Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 14747–14764.
DOI: 10.3934/math.2024717
Received: 28 February 2024
Revised: 06 April 2024
Accepted: 09 April 2024
Published: 23 April 2024

Research article

Existence of solution for fractional differential equations involving symmetric fuzzy numbers

Muhammad Sarwar[1,2,*], Noor Jamal[1], Kamaleldin Abodayeh[2], Manel Hleili[3], Thanin Sitthiwirattham[4,6] and Chanon Promsakon[5,6,*]

[1] Department of Mathematics, Univdersity of malakand, Dir Lower, Pakistan, sarwarswati@gmail.com, noorjamalmphil791@gmail.com

[2] Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia, kamal@psu.edu.sa

[3] Department of mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk, 71491, Saudi Arabia, mhleili@ut.edu.sa

[4] Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, 10300, Thailand, thanin_sit@dusit.ac.th

[5] Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand, chanon.p@sci.kmutnb.ac.th

[6] Research Group for Fractional Calculus Theory and Applications, Science and Technology Research Institute, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand

* **Correspondence:** Email: sarwar@uom.edu.pk, chanon.p@sci.kmutnb.ac.th.

Abstract: Linear correlated fractional fuzzy differential equations (LCFFDEs) are one of the best tools for dealing with physical problems with uncertainty. The LCFFDEs mostly do not have unique solutions, especially if the basic fuzzy number is symmetric. The LCFFDEs of symmetric basic fuzzy numbers extend to the new system by extension and produce many solutions. The existing literature does not have any criteria to ensure the existence of unique solutions to LCFFDEs. In this study, we will explore the main causes of the extension and the unavailability of unique solutions. Next, we will discuss the existence and uniqueness conditions of LCFFDEs by using the concept of metric fixed point theory. For the useability of established results, we will also provide numerical examples and discuss their unique solutions. To show the authenticity of the solutions, we will also provide 2D and 3D plots of the solutions.

Keywords: fuzzy numbers; existence; uniqueness; symmetric fuzzy numbers; non-symmetric fuzzy numbers; uncertainty

Mathematics Subject Classification: 35R13, 26E50, 34K37

1. Introduction

Zadeh [1] introduced the concept of fuzzy numbers(F-numbers) and fuzzy operations. The fuzzy set theory is a robust mathematical framework for addressing uncertainty in practical situations. The physical models, network problems [2, 3], etc. are facing uncertainty. The uncertainty is dealt with easily in the fuzzy models. The fuzzy concept is important to optimize various problems, such as robot routing and energy consumption problems [4]. In the decision-making models for the operating system and human-computer interaction [5], data envelopment analysis in the banking industry [6], medical resources allocation [7], etc., fuzzy concepts have significant importance.

The operations of F-numbers do not indicate any interaction due to the random variables used in the arithmetic operations. Therefore, Carlsson et al. [8] introduced interactive F-numbers, whose interactivities were based on some joint distribution functions. To avoid the need for joint distribution functions, Barros and Pedro [9] introduced linear correlated fuzzy numbers (LCF-numbers). Esmi et al. [10] defined an operator from 2D Euclidean space to space of LCF-numbers $R_{F(\mathbb{A})}$. The spaces of LCF-numbers $R_{F(\mathbb{A})}$ are based on fuzzy numbers $\mathbb{A}$, known as basic fuzzy numbers(BF-numbers). If the BF-number is non-symmetric, then the space of LCF-numbers is linear and given by $R^n_{F(\mathbb{A})}$. Therefore, the operator from 2D Euclidean space to space of LCF-numbers $R^n_{F(\mathbb{A})}$ is a bijection. The addition and scalar multiplication in $R^n_{F(\mathbb{A})}$ are defined by the linear isomorphism. While fuzzy difference $\ominus_{\mathbb{A}}$ is defined by addition and scalar multiplication. But if the BF-number is symmetric, then the space of LCF-numbers is not linear and is given by $R^s_{F(\mathbb{A})}$. The operator from 2D Euclidean space to the space of LCF-numbers $R^s_{F(\mathbb{A})}$ is not a bijection. Therefore, the fuzzy operations in space $R^s_{F(\mathbb{A})}$ are not possible by the process performed for space $R^n_{F(\mathbb{A})}$. Shen's [11] defined equivalence relations and conical representation to address this complicity in the space $R^s_{F(\mathbb{A})}$ by defining a bijection operator from the set of equivalence classes associated with the equivalence relation in R^2 to the space $R_{F(\mathbb{A})}$. Therefore, the fuzzy operations were defined by the linear isomorphism and developed calculus for both non-symmetric and symmetric fuzzy numbers. But $C \ominus_{\mathbb{A}} C \neq 0$ if $C \in R_F \setminus R$ is symmetric also the space $R^s_{F(\mathbb{A})}$ is not liner. To avoid these difficulties, Shen's [12] introduced LC-difference and LC-differentiability, which always exist in both spaces $R^n_{F(\mathbb{A})}$ and $R^s_{F(\mathbb{A})}$. LC-differentiability is equivalent to Fréchet differentiability [10] and differentiability of [11] in the spaces $R^n_{F(\mathbb{A})}$. Also in the spaces, $R^s_{F(\mathbb{A})}$ LC-differentiability and gH-differentiability [13] are equivalent. The linear correlated fuzzy differential equations (LCFDEs) with LC-differentiability were discussed by [14]. These LCFDEs mostly have many solutions due to the extension process and sometimes do not have solutions. Therefore, Jamal et al. [15] discussed the existence and uniqueness conditions for unique solutions of LCFDEs. In their paper [16], the authors discussed the existence for unique solutions of LCFDEs in conical form and their practical applications.

