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Research article

Exponential stability of periodic solution for stochastic neural networks involving multiple time-varying delays

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Abstract: This paper discusses the exponential stability of periodic solutions for stochastic neural networks with multiple time-varying delays. For these networks, sufficient conditions in the linear matrix inequality forms are rare in the literature. We constructed an appropriate Lyapunov-Krasovskii functional to eliminate the items with multiple delays and establish some sufficient conditions in linear matrix inequality forms, to ensure exponential stability of the periodic solutions. Several examples are provided to demonstrate that our results are effective and less conservative than previous ones.

Keywords: stochastic neural network; multiple time-varying delays; periodic solution; exponential stability
Mathematics Subject Classification: 32D40

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1. Introduction

In the past two decades, neural networks have received considerable attention, having been successfully applied in signal processing, parallel computing, and combinational optimization. In the implementation and application of neural networks, time delay is unavoidable, which may bring about instability, oscillations, bifurcation, and chaos for systems. In the application of neural networks with delays, it is often required that the networks have a unique and stable equilibrium point. Therefore, the stability analysis of equilibrium points for neural networks with delays is widely relevant [1–9].

Besides the stability analysis of equilibrium points, the research on neural networks with delays involves the stability analysis of periodic solutions. The properties of periodic solutions are of great interest, and these have been successfully applied in many biological and cognitive fields. For example, periodic solutions can be used to store acoustic characteristics in speech recognition, because they can contain information for a time series. In practice, a time series of audio signal is fed as input into the network if the weight of the network meet established criteria; then, the output is the periodic solution of the time series. We calculate the average power spectral density (PSD) of the periodic solution and compare it with the PSD of the original audio signal. If the difference between them is small, then the periodic solution can be considered as effectively storing the acoustic characteristics. In addition, an equilibrium point can be considered as a special periodic solution of neural networks with an arbitrary period. Therefore, the stability analysis of periodic solutions for neural networks is more general and interesting than that of equilibrium points. Recently, there have been a lot of work on the stabilization of periodic solutions; see [10–18].

As pointed out in [19], the synaptic transmission in real nervous systems is a noisy process brought on by random fluctuations in the release of neurotransmitters, and other probabilistic causes. Neural networks could be stabilized or destabilized by some stochastic inputs. Therefore, it is of prime importance to consider the stability of stochastic neural networks with delays [20–26].

The system discussed in this paper cannot be transformed into the vector-matrix form because of multiple delays $\tau_{ij}(t)$. For neural networks with delays $\tau_{ij}(t)$, the common methods in the literature include fixed point principles, differential inequalities, Lyapunov functional, and Halanay inequality. These methods do not include linear matrix inequalities. Therefore, to the best of our knowledge, for stochastic neural networks with delays $\tau_{ij}(t)$, there is a lack of sufficient conditions in linear matrix inequality forms.

The innovations of this paper are listed as follows:

(1) By constructing an appropriate Lyapunov-Krasovskii functional and using linear matrix inequality, some sufficient conditions in the linear matrix inequality forms for the exponential stability of periodic solutions of stochastic neural networks with delays $\tau_{ij}(t)$ are established.

(2) It is confirmed that the Lyapunov-Krasovskii functional and linear matrix inequality can be applied to stochastic neural networks with delays $\tau_{ij}(t)$ that cannot be written in the vector-matrix form.

(3) Compared with the sufficient criteria established in [22,27,28], our sufficient criteria are delay-dependent and less conservative.

The rest of the paper is organized as follows. In Section 2, the stochastic neural networks with multiple time-varying delays and assumptions are introduced. In Sections 3, some sufficient conditions in the linear matrix inequality forms are given to ensure exponential stability of periodic solutions. In Section 4, several examples are given to demonstrate that our criteria are effective and less conservative. The conclusions are drawn in Section 5.

Notation:

(1) Let $\mathbf{w}(t) = (w_1(t), \cdots, w_n(t))^T$ be n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by $\{\mathbf{w}(t)\}$, in which Ω is the canonical space generated by $w(t)$ and $\mathcal{F}$ is the associated σ-algebra generated by $\{\mathbf{w}(s) : 0 \leq s \leq t\}$ with the probability measure $\mathbb{P}$.

(2) $\mathbf{A} < 0$ means that matrix $\mathbf{A}$ is symmetric negative definite; $\mathbf{A}^T$ denotes the transpose of the matrix $\mathbf{A}$; $*$ means the symmetric terms of a symmetric matrix, and $\mathbf{I}$ denotes identity matrix.

(3) $C([-\tau, 0]; \mathbf{R}^n)$ denotes the Banach space of all continuous $\mathbf{R}^n$-valued functions defined on $[-\tau, 0]$ satisfying $\sup_{-\tau\leq t\leq 0} \mathbb{E}\|\xi(t)\|^2 < \infty$; $\mathbb{E}$ denotes the mathematical expectation; $\|\cdot\|$ denotes the Euclidean norm.

2. Preliminaries

The stochastic neural networks with multiple time-varying delays studied in this paper can be described as the following mathematical expression

$$\begin{aligned} dx_i(t) &= [-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij}(t))) + u_i(t)]dt \\ &\quad + \sum_{j=1}^{n} \sigma_{ij}(x_j(t), x_j(t-\tau_{ij}(t)))dw_j(t), i = 1, \cdots, n, t \geq 0, \end{aligned} \tag{2.1}$$

where a_{ij}, b_{ij}, and c_i are some constants and $c_i > 0$, $f_i(\cdot)$ and $g_i(\cdot)$ are activation functions, $\sigma_{ij}(\cdot,\cdot)$ are diffusion functions. Obviously, the mathematical expression of system (2.1) cannot be transformed into a vector-matrix system.

The following assumption is added in order to discuss the stability of periodic solutions of the system (2.1):

There exist some constants $F_i^-, F_i^+, G_i^-, G_i^+, L_{ij} > 0, M_{ij} > 0, \tau$ and $\tilde{\tau}$ such that for all $z_1, z_2, z_3, z_4 \in R$, $F_i^- \leq F_i^+, G_i^- \leq G_i^+$ and $t \geq 0$,

$$F_i^- \leq \frac{f_i(z_1) - f_i(z_2)}{z_1 - z_2} \leq F_i^+, G_i^- \leq \frac{g_i(z_1) - g_i(z_2)}{z_1 - z_2} \leq G_i^+, z_1 \neq z_2, \tag{2.2}$$

$$0 \leq \tau_{ij}(t) \leq \tau, \dot{\tau}_{ij}(t) \leq \tilde{\tau} < 1, \tag{2.3}$$

$$|\sigma_{ij}(z_1, z_2) - \sigma_{ij}(z_3, z_4)| \leq L_{ij}|z_1 - z_3| + M_{ij}|z_2 - z_4|. \tag{2.4}$$

The initial condition $x_i\ (s) = \xi_i\ (s), s \in [-\tau, 0]$, and $\xi = \{(\xi_1(s), \cdots, \xi_1(s))^T : -\tau \leq s \leq 0\}$ is $C([-\tau, 0]; \mathbf{R}^n)$-valued function and $\mathcal{F}_0$-measurable $\mathbf{R}^n$-valued random variable.

