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Research article

Analysis of peakon-like soliton solutions: (3+1)-dimensional Fractional Klein-Gordon equation

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Abstract: In this study, we investigate the fundamental properties of $(3 + 1)$-D Fractional Klein-Gordon equation using the sophisticated techniques of Riccatti-Bornoulli sub-ODE approach with Backlund transformation. Using a more stringent criterion, our study reveals new soliton solutions that have peakon-like properties and unique cusp features. This research provides significant understanding of the dynamic behaviours and odd events related to these solutions. This work is important because it helps to elucidate the complex dynamics that exist within physical systems, which will benefit many different scientific fields. Our method is used to examine the existence and stability of compactons and kinks in the context of actual physical systems. Under a double-well on-site potential, these structures are made up of a network of connected nonlinear pendulums. Both $2D$ and contour plots produced by parameter changes provide as clear examples of the efficiency, simplicity, and conciseness of the computational method used. The results highlight how flexible this approach is, and demonstrate how symbolic calculations broaden its application to more complex events. This work offers a useful framework and studying intricate physical systems, as well as a flexible computational tool that may be used in a variety of scientific fields.

1. Introduction

Much focus in the last few years has been on the nonlinear phenomena, which appears in various fields of mathematical and physical sciences, such as physics, biology, and chemistry. In the

mathematical and physical domains, nonlinear evolution equations are essential for modeling a wide range of physical processes, making the analytical solutions to related evolution equations important. Since nonlinear equations are used to describe many physical processes, it becomes vital to investigate analytical solutions, especially those in the form of solitons [1]. One characteristic that sets soliton solutions apart from other types of solutions is that they depend only on one set of variables. A soliton is defined as a self-reinforcing single wave or pulse in mathematics and physics that keeps its shape while moving at a steady speed. The inception of the soliton phenomena may be traced back to the groundbreaking observations made by John Scott Russell who at first noticed a single wave in the Union Canal in Scotland [2–6].

In fact, the framework designated as fractional partial differential equations (FPDEs) provides accurate representation for a wide range of technical and physical phenomena [7–9]. Applications of FPDEs in a variety of scientific and engineering fields demonstrate how versatile it is; these include financial modeling [10], wave modeling [11], signal control systems [12], earthquake modeling [13], and more. Because Fractional partial differential equations appear so frequently in various applications, they become more significant and has drawn the interest of scholars who understand how effective they are at handling a wide range of complex events. Because fractional partial differential equations have been used in so many different scientific and technical fields, mathematicians have created a variety of numerical and analytical methods to solve fractional-related problems [14, 15]. It is noteworthy that fractional variational iteration method (FVIM) has been utilized by researchers to handle particular cases of fractional partial differential equations, as J. Song et al. in [16] illustrates. Furthermore, taking into account both beginning and boundary conditions, Duan et al. have effectively applied Adomian decomposition method (ADM) to obtain series-form solutions for various partial differential equations, including those involving fractional partial differential equations [17]. Furthermore, the Laplace ADM has been applied to solve problems with the Fractional partial differential equation [18]. Fractional k-dV-Burger equations were analytically derived using the homotopy perturbation method by Wang [19]. Meanwhile, the extend tanh-method has been employed by Raslan, et al. [20] to solve fractional equal width wave and modified equal wave equations and other different techniques [21–25].

Also, embedded inside the framework of the presented methodology [26–28], which is known for its ability to successfully deal with intricate algebraic manipulations, is the ability to find answers to a wide range of events in biology, chemistry, physics, fluid mechanics, and optical fibers. It may improve optimization and prediction approaches, and provide adequate knowledge about the behavior of fluid flow when exploited in biological systems, commercial process functions, and environmental fluid dynamics processes. This further helps to elucidate the essential aspects embedded in the intricate dynamics associated with such systems. Meng et al. presented an observer design method tailored for nonlinear generalized systems with nonlinear algebraic constraints, offering practical applications in system control and optimization [29]. Meanwhile, Cai et al. explored the dynamic manipulation of terahertz wavefronts using cascaded metasurfaces, unveiling possibilities for advanced photonics applications [30]. Ali et al. introduced a new class of digital integrators based on trigonometric quadrature rules, contributing to the advancement of industrial electronics and digital signal processing techniques [31]. Guo and Hu proposed a time based generator-based approach for the practical predefined-time stabilization of high-order systems in the presence of unknown disturbances, offering robust solutions for circuit and system design challenges [32]. Additionally,

Kai and Ji investigated the generalization of regularized long-wave equations [33], while Kai and Yin delved into the linear structure and soliton molecules of the Sharma-Tasso-Olver-Burgers equation, shedding light on fundamental aspects of nonlinear dynamics and mathematical physics [34].

Our goal in this work is to solve the fractional-order Klein-Gordon equation using the sub-ODE technique of Riccatti-Bornoulli. The overarching expression for the (3 + 1)-*D* Fractional order Klein-Gordon equation is provided as follows [35]:

$$D_t^{2\alpha} F - \left(D_x^{2\beta} F + D_y^{2\gamma} F \right) + m^2 F + \lambda |F|^2 F = 0, \quad 0 < \alpha, \beta, \gamma \le 1. \tag{1.1}$$

Here, (m) and (λ) are constants. Conformable fractional derivatives are used in this analysis based on the following properties [36]. The operator representing derivatives of order (α) is defined as follows:

$$D_\theta^\alpha q(\theta) = \lim_{m \to 0} \frac{q(m(\theta)^{1-\alpha} - q(\theta))}{m}, 0 < \alpha \le 1. \tag{1.2}$$

$$\begin{cases} D_\theta^\alpha \theta^j = j\theta^{j-\alpha}, j \in R \\ D_\theta^\alpha \left(j_1 \eta(\theta) \pm j_2 m(\theta) \right) = j_1 D_\theta^\alpha(\eta(\theta)) \pm j_2 D_\theta^\alpha(m(\theta)). \\ D_\theta^\alpha \left[fog \right] = \theta^{1-\alpha} g(\theta) D_\theta^\alpha f(g(\theta)) \end{cases} \tag{1.3}$$

Quantum dynamics, the spatial distribution of physical plasma, and the propagation of different waves are some of the fascinating combinations of mathematics and physics that the Klein-Gordon equation presents. It is amazing how many methods have emerged lately to solve the Klein-Gordon models [37–45]. The recommended approach is different from the others in that it approaches problem-solving directly. Unlike other methods, this strategy navigates the complexity of the Klein-Gordon equation without resorting to discretization or linearization, hence avoiding illogical assumptions. The method's versatility and efficacy in resolving complex problems are enhanced by the broad range of families of solutions covered by the analytical solutions that are generated.