Fractional calculus is the globalization of classical calculus. The fractional models have significant use in dealing with real-life problem; therefore, existence [17], stability [18, 19] and control [20, 21] are discussed by the authors of these papers and many others. The fuzzy fractional differential equations have also wide application in the population dynamic [22], evaluating weapon systems [23], electro-hydraulic servo systems [24], security control in response to cyber attacks [25] and other scientific area (see [26, 27]) etc. The linear correlated fuzzy fractional differential equations (LCFFDEs) with Caputo's fractional LC-differentiability are discussed in [28]. They discussed the stability of LCFFDEs by considering unique solutions to LCFFDEs. The LCFFDEs mostly do not have unique solutions due to the extension process and produce many continuous and differentiable solutions. The existing literature does not have any criteria to ensure the existence of unique solutions for LCFFDEs.

Motivated by the above deficiency in the existing literature, in this paper, we will discuss the existence and uniqueness conditions for solutions to LCFFDEs. LCFFDEs mostly have many solutions due to the extension process. The main case of the extension of a system into a new system is the form of LCFFDEs discussed in [28]. Therefore, we will discuss the following LCFFDEs to avoid the extension of systems.

$$\begin{cases} {}^{C}_{LC}D^{\theta}_{a}\vartheta(\gimel) = \zeta(\gimel, \vartheta(\gimel)), \ \gimel \in I, \\ \qquad \vartheta(a) = \vartheta_0, \ \vartheta_0 \in R_{F(\mathbb{A})}, \end{cases} \tag{1.1}$$

where, $\zeta : I \times R_{F(\mathbb{A})} \to R_{F(\mathbb{A})}$ is a LCF-numbers valued function. At the points at which the extension condition holds, the LCFFDEs of [28] will extend and produce many solutions, but Eq (1.1) does not extend at these points therefore, we preferred to discuss Eq (1.1). For the authenticity of established results, we will provide numerical examples. To show the validity of our findings, we will compare the solutions of numerical examples with the results of [28].

2. Preliminaries

Throughout this manuscript, R_F, $R_{F(\mathbb{A})}$ and R denote the space of F-numbers, LCF-numbers and real numbers, respectively.

Definition 2.1. *[29] The mapping $D : R \to [0,1]$ is a fuzzy number if D following conditions:*

(i) *D exhibits upper semi-continuity;*
(ii) *For all $a, b \in R$ and $\mu \in [0,1]$, $D(\mu a + (1-\mu)b) \geq \min D(a), D(b)$;*
(iii) *D is normal, implying there exists $c \in R$ such that $k(c) = 1$;*
(iv) *The closure of $\{a \in R \mid D(a) > 0\}$ is compact.*

The space R_F contains all fuzzy numbers.

Definition 2.2. *[30] The set $\{a \in R \mid D(a) \geq \alpha\}$ with $\alpha \in [0,1]$ is called $\alpha-$level set of F-number, $D \in R_F$.*

Where the $\alpha-$level set has lower and upper bounds $\underline{D}(a)$ and $\overline{D}(a)$, respectively.

The $\alpha-$level sets of triangular F-numbers $D = (x; y; \vartheta)$ and trapezoidal $E = (p; x; y; \vartheta)$ where $p \leq x \leq y \leq \vartheta$ are given by

$$[D]_\alpha = [x + (y-x)\alpha, \vartheta - (\vartheta - y)\alpha], \quad [E]_\alpha = [p + (x-p)\alpha, \vartheta - (\vartheta - y)\alpha].$$

The diameter of a fuzzy number D is the length of the support set $\{a \in R \mid D(a) \geq 0\}$.

Definition 2.3. *[10] If for unique $a_0 \in R$, $D(a + a_0) = D(a_0 - a)$ for all $a \in R$, then D is a symmetric F-number with respect to a_0; otherwise, D is non-symmetric.*

Let $\Psi_{\mathbb{A}} : R^2 \to R_{F(\mathbb{A})}$ be an operator such that for all $(g,h) \in R^2$, there exists $D \in R_{F(\mathbb{A})}$ where $D = \Psi_{\mathbb{A}}(g,h)$. Then, $R_{F(\mathbb{A})} = \left\{\Psi_{\mathbb{A}}(g,h) \mid (g,h) \in R^2\right\}$ is the space of LCF-numbers of BF-number $\mathbb{A}$. If BF-number $\mathbb{A}$ is non-symmetric, then the space of LCF-numbers is denoted by $R^n_{F(\mathbb{A})}$, while if BF-number $\mathbb{A}$ is symmetric, then the space of LCF-numbers is denoted by $R^s_{F(\mathbb{A})}$. Let for $(g,h) \in R^2$ there exist $D = \Psi_{\mathbb{A}}(g,h) = gA + h$ then D is called LCF-number. Moreover, the α-level set is defined by $[\Psi_{\mathbb{A}}(g,h)]_\alpha = \left\{gt + h \in R \mid t \in [\mathbb{A}]_\alpha\right\} = p[\mathbb{A}]_\alpha + s$. Clearly, if $\mathbb{A}$ is a non-symmetric F-number, then $\Psi_{\mathbb{A}}$ is one-one and onto, therefore $(R_{F(\mathbb{A})}, \oplus_{\mathbb{A}}, \odot_{\mathbb{A}})$ is a linear space, where $\oplus_{\mathbb{A}}$ and $\odot_{\mathbb{A}}$ are defined as