Let $\mathbf{x}(t, \xi)$ and $\mathbf{y}(t, \psi)$ be the solutions of system (2.1) with arbitrary initial conditions ξ and ψ, respectively. If it will not cause any misunderstanding, $\mathbf{x}(t, \xi)$ and $\mathbf{y}(t, \psi)$ are denoted by $\mathbf{x}(t)$ and $\mathbf{y}(t)$, respectively. Then, for $i = 1, \cdots, n, t \geq 0$, we have

$$\begin{aligned} d[x_i(t) - y_i(t)] &= \Big(- c_i[x_i(t) - y_i(t)] + \sum_{j=1}^{n} a_{ij}[f_j(x_j(t)) - f_j(y_j(t))] \\ &\quad + \sum_{j=1}^{n} b_{ij}[g_j(x_j(t-\tau_{ij}(t))) - g_j(y_j(t-\tau_{ij}(t)))]\Big)dt \\ &\quad + \sum_{j=1}^{n}[\sigma_{ij}(x_j(t), x_j(t-\tau_{ij}(t))) - \sigma_{ij}(y_j(t), y_j(t-\tau_{ij}(t)))]dw_j(t). \end{aligned} \tag{2.5}$$

3. Main results

Theorem 1. There exist two positive constants λ and $K > 1$ such that

$$\mathbb{E}\|\mathbf{x}(t) - \mathbf{y}(t)\|^2 \leq Ke^{-\lambda t}\mathbb{E}\|\xi - \psi\|^2, t \geq 0, \tag{3.1}$$

if there exist some positive constants $p_1, \cdots, p_n, u_{i1}, \cdots, u_{in}$ $(i = 1, 2)$ such that

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_1 & \mathbf{PA} + \mathbf{U}_1\bar{\mathbf{F}} & \mathbf{U}_2\bar{\mathbf{G}} \\ * & -2\mathbf{U}_1 & 0 \\ * & * & -2\mathbf{U}_2 + \frac{\mathbf{B}_2}{1-\tilde{\tau}} \end{pmatrix} < 0, \tag{3.2}$$

where

$$\mathbf{\Sigma}_1 = -2\mathbf{PC} + \mathbf{PB}_1 + \mathbf{L} + \frac{1}{1-\tilde{\tau}}\mathbf{M} - 2\mathbf{U}_1\tilde{\mathbf{F}} - 2\mathbf{U}_2\tilde{\mathbf{G}}, \mathbf{P} = diag\{p_1, \cdots, p_n\},$$
$$\mathbf{A} = (a_{ij})_{n\times n}, \mathbf{C} = diag\{c_1, \cdots, c_n\}, \mathbf{U}_i = diag\{u_{i1}, \cdots, u_{in}\}(i = 1, 2),$$
$$\mathbf{B}_1 = diag\{\sum_{j=1}^{n}|b_{1j}|, \cdots, \sum_{j=1}^{n}|b_{nj}|\}, \mathbf{B}_2 = diag\{\sum_{j=1}^{n}p_j|b_{j1}|, \cdots, \sum_{j=1}^{n}p_j|b_{jn}|\},$$
$$\mathbf{L} = diag\{\sum_{j=1}^{n}2p_jL_{j1}^2, \cdots, \sum_{j=1}^{n}2p_jL_{jn}^2\}, \mathbf{M} = 2diag\{\sum_{j=1}^{n}2p_jM_{j1}^2, \cdots, \sum_{j=1}^{n}2p_jM_{jn}^2\},$$
$$\tilde{\mathbf{F}} = diag\{F_1^-F_1^+, \cdots, F_n^-F_n^+\}, \bar{\mathbf{F}} = diag\{F_1^- + F_1^+, \cdots, F_n^- + F_n^+\},$$
$$\tilde{\mathbf{G}} = diag\{G_1^-G_1^+, \cdots, G_n^-G_n^+\}, \bar{\mathbf{G}} = diag\{G_1^- + G_1^+, \cdots, G_n^- + G_n^+\}.$$

Proof. It follows from (3.2) that there exists a constant $\lambda > 0$ such that

$$\bar{\mathbf{\Sigma}} = \begin{pmatrix} \bar{\mathbf{\Sigma}}_1 & \mathbf{PA} + \mathbf{U}_1\bar{\mathbf{F}} & \mathbf{U}_2\bar{\mathbf{G}} \\ * & -2\mathbf{U}_1 & 0 \\ * & * & -2\mathbf{U}_2 + \frac{e^{\lambda\tau}}{1-\tilde{\tau}}\mathbf{B}_2 \end{pmatrix} < 0, \tag{3.3}$$

where

$$\bar{\mathbf{\Sigma}}_1 = \lambda\mathbf{P} - 2\mathbf{PC} + \mathbf{PB}_1 + \mathbf{L} + \frac{e^{\lambda\tau}}{1-\tilde{\tau}}\mathbf{M} - 2\mathbf{U}_1\tilde{\mathbf{F}} - 2\mathbf{U}_2\tilde{\mathbf{G}}.$$

The Lyapunov-Krasovskii functional is defined as follows:

$$V(t) = V_1(t) + V_2(t), \tag{3.4}$$

where

$$V_1(t) = e^{\lambda t}\sum_{i=1}^{n}p_i[x_i(t) - y_i(t)]^2,$$

$$V_2(t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}\frac{e^{\lambda(s+\tau)}p_i}{1-\tilde{\tau}}\Big(|b_{ij}|[g_j(x_j(s)) - g_j(y_j(s))]^2 + 2M_{ij}^2[x_j(s) - y_j(s)]^2\Big)ds.$$

Applying Itô formula in [20] to $V(t)$ along with system (2.5), we obtain

$$\begin{aligned}
dV_1(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} W_{ij}(t)dw_j(t) + \lambda e^{\lambda t}\sum_{i=1}^{n} p_i[x_i(t)-y_i(t)]^2 dt \\
&\quad +2e^{\lambda t}\sum_{i=1}^{n} p_i[x_i(t)-y_i(t)]\Big(- c_i[x_i(t)-y_i(t)] + \sum_{j=1}^{n} a_{ij}[f_j(x_j(t)) - f_j(y_j(t))] \\
&\quad + \sum_{j=1}^{n} b_{ij}[g_j(x_j(t-\tau_{ij}(t))) - g_j(y_j(t-\tau_{ij}(t)))] \\
&\quad +e^{\lambda t}\sum_{i=1}^{n} p_i \sum_{j=1}^{n}[\sigma_{ij}(x_j(t), x_j(t-\tau_{ij}(t))) - \sigma_{ij}(y_j(t), y_j(t-\tau_{ij}(t)))]^2\Big)dt \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} W_{ij}(t)dw_j(t) + e^{\lambda t}\sum_{i=1}^{n}\Big(p_i[\lambda - 2c_i][x_i(t)-y_i(t)]^2 \\
&\quad + \sum_{j=1}^{n} 2a_{ij}p_i[x_i(t)-y_i(t)][f_j(x_j(t)) - f_j(y_j(t))] + \sum_{j=1}^{n}|b_{ij}|p_i[x_i(t)-y_i(t)]^2 \\
&\quad + \sum_{j=1}^{n}|b_{ij}|p_i[g_j(x_j(t-\tau_{ij}(t))) - g_j(y_j(t-\tau_{ij}(t)))]^2 \\
&\quad +e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} 2p_i L_{ij}^2 (x_j(t)-y_j(t))^2 \\
&\quad +e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} 2p_i M_{ij}^2 (x_j(t-\tau_{ij}(t)) - y_j(t-\tau_{ij}(t)))^2\Big)dt, \qquad (3.5)
\end{aligned}$$