2. Methodology

Consider the following fractional partial differential equation (FPDE):

$$Q_1 \left(f, D_t^\alpha(f), D_{x_1}^\beta(f), D_{x_2}^\gamma(f), f D_{x_1}^\beta(f), \ldots \right) = 0, \quad 0 < \alpha, \beta, \gamma \le 1. \tag{2.1}$$

Let f stand for an unknown function in the given context that depends on $(x_1, x_2, x_3 \ldots)$ and (t). f and its fractional partial derivatives are involved in the polynomial equation (Q). The fractional partial differential equation represented by equation (1.3) can be solved by the following steps in procedure: (i) It is possible to define the variable (x) in a variety of ways when using the variable transformation $f(t, x_1, x_2, x_3, \ldots x_n) = f(x)$. In this particular case, a sophisticated form of transformation has been utilized:

$$\psi = p \frac{x_1^\beta}{\beta} + q \frac{x_2^\gamma}{\gamma} + r \frac{t^\alpha}{\alpha} \ldots + \psi_o. \tag{2.2}$$

Here, $p, q, r, \ldots \psi_o$ are constants.
(ii) Equation (1.3) undergoes this transformation becoming a nonlinear ordinary differential equation

(ODE) with the following structure:

$$Q_2\left(f, \frac{df}{d\psi}, \frac{d^2 f}{d\psi^2}, f\frac{df}{d\psi}, \ldots\right) = 0. \tag{2.3}$$

(iii) The following is an expression for the solution to equation (2.2) indicated by the Riccatti-Bornoulli sub-ODE approach. The constants b_i must be found, subject to the restriction that $b_m \neq 0$ and $b_{-m} \neq 0$ occur simultaneously.

$$f(\psi) = \sum_{i=-m}^{m} b_i \varphi(\psi)^i. \tag{2.4}$$

(iv) Simultaneously, the function is acquired through the following transformation:

$$\varphi(\psi) = \frac{-\Gamma B + A\phi(\psi)}{A + B\phi(\psi)}. \tag{2.5}$$

Let us consider the constants (Γ), (A), and (B), given the constraint that $B \neq 0$. Furthermore, let us assume that $\phi(\psi)$ is a function with the following definition:

$$\frac{d\phi}{d\psi} = \Gamma + \phi(\psi)^2. \tag{2.6}$$

It is commonly acknowledged [46] that the following are the solutions to equation (2.5):

(*i*) If $\Gamma < 0$, then $\phi(\psi) = -\sqrt{-\Gamma}\tanh(\sqrt{-\Gamma}\psi)$, or $\phi(\psi) = -\sqrt{-\Gamma}coth(\sqrt{-\Gamma}\psi)$. (2.7)

(*ii*) If $\Gamma > 0$, then $\phi(\psi) = \sqrt{\Gamma}\tan(\sqrt{\Gamma}\psi)$, or $\phi(\psi) = -\sqrt{\Gamma}cot(\sqrt{\Gamma}\psi)$. (2.8)

(*iii*) If $\Gamma = 0$, then $\phi(\psi) = \dfrac{-1}{\psi}$. (2.9)

(v) Achieving a homogenous balance between the highest-order derivative terms and the nonlinear factors in Eq (2.2) yields the equilibrium value. More specifically, the following formula [47] can be used to get the balance number:

$$D\left[\frac{d^k f}{d\psi^k}\right] = N + k, \quad D\left[f^J \frac{d^k f}{d\psi^k}\right]^s = NJ + s(k + N). \tag{2.10}$$

(vi) We group the terms with the same power of $\varphi^i(\psi)$, and substitute Eq (2.3) into Eq (2.2).

(vii) In the resulting polynomial, we set all of the coefficients of $\phi^i(\psi) = 0$, resulting in a system of algebraic equations.

(viii) The software Maple is utilized to solve the resultant system in order to ascertain the unknown coefficients.

(ix) The answer $f(\psi)$ for Eq (1.3) is then obtained by substituting the derived values from step (viii) back into Eq (2.3).

3. Evaluation of the problem

By using the approach described in Section 2, we methodically solve the (3+1)-order fractional Klein-Gordon equation (1.1), concentrating on single wave solutions. By applying the following complex transformation, we consider the exact solution for the proposed fractional model:

$$\psi = p\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} + a\frac{t^\alpha}{\alpha}.$$
$$\eta = k\frac{x^\beta}{\beta} + \frac{y^\gamma}{\gamma} - \omega\frac{t^\alpha}{\alpha}. \tag{3.1}$$
$$F(x, y, t) = e^{i\eta} f(\psi).$$

Here the function $\eta(x, y, t)$ describes the transformation of propagating waves. The dynamics of the traveling waves are given unique qualities by the presence of the non-zero parameter (ω).

$$D_t^{2\alpha} F = \omega^2 F - i(2\omega a) F' + a^2 F''. \tag{3.2}$$

$$D_x^{2\beta} F = -k^2 F + i(2kp) F' + p^2 F''. \tag{3.3}$$

$$D_y^{2\gamma} F = -F + i2F' + F''. \tag{3.4}$$

Equation (1.1) is transformed into the following ordinary differential equation (ODE) by using the previously indicated complex transformation.

$$-(p^2 + 1)F'' - 2i(\omega a + kp + 1) F' + \left(\omega^2 + m^2 + k^2 + 1\right) F + \lambda F^3 = 0. \tag{3.5}$$

We will now focus our efforts on solving the real part of the equation in order to arrive at the desired solution after removing the following constraint equation from the imaginary part of the equation:

$$\omega = \frac{-(kp + 1)}{a} \tag{3.6}$$

We now address the real part of the equation, which is post-elimination of the constraint.

$$(p^2 + 1)F'' - \left(\omega^2 + m^2 + k^2 + 1\right) F - \lambda F^3 = 0. \tag{3.7}$$