$$D_1 \oplus_{\mathbb{A}} D_2 = \Psi_{\mathbb{A}}({\Psi_{\mathbb{A}}}^{-1}(D_1) + {\Psi_{\mathbb{A}}}^{-1}(D_2)) \text{ and } \beta \odot_{\mathbb{A}} D = \Psi_{\mathbb{A}}(\beta {\Psi_{\mathbb{A}}}^{-1}(D)).$$

Moreover, if $\mathbb{A} \in R_F \backslash R$ is symmetric, then $\Psi_{\mathbb{A}} : R^2 \to R_{F(\mathbb{A})}$ is not one-one because $\Psi_{\mathbb{A}}(g,h) = \Psi_{\mathbb{A}}(-g, 2gt + h)$, where t is a symmetric point.

Definition 2.4. *[11] The equivalence relation $\equiv_{\mathbb{A}}$ of any $(g,h),(j,k) \in R^2$ is define by $(g,h) \equiv_{\mathbb{A}} (j,h)$ if and only if $(g,h) = (j,k)$ or $(g,h) = (-j, 2jt + k)$.*

The set of equivalence classes associated with an equivalence relation $\equiv_{\mathbb{A}}$ is a quotient in R^2, defined by $R^2/\equiv_{\mathbb{A}} = \left\{[(g,h)]_{\equiv_{\mathbb{A}}} \mid (g,h) \in R^2\right\}$ where, $[(g,h)]_{\equiv_{\mathbb{A}}} = \left\{(g,h), (-g, 2gt + h)\right\}$ is equivalence class. The fuzzy operations $\oplus_{\mathbb{A}}$ and $\odot_{\mathbb{A}}$ in $R^2/\equiv_{\mathbb{A}}$ are defined as $[(g,h)]_{\equiv_{\mathbb{A}}} \oplus_{\mathbb{A}} [(j,k)]_{\equiv_{\mathbb{A}}} = [(g+j, h+k)]_{\equiv_{\mathbb{A}}}$ and

$$\beta \odot_{\mathbb{A}} [(g,h)]_{\equiv_{\mathbb{A}}} = \begin{cases} [(\beta g, \beta h)]_{\equiv_{\mathbb{A}}}, \beta \geq 0, \\ [(-\beta g, 2\beta g t + \beta h)]_{\equiv_{\mathbb{A}}}, \beta < 0. \end{cases}$$

Now, if $\Psi_{\mathbb{A}}([(g,h)]_{\equiv_{\mathbb{A}}}) = \widetilde{g}\mathbb{A} + \widetilde{h}$ is canonical form where $\widetilde{g} = g$ or $\widetilde{g} = -g$ and $\widetilde{h} = h$ or $\widetilde{h} = 2gt + h$. Now the operator $\Psi_{\mathbb{A}}$ is a bijection from $R^2/\equiv_{\mathbb{A}}$ to $R_{F(\mathbb{A})}$. The difference $\ominus_{\mathbb{A}}$ is defined from the operation of $\oplus_{\mathbb{A}}$ and $\odot_{\mathbb{A}}$ in $R^2/\equiv_{\mathbb{A}}$ as $D_1 \ominus_{\mathbb{A}} D_2 = D_1 \oplus_{\mathbb{A}} (-1) \odot_{\mathbb{A}} D_2$. From Proposition 3.5 in [11], $C \ominus_{\mathbb{A}} C \neq 0$, if $\in R_F \backslash R$ is symmetric. To remove this drawback, linear correlated fuzzy difference (LC-difference) is introduced in the literature.

Definition 2.5. *[12] Let $D_1, D_2 \in R^n_{F(\mathbb{A})}$ then LC-difference is defined as*

$$D_1 \boxminus_{\mathbb{A}} D_2 = \Psi_{\mathbb{A}}(g,h) \boxminus_{\mathbb{A}} \Psi_{\mathbb{A}}(j,k) = \Psi_{\mathbb{A}}(g-j, h-k) = (g-j)\mathbb{A} + h - k.$$

Moreover, if $D_1, D_2 \in R^s_{F(\mathbb{A})}$ where BF-number $\mathbb{A}$ has point of symmetry t, then LC-difference is defined as

$$D_1 \boxminus_{\mathbb{A}} D_2 = \Psi_{\mathbb{A}}([(g,h)]_{\equiv_{\mathbb{A}}}) \boxminus_{\mathbb{A}} \Psi_{\mathbb{A}}([(j,k)]_{\equiv_{\mathbb{A}}}) = \Psi_{\mathbb{A}}([(g,h)]_{\equiv_{\mathbb{A}}} \boxminus_{\mathbb{A}} [(j,k)]_{\equiv_{\mathbb{A}}}),$$

$$\Psi_{\mathbb{A}}([(g,h)]_{\equiv_{\mathbb{A}}} \boxminus_{\mathbb{A}} [(j,k)]_{\equiv_{\mathbb{A}}}) = \begin{cases} (g-j)\mathbb{A} + h - k, g \geq j, \\ (j-g)\mathbb{A} + 2(g-j)y + h - k, g < j. \end{cases}$$