where

$$W_{ij}(t) = 2e^{\lambda t}p_i[x_i(t)-y_i(t)][\sigma_{ij}(x_j(t), x_j(t-\tau_{ij}(t))) - \sigma_{ij}(y_j(t), y_j(t-\tau_{ij}(t)))],$$

and

$$\begin{aligned}
dV_2(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e^{\lambda(t+\tau)}p_i}{1-\tilde{\tau}}\Big(|b_{ij}|[g_j(x_j(t)) - g_j(y_j(t))]^2 + 2M_{ij}^2[x_j(t)-y_j(t)]^2\Big)dt \\
&\quad -(1-\dot{\tau}_{ij}(t))\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e^{\lambda(t-\tau_{ij}(t))+\tau)}p_i}{1-\tilde{\tau}}\Big(|b_{ij}|[g_j(x_j(t-\tau_{ij}(t)))) - g_j(y_j(t-\tau_{ij}(t))))]^2 \\
&\quad +2M_{ij}^2[x_j(t-\tau_{ij}(t))) - y_j(t-\tau_{ij}(t)))]^2\Big)dt \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e^{\lambda(t+\tau)}p_i}{1-\tilde{\tau}}\Big(|b_{ij}|[g_j(x_j(t)) - g_j(y_j(t))]^2 + 2M_{ij}^2[x_j(t)-y_j(t)]^2\Big)dt \\
&\quad -\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\lambda t}p_i\Big(|b_{ij}|[g_j(x_j(t-\tau_{ij}(t)))) - g_j(y_j(t-\tau_{ij}(t))))]^2
\end{aligned}$$

$$+2M_{ij}^2[x_j(t-\tau_{ij}(t))) - y_j(t-\tau_{ij}(t)))]^2\Big)dt. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned}
dV(t) &= dV_1(t) + dV_2(t) \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n} W_{ij}(t)dw_j(t) + e^{\lambda t}\sum_{i=1}^{n}\Big([\lambda p_i - 2c_ip_i + \sum_{j=1}^{n}|b_{ij}|p_i \\
&\quad + \sum_{j=1}^{n} p_j(2L_{ji}^2 + e^{\lambda\tau}\frac{2M_{ji}^2}{1-\tilde{\tau}})][x_i(t)-y_i(t)]^2 \\
&\quad + \sum_{j=1}^{n} 2a_{ij}p_i[x_i(t)-y_i(t)][f_j(x_j(t)) - f_j(y_j(t))]\Big)dt \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} e^{\lambda(t+\tau)}\frac{|b_{ji}|p_j}{1-\tilde{\tau}}[g_i(x_i(t)) - g_i(y_i(t))]^2dt \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} W_{ij}(t)dw_j(t) + 2[\mathbf{x}(t)-\mathbf{y}(t)]^T\mathbf{PA}[\mathbf{f}(\mathbf{x}(t)) - \mathbf{f}(\mathbf{y}(t))]dt \\
&\quad + e^{\lambda t}\Big\{[\mathbf{x}(t)-\mathbf{y}(t)]^T\Big(\lambda\mathbf{P} - 2\mathbf{PC} + \mathbf{PB}_1 + \mathbf{L} + \frac{e^{\lambda\tau}}{1-\tilde{\tau}}\mathbf{M}\Big)[\mathbf{x}(t)-\mathbf{y}(t)]dt \\
&\quad + \frac{e^{\lambda\tau}}{1-\tilde{\tau}}[\mathbf{g}(\mathbf{x}(t)) - \mathbf{g}(\mathbf{y}(t))]^T\mathbf{B}_2[\mathbf{g}(\mathbf{x}(t)) - \mathbf{g}(\mathbf{y}(t))]dt\Big\},
\end{aligned} \tag{3.7}$$

where

$$\mathbf{x}(t) - \mathbf{y}(t) = (x_1(t) - y_1(t), \cdots, x_n(t) - y_n(t))^T,$$
$$\mathbf{f}(\mathbf{x}(t)) - \mathbf{f}(\mathbf{y}(t)) = (f_1(x_1(t)) - f_1(y_1(t)), \cdots, f_n(x_n(t)) - f_n(y_n(t)))^T,$$
$$\mathbf{g}(\mathbf{x}(t)) - \mathbf{g}(\mathbf{y}(t)) = (g_1(x_1(t)) - g_1(y_1(t)), \cdots, g_n(x_n(t)) - g_n(y_n(t)))^T.$$

Simultaneously, by using (2.2), we can deduce

$$\begin{aligned}
0 &\le -2\sum_{i=1}^{n} u_{1i}[(f_i(x_i(t)) - f_i(y_i(t))) - F_i^+(x_i(t)-y_i(t))][(f_i(x_i(t)) - f_i(y_i(t))) \\
&\quad - F_i^-(x_i(t)-y_i(t))] \\
&= -2\sum_{i=1}^{n} u_{1i}(f_i(x_i(t)) - f_i(y_i(t)))^2 - 2\sum_{i=1}^{n} u_{1i}F_i^+F_i^-(x_i(t)-y_i(t))^2 \\
&\quad + 2(F_i^+ + F_i^-)\sum_{i=1}^{n} u_{1i}(f_i(x_i(t)) - f_i(y_i(t)))(x_i(t)-y_i(t)) \\
&= -2[\mathbf{f}(\mathbf{x}(t)) - \mathbf{f}(\mathbf{y}(t))]^TU_1[\mathbf{f}(\mathbf{x}(t)) - \mathbf{f}(\mathbf{y}(t))] - 2[\mathbf{x}(t)-\mathbf{y}(t)]^T\mathbf{U}_1\tilde{\mathbf{F}}[\mathbf{x}(t)-\mathbf{y}(t)] \\
&\quad + 2[\mathbf{f}(\mathbf{x}(t)) - \mathbf{f}(\mathbf{y}(t))]^T\mathbf{U}_1\bar{\mathbf{F}}[\mathbf{x}(t)-\mathbf{y}(t)],
\end{aligned} \tag{3.8}$$

