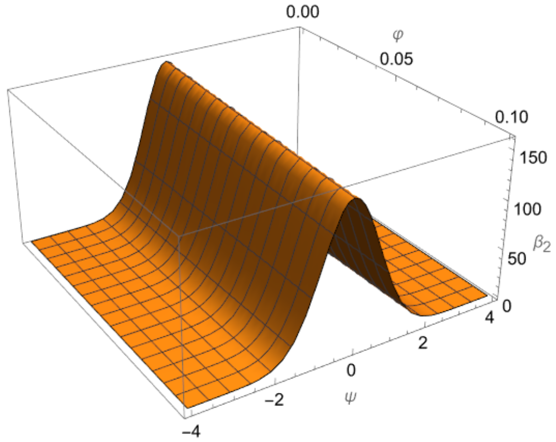

Figure 7. Approximate solution of $\beta_2(\psi, \varphi)$ via ATIM for $p = 1$.

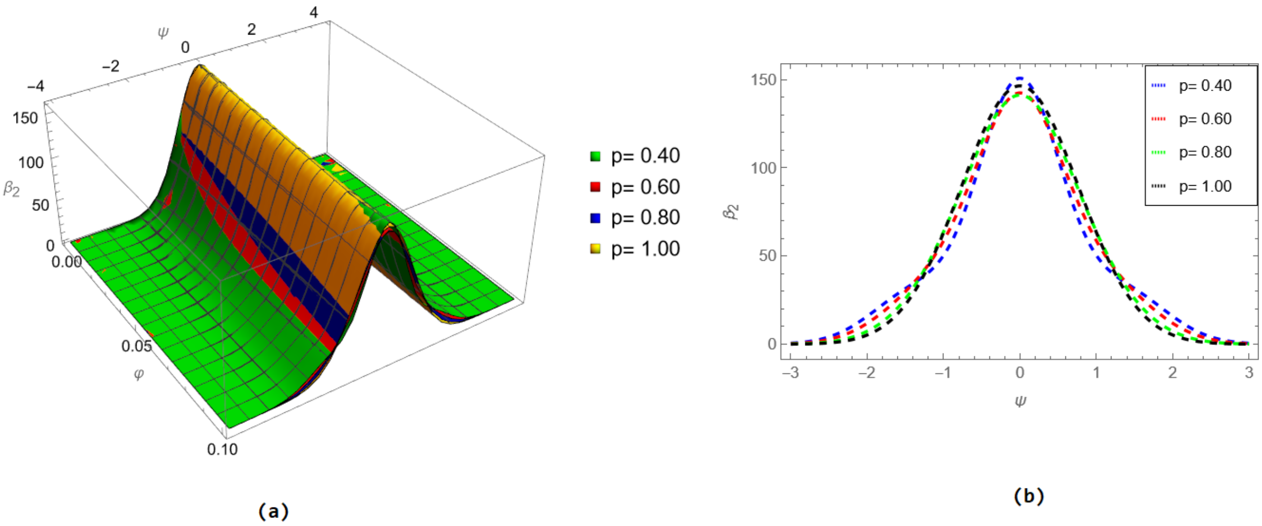

Figure 8. $\beta_2(\psi, \varphi)$: (a) shows three-dimensional analysis of different values of p; (b) shows two-dimensional analysis of different values of p at $\varphi = 0.1$ via ATIM.

Table 3. Analysis of various values of fractional order of ATIM of Problem 1 of $\beta_1(\psi, \varphi)$.

φ	$p = 0.40$	$p = 0.60$	$p = 0.80$	$p = 1.00$
0.2	0.0048028	0.0022338	0.00098416	0.00043033
0.4	0.0082095	0.0049226	0.00273345	0.00144403
0.6	0.0112579	0.0078698	0.00506043	0.00305912
0.8	0.0140964	0.0110042	0.00787818	0.00527558
1.0	0.0167892	0.0142880	0.01113370	0.00809342

Table 4. Analysis of various values of fractional order of ATIM of Problem 1 of $\beta_2(\psi, \varphi)$.