Moreover, the metric $d_{\Psi_{\mathbb{A}}}$ with LC-difference in the space $R^n_{F(\mathbb{A})}$ is defined as

$$d_{\Psi_{\mathbb{A}}}(D,E) = \|D \boxminus_{\mathbb{A}} D\|_{\Psi_{\mathbb{A}}}, \text{ where } D, E \in R^n_{F(\mathbb{A})}.$$

While, the norm $\|.\|_{\Psi_{\mathbb{A}}}$ is defined as $\|D\|_{\Psi_{\mathbb{A}}} = \|{\Psi_{\mathbb{A}}}^{-1}(D)\|_\infty$, with $D \in R^n_{F(\mathbb{A})}$.
Furthermore, the metric $d_{\Psi_{\mathbb{A}}}$ with LC-difference in the space $R^s_{F(\mathbb{A})}$ is defined as

$$d_{\Psi_{\mathbb{A}}}(D,E) = \|D \boxminus_{\mathbb{A}} E\|_{\Psi_{\mathbb{A}}} = \|[(g,h)]_{\equiv_{\mathbb{A}}} \boxminus_{\mathbb{A}} [(j,k)]_{\equiv_{\mathbb{A}}}\|_{\Psi_{\mathbb{A}}}, \text{ where } D, E \in R^s_{F(\mathbb{A})}.$$

Where, norm $\|.\|_{\Psi_{\mathbb{A}}}$ is defined as $\|D\|_{\Psi_{\mathbb{A}}} = \|{\Psi_{\mathbb{A}}}^{-1}(D)\|_\infty = \|[(g,h)]_{\equiv_{\mathbb{A}}}\|_\infty$, with $[(g,h)]_{\equiv_{\mathbb{A}}} \in R^2/\equiv_{\mathbb{A}}$.

$$\|[(g,h)]_{\equiv_{\mathbb{A}}}\|_\infty = \max\{\|(g,h)\|_\infty, (-g, 2gt + h)\|_\infty\}.$$

Lemma 2.6. *[12] Let $\mathbb{A} \in R_F$ be non-symmetric and the continuous function $f : I \to R_{F(\mathbb{A})}$ is defined by $f(a) = g(a)\mathbb{A} + h(a)$. If $g(a)$ and $h(a)$ are differentiable, then f is LC-differentiable, and $f^{'}(t) = g^{'}(a)\mathbb{A} + h^{'}(a)$.*

Lemma 2.7. *[12] Let the function $f : I \to R_{F(\mathbb{A})}$ be continuous, and its canonical form is given by*

$$f(a) = \Psi_{\mathbb{A}}([(g,h)]_{\equiv_{\mathbb{A}}}) = \widetilde{g}\mathbb{A} + \widetilde{h} = \begin{cases} g\mathbb{A} + h, g \geq 0, \\ -g\mathbb{A} + 2gt + h, g < 0, \end{cases}$$

where, $\mathbb{A} \in R_F \backslash R$ is symmetric with symmetric point t then $f(a)$ is LC-differentiable if the following conditions holds:
$\widetilde{g'}_-(a) = \widetilde{g'}_+(a), \widetilde{h'}_-(a) = \widetilde{h'}_+(a)$, or $\widetilde{g'}_-(a) = -\widetilde{g'}_+(a), \widetilde{h'}_-(a) = 2\widetilde{g'}_+(a)t + \widetilde{h'}_+(a)$,
where, the right and left derivatives of $\widetilde{g}, \widetilde{h}$ are denoted by $\widetilde{g'}_+, \widetilde{h'}_+$ and $\widetilde{g'}_-, \widetilde{h'}_-$ respectively.

Definition 2.8. *[28] The Caputo fractional linear correlated derivative of order $\theta \in (0, 1]$ of $\vartheta : I \to R_{F(\mathbb{A})}$, where $\vartheta(\gimel) = r(\gimel)\mathbb{A} + s(\gimel)$ and $r, s : I \to R$, is defined as*

$${}^{C}_{LC}D^{\theta}_{a^+}\vartheta(\gimel) = \Psi_{\mathbb{A}}\Big({}^{C}_{LC}D^{\theta}_{a^+}r(\gimel), {}^{C}_{LC}D^{\theta}_{a^+}s(\gimel)\Big),$$

where for function $y(\gimel)$ we have

$${}^{C}_{LC}D^{\theta}_{a^+}y(\gimel) = \frac{1}{\Gamma(n-\theta)}\int_a^{\gimel}(\gimel - \gimel')^{n-\theta-1}y^n(\gimel')d\gimel'.$$

Definition 2.9. *[28] The Riemann-Liouville linear correlated fractional integral of order $\theta \in (0, 1]$ of $\vartheta : I \to R_{F(\mathbb{A})}$, defined by $\vartheta(\gimel) = r(\gimel)\mathbb{A} + s(\gimel)$ where $r, s : I \to R$,*

$${}^{RL}_{LC}I^{\theta}_{a^+}\vartheta(\gimel) = \Psi_{\mathbb{A}}\Big({}^{RL}_{LC}I^{\theta}_{a^+}r(\gimel), {}^{RL}_{LC}I^{\theta}_{a^+}s(\gimel)\Big),$$

where the fractional integral of the function say $y(\gimel)$ is defined as

$${}^{RL}_{LC}I^{\theta}_{a^+}y(\gimel) = \frac{1}{\Gamma(\theta)}\int_a^{\gimel}(\gimel - \gimel')^{\theta-1}y(\gimel')d\gimel',$$

such that the antiderivative converges to some value.