Finding the point of homogeneous equilibrium $(N = 1)$ entails striking a harmonious balance between the nonlinear term and the highest order derivatives in the given equation. After equating the coefficients of $\phi^i(\psi) = 0$ and substituting Eq (2.2) and Eq (2.3) into Eq (3.7), we are left with an algebraic system of equations. The following results form the solutions to this system, which were found using the Maple software.
Case 1 :

$$b_0 = 0, b_1 = 0, b_{-1} = \sqrt{(2\lambda P^2 + 2\lambda)^{-1}} \left(k^2 + m^2 + \omega^2 + 1\right), k = k, \Gamma = 1/2\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}, p = p, a = a. \tag{3.8}$$

Case 2:

$$b_0 = 0, b_1 = \sqrt{-\frac{-2-2P^2}{\lambda}}, b_{-1} = 0, k = k, \Gamma = 1/2\,\frac{k^2+m^2+\omega^2+1}{P^2+1}, p = p, a = a. \tag{3.9}$$

Case 3:

$$b_0 = 0, b_1 = \sqrt{-\frac{-2-2P^2}{\lambda}}, b_{-1} = -1/4\left(k^2+m^2+\omega^2+1\right)\lambda^{-1}\frac{1}{\sqrt{-\frac{-2-2P^2}{\lambda}}}, k = k,$$

$$\Gamma = 1/8\,\frac{k^2+m^2+\omega^2+1}{P^2+1}, p = p, a = a. \tag{3.10}$$

The solution set corresponding to the given values of (ψ) and (η) is derived under the assumptions of Case 1.

$$\psi = \frac{px^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{at^\alpha}{\alpha},$$

$$\eta = \frac{kx^\beta}{\beta} + \frac{y^\gamma}{\gamma} - \frac{\omega t^\alpha}{\alpha}. \tag{3.11}$$

Solution Set. 1: When $\Gamma < 0$, Eq (1.1) yields the resultant single-wave solutions:

$$F_1(x,y,t) = \sqrt{(2\lambda P^2+2\lambda)^{-1}}\left(k^2+m^2+\omega^2+1\right)$$

$$\left(A - 1/2\,B\sqrt{-\frac{2k^2+2m^2+2\omega^2+2}{P^2+1}}\tanh\left(1/2\,\sqrt{-\frac{2k^2+2m^2+2\omega^2+2}{P^2+1}}\psi\right)\right)$$

$$e^{i\eta}\left(-1/2\,\frac{\left(k^2+m^2+\omega^2+1\right)B}{P^2+1} - 1/2\,A\sqrt{-\frac{2k^2+2m^2+2\omega^2+2}{P^2+1}}\tanh\left(1/2\,\sqrt{-\frac{2k^2+2m^2+2\omega^2+2}{P^2+1}}\psi\right)\right)^{-1}. \tag{3.12}$$

or

$$F_2(x,y,t) = \sqrt{(2\lambda P^2+2\lambda)^{-1}}\left(k^2+m^2+\omega^2+1\right)$$

$$\left(A - 1/2\,B\sqrt{-\frac{2k^2+2m^2+2\omega^2+2}{P^2+1}}\coth\left(1/2\,\sqrt{-\frac{2k^2+2m^2+2\omega^2+2}{P^2+1}}\psi\right)\right)$$

$$e^{i\eta}\left(-1/2\,\frac{\left(k^2+m^2+\omega^2+1\right)B}{P^2+1} - 1/2\,A\sqrt{-\frac{2k^2+2m^2+2\omega^2+2}{P^2+1}}\coth\left(1/2\,\sqrt{-\frac{2k^2+2m^2+2\omega^2+2}{P^2+1}}\psi\right)\right)^{-1}. \tag{3.13}$$

Solution Set. 2: When $\Gamma > 0$, Eq (1.1) yields the resultant single-wave solutions:

$$F_3(x, y, t) = \sqrt{(2\lambda P^2 + 2\lambda)^{-1}\left(k^2 + m^2 + \omega^2 + 1\right)}$$

$$\left(A + 1/2\,B\,\sqrt{2}\,\sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\,\tan\left(1/2\,\sqrt{2}\,\sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\,\psi\right)\right)$$

$$e^{i\eta}\left(-1/2\,\frac{\left(k^2 + m^2 + \omega^2 + 1\right)B}{P^2 + 1} + 1/2\,A\,\sqrt{2}\,\sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\,\tan\left(1/2\,\sqrt{2}\,\sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\,\psi\right)\right)^{-1}.$$

(3.14)

or

$$F_4(x, y, t) = \sqrt{(2\lambda P^2 + 2\lambda)^{-1}\left(k^2 + m^2 + \omega^2 + 1\right)}$$

$$\left(A - 1/2\,B\,\sqrt{2}\,\sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\,\cot\left(1/2\,\sqrt{2}\,\sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\,\psi\right)\right)$$

$$e^{i\eta}\left(-1/2\,\frac{\left(k^2 + m^2 + \omega^2 + 1\right)B}{P^2 + 1} - 1/2\,A\,\sqrt{2}\,\sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\,\cot\left(1/2\,\sqrt{2}\,\sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\,\psi\right)\right)^{-1}.$$

(3.15)

Solution Set. 3: When $\Gamma = 0$, equation (1.1) yields the resultant single-wave solutions:

$$F_5(x, y, t) = \sqrt{(2\lambda P^2 + 2\lambda)^{-1}\left(k^2 + m^2 + \omega^2 + 1\right)}\left(A - \frac{B}{\psi}\right)e^{i\eta}\left(-1/2\,\frac{\left(k^2 + m^2 + \omega^2 + 1\right)B}{P^2 + 1} - \frac{A}{\psi}\right)^{-1}.$$

(3.16)

The solution set corresponding to the given values of (ψ) and (η) is derived under the assumption of case 2.

$$\psi = \frac{px^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{at^\alpha}{\alpha}.$$

$$\eta = \frac{kx^\beta}{\beta} + \frac{y^\gamma}{\gamma} - \frac{\omega t^\alpha}{\alpha}.$$

(3.17)

Solution Set. 4: When $\Gamma < 0$, Eq (1.1) yields the resultant single-wave solutions:

$$F_6(x, y, t) = \sqrt{-\frac{-2 - 2P^2}{\lambda}} \times$$

$$\left(-1/2\,\frac{\left(k^2 + m^2 + \omega^2 + 1\right)B}{P^2 + 1} - 1/2\,A\,\sqrt{-\frac{2k^2 + 2m^2 + 2\omega^2 + 2}{P^2 + 1}}\,\tanh\left(1/2\,\sqrt{-\frac{2k^2 + 2m^2 + 2\omega^2 + 2}{P^2 + 1}}\,\psi\right)\right)$$

$$e^{i\eta}\left(A - 1/2\,B\,\sqrt{-\frac{2k^2 + 2m^2 + 2\omega^2 + 2}{P^2 + 1}}\,\tanh\left(1/2\,\sqrt{-\frac{2k^2 + 2m^2 + 2\omega^2 + 2}{P^2 + 1}}\,\psi\right)\right)^{-1}.$$