φ	$p = 0.40$	$p = 0.60$	$p = 0.80$	$p = 1.00$
0.2	0.121748	0.054481	0.022376	0.0086342
0.4	0.211336	0.124308	0.066842	0.0334890
0.6	0.291899	0.201634	0.127178	0.0745776
0.8	0.367118	0.284297	0.200924	0.1319000
1.0	0.438600	0.371182	0.286602	0.2054560

In order to facilitate a comprehensive comparison, Tables 5 and 6 juxtapose the results obtained from both ARPSM and ATIM for $\beta_1(\psi, \varphi)$ and $\beta_2(\psi, \varphi)$ in Problem 1. These tables provide a nuanced understanding of the performance of each method, shedding light on their strengths and limitations in solving the given fractional-order equations. The two most basic approaches to solving fractional differential equations are the ATIM and the ARPSM, as stated in [52, 53] and [49–51], respectively. These techniques provide numerical solutions to PDEs that do not need discretization or linearization, making the symbolic terms in analytical solutions instantly visible. The primary objective of this study is to compare and contrast the performance of ARPSM and ATIM in solving the Keller-Segel (KS) model. It is worth mentioning that several linear and nonlinear fractional differential problems have been solved using these two approaches.

Table 5. Problem 1: comparison of both methods for $\beta_1(\psi, \varphi)$.

	$p = 1.00$		$p = 0.80$		$p = 0.60$		$p = 0.40$	
φ	ARPSM	ATIM	ARPSM	ATIM	ARPSM	ATIM	ARPSM	ATIM
0.2	0.00043033	0.00043033	0.00098417	0.00098416	0.0022338	0.0022338	0.0048028	0.0048028
0.4	0.00144405	0.00144403	0.00273349	0.00273345	0.0049227	0.0049226	0.0082096	0.0082095
0.6	0.00305916	0.00305912	0.00506050	0.00506043	0.0078699	0.0078698	0.0112581	0.0112579
0.8	0.00527565	0.00527558	0.00787829	0.00787818	0.0110044	0.0110042	0.0140966	0.0140964
1.0	0.00809353	0.00809342	0.01113380	0.01113370	0.0142882	0.0142880	0.0167894	0.0167892

Table 6. Problem 1: comparison of both methods for $\beta_2(\psi, \varphi)$.

	$p = 1.00$		$p = 0.80$		$p = 0.60$		$p = 0.40$	
φ	ARPSM	ATIM	ARPSM	ATIM	ARPSM	ATIM	ARPSM	ATIM
0.2	0.0086336	0.0086342	0.022375	0.022376	0.054477	0.054481	0.121739	0.121748
0.4	0.0334867	0.0334890	0.066837	0.066842	0.124300	0.124308	0.211321	0.211336
0.6	0.0745724	0.0745776	0.127169	0.127178	0.201620	0.201634	0.291879	0.291899
0.8	0.1318910	0.1319000	0.200911	0.200924	0.284277	0.284297	0.367092	0.367118
1.0	0.2054420	0.2054560	0.286582	0.286602	0.371156	0.371182	0.438569	0.438600

In Problem 2, the analysis of solutions $\beta_1(\psi, \varphi)$ and $\beta_2(\psi, \varphi)$ is carried out using the ARPSM and the ATIM. For $\beta_1(\psi, \varphi)$, Figure 9 illustrates the approximate solution via ARPSM for $p = 1$. Subsequently, Figure 10 presents 3D and 2D analyses, demonstrating the influence of varying p on the solution at $\varphi = 0.1$. Analogously, Figures 11 and 12 provide the corresponding analyses for $\beta_2(\psi, \varphi)$. Complementing the graphical exploration, Table 7 details the fractional order analysis for ARPSM of $\beta_1(\psi, \varphi)$, and Table 8 does the same for $\beta_2(\psi, \varphi)$.

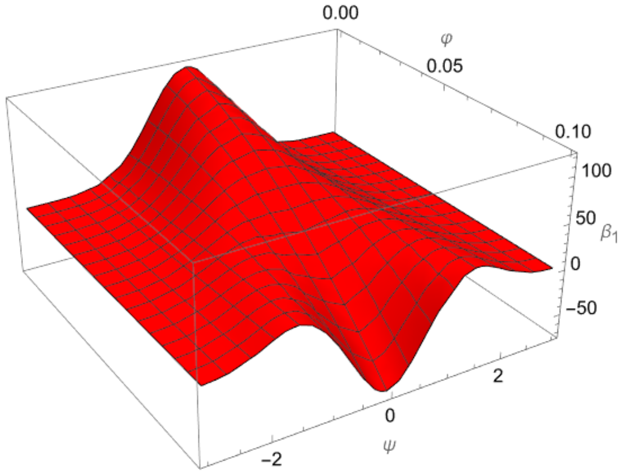

Figure 9. Approximate solution of $\beta_1(\psi, \varphi)$ via ARPSM for $p = 1$.