Definition 2.10. *[31] Mittag-Leffler type two-parametric function is defined by $E_{a,b}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(ak+b)}$, where $\Gamma(x) = \int_0^{\infty} e^{-\gimel}\gimel^{x-1}dt$ $x, a, b \in C$ where real part of $a, b > 0$.*

The Mittag-Leffler type two-parametric function has the following properties;

(i) $x^{\theta}E_{1,\theta+1}(\alpha x) = \frac{1}{\Gamma\theta}\int_0^x(x - x')^{\theta-1}e^{\alpha x'}dx'$ *where* $\theta > 0$;

(ii) $E_{1,m}(x) = \frac{1}{x^{m-1}}\Big\{e^x - \sum_{k=0}^{m-2}\frac{x^k}{k!}\Big\}$.

Definition 2.11. *[14]*

If Problem (1.1) possesses a solution, denoted as $\vartheta(\gimel)$ in the space $C(I, R_F(\mathbb{A}))$, then there exists a vector (r, s) such that $\vartheta(\gimel) = r(\gimel)\mathbb{A} + s(\gimel)$ and $\vartheta(\gimel)$ satisfies the equation of Problem (1.1). Additionally, for any $\vartheta_0 \in C(I, R_F(\mathbb{A}))$, there exist s_0 and r_0 such that $\vartheta_0 = r_0\mathbb{A} + s_0$.

3. LCFFDEs in the LC-spaces of non-symmetric basic fuzzy numbers

In this section, we discuss the existence and uniqueness of the solution for Problem (1.1) with a non-symmetric BF-number, denoted as $\mathbb{A} \in R^n_{F(\mathbb{A})}$.

$$\begin{cases} {}^{C}_{LC}D^{\theta}_{a^+}\vartheta(\gimel) = \zeta(\gimel, \vartheta(\gimel)), \\ \vartheta(\gimel_0) = r_0\mathbb{A} + s_0. \end{cases}$$

One can easily express this problem in the following equivalent systems of equations:

$$\begin{cases} {}^{C}_{LC}D^{\theta}_{a^+}r(\gimel) = \zeta(\gimel, r(\gimel)), \\ {}^{C}_{LC}D^{\theta}_{a^+}s(\gimel) = \zeta(\gimel, s(\gimel)), \\ r(\gimel_0) = r_0, s(\gimel_0) = s_0. \end{cases}$$

From this, the following equivalent system of integral equations is obtained:

$$\begin{cases} r(\gimel) = & r_0 + \frac{1}{\Gamma\theta}\int_a^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel, r(\gimel))d\gimel', \\ s(\gimel) = & s_0 + \frac{1}{\Gamma\theta}\int_a^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel, s(\gimel))d\gimel'. \end{cases} \tag{3.1}$$

In the following theorem, we discuss the existence and uniqueness of the solution for Problem (1.1) for non-symmetric BF-numbers.

Theorem 3.1. *Let* $|\zeta(\gimel, r(\gimel))| \leq \alpha|r(\gimel)|$ *and* $|\zeta(\gimel, s_1(\gimel))-\zeta(\gimel, s_2(\gimel))| \leq |s_1(\gimel)-s_2(\gimel)|$ *such that* $0 < \left[\frac{1-e^{-\alpha\gimel}}{(\alpha)^{\theta-1}}\right] < 1$ *then Problem (1.1) has a unique solution* $\vartheta(\gimel) \in C(I, R^n_{F(\mathbb{A})})$.

Proof. Let us define the metric $\alpha \geq 1$

$$d(\vartheta_1(\gimel), \vartheta_2(\gimel)) = \sup_{\gimel\in I}\{||\vartheta_1(\gimel) - \vartheta_2(\gimel)||e^{-\alpha\gimel}\}, \alpha \geq 1, \vartheta_1, \vartheta_2 \in C(I, R^n_{F(\mathbb{A})}). \tag{3.2}$$

The operator, $T : R^n_{F(\mathbb{A})} \to R^n_{F(\mathbb{A})}$ is defined as $T(\vartheta(\gimel)) = \vartheta(\gimel)$, where $\vartheta(\gimel) = \psi_{\mathbb{A}}(r(\gimel), s(\gimel))$. Therefore, by using Eq (3.1), we have

$$T(\psi_{\mathbb{A}}(r(\gimel), s(\gimel))) = \psi_{\mathbb{A}}(r_0 + \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel', r(\gimel'))d\gimel', s_0 + \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel', s(\gimel'))d\gimel').$$

Now, we have to prove that T is a contraction.