(3.18)

or

$$F_7(x, y, t) = \sqrt{-\frac{-2 - 2P^2}{\lambda}} \times$$

$$\left(-1/2 \frac{\left(k^2 + m^2 + \omega^2 + 1\right)B}{P^2 + 1} - 1/2 A \sqrt{-\frac{2k^2 + 2m^2 + 2\omega^2 + 2}{P^2 + 1}} \coth\left(1/2 \sqrt{-\frac{2k^2 + 2m^2 + 2\omega^2 + 2}{P^2 + 1}}\psi\right) \right)$$

$$e^{i\eta}\left(A - 1/2 B \sqrt{-\frac{2k^2 + 2m^2 + 2\omega^2 + 2}{P^2 + 1}} \coth\left(1/2 \sqrt{-\frac{2k^2 + 2m^2 + 2\omega^2 + 2}{P^2 + 1}}\psi\right) \right)^{-1}.$$

(3.19)

Solution Set. 5: When $\Gamma > 0$, Eq (1.1) yields the resultant single-wave solutions:

$$F_8(x, y, t) = \sqrt{-\frac{-2 - 2P^2}{\lambda}} \times$$

$$\left(-1/2 \frac{\left(k^2 + m^2 + \omega^2 + 1\right)B}{P^2 + 1} + 1/2 A \sqrt{2} \sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}} \tan\left(1/2 \sqrt{2} \sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\psi\right) \right)$$

$$e^{i\eta}\left(A + 1/2 B \sqrt{2} \sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}} \tan\left(1/2 \sqrt{2} \sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\psi\right) \right)^{-1}.$$

(3.20)

or

$$F_9(x, y, t) = \sqrt{-\frac{-2 - 2P^2}{\lambda}} \times$$

$$\left(-1/2 \frac{\left(k^2 + m^2 + \omega^2 + 1\right)B}{P^2 + 1} - 1/2 A \sqrt{2} \sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}} \cot\left(1/2 \sqrt{2} \sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\psi\right) \right)$$

$$e^{i\eta}\left(A - 1/2 B \sqrt{2} \sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}} \cot\left(1/2 \sqrt{2} \sqrt{\frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}}\psi\right) \right)^{-1}.$$

(3.21)

Solution Set. 6: When $\Gamma = 0$, Eq (1.1) yields the resultant single-wave solutions:

$$F_{10}(x, y, t) = \sqrt{-\frac{-2 - 2P^2}{\lambda}}\left(-1/2 \frac{\left(k^2 + m^2 + \omega^2 + 1\right)B}{P^2 + 1} - \frac{A}{\psi} \right) e^{i\eta}\left(A - \frac{B}{\psi} \right)^{-1}.$$

(3.22)

The solution set corresponding to the given values of (ψ, η) and (Γ) is derived under the assumptions of Case 3.

$$\psi = \frac{px^\beta}{\beta} + \frac{y^\gamma}{\gamma} + \frac{at^\alpha}{\alpha},$$

$$\eta = \frac{kx^\beta}{\beta} + \frac{y^\gamma}{\gamma} - \frac{\omega t^\alpha}{\alpha}. \tag{3.23}$$

$$\Gamma = 1/8 \frac{k^2 + m^2 + \omega^2 + 1}{P^2 + 1}.$$

Solution Set. 7: When $\Gamma < 0$, Eq (1.1) yields the resultant single-wave solutions:

$$F_{11}(x, y, t) = -1/4 \left(k^2 + m^2 + \omega^2 + 1\right)\left(A - B \sqrt{-\Gamma} \tanh\left(\sqrt{-\Gamma}\psi\right)\right) e^{i\eta} \lambda^{-1} \frac{1}{\sqrt{-\frac{-2-2P^2}{\lambda}}}$$

$$\left(-(\Gamma) B - A \sqrt{-\Gamma} \tanh\left(\sqrt{-\Gamma}\psi\right)\right)^{-1} + \sqrt{-\frac{-2-2P^2}{\lambda}} \left(-(\Gamma) B - A \sqrt{-\Gamma} \tanh\left(\sqrt{-\Gamma}\psi\right)\right)$$

$$e^{i\eta} \left(A - B \sqrt{-\Gamma} \tanh\left(\sqrt{-\Gamma}\psi\right)\right)^{-1}. \tag{3.24}$$

or

$$F_{12}(x, y, t) = -1/4 \left(k^2 + m^2 + \omega^2 + 1\right)\left(A - B \sqrt{-\Gamma} \coth\left(\sqrt{-\Gamma}\psi\right)\right)$$

$$e^{i\eta} \lambda^{-1} \frac{1}{\sqrt{-\frac{-2-2P^2}{\lambda}}} \left(-(\Gamma) B - A \sqrt{-\Gamma} \coth\left(\sqrt{-\Gamma}\psi\right)\right)^{-1}$$

$$+ \sqrt{-\frac{-2-2P^2}{\lambda}} \left(-(\Gamma) B - A \sqrt{-\Gamma} \coth\left(\sqrt{-\Gamma}\psi\right)\right) e^{i\eta} \left(A - B \sqrt{-\Gamma} \coth\left(\sqrt{-\Gamma}\psi\right)\right)^{-1}. \tag{3.25}$$