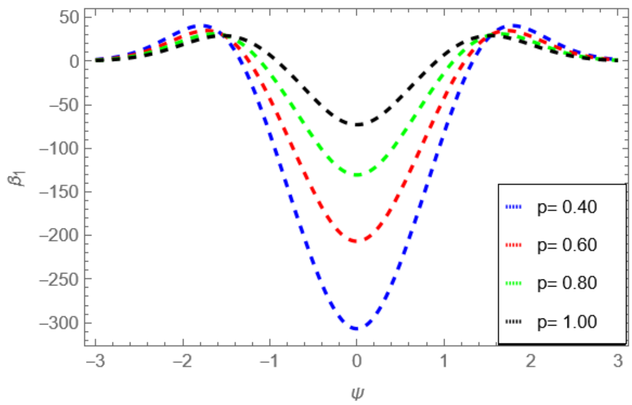

Figure 10. 2D analysis of different values of p at $\varphi = 0.1$.

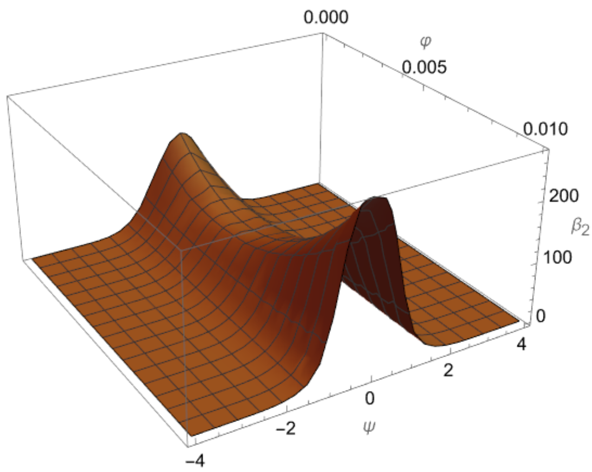

Figure 11. Approximate solution of $\beta_2(\psi, \varphi)$ via ARPSM for $p = 1$.

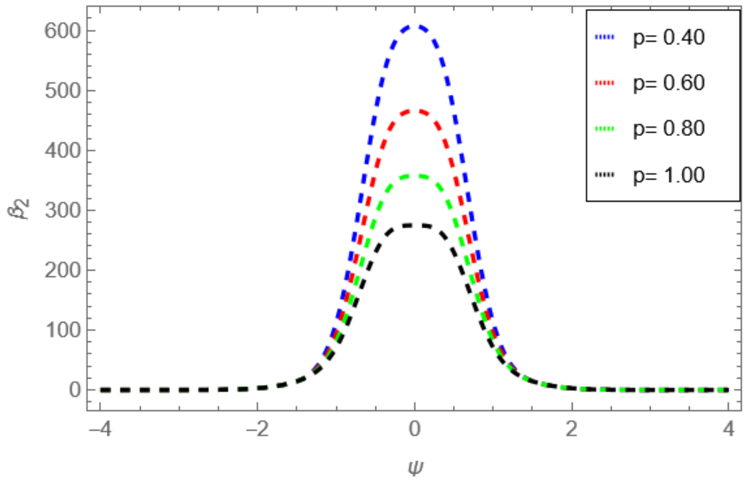

Figure 12. 2D analysis of different values of p at $\varphi = 0.1$.

Table 7. Analysis of various values of fractional order of ARPSM of Problem 2 of $\beta_1(\psi, \varphi)$.

φ	$p = 0.40$	$p = 0.60$	$p = 0.80$	$p = 1.00$
0.2	0.0048027	0.0022338	0.00098415	0.00043032
0.4	0.0082094	0.0049226	0.00273344	0.00144403
0.6	0.0112578	0.0078698	0.00506040	0.00305910
0.8	0.0140963	0.0110042	0.00787813	0.00527555
1.0	0.0167891	0.0142880	0.01113360	0.00809337

Table 8. Analysis of various values of fractional order of ARPSM of Problem 2 of $\beta_2(\psi, \varphi)$.

φ	$p = 0.40$	$p = 0.60$	$p = 0.80$	$p = 1.00$
0.2	0.121739	0.054477	0.022375	0.0086336
0.4	0.211321	0.124300	0.066837	0.0334867
0.6	0.291879	0.201620	0.127169	0.0745724
0.8	0.367092	0.284277	0.200911	0.1318910
1.0	0.438569	0.371156	0.286582	0.2054420

Shifting focus to the ATIM method, Figures 13 and 15 display the approximate solutions of $\beta_1(\psi, \varphi)$ and $\beta_2(\psi, \varphi)$ for $p = 1$, respectively. Figures 14 and 16 further extend the analysis, offering insights into the impact of varying p at $\varphi = 0.1$. The fractional order sensitivity is examined through Tables 9 and 10 for $\beta_1(\psi, \varphi)$ and $\beta_2(\psi, \varphi)$ under ATIM.