$$\begin{aligned} d(T\vartheta_1(\gimel), T\vartheta_2(\gimel)) &= \sup_{\gimel\in I}\{||\vartheta_1(\gimel) - \vartheta_2(\gimel)||_{\psi_{\mathbb{A}}}e^{-\alpha\gimel}\} = \sup_{\gimel\in I}\{||\psi_{\mathbb{A}}(r_1(\gimel), s_1(\gimel)) - \psi_{\mathbb{A}}(r_2(\gimel), s_2(\gimel))||_{\psi_{\mathbb{A}}}e^{-\alpha\gimel}\} \\ &= \sup_{\gimel\in I}\{||\psi_{\mathbb{A}}(r_0 + \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel', r_1(\gimel'))d\gimel', s_0 + \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel', s_1(\gimel'))d\gimel') - \psi_{\mathbb{A}}(r_0 \\ &\quad + \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel', r_2(\gimel'))d\gimel', s_0 + \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel', s_2(\gimel'))d\gimel')||_{\psi_{\mathbb{A}}}e^{-\alpha\gimel}\} \end{aligned}$$

$$= \sup_{\gimel\in I}\{\|\psi_{\mathbb{A}}(\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',r_1(\gimel'))d\gimel' - \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',r_2(\gimel'))d\gimel', \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}$$
$$\zeta(\gimel',s_1(\gimel'))d\gimel') - \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',s_2(\gimel'))d\gimel')\|_{\psi_{\mathbb{A}}}e^{-\alpha\gimel}\}$$
$$= \sup_{\gimel\in I}\{\|\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\psi_{\mathbb{A}}(\zeta(\gimel',r_1(\gimel')) - \zeta(\gimel',r_2(\gimel')), \zeta(\gimel',s_1(\gimel')) - \zeta(\gimel',s_2(\gimel'))d\gimel')\|_{\psi_{\mathbb{A}}}e^{-\alpha\gimel}\}$$

$$\leq \sup_{\gimel\in I}\{\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\|\psi_{\mathbb{A}}(\zeta(\gimel',r_1(\gimel')) - \zeta(\gimel',r_2(\gimel')), \zeta(\gimel',s_1(\gimel')) - \zeta(\gimel',s_2(\gimel')))\|_{\psi_{\mathbb{A}}}d\gimel' e^{-\alpha\gimel}\}$$
$$\leq \sup_{\gimel\in I}\{\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\|\psi_{\mathbb{A}}(\alpha r_1(\gimel') - \alpha r_2(\gimel'), s_1(\gimel')) - s_2(\gimel')))\|_{\psi_{\mathbb{A}}}d\gimel' e^{-\alpha\gimel}\}$$
$$\leq \sup_{\gimel\in I}\{\frac{\alpha}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\|\psi_{\mathbb{A}}(r_1(\gimel'), s_1(\gimel')) - \psi_{\mathbb{A}}(r_2(\gimel'), s_2(\gimel'))\|_{\psi_{\mathbb{A}}}d\gimel' e^{-\alpha\gimel}\}$$
$$= \sup_{\gimel\in I}\{\frac{\alpha}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\|_{\psi_{\mathbb{A}}}\vartheta_1(\gimel') - \vartheta_2(\gimel'))\|_{\psi_{\mathbb{A}}}e^{-\alpha\gimel'}e^{\alpha\gimel'}d\gimel' e^{-\alpha\gimel}\}$$
$$= \sup_{\gimel\in I}\{\frac{\alpha}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}e^{\alpha\gimel'}d\gimel' e^{-\alpha\gimel}d(\vartheta_1(\gimel),\vartheta_2(\gimel))\}.$$

Using the two parameter Mittag-Leffler function $E_{1,\theta+1}$ as follows:

$$= \sup_{\gimel\in I}\{\left[\gimel^{\theta}E_{1,\theta+1}(\alpha\gimel)e^{-\alpha_2\gimel}\right]\}e^{-\alpha\gimel}\alpha d(\vartheta_1(\gimel),\vartheta_2(\gimel)).$$

Using the series expression of Mittag-Leffler function $E_{1,\theta+1}$ as follows:

$$= \sup_{\gimel\in I}\left[\gimel^{\theta}\frac{1}{(\alpha\gimel)^{\theta}}\left\{e^{\alpha\gimel} - \sum_{k=0}^{\theta-1}\frac{(\alpha\gimel)^k}{k!}\right\}e^{-\alpha\gimel}\right]\alpha d(\vartheta_1(\gimel),\vartheta_2(\gimel)) = \left[\frac{1-e^{-\alpha\gimel_0}}{(\alpha)^{\theta}}\right]\alpha d(\vartheta_1(\gimel),\vartheta_2(\gimel)).$$

Hence, $d(T\vartheta_1(\gimel), T\vartheta_2(\gimel)) \leq kd(\vartheta_1(\gimel),\vartheta_2(\gimel))$ where, $k = \left[\frac{1-e^{-\alpha\gimel}}{(\alpha)^{\theta-1}}\right]$. If $\alpha \geq 1$ is choose enough large that $0 < k = \left[\frac{1-e^{-\alpha\gimel_0}}{(\alpha)^{\theta-1}}\right] < 1$ then T is a contraction and Problem (1.1) has a unique solution $\vartheta(\gimel) \in C(I, R^n_{F(\mathbb{A})})$. □