and

$$0 \le -2\sum_{i=1}^{n} u_{2i}[(g_i(x_i(t)) - g_i(y_i(t))) - G_i^+(x_i(t)-y_i(t))][(g_i(x_i(t)) - g_i(y_i(t)))$$

$$
\begin{aligned}
& -G_i^-(x_i(t)-y_i(t))] \\
= \;& -2[\mathbf{g}(\mathbf{x}(t))-\mathbf{g}(\mathbf{y}(t))]^T\mathbf{U}_2[\mathbf{g}(\mathbf{x}(t))-\mathbf{g}(\mathbf{y}(t))]-2[\mathbf{x}(t)-\mathbf{y}(t)]^T\mathbf{U}_2\tilde{\mathbf{G}}[\mathbf{x}(t)-\mathbf{y}(t)] \\
& +2[\mathbf{g}(\mathbf{x}(t))-\mathbf{g}(\mathbf{y}(t))]^T\mathbf{U}_2\bar{\mathbf{G}}[\mathbf{x}(t)-\mathbf{y}(t)]. \qquad (3.9)
\end{aligned}
$$

From (3.4) and (3.7)–(3.9), we deduce

$$
\begin{aligned}
& e^{\lambda t}\min_{1\le i\le n}\{p_i\}\|\mathbf{x}(t)-\mathbf{y}(t)\|^2 \le V(t) \\
\le\;& V(0)+\int_0^t e^{\lambda s}\mathbf{h}^T(s)\bar{\mathbf{\Sigma}}_1\mathbf{h}(s)ds+\int_0^t\sum_{i=1}^n\sum_{j=1}^n W_{ij}(s)dw_j(s) \\
\le\;& \max_{1\le i\le n}\{p_i\}\|\mathbf{x}(0)-\mathbf{y}(0)\|^2+\int_0^t\sum_{i=1}^n\sum_{j=1}^n W_{ij}(s)dw_j(s) \\
& +\sum_{i=1}^n\sum_{j=1}^n\int_{-\tau}^0\frac{e^{\lambda(s+\tau)}p_i}{1-\tilde{\tau}}\Big(|b_{ij}|[g_j(x_j(s))-g_j(y_j(s))]^2+2M_{ij}^2[x_j(s)-y_j(s)]^2\Big)ds \\
\le\;& \max_{1\le i\le n}\{p_i\}\|\mathbf{x}(0)-\mathbf{y}(0)\|^2+\int_0^t\sum_{i=1}^n\sum_{j=1}^n W_{ij}(s)dw_j(s) \\
& +\sum_{i=1}^n\sum_{j=1}^n\int_{-\tau}^0\frac{e^{\lambda(s+\tau)}p_i}{1-\tilde{\tau}}\Big(|b_{ij}|G_j^2+2M_{ij}^2\Big)[x_j(s)-y_j(s)]^2ds, \qquad (3.10)
\end{aligned}
$$

where $\mathbf{h}(t)=([\mathbf{x}(t)-\mathbf{y}(t)]^T,[\mathbf{f}(\mathbf{x}(t))-\mathbf{f}(\mathbf{y}(t))]^T,[\mathbf{g}(\mathbf{x}(t))-\mathbf{g}(\mathbf{y}(t))]^T)^T, G_i=\max\{|G_i^-|,|G_i^+|\}, i=1,2,\cdots,n$.

Taking mathematical expectations for both sides of (3.10), we obtain

$$
\begin{aligned}
& e^{\lambda t}\min_{1\le i\le n}\{p_i\}\mathbb{E}\|\mathbf{x}(t)-\mathbf{y}(t)\|^2 \le \mathbb{E}V(t) \qquad (3.11) \\
\le\;& \left(\max_{1\le i\le n}\{p_i\}+\frac{e^{\lambda\tau}\tau}{1-\tilde{\tau}}\max_{1\le i\le n}\{\sum_{j=1}^n p_j[|b_{ji}|G_i^2+2M_{ji}^2]\}\right)\sup_{-\tau\le t\le 0}\mathbb{E}\|\mathbf{x}(t)-\mathbf{y}(t)\|^2,
\end{aligned}
$$

which can deduce (3.1).

Remark 1. Obviously, the right-hand side of (3.10) is a non-negative semimartingale. From the non-negative semimartingale convergence theorem [20] and (3.10), we obtain

$$
\limsup_{t\to\infty} e^{\lambda t}\min_{1\le i\le n}\{p_i\}\|\mathbf{x}(t)-\mathbf{y}(t)\|^2<+\infty, \mathbb{P}-\textit{almost surely},
$$

which implies

$$
\limsup_{t\to\infty}\frac{\ln\|\mathbf{x}(t)-\mathbf{y}(t)\|^2}{t}<-\lambda, \mathbb{P}-\textit{almost surely}.
$$

Remark 2. Inequality (3.2) is the sufficient condition in the linear matrix inequality form. However, the matrices $\mathbf{P}$, $\mathbf{U}_1$, and $\mathbf{U}_2$ in (3.2) cannot be solved by MATLAB because the matrices $\mathbf{B}_2, \mathbf{L}$, and $\mathbf{M}$ contain the constants $p_1,\cdots,p_n$. In order to obtain the easily verified condition, we take $p_i=p$ ($i=1,\cdots,n$), then (3.2) transforms into (3.12).

Corollary 1. Inequality (3.1) holds if there exist some positive constants $p, u_{i1}, \cdots, u_{in}$ $(i = 1, 2)$ such that

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_1 & p\mathbf{A} + \mathbf{U}_1\bar{\mathbf{F}} & \mathbf{U}_2\bar{\mathbf{G}} \\ * & -2\mathbf{U}_1 & 0 \\ * & * & -2\mathbf{U}_2 + \frac{p}{1-\tilde{\tau}}\bar{\mathbf{B}}_2 \end{pmatrix} < 0, \tag{3.12}$$

where

$$\mathbf{\Sigma}_1 = -2p\mathbf{C} + p\mathbf{B}_1 + p\bar{\mathbf{L}} + \frac{p}{1-\tilde{\tau}}\bar{\mathbf{M}} - 2\mathbf{U}_1\tilde{\mathbf{F}} - 2\mathbf{U}_2\tilde{\mathbf{G}}, \bar{\mathbf{B}}_2 = diag\{\sum_{j=1}^{n} |b_{j1}|, \cdots, \sum_{j=1}^{n} |b_{jn}|\},$$

$$\bar{\mathbf{L}} = diag\{\sum_{j=1}^{n} 2L_{j1}^2, \cdots, \sum_{j=1}^{n} 2L_{jn}^2\}, \bar{\mathbf{M}} = diag\{\sum_{j=1}^{n} 2M_{j1}^2, \cdots, \sum_{j=1}^{n} 2M_{jn}^2\},$$

the other symbols are the same as Theorem 1.

Certainly, we can also slightly modify the proof of Theorem 1 so that these matrices $\mathbf{B}_2, \mathbf{L}$, and $\mathbf{M}$ do not contain the constants $p_1, \cdots, p_n$.