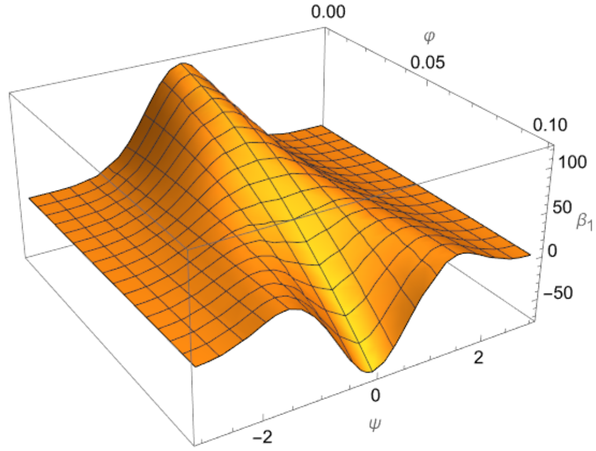

Figure 13. Approximate solution of $\beta_1(\psi, \varphi)$ via ATIM for $p = 1$.

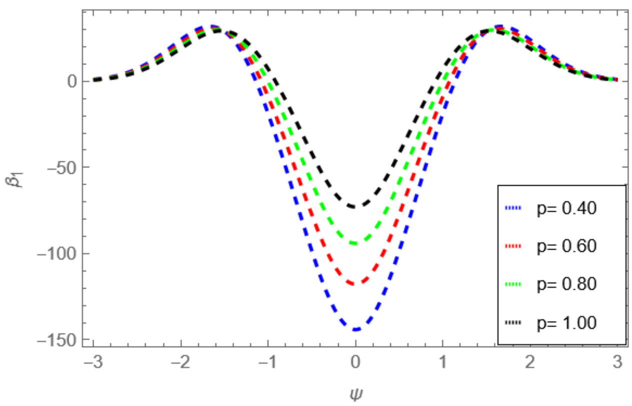

Figure 14. 2D analysis of different values of p at $\varphi = 0.1$.

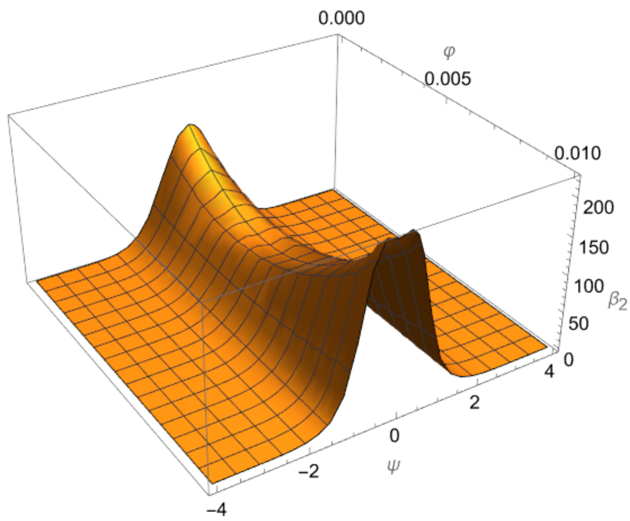

Figure 15. Approximate solution of $\beta_2(\psi, \varphi)$ via ATIM for p=1.

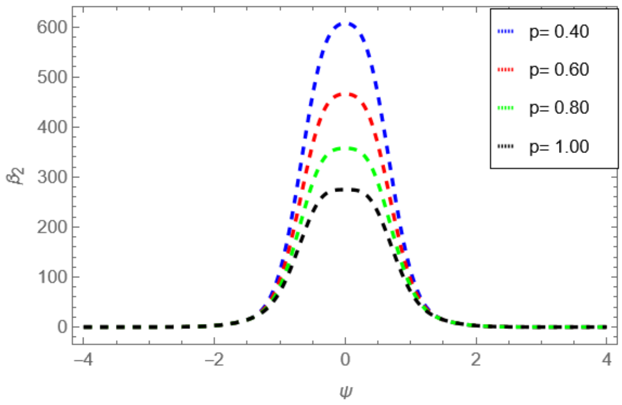

Figure 16. 2D analysis of different values of p at $\varphi = 0.1$.

Table 9. Analysis of various values of fractional order of ATIM of Problem 2 of $\beta_1(\psi, \varphi)$.