Example 3.2. *If with a non-symmetric BF-number,* $\mathbb{A} = (0; 2; 3)$, *one can take the FDEs*

$$ {}^{C}_{LC}D^{\frac{1}{2}}_{0^+}\vartheta(\gimel) = \frac{16}{3\sqrt{\pi}\sqrt{\gimel}} \odot_{\mathbb{A}} \vartheta(\gimel) \boxminus_{\mathbb{A}} \frac{10\sqrt{\gimel}}{3\sqrt{\pi}}, \tag{3.3}$$

The Eq (3.3) satisfying the conditions of Theorem (3.1) by choosing $\alpha \geq 1$ *is sufficiently large that* $|\zeta(\gimel, r(\gimel))| = \left|\frac{16\gimel\sqrt{\gimel}}{3\sqrt{\pi}}\right| \leq \alpha|r(\gimel)| = \alpha|\gimel^2|$ *and* $0 < k = \left[\frac{1-e^{-\alpha\gimel}}{(\alpha)^{-\frac{1}{2}}}\right] < 1$ *for all* $\gimel \in (0,\infty)$.

Hence, Eq (3.3) has a unique solution $\vartheta(\gimel) = \gimel^2\mathbb{A} + \gimel$, *for all* $\gimel \in (0,\infty)$. *Figure 1 show* 2D *and* 3D *plots of this solution.*

Moreover, $\vartheta(\gimel) = \gimel^2\mathbb{A} + \gimel$ *is also the solution of the following equation discussed in the example (3.1) of the paper [28].*

$$ {}^{C}_{LC}D^{\frac{1}{2}}_{0^+}\vartheta(\gimel) = \frac{16\gimel\sqrt{\gimel}}{3\sqrt{\pi}}\mathbb{A} + \frac{2\sqrt{\gimel}}{\sqrt{\pi}}. \tag{3.4}$$

But Eq (3.4) does not have a unique solution, while Eq (3.3) has a unique solution.

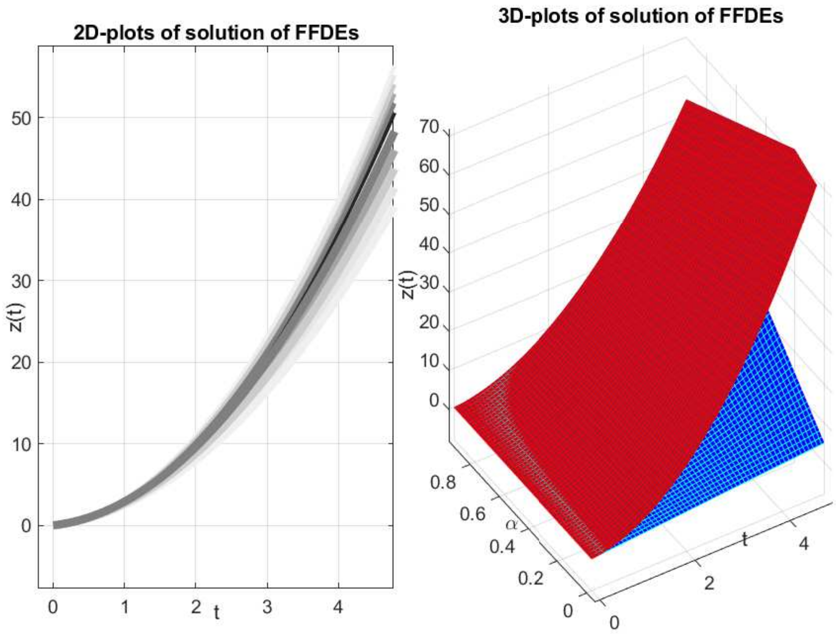

Figure 1. Show 2D-fuzzy plots and 3D-fuzzy plots of the solution of Example (3.2).

Example 3.3. *If with non-symmetric BF-number,* $\mathbb{A} = (-1; 0; 2)$*, one can take the LCFFDEs*

$$\begin{cases} {}^{C}_{LC}D^{\frac{1}{2}}_{0^+}\vartheta(\gimel) = \dfrac{2\sqrt{\gimel}}{\sqrt{\pi}(\gimel+1)} \odot_{\mathbb{A}} \vartheta(\gimel) \boxminus_{\mathbb{A}} \dfrac{2\sqrt{\gimel}(\gimel+3)}{3\sqrt{\pi}}, \\ \vartheta(0) = \mathbb{A} + 1. \end{cases} \tag{3.5}$$

The following condition can be readily demonstrated $|\zeta(\gimel, r(\gimel))| = \left|\frac{2\sqrt{\gimel}}{\sqrt{\pi}}\right| \leq \alpha|r(\gimel)| = \alpha|(\gimel+1)|$*, Clearly for large value of* $\gimel$ *we need a large* α *to hold this condition, and the contraction condition* $0 < k = \left[\frac{1-e^{-\alpha\gimel}}{(\alpha)^{-\frac{1}{2}}}\right] < 1$ *holds for all* $\gimel \in [0, \infty)$ *and* $\alpha \geq 1$.