Theorem 2. Inequality (3.1) holds if there exist some positive constants $p_1, \cdots, p_n, q, u_{i1}, \cdots, u_{in}$ $(i = 1, 2)$ such that $q > \max_{1\le i\le n}\{p_i\}$ and

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_1 & \mathbf{PA} + \mathbf{U}_1\bar{\mathbf{F}} & \mathbf{U}_2\bar{\mathbf{G}} \\ * & -2\mathbf{U}_1 & 0 \\ * & * & -2\mathbf{U}_2 + \frac{1}{1-\tilde{\tau}}\bar{\mathbf{B}}_2 \end{pmatrix} < 0, \tag{3.13}$$

where

$$\mathbf{\Sigma}_1 = -2\mathbf{PC} + \mathbf{P}^2\mathbf{B}_1 + q\bar{\mathbf{L}} + \frac{q}{1-\tilde{\tau}}\bar{\mathbf{M}} - 2\mathbf{U}_1\tilde{\mathbf{F}} - 2\mathbf{U}_2\tilde{\mathbf{G}},$$

the other symbols are the same as Theorem 1 and Corollary 1.

Proof. From (3.13), we can obtain a constant $\lambda > 0$ such that

$$\bar{\mathbf{\Sigma}} = \begin{pmatrix} \bar{\mathbf{\Sigma}}_1 & \mathbf{PA} + \mathbf{U}_1\bar{\mathbf{F}} & \mathbf{U}_2\bar{\mathbf{G}} \\ * & -2\mathbf{U}_1 & 0 \\ * & * & -2\mathbf{U}_2 + \frac{e^{\lambda\tau}}{1-\tilde{\tau}}\bar{\mathbf{B}}_2 \end{pmatrix} < 0, \tag{3.14}$$

where

$$\bar{\mathbf{\Sigma}}_1 = \lambda\mathbf{P} - 2\mathbf{PC} + \mathbf{P}^2\mathbf{B}_1 + q\bar{\mathbf{L}} + \frac{qe^{\lambda\tau}}{1-\tilde{\tau}}\bar{\mathbf{M}} - 2\mathbf{U}_1\tilde{\mathbf{F}} - 2\mathbf{U}_2\tilde{\mathbf{G}}.$$

The Lyapunov-Krasovskii functional is defined as follows:

$$\begin{aligned} V(t) &= e^{\lambda t}\sum_{i=1}^{n} p_i[x_i(t) - y_i(t)]^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t} \frac{e^{\lambda(s+\tau)}}{1-\tilde{\tau}} \\ &\quad \Big(|b_{ij}|[g_j(x_j(s)) - g_j(y_j(s))]^2 + 2qM_{ij}^2[x_j(s) - y_j(s)]^2\Big)ds. \end{aligned}$$

From (3.5) and $q > \max_{1\le i\le n}\{p_i\}$, we have

$$\begin{aligned}
dV(t) \le\ & \sum_{i=1}^{n}\sum_{j=1}^{n} W_{ij}(t)dw_j(t) + e^{\lambda t}\sum_{i=1}^{n}\Big(p_i[\lambda - 2c_i][x_i(t)-y_i(t)]^2 \\
& + \sum_{j=1}^{n} 2a_{ij}p_i[x_i(t)-y_i(t)][f_j(x_j(t)) - f_j(y_j(t))] + \sum_{j=1}^{n}|b_{ij}|p_i^2[x_i(t)-y_i(t)]^2 \\
& + \sum_{j=1}^{n}|b_{ij}|[g_j(x_j(t-\tau_{ij}(t))) - g_j(y_j(t-\tau_{ij}(t)))]^2 \\
& + e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} 2qL_{ij}^2(x_j(t)-y_j(t))^2 \\
& + e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} 2qM_{ij}^2(x_j(t-\tau_{ij}(t)) - y_j(t-\tau_{ij}(t)))^2\Big)dt \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e^{\lambda(t+\tau)}}{1-\tilde{\tau}}\Big(|b_{ij}|[g_j(x_j(t)) - g_j(y_j(t))]^2 + 2qM_{ij}^2[x_j(t)-y_j(t)]^2\Big)dt \\
& - \sum_{i=1}^{n}\sum_{j=1}^{n} e^{\lambda t}\Big(|b_{ij}|[g_j(x_j(t-\tau_{ij}(t)))) - g_j(y_j(t-\tau_{ij}(t))))]^2 \\
& + 2qM_{ij}^2[x_j(t-\tau_{ij}(t))) - y_j(t-\tau_{ij}(t)))]^2\Big)dt \\
\le\ & \sum_{i=1}^{n}\sum_{j=1}^{n} W_{ij}(t)dw_j(t) + e^{\lambda t}\sum_{i=1}^{n}\Big([\lambda p_i - 2c_ip_i + \sum_{j=1}^{n}|b_{ij}|p_i^2 + \sum_{j=1}^{n} q(2L_{ji}^2 + 2M_{ji}^2\frac{e^{\lambda\tau}}{1-\tilde{\tau}})] \\
& [x_i(t)-y_i(t)]^2 + \sum_{j=1}^{n} 2a_{ij}p_i[x_i(t)-y_i(t)][f_j(x_j(t)) - f_j(y_j(t))) \\
& + \sum_{j=1}^{n}\frac{e^{\lambda\tau}|b_{ji}|}{1-\tilde{\tau}}[g_i(x_i(t)) - g_i(y_i(t))]^2\Big)dt \\
=\ & \sum_{i=1}^{n}\sum_{j=1}^{n} W_{ij}(t)dw_j(t) + e^{\lambda t}\Big\{[\mathbf{x}(t)-\mathbf{y}(t)]^T\Big(\lambda\mathbf{P} - 2\mathbf{PC} + \mathbf{P}^2\mathbf{B}_1 + q\bar{\mathbf{L}} + \frac{qe^{\lambda\tau}}{1-\tilde{\tau}}\bar{\mathbf{M}}\Big) \\
& [\mathbf{x}(t)-\mathbf{y}(t)] + 2[\mathbf{x}(t)-\mathbf{y}(t)]^T\mathbf{PA}[\mathbf{f}(\mathbf{x}(t)) - \mathbf{f}(\mathbf{y}(t))] \\
& + \frac{e^{\lambda\tau}}{1-\tilde{\tau}}[\mathbf{g}(\mathbf{x}(t)) - \mathbf{g}(\mathbf{y}(t))]^T\bar{\mathbf{B}}_2[\mathbf{g}(\mathbf{x}(t)) - \mathbf{g}(\mathbf{y}(t))]\Big\}dt.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 1 and so is omitted.