Solution Set. 8: When $\Gamma > 0$, Eq (1.1) yields the resultant single-wave solutions:

$$F_{13}(x, y, t) = -1/4 \left(k^2 + m^2 + \omega^2 + 1\right)\left(A + B \sqrt{\Gamma} \tan\left(\sqrt{\Gamma}\psi\right)\right) e^{i\eta} \lambda^{-1} \frac{1}{\sqrt{-\frac{-2-2P^2}{\lambda}}} \left(-(\Gamma) B + A \sqrt{\Gamma} \tan\left(\sqrt{\Gamma}\psi\right)\right)^{-1}$$

$$+ \sqrt{-\frac{-2-2P^2}{\lambda}} \left(-(\Gamma) B + A \sqrt{\Gamma} \tan\left(\sqrt{\Gamma}\psi\right)\right) e^{i\eta} \left(A + B \sqrt{\Gamma} \tan\left(\sqrt{\Gamma}\psi\right)\right)^{-1}. \tag{3.26}$$

or

$$F_{14}(x, y, t) = -1/4 \left(k^2 + m^2 + \omega^2 + 1\right)\left(A - B \sqrt{\Gamma} \cot\left(\sqrt{\Gamma}\psi\right)\right) e^{i\eta} \lambda^{-1} \frac{1}{\sqrt{-\frac{-2-2P^2}{\lambda}}} \left(-(\Gamma) B - A \sqrt{\Gamma} \cot\left(\sqrt{\Gamma}\psi\right)\right)^{-1}$$

$$+ \sqrt{-\frac{-2-2P^2}{\lambda}} \left(-(\Gamma) B - A \sqrt{\Gamma} \cot\left(\sqrt{\Gamma}\psi\right)\right) e^{i\eta} \left(A - B \sqrt{\Gamma} \cot\left(\sqrt{\Gamma}\psi\right)\right)^{-1}. \tag{3.27}$$

Solution Set. 9: When $\Gamma = 0$, Eq (1.1) yields the resultant single-wave solutions:

$$
F_{15}(x, y, t) = -1/4 \left(k^2 + m^2 + \omega^2 + 1 \right) \left(A - \frac{B}{\psi} \right)
$$

$$
e^{i\eta} \lambda^{-1} \frac{1}{\sqrt{-\frac{-2 - 2 P^2}{\lambda}}} \left(-1/8 \left[k^2 + \frac{1}{64} \frac{\left(k^2 + m^2 + \omega^2 + 1 \right)^2}{\left(P^2 + 1 \right)^2} + \omega^2 + 1 \right] B \left(P^2 + 1 \right)^{-1} - \frac{A}{\psi} \right)^{-1}
$$

$$
+ \sqrt{-\frac{-2 - 2 P^2}{\lambda}} \left(-1/8 \left[k^2 + \frac{1}{64} \frac{\left(k^2 + m^2 + \omega^2 + 1 \right)^2}{\left(P^2 + 1 \right)^2} + \omega^2 + 1 \right] B \left(P^2 + 1 \right)^{-1} - \frac{A}{\psi} \right) e^{i\eta} \left(A - \frac{B}{\psi} \right)^{-1}.
$$

$$
(3.28)
$$

4. Results and discussion

Compared with traditional analytical methods, the Riccatti-Bornoulli sub-ODE approach shows remarkable analytical efficiency by producing a large number of periodic and single traveling wave solutions with parameterizable characteristics. The results of this novel approach reveal the fundamental principles of physical phenomena and provide exact answers for many nonlinear structures in science. In addition to its direct uses in physics, this method yields series-based answers that are useful as standards for numerical solvers, allowing them to assess the precision of their output and simplifying stability studies.

The development of solitons in our suggested methodology is a manifestation of the delicate equilibrium between linear and nonlinear effects. Solitons are defined as solitary waves that dissipate very little energy while propagating and continue to preserve their original form and velocity even after colliding with other similar waves. Our novel strategy produces three important and different families of solitary wave solutions, specifically for the fractional Klein-Gordon equation, which are hyperbolic, rational, and periodic solitary wave solutions. This wide range of solutions increases the generalizability of our approach in different circumstances and advances our understanding of soliton dynamics in those contexts. To clarify what sets different periodic and single solutions apart from each other, we turn to the MATLAB, adjusting certain parameter values to the solutions themselves. As such, the images below represent the graphical representations that make it possible for one to understand such solutions features and behavior as a vivid example.

Figure 1: Visualization of the detailed dynamics of the real and imaginary parts of the solution. In this figure, we observe the dromion structures which describe the dynamics and variations of the solution $F_1(x, y, t)$ at different scales in space and time. This visualization quantitatively elucidates the complexities and variations of the system.

Figure 2: Visualization of the lump-type kink soliton in the $F_8(x, y, t)$. The evolution about the semi-axes types of kink soliton in the lump solution is documented in the figure.

Figure 3: Interplay of the real and imaginary parts of the solution $F_{10}(x, y, t)$ on several levels of detail. With the help of lump-type structures over a periodic background representation together, it is possible to convey the comprehensive picture of the underlying attributes and temporal behavior in the different type of phases and scales.

Figure 4: This figure shows the development of the cuspon kink soliton, in which the space and time

differ somewhat between the complex parts of its solution, $F_{11}(x, y, t)$. Neighborhoods of contrasting resolution with the feasible coefficients were actively resolved. The presented cuspon kink structures show the mixing spatial-temporal behavior of $F_{11}(x, y, t)$, at different spatial and time scales.
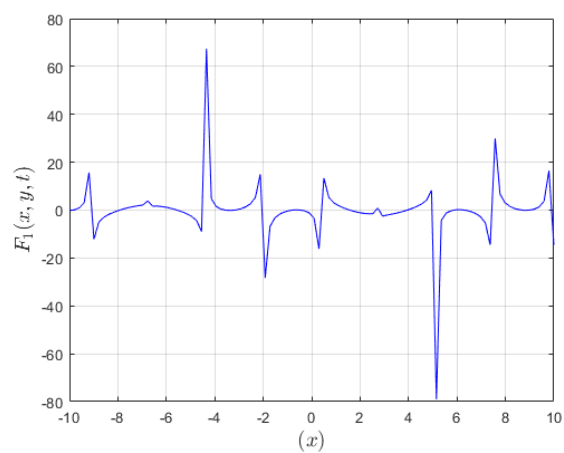
Figure 5: Similar to Figure 2, this figure illustrates the dynamics of the complex components of the lump-type kink soliton $F_{15}(x, y, t)$.
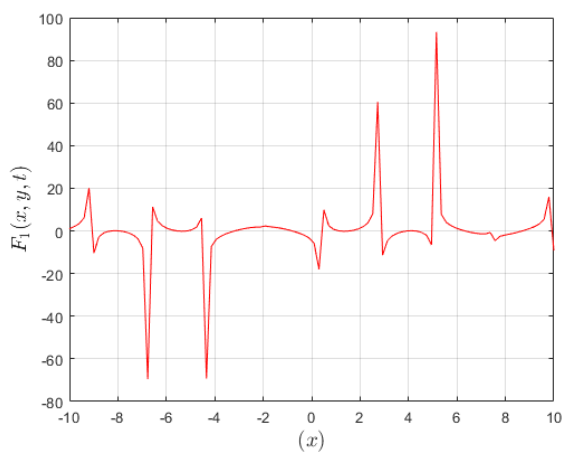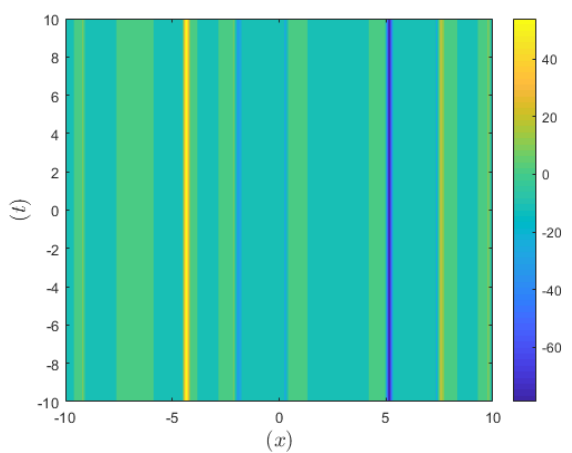