φ	$p = 0.40$	$p = 0.60$	$p = 0.80$	$p = 1.00$
0.2	0.0047363	0.00220436	0.00097225	0.00042586
0.4	0.0080934	0.00485484	0.00269728	0.00142614
0.6	0.0110971	0.00775928	0.00499109	0.00301877
0.8	0.0138937	0.01084780	0.00776810	0.00520373
1.0	0.0165466	0.01408330	0.01097610	0.00798096

Table 10. Analysis of various values of fractional order of ATIM of Problem 2 of $\beta_2(\psi, \varphi)$.

φ	$p = 0.40$	$p = 0.60$	$p = 0.80$	$p = 1.00$
0.2	0.121748	0.054481	0.0223765	0.0086342
0.4	0.211336	0.124308	0.0668425	0.0334890
0.6	0.291899	0.201634	0.1271780	0.0745776
0.8	0.367118	0.284297	0.2009240	0.1319000
1.0	0.438600	0.371182	0.2866020	0.2054560

To facilitate a comprehensive comparison, Tables 11 and 12 juxtapose the results obtained from both ARPSM and ATIM for $\beta_1(\psi, \varphi)$ and $\beta_2(\psi, \varphi)$ in Problem 2. These tables provide a nuanced understanding of the performance of each method, shedding light on their strengths and limitations in solving the given fractional-order equations in the context of Problem 2.

Table 11. Problem 2 comparison of both methods for $\beta_1(\psi, \varphi)$.

	$p = 1.00$		$p = 0.80$		$p = 0.60$		$p = 0.40$	
φ	ARPSM	ATIM	ARPSM	ATIM	ARPSM	ATIM	ARPSM	ATIM
0.2	0.0004303	0.0004258	0.0009841	0.0009722	0.0022338	0.0022043	0.004802	0.0047363
0.4	0.0014440	0.0014261	0.0027334	0.0026972	0.0049226	0.0048548	0.008209	0.0082096
0.6	0.0030591	0.0030187	0.0050604	0.0049910	0.0078698	0.0077592	0.011257	0.0110971
0.8	0.0052755	0.0052037	0.0078781	0.0077681	0.0110042	0.0108478	0.014096	0.0138937
1.0	0.0080933	0.0079809	0.0111336	0.0109761	0.0142880	0.0140833	0.016789	0.0165466

Table 12. Problem 2 comparison of both methods for $\beta_2(\psi, \varphi)$.

	$p = 1.00$		$p = 0.80$		$p = 0.60$		$p = 0.40$	
φ	ARPSM	ATIM	ARPSM	ATIM	ARPSM	ATIM	ARPSM	ATIM
0.2	0.0086336	0.0086342	0.022375	0.022376	0.054477	0.054481	0.121739	0.121748
0.4	0.0334867	0.0334890	0.066837	0.066842	0.124300	0.124308	0.211321	0.211336
0.6	0.0745724	0.0745776	0.127169	0.127178	0.201620	0.201634	0.291879	0.291899
0.8	0.1318910	0.1319000	0.200911	0.200924	0.284277	0.284297	0.367092	0.367118
1.0	0.2054420	0.2054560	0.286582	0.286602	0.371156	0.371182	0.438569	0.438600

7. Conclusions

In summary, this research has focused on improving numerical methods designed for solving the fractional Keller-Segel (KS) model, which is a crucial framework for studying chemotaxis phenomena. By utilizing the Caputo operator framework, we have employed two distinct methodologies: the Aboodh residual power series method (ARPSM) and the Aboodh transform iteration method (ATIM). These approaches have enabled us to obtain accurate solutions to the fractional KS equation, contributing to a better understanding of chemotactic behavior in biological systems. Through a comparative analysis of ARPSM and ATIM, we have revealed their individual strengths and applications in addressing complex fractional models. This work not only advances numerical techniques tailored for fractional differential equations but also improves our understanding of chemotaxis dynamics through precise modeling.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The researcher would like to thank the Deanship of Scientific Research, Shaqra University, for funding the publication of this project.

Conflict of interest

The authors declare that they have no competing interests.

References

1. A. Carpinteri, F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, Vienna: Springer, https://doi.org/10.1007/978-3-7091-2664-6
2. L. Debnath, Recent applications of fractional calculus to science and engineering, *Int. J. Math. Math. Sci.*, **2003** (2003), 753601. https://doi.org/10.1155/S0161171203301486
3. J. T. Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus, *Commun. Nonlinear Sci. Numer. Simul.*, **16** (2011), 1140–1153. https://doi.org/10.1016/j.cnsns.2010.05.027

4. A. A. M. Arafa, S. Z. Rida, H. Mohamed, Approximate analytical solutions of Schnakenberg systems by homotopy analysis method, *Appl. Math. Model.*, **36** (2012), 4789–4796. https://doi.org/10.1016/j.apm.2011.12.014