Remark 3. When $p_i = p$ $(i = 1,\cdots,n)$, (3.3) becomes

$$\bar{\mathbf{\Sigma}} = \begin{pmatrix} \bar{\Sigma}_1 & p\mathbf{A} + \mathbf{U}_1\bar{\mathbf{F}} & \mathbf{U}_2\bar{\mathbf{G}} \\ * & -2\mathbf{U}_1 & 0 \\ * & * & -2\mathbf{U}_2 + \frac{e^{\lambda\tau}}{1-\tilde{\tau}}p\bar{\mathbf{B}}_2 \end{pmatrix} < 0, \tag{3.15}$$

where

$$\bar{\mathbf{\Sigma}}_1 = \lambda p\mathbf{I} - 2p\mathbf{C} + p\mathbf{B}_1 + p\bar{\mathbf{L}} + \frac{e^{\lambda\tau}}{1-\tilde{\tau}}p\bar{\mathbf{M}} - 2\mathbf{U}_1\tilde{\mathbf{F}} - 2\mathbf{U}_2\tilde{\mathbf{G}}.$$

It is easy to see that the stability criteria (3.12)–(3.15) are delay-dependent.

Remark 4. In [22,27,28], delay-independent stability criteria of periodic solutions are established by using the fixed-point principle, Gronwall-Bellman inequality, Lyapunov functional, Halanay inequality, and differential inequality technique (see Remarks 5–7). It is generally agreed that delay-dependent stability criteria are less conservative than delay-independent ones [7]. Therefore, our sufficient criteria are less conservative than those in [22,27,28].

Theorem 3. Let $u_i(t)$ and $\tau_{ij}(t)(i,j = 1,\cdots,n)$ be ω-periodic functions. If the condition (3.12) (or (3.13)) holds, then system (2.1) has a unique periodic solution, which is globally exponentially stable in the mean square.

Proof. If the condition (3.12) (or (3.13)) holds, then inequality (3.1) holds. We can choose a positive integer m such that

$$Ke^{-\lambda m\omega} \leq \frac{1}{16}. \tag{3.16}$$

We define a Poincare mapping P:$C([-\tau,0];\mathbf{R}^n) \to C([-\tau,0];\mathbf{R}^n)$ by $P\xi = \mathbf{x}_\omega(\xi) = \mathbf{x}(\omega+\theta,\xi), \theta \in [-\tau,0), t \geq 0$. Then, from (3.1) and (3.16), we have

$$\mathbb{E}\|\mathbf{x}(t)-\mathbf{y}(t)\|^2 = \mathbb{E}\|P^m\xi - P^m\psi\|^2 \leq \frac{1}{8}\mathbb{E}\|\xi-\psi\|^2.$$

By the integral property of measurable functions, we have

$$\|P^m\xi - P^m\psi\|^2 \leq \frac{1}{16}\|\xi-\psi\|^2, a.e.$$

that is,

$$\|P^m\xi - P^m\psi\| \leq \frac{1}{4}\|\xi-\psi\|, a.e.$$

So P^m is a contraction mapping and there exists a unique fixed point $\xi^* \in C([-\tau,0];\mathbf{R}^n)$ such that $P^m\xi^* = \xi^*, a.e.$ Note that

$$P^m(P\xi^*) = P(P^m\xi^*) = P\xi^*, a.e.$$

We know that $P\xi^* \in C([-\tau,0];\mathbf{R}^n)$ is also a fixed point of P^m and obtain

$$\mathbf{x}_\omega(\xi^*) = P\xi^* = \xi^*, a.e.$$

Let $\mathbf{x}(t,\xi^*)$ be the solution of (2.1) through $(0,\xi^*)$. Then, $\mathbf{x}(t+\omega,\xi^*)$ is also a solution of (2.1) and for $t \geq 0$,

$$\mathbf{x}_{t+\omega}(\xi^*) = \mathbf{x}_t(x_\omega(\xi^*)) = \mathbf{x}_t(\xi^*), a.e.$$

So we have $\mathbf{x}(t+\omega,\xi^*) = \mathbf{x}(t,\xi^*), a.e.$ which shows $\mathbf{x}(t,\xi^*)$ is one ω-periodic solution of (2.1). From (3.1), we know all other solutions converge exponentially to $\mathbf{x}(t,\xi^*)$ in the mean square as $t \to \infty$.

4. Several examples and comparison

Example 1. Consider system (2.1) with the following parameters and functions:

$$\mathbf{A}=\begin{pmatrix}1&-1&-1&1\\-1&1&-1&-1\\1&1&-1&1\\-1&-1&-1&-1\end{pmatrix},\mathbf{B}=\begin{pmatrix}-1&1&-1&1\\-1&-1&1&-1\\1&-1&-1&-1\\-1&-1&-1&1\end{pmatrix},$$

$\mathbf{C}=diag\{6,6,5,5\}, f_i(x)=tanh(x_i), g_i(x_i)=0.8tanh(x_i), u_i=sint, \tau_{ii}(t)=0.2sint+0.2, \tau_{ij}(t)=0.2cost+0.2(i\neq j), \sigma_{ij}(x_j(t),x_j(t-\tau_{ij}(t)))=0.05x_j(t)+0.05x_j(t-\tau_{ij}(t)), i,j=1,2,3,4.$

We calculate that $\mathbf{B}_1=\bar{\mathbf{B}}_2=4\mathbf{I}, \tilde{\mathbf{F}}=\tilde{\mathbf{G}}=0, \bar{\mathbf{F}}=\mathbf{I}, \bar{\mathbf{G}}=0.8\mathbf{I}, L_{ij}=M_{ij}=0.005, \bar{\mathbf{L}}=\bar{\mathbf{M}}=0.02\mathbf{I}, \tilde{\tau}=0.2$.

By using MATLAB LMI Control Toolbox, we obtain $\mathbf{P}=0.6044I$, $\mathbf{U}_1=diag\{1.5261,1.5243,1.5127,1.6004\}$, $\mathbf{U}_2=diag\{2.9135,2.9018,2.9970,2.9887\}$ to satisfy the condition (3.12) of Corollary 1, and obtain $\mathbf{P}=diag\{0.5191,0.5209,0.5726,0.5705\}$, $q=1.7100$, $\mathbf{U}_1=diag\{1.2485,1.2521,1.2398,1.3223\}$, $\mathbf{U}_2=diag\{2.3854,2.3835,2.6177,2.6148\}$ to satisfy the condition (3.13) of Theorem 2. Therefore, Corollary 1 and Theorem 2 are effective. Figure 1 shows the solution trajectories of system (2.1) with the initial value $(-0.4,-0.2,0.4,0.6)^T$ appearing as periodic motions after a few seconds.

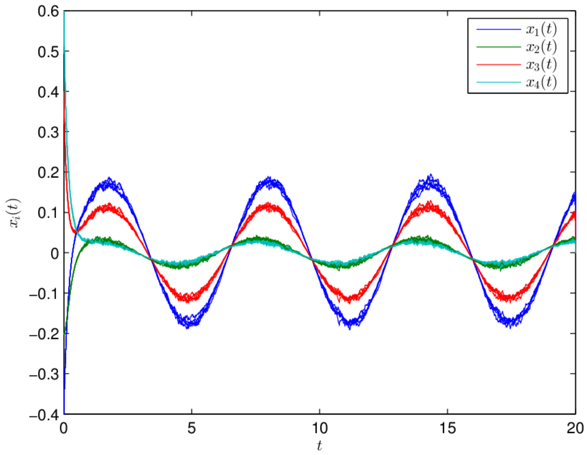

Figure 1. The solution trajectories of system (2.1) with initial value $(-0.4,-0.2,0.4,0.6)^T$.