(a) A three-dimensional plot is provided to visualize the real component of $F_1(x, y, t)$.
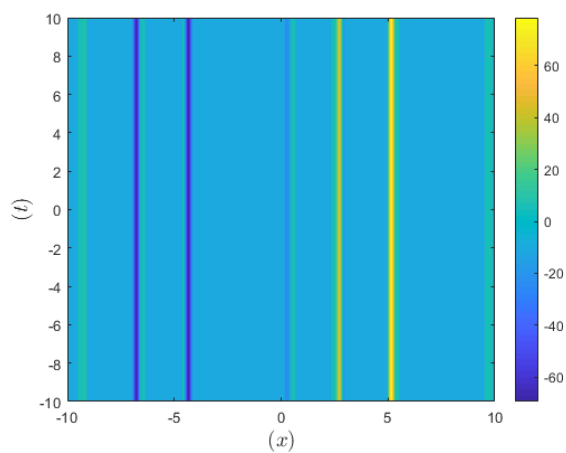
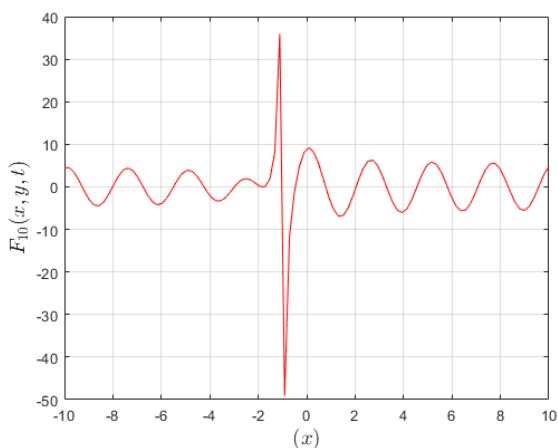(b) A contour representation is provided to visualize the real component of F_1.

(c) A three-dimensional plot is provided to visualize the imaginary part of F_1.
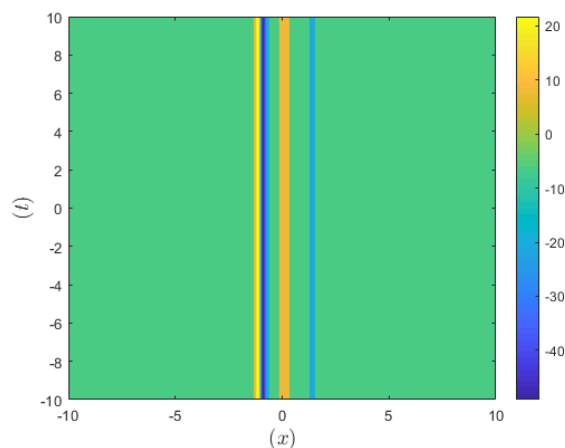
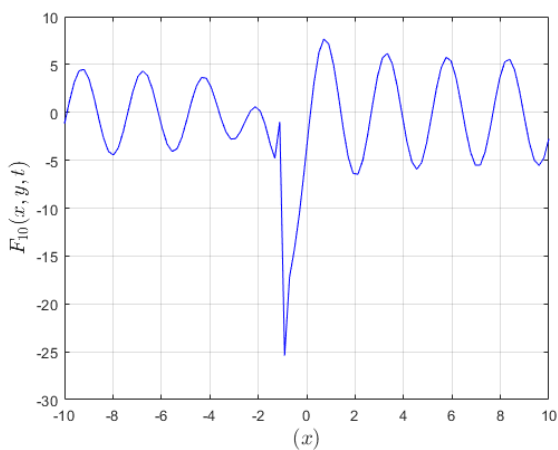(d) A contour representation is provided to visualize the imaginary part of F_1.