5. P. Sunthrayuth, A. M. Zidan, S. W. Yao, R. Shah, M. Inc, The comparative study for solving fractional-order Fornberg Whitham equation via ρ-Laplace transform, *Symmetry*, **13** (2021), 784. https://doi.org/10.3390/sym13050784

6. R. Shah, H. Khan, D. Baleanu, Fractional Whitham Broer Kaup equations within modified analytical approaches, *Axioms*, **8** (2019), 125. https://doi.org/10.3390/axioms8040125

7. H. M. Srivastava, R. Shah, H. Khan, M. Arif, Some analytical and numerical investigation of a family of fractional-order Helmholtz equations in two space dimensions, *Math. Methods Appl. Sci.*, **43** (2020), 199–212. https://doi.org/10.1002/mma.5846

8. H. Yasmin, N. H. Aljahdaly, A. M. Saeed, Investigating symmetric soliton solutions for the fractional coupled konno onno system using improved versions of a novel analytical technique, *Mathematics*, **11** (2023), 2686. https://doi.org/10.3390/math11122686

9. M. M. Al-Sawalha, R. Shah, A. Khan, O. Y. Ababneh, T. Botmart, Fractional view analysis of Kersten-Krasil'shchik coupled KdV-mKdV systems with non-singular kernel derivatives, *AIMS Mathematics*, **7** (2022), 18334-18359. https://doi.org/10.3934/math.20221010

10. A. A. Alderremy, R. Shah, N. Iqbal, S. Aly, K. Nonlaopon, Fractional series solution construction for nonlinear fractional reaction-diffusion Brusselator model utilizing Laplace residual power series, *Symmetry*, **14** (2022), 1944. https://doi.org/10.3390/sym14091944

11. S. Alshammari, M. M. Al-Sawalha, R. Shah, Approximate analytical methods for a fractional-order nonlinear system of Jaulent Miodek equation with energy-dependent Schrodinger potential, *Fractal Fract.*, **7** (2023), 140. https://doi.org/10.3390/fractalfract7020140

12. E. M. Elsayed, R. Shah, K. Nonlaopon, The analysis of the fractional-order Navier-Stokes equations by a novel approach, *J. Funct. Spaces*, **2022** (2022), 8979447. https://doi.org/10.1155/2022/8979447

13. M. Alqhtani, K. M. Saad, W. Weera, W. M. Hamanah, Analysis of the fractional-order local Poisson equation in fractal porous media, *Symmetry*, **14** (2022), 1323. https://doi.org/10.3390/sym14071323

14. H. Yasmin, A. S. Alshehry, A. H. Ganie, A. M. Mahnashi, Perturbed Gerdjikov Ivanov equation: Soliton solutions via Backlund transformation, *Optik*, **298** (2024), 171576. http://dx.doi.org/10.1016/j.ijleo.2023.171576

15. N. A. Pirim, F. Ayaz, A new technique for solving fractional order systems: Hermite collocation method, *Appl. Math.*, **7** (2016), 2307–2323. http://dx.doi.org/10.4236/am.2016.718182

16. V. Marinca, N. Herisanu, Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer, *Int. Commun. Heat Mass Transfer*, **35** (2008), 710–715. https://doi.org/10.1016/j.icheatmasstransfer.2008.02.010

17. J. S. Duan, R. Rach, D. Baleanu, A. M. Wazwaz, A review of the Adomian decomposition method and its applications to fractional differential equations, *Commun. Frac. Calc.*, **3** (2012), 73–99.

18. M. Khan, M. A. Gondal, I. Hussain, S. K. Vanani, A new comparative study between homotopy analysis transform method and homotopy perturbation transform method on a semi infinite domain, *Math. Comput. Model.*, **55** (2012), 1143–1150. https://doi.org/10.1016/j.mcm.2011.09.038

19. A. Jabbari, H. Kheiri, A. Yildirim, Homotopy analysis and homotopy Pade methods for (1+1) and (2+1) dimensional dispersive long wave equations, *Internat. J. Numer. Methods Heat Fluid Flow*, **23** (2013), 692–706. http://dx.doi.org/10.1108/09615531311323818

20. R. K. Gazizov, A. A. Kasatkin, Construction of exact solutions for fractional order differential equations by the invariant subspace method, *Comput. Math. Appl.*, **66** (2013), 576–584. https://doi.org/10.1016/j.camwa.2013.05.006

21. A. Prakash, P. Veeresha, D. G. Prakasha, M. Goyal, A new efficient technique for solving fractional coupled Navier-Stokes equations using q-homotopy analysis transform method, *Pramana-J. Phys.*, **93** (2018), 6. https://doi.org/10.1007/s12043-019-1763-x