By changing some functions in system (2.1), we obtain the following systems considered in [22,27,28] :

$$\begin{aligned}dx_i(t) &= [-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau_{ij}(t)))+u_i(t)]dt\\&\quad+\sum_{j=1}^{n}\sigma_{ij}(x_j(t))dw_j(t),\end{aligned}\tag{4.1}$$

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_j(t))) + u_i(t), \tag{4.2}$$

and

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_{ij}(t))) + u_i(t). \tag{4.3}$$

Remark 5. In [22], fixed-point principle and Gronwall-Bellman inequality are used to establish the stability condition of the periodic solutions of system (4.1), that is, $\rho = max_{1\leq i\leq n}\{3\theta_i/c_i^2\} < 1$, where $\theta_i = (\sum_{j=1}^{n} |a_{ij}|F_j + |b_{ij}|G_j)^2 + (\sum_{j=1}^{n} L_{ij})^2 + (\sum_{j=1}^{n} \tilde{L}_{ij})^2$, $F_j = \max\{|F_j^-|, |F_j^+|\}, G_j = \max\{|G_j^-|, |G_j^+|\}, |\sigma_{ij}(\cdot)| \leq \tilde{L}_{ij}$ in this paper. However, for system (4.1) in Example 2, we calculate $\rho > 1$, which implies that the condition in [22] is not applicable.

Example 2. Consider system (4.1) with $\sigma_{ij}(x_j(t)) = 0.1 + 0.05\sin(x_j(t)), i, j = 1, 2, 3, 4$, other symbols are the same as in Example 1.

We calculate that $\mathbf{B}_1 = \bar{\mathbf{B}}_2 = 4\mathbf{I}, \tilde{\mathbf{F}} = \tilde{\mathbf{G}} = 0, \bar{\mathbf{F}} = I, \bar{\mathbf{G}} = 0.8I, L_{ij} = 0.05, M_{ij} = 0, \bar{\mathbf{L}} = 0.02I, \bar{\mathbf{M}} = 0, \tilde{\tau} = 0.2$.

By using MATLAB LMI Control Toolbox, we obtain $\mathbf{P} = 0.1563I$, $\mathbf{U}_1 = diag\{0.4515, 0.4500, 0.4772, 0.4761\}$, $\mathbf{U}_2 = diag\{0.7289, 0.7291, 0.7314, 0.7315\}$ to satisfy the condition (3.12) of Corollary 1, and obtain $\mathbf{P} = diag\{0.1765, 0.1751, 0.1993, 0.1970\}$, $q = 0.8439$, $\mathbf{U}_1 = diag\{0.4678, 0.4659, 0.4929, 0.4932\}$, $\mathbf{U}_2 = diag\{0.7652, 0.7624, 0.8485, 0.8434\}$ to satisfy the condition (3.13) of Theorem 2. Therefore, Corollary 1 and Theorem 2 are effective. Figure 2 shows the solution trajectories of system (4.1) with the initial value $(-0.4, -0.2, 0.4, 0.6)^T$ appearing as periodic motions after a few seconds.

On the other hand, based on Theorem 3.1 in [22], we calculate that $F_j = 1, G_j = 0.8, |a_{ij}| = |b_{ij}| = 1, L_{ij} = 0.05, \tilde{L}_{ij} = 0.15$ $\theta_i = 52.24$ and $\rho = 52.24 \times 3/25 = 6.2688 > 1$. Therefore, Theorem 3.1 in [22] is not applicable to system (4.1) in this example.

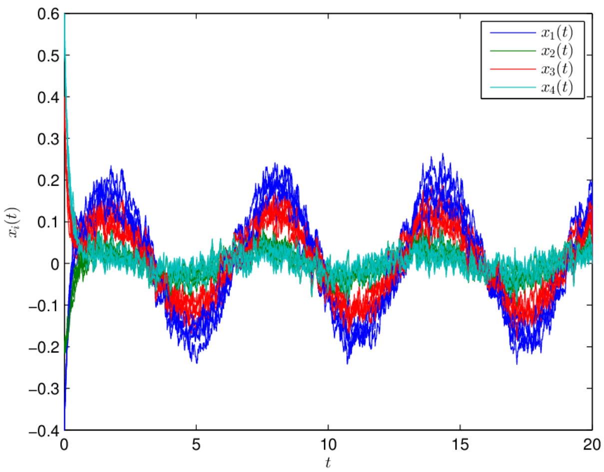

Figure 2. The solution trajectories of system (4.1) with initial value $(-0.4, -0.2, 0.4, 0.6)^T$.

Remark 6. In [27], fixed-point principle and differential inequality technique are used to establish the stability condition of the periodic solutions of system (4.2), that is, $F = \frac{\xi\|\mathbf{A}\|_2+\eta\|\mathbf{B}\|_2}{c_0} < 1$, where $\xi = \max_{1\le i\le n}\{\sup_{x_i\neq 0}\frac{f_i(x_i)}{x_i}\}, \eta = \max_{1\le i\le n}\{\sup_{x_i\neq 0}\frac{g_i(x_i)}{x_i}\}, c_0 = \min_{1\le i\le n}\{c_i\}, \|\mathbf{A}\|_2$ denotes the square root of the largest eigenvalue of $\mathbf{A}^T\mathbf{A}$. However, for system (4.2) in Example 3, we calculate $F > 1$, which implies the condition in [27] is not applicable.

Example 3. Consider system (4.2) with $\tau_j(t) = 0.2sint + 0.2$, $j = 1, 2, 3, 4$, other symbols are the same as in Example 1.

We calculate that $\mathbf{B}_1 = \bar{\mathbf{B}}_2 = 4\mathbf{I}, \tilde{\mathbf{F}} = \tilde{\mathbf{G}} = 0, \bar{\mathbf{F}} = I, \bar{\mathbf{G}} = 0.8I, \bar{\mathbf{L}} = \bar{\mathbf{M}} = 0, \tilde{\tau} = 0.2$.

By using MATLAB LMI Control Toolbox, we obtain $\mathbf{P} = 0.1564I$, $\mathbf{U}_1 = diag\{0.4524, 0.4509, 0.4781, 0.4770\}$, $\mathbf{U}_2 = diag\{0.7300, 0.7301, 0.7325, 0.7326\}$ to satisfy the condition (3.12) of Corollary 1, and obtain $\mathbf{P} = diag\{0.1829, 0.1814, 0.2069, 0.2045\}$, $\mathbf{U}_1 = diag\{0.4908, 0.4887, 0.5160, 0.5162\}$, $\mathbf{U}_2 = diag\{0.7981, 0.7951, 0.8849, 0.8795\}$ to satisfy the condition (3.13) of Theorem 2. Therefore, Corollary 1 and Theorem 2 are effective. Figure 3 shows the solution trajectories of system (4.2) with the initial value $(1, -1, -0.5, 0.5)^T$ appearing as periodic motions after a few seconds.