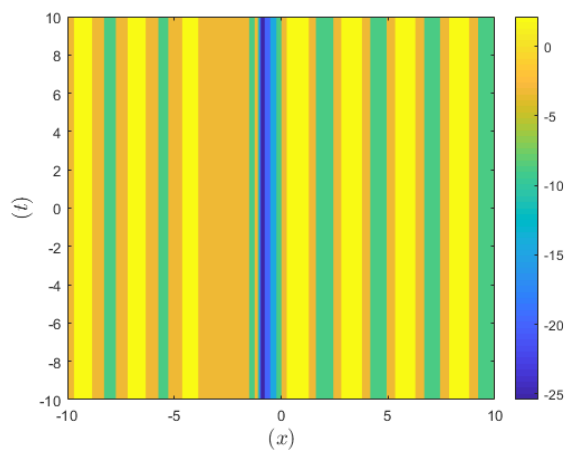**Figure 1.** These visual representations illustrate fluctuations in the complex dynamics of the solution F_1.
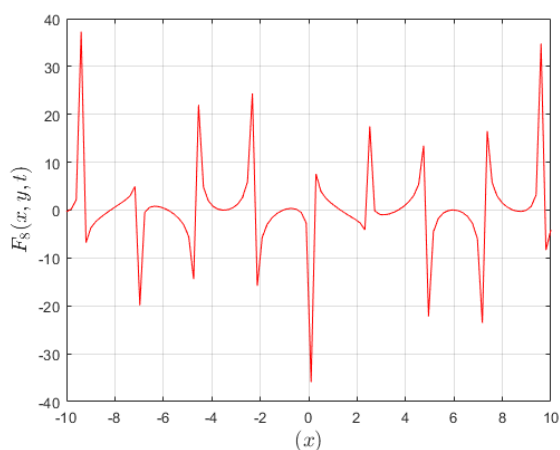
(a) A three-dimensional plot is provided to visualize the real component of F_8.
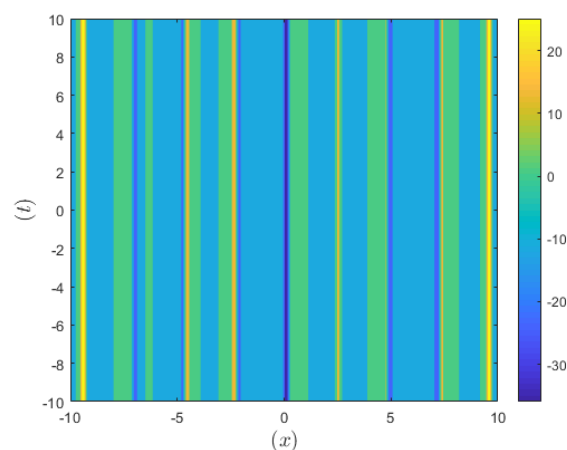
(b) A contour representation is provided to visualize the real component of F_8.
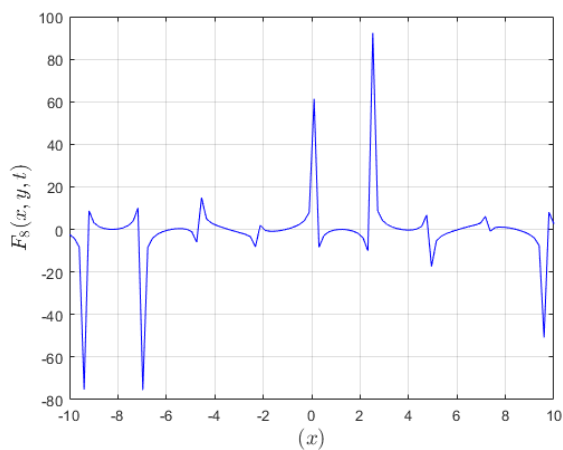
(c) A three-dimensional plot is provided to visualize the imaginary part of F_8.
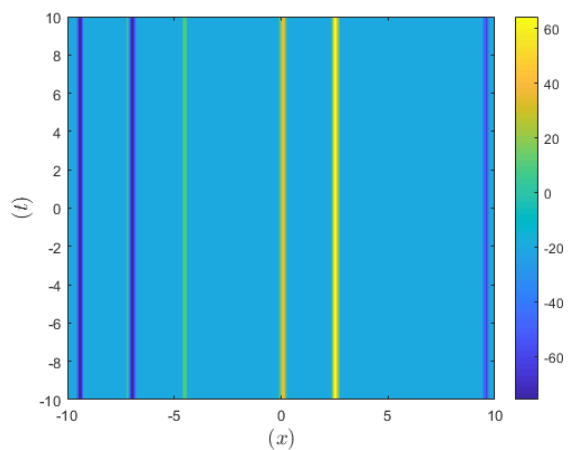
(d) A contour representation is provided to visualize the imaginary part of F_8.

Figure 2. These visual representation illustrate fluctuations in the complex dynamics of the solution F_8.

(a) A three-dimensional plot is provided to visualize the real component of F_{10}.

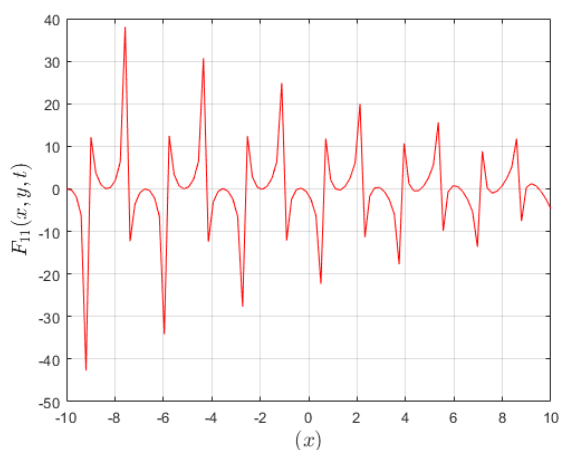(b) A contour representation is provided to visualize the real component of F_{10}.

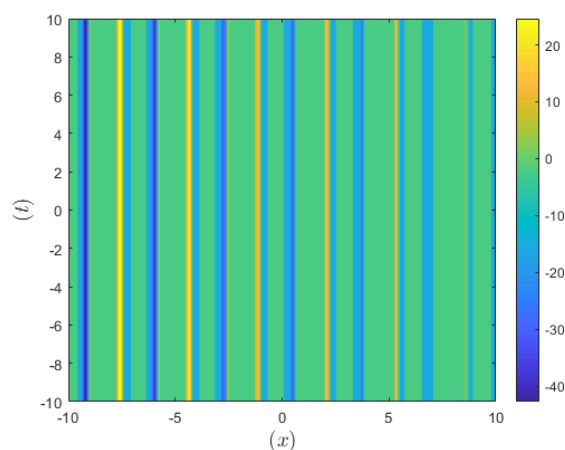(c) A three-dimensional plot is provided to visualize the imaginary part of F_{10}.

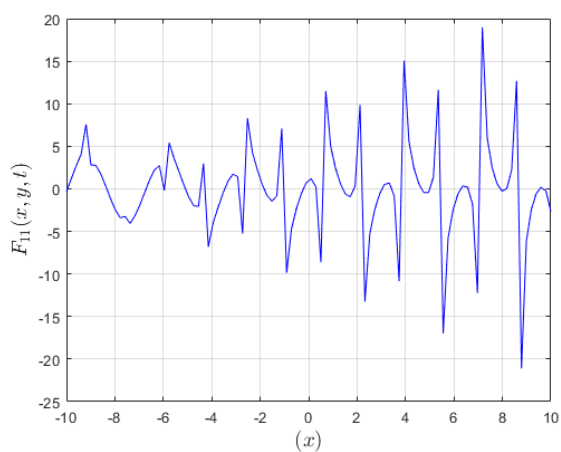(d) A contour representations is provided to visualize the imaginary part of F_{10}.