22. R. K. Pandey, H. K. Mishra, Homotopy analysis Sumudu transform method for time-fractional third order dispersive partial differential equation, *Adv. Comput. Math.*, **43** (2017), 365–383. https://doi.org/10.1007/s10444-016-9489-5

23. Z. H. Guo, O. Acan, S. Kumar, Sumudu transform series expansion method for solving the local fractional Laplace equation in fractal thermal problems, *Thermal Sci.*, **20** (2016), 739–742. http://dx.doi.org/10.2298/TSCI16S3739G

24. K. K. Ali, M. Maneea, M. S. Mohamed, Solving nonlinear fractional models in superconductivity using the q-Homotopy analysis transform method, *J. Math.*, **2023** (2023), 6647375. https://doi.org/10.1155/2023/6647375

25. Z. Y. Fan, K. K. Ali, M. Maneea, M. Inc, S. W. Yao, Solution of time fractional Fitzhugh-Nagumo equation using semi analytical techniques, *Results Phys.*, **51** (2023), 106679. https://doi.org/10.1016/j.rinp.2023.106679

26. K. K. Ali, F. E. A. Elbary, M. Maneea, Efficient techniques for nonlinear dynamics: A study of fractional generalized quintic Ginzburg-Landau equation, *J. Taibah Univ. Sci.*, **18** (2024), 2333593. https://doi.org/10.1080/16583655.2024.2333593

27. M. A. El-Tawil, S. N. Huseen, The q-homotopy analysis method (q-HAM), *Int. J. Appl. Math. Mech.*, **8** (2012), 51–75.

28. M. A. El-Tawil, S. N. Huseen, On convergence of the q-homotopy analysis method, *Int. J. Contemp. Math. Sci.*, **8** (2013), 481–497.

29. Z. J. Liu, M. Y. Adamu, E. Suleiman, J. H. He, Hybridization of homotopy perturbation method and Laplace transformation for the partial differential equations, *Thermal Sci.*, **21** (2017), 1843–1846. http://dx.doi.org/10.2298/TSCI160715078L

30. A. Prakash, H. Kaur, q-homotopy analysis transform method for space and time-fractional KdV-Burgers equation, *Nonlinear Sci. Lett. A*, **9** (2018), 44–61.

31. E. F. Keller, L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, **26** (1970), 399–415. https://doi.org/10.1016/0022-5193(70)90092-5

32. A. Atangana, Extension of the Sumudu homotopy perturbation method to an attractor for one-dimensional Keller-Segel equations, *Appl. Math. Model.*, **39** (2015), 2909–2916. https://doi.org/10.1016/j.apm.2014.09.029

33. A. Atangana, B. S. T. Alkahtani, Analysis of the Keller-Segel model with a fractional derivative without singular kernel, *Entropy*, **17** (2015), 4439–4453. https://doi.org/10.3390/e17064439

34. A. Atangana, E. Alabaraoye, Solving a system of fractional partial differential equations arising in the model of HIV infection of CD4+ cells and attractor one-dimensional Keller-Segel equations, *Adv. Differ. Equ.*, **2013** (2013), 94. https://doi.org/10.1186/1687-1847-2013-94

35. M. Zayernouri, A. Matzavinos, Fractional Adams-Bashforth/Moulton methods: An application to the fractional Keller-Segel chemotaxis system, *J. Comput. Phys.*, **317** (2016), 1–14. https://doi.org/10.1016/j.jcp.2016.04.041

36. S. Kumar, A. Kumar, I. K. Argyros, A new analysis for the Keller-Segel model of fractional order, *Numer. Algorithms*, **75** (2017), 213–228. https://doi.org/10.1007/s11075-016-0202-z

37. M. A. Dokuyucu, D. Baleanu, E. Çelik, Analysis of Keller-Segel model with Atangana-Baleanu fractional derivative, *Filomat*, **32** (2018), 5633–5643. http://dx.doi.org/10.2298/FIL1816633D

38. X. Luo, M. Nadeem, M. Inc, S. Dawood, Fractional complex transform and homotopy perturbation method for the approximate solution of Keller-Segel model, *J. Funct. Spaces*, **2022** (2022), 9637098. https://doi.org/10.1155/2022/9637098

39. O. A. Arqub, Series solution of fuzzy differential equations under strongly generalized differentiability, *J. Adv. Res. Appl. Math.*, **5** (2013), 31–52. http://dx.doi.org/10.5373/jaram.1447.051912