On the other hand, according to Theorem 3 in [27], we calculate $\xi = 1, \eta = 0.8, \|A\|_2 = \sqrt{8}, \|B\|_2 = \sqrt{7.4641}, c_0 = \min_{1\le i\le 4}\{c_i\} = 5$ and

$$F = \frac{\xi\|\mathbf{A}\|_2 + \eta\|\mathbf{B}\|_2}{c_0} = \frac{5.014}{5} > 1.$$

Therefore, Theorem 3 in [27] is not applicable to system (4.2) in this example.

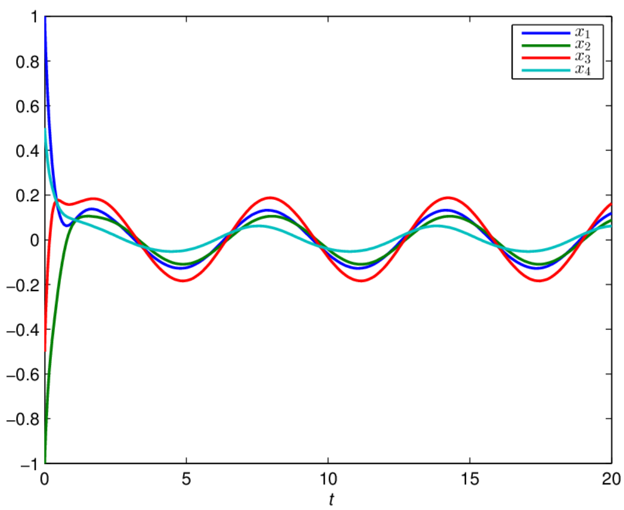

Figure 3. The solution trajectories of system (4.2) with initial value $(1, -1, -0.5, 0.5)^T$.

Remark 7. In [28], Lyapunov functional and Halanay inequality are used to establish the stability conditions of the periodic solutions of system (4.3) : $d_i - L_i\sum_{j=1}^n(|a_{ji}| + |b_{ji}|) > 0$ or $d_i - \sum_{j=1}^n(|a_{ij}| + |b_{ij}|)L_j > 0, i = 1, \cdots, n$, where $d_i = c_i$ and $L_j = \max\{|F_j^-|, |F_j^+|\}$ in this paper. However, for system (4.3) in Example 4, we calculate $d_i - L_i\sum_{j=1}^n(|a_{ji}| + |b_{ji}|) = d_i - \sum_{j=1}^n(|a_{ij}| + |b_{ij}|)L_j = 0 (i = 1, \cdots, n)$, which implies the criteria in [28] are not applicable.

Example 4. Consider system (4.3) with $\mathbf{C} = diag\{4,4,4,4\}, f_i(x_i) = 0.5tanh(x_i), \tau_{ii}(t) = 0.2sint + 0.2, \tau_{ij}(t) = 0.2cost + 0.2 (i \neq j), i, j = 1,2,3,4$, the matrices **A** and **B** are the same as in Example 1.

We calculate that $\bar{\mathbf{B}}_1 = \bar{\mathbf{B}}_2 = 4I, \tilde{\mathbf{F}} = \tilde{\mathbf{G}} = \bar{\mathbf{L}} = \bar{\mathbf{M}} = 0, \bar{\mathbf{F}} = \bar{\mathbf{G}} = 0.5\mathbf{I}, \tilde{\tau} = 0.2$.

By using MATLAB LMI Control Toolbox, we obtain $\mathbf{P} = 2.2433I$, $\mathbf{U}_1 = diag\{9.8971, 9.8971, 10.7128, 10.7128\}$, $\mathbf{U}_2 = 14.3836I$ to satisfy the condition (3.12) of Corollary 1, and obtain $\mathbf{P} = diag\{1.6333, 1.6333, 2.2619, 2.2619\}$, $\mathbf{U}_1 = diag\{7.6665, 7.6665, 8.3747, 8.3747\}$, $\mathbf{U}_2 = diag\{10.9331, 10.9331, 12.0760, 12.0760\}$ to satisfy the condition (3.13) of Theorem 2. Therefore, Corollary 1 and Theorem 2 are effective. Figure 4 shows that the solution trajectories of system (4.3) with the initial value $(-0.6, 0.4, -0.4, 0.8)^T$ appearing as periodic motions after a few seconds.

On the other hand, based on the results proposed in [28], we calculate that $d_i = 4, L_i = 0.5, d_i - L_i \sum_{j=1}^{4}(|a_{ji}| + |b_{ji}|) = d_i - \sum_{j=1}^{4}(|a_{ij}| + |b_{ij}|)L_j = 0, i = 1,2,3,4$. Therefore, the sufficient criteria in [28] are not applicable to system (4.3) in this example.

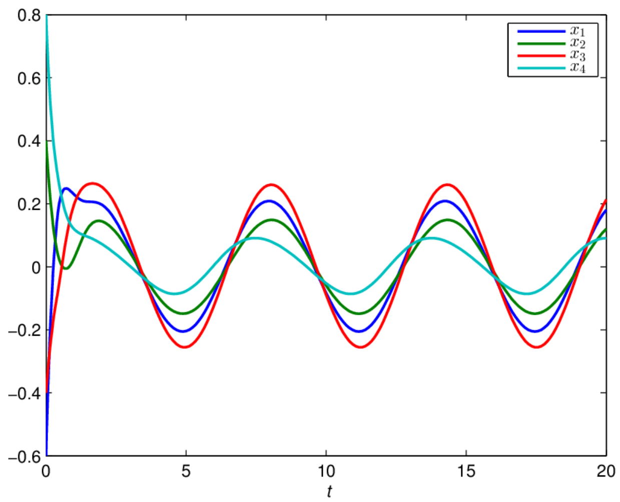

Figure 4. The solution trajectories of system (4.3) with initial value $(-0.6, 0.4, -0.4, 0.8)^T$.

5. Conclusions

This paper has investigated the global exponential periodicity of stochastic neural networks with multiple time-varying delays $\tau_{ij}(t)$. Such stochastic neural networks cannot be transformed into the vector-matrix form because of the multiple time-varying delays $\tau_{ij}(t)$. For neural networks with delays $\tau_{ij}(t)$, the commonly used methods do not include linear matrix inequality. Therefore, the stability conditions in the linear matrix inequality forms are rare for networks with delays $\tau_{ij}(t)$. In this paper, we establish several sets of easily verified conditions in the linear matrix inequality forms to ensure that the stochastic networks have a unique and exponentially stable periodic solution. Several examples are given to demonstrate that our sufficient criteria are effective and less conservative than those in the existing references.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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