Figure 3. These visual representations illustrate fluctuations in the complex dynamics of the solution F_{10}.
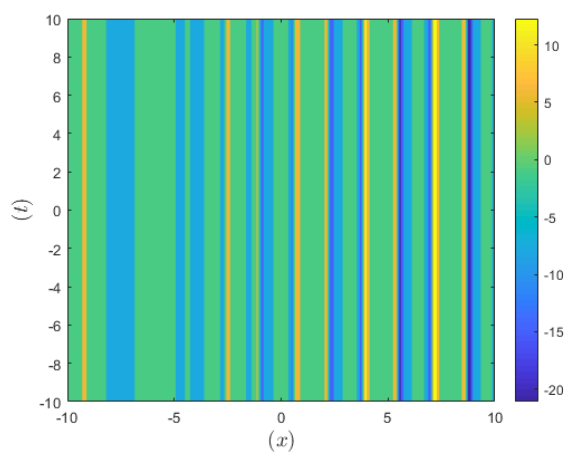
(a) A three-dimensional plot is provided to visualize the real component of F_{11}.

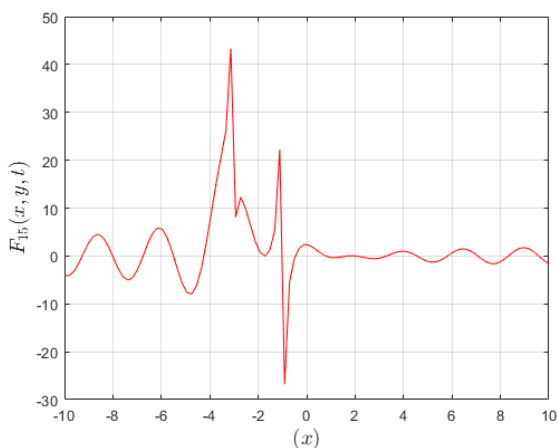(b) A contour representation is provided to visualize the real component of F_{11}.

(c) A three-dimensional plot is provided to visualize the imaginary part of F_{11}.
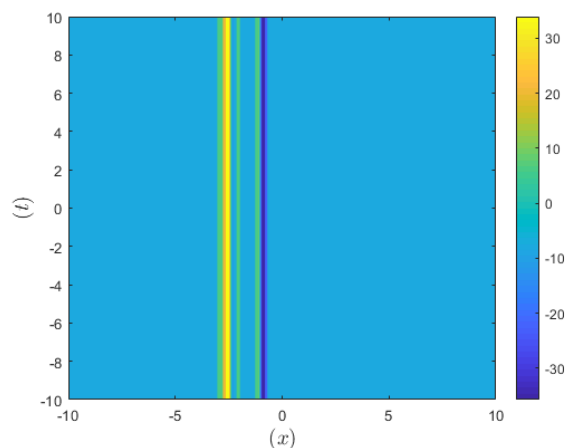

(d) A contour representation is provided to visualize the imaginary part of F_{11}.
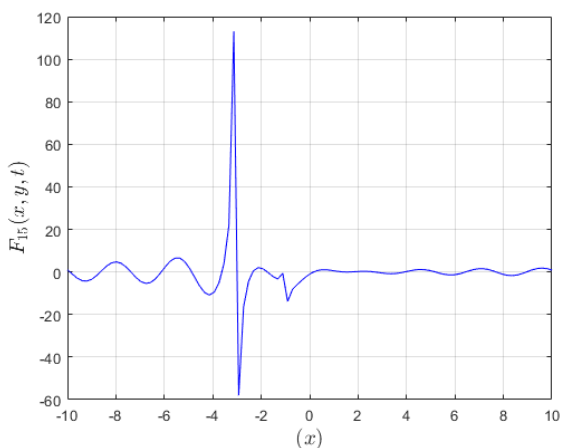
Figure 4. These visual representations illustrate fluctuations in the complex dynamics of the solution F_{11}.
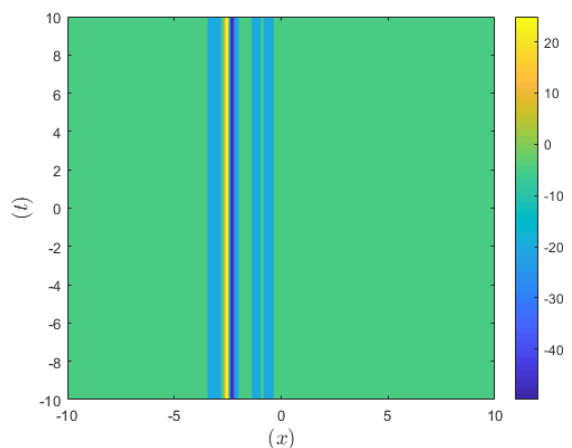
(a) A three-dimensional plot is provided to visualize the real component of F_{15}.

(b) A contour representation is provided to visualize the real part of F_{15}.

(c) A three-dimensional plot is provided to visualize the imaginary part of F_{15}.

(d) A contour representation is provided to visualize the imaginary part of F_{15}.

Figure 5. These visual representations illustrate fluctuations in the complex dynamics of the solution F_{15}.

5. Conclusions

The presented approach is used in this study to get analytical solutions for fractional Klein-Gordon equation, for which the fractional derivative is defined in a conformable context. To convert the provided fractional partial differential equations into ordinary differential equations, a complex transformation is used. Then, using unknown coefficients, the proposed approach is applied in series form. This system of equations has a solution that can be found using the Maple software, which makes it possible to identify the unknown coefficients. As a result, the the Riccatti-Bornoulli sub-ODE algorithm produces solutions for the given problems, which fall into the rational,

hyperbolic, and trigonometric families. This methodology enables an extensive investigation of analytical solutions over various mathematical structures in the framework of Fractional Klein-Gordon equation. Compared to previous methods, the suggested method is an effective analytical technique since it may provide several periodic and single traveling wave solutions, each with tunable parameters. The outcomes of this approach are crucial in revealing the underlying principles that regulate physical events, providing unique solutions for many physical structures in the field of nonlinear research. In addition to its direct uses in the physical sciences, this method yields series-form solutions that are useful resources for numerical solvers. By facilitating stability analyses and allowing practitioners to assess the quality of their numerical results, these solutions help researchers gain a deeper understanding of the systems they are studying.

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Conflict of interest

The authors declare no conflicts of interest.

Use of AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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