40. O. A. Arqub, Z. Abo-Hammour, R. Al-Badarneh, S. Momani, A reliable analytical method for solving higher-order initial value problems, *Discrete Dyn. Nat. Soc.*, **2013** (2013), 673829. http://dx.doi.org/10.1155/2013/673829

41. O. A. Arqub, A. El-Ajou, Z. A. Zhour, S. Momani, Multiple solutions of nonlinear boundary value problems of fractional order: A new analytic iterative technique, *Entropy*, **16** (2014), 471–493. https://doi.org/10.3390/e16010471

42. A. El-Ajou, O. A. Arqub, S. Momani, Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: A new iterative algorithm, *J. Comput. Phys.*, **293** (2015), 81–95. https://doi.org/10.1016/j.jcp.2014.08.004

43. S. Rida, A. Arafa, A. Abedl-Rady, H. Abdl-Rahaim, Fractional physical differential equations via natural transform, *Chinese J. Phys.*, **55** (2017), 1569–1575. https://doi.org/10.1016/j.cjph.2017.05.004

44. J. Zhang, Z. Wei, L. Li, C. Zhou, Least-squares residual power series method for the time-fractional differential equations, *Complexity*, **2019** (2019), 6159024. https://doi.org/10.1155/2019/6159024

45. Y. Xie, I. Ahmad, T. I. S. Ikpe, E. F. Sofia, H. Seno, What influence could the acceptance of visitors cause on the epidemic dynamics of a Reinfectious disease?: A mathematical model, *Acta Biotheor.*, **72** (2024), 3. https://doi.org/10.1007/s10441-024-09478-w

46. I. Jaradat, M. Alquran, K. Al-Khaled, An analytical study of physical models with inherited temporal and spatial memory, *Eur. Phys. J. Plus*, **133** (2018), 162. https://doi.org/10.1140/epjp/i2018-12007-1
47. M. Alquran, K. Al-Khaled, S. Sivasundaram, H. M. Jaradat, Mathematical and numerical study of existence of bifurcations of the generalized fractional Burgers-Huxley equation, *Nonlinear Stud.*, **24** (2017), 235–244.
48. I. Ahmad, H. Seno, An epidemic dynamics model with limited isolation capacity, *Theory Biosci.*, **142** (2023), 259–273. https://doi.org/10.1007/s12064-023-00399-9
49. G. O. Ojo, N. I. Mahmudov, Aboodh transform iterative method for spatial diffusion of a biological population with fractional-order, *Mathematics*, **9** (2021), 155. https://doi.org/10.3390/math9020155
50. M. A. Awuya, G. O. Ojo, N. I. Mahmudov, Solution of space-time fractional differential equations using Aboodh transform iterative method, *J. Math.*, **2022** (2022), 4861588. https://doi.org/10.1155/2022/4861588
51. M. A. Awuya, D. Subasi, Aboodh transform iterative method for solving fractional partial differential equation with Mittag-Leffler Kernel, *Symmetry*, **13** (2021), 2055. https://doi.org/10.3390/sym13112055
52. M. I. Liaqat, S. Etemad, S. Rezapour, C. Park, A novel analytical Aboodh residual power series method for solving linear and nonlinear time-fractional partial differential equations with variable coefficients, *AIMS Mathematics*, **7** (2022), 16917–16948. http://dx.doi.org/10.3934/math.2022929
53. M. I. Liaqat, A. Akgul, H. Abu-Zinadah, Analytical investigation of some time-fractional Black-Scholes models by the Aboodh residual power series method, *Mathematics*, **11** (2023), 276. https://doi.org/10.3390/math11020276
54. K. S. Aboodh, The new integral transform'Aboodh transform, *Glob. J. Pure Appl. Math.*, **9** (2013), 35–43.
55. S. Aggarwal, R. Chauhan, A comparative study of Mohand and Aboodh transforms, *Int. J. Res. Adv. Technol.*, **7** (2019), 520–529.
56. M. E. Benattia, K. Belghaba, Application of the Aboodh transform for solving fractional delay differential equations, *Univ. J. Math. Appl.*, **3** (2020), 93–101. https://doi.org/10.32323/ujma.702033
57. B. B. Delgado, J. E. Macias-Diaz, On the general solutions of some non-homogeneous Div-curl systems with Riemann-Liouville and Caputo fractional derivatives, *Fractal Fract.*, **5** (2021), 117. https://doi.org/10.3390/fractalfract5030117
58. S. Alshammari, M. Al-Smadi, I. Hashim, M. A. Alias, Residual power series technique for simulating fractional Bagley-Torvik problems emerging in applied physics, *Appl. Sci.*, **9** (2019), 5029. https://doi.org/10.3390/app9235029

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