Thus, the condition of Theorem (3.1) holds for all $\gimel \in (0, \infty)$*. Consequently, the FDEs (3.5) possess a unique solution, given by* $\vartheta(\gimel) = (\gimel+1)\mathbb{A} + 1 - \gimel^2$*. Figure 2 show* 2*D and* 3*D plots of this solution.*

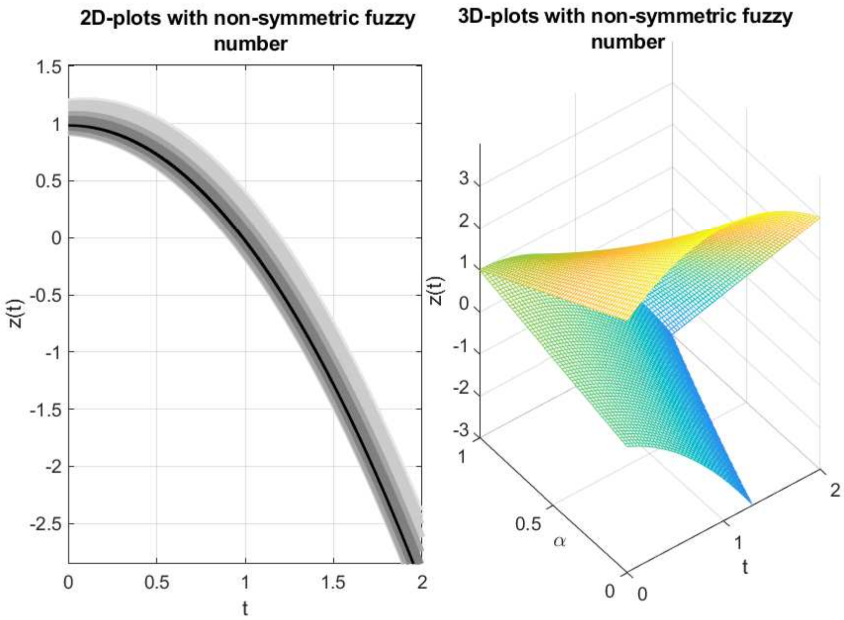

Figure 2. Show 2D-fuzzy plots and 3D-fuzzy plots of the solution of Example (3.3).

4. LCFFDEs in the LC-spaces of symmetric basic fuzzy numbers

In this section, we discuss the existence and uniqueness of the solution for Problem (1.1) in the space of symmetric BF-number $R_{F(\mathbb{A})}$.

If $\mathbb{A} \in R_F \backslash R$ is a symmetric BF-number with symmetric point x and $\Psi_{\mathbb{A}} : R^2/\equiv \mathbb{A} \to R^s_{F(\mathbb{A})}$ is bijection, then the canonical form of $\vartheta(\daleth) = \Psi_{\mathbb{A}}([\widetilde{r}(\daleth), \widetilde{s}(\daleth)]_{\equiv \mathbb{A}}) = \widetilde{r}(\daleth)\mathbb{A} + \widetilde{s}(\daleth)$
and ${}^{C}_{LC}D^{\theta}_{a}\vartheta(\daleth) = \Psi_{\mathbb{A}}([|{}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth)|, {}^{C}_{LC}D^{\theta}_{a}\widetilde{s}_c(\daleth)]_{\equiv\mathbb{A}}) = |{}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth)|\mathbb{A} + {}^{C}_{LC}D^{\theta}_{a}\widetilde{s}_c(\daleth)$, can be expressed as

$$
{}^{C}_{LC}D^{\theta}_{a}\vartheta(\daleth) = |{}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth)|\mathbb{A} + {}^{C}_{LC}D^{\theta}_{a}\widetilde{s}_c(\daleth) = \begin{cases} {}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth)\mathbb{A} + {}^{C}_{LC}D^{\theta}_{a}\widetilde{s}(\daleth) & \text{if } {}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth) \geq 0, \\ {}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth)\mathbb{A} + 2{}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth)x + {}^{C}_{LC}D^{\theta}_{a}\widetilde{s}(\daleth), & \text{if } {}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth) < 0. \end{cases}
$$

Therefore, Problem (1.1) has the following form:

$$
{}^{C}_{LC}D^{\theta}_{a}\vartheta(\daleth) = \Psi_{\mathbb{A}}([|{}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth)|, {}^{C}_{LC}D^{\theta}_{a}\widetilde{s}_c(\daleth)]_{\equiv\mathbb{A}}) = \begin{cases} \zeta(\daleth, \Psi_{\mathbb{A}}([\widetilde{r}(\daleth), \widetilde{s}_c(\daleth)]_{\equiv\mathbb{A}})), \\ \vartheta(\daleth_0) = \vartheta_0 = \widetilde{r}_0\mathbb{A} + \widetilde{s}. \end{cases} \tag{4.1}
$$

This produces the following equivalent system of equations

$$
\begin{cases} {}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth) = \zeta(\daleth, \widetilde{r}(\daleth)), \text{for } {}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth) \geq 0, \\ {}^{C}_{LC}D^{\theta}_{a}\widetilde{s}(\daleth) = \zeta(\daleth, \widetilde{s}(\daleth)), \end{cases} \text{and} \begin{cases} -{}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth) = \zeta(\daleth, \widetilde{r}(\daleth)), \text{for } {}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth) < 0 \\ 2{}^{C}_{LC}D^{\theta}_{a}\widetilde{r}(\daleth)x + {}^{C}_{LC}D^{\theta}_{a}\widetilde{s}(\daleth) = \zeta(\daleth, \widetilde{s}(\daleth)). \end{cases} \tag{4.2}
$$

Eq (4.2) can be expressed in the following integral form:

$$
\begin{cases} \widetilde{r}(\daleth) = \widetilde{r}_0 + \dfrac{1}{\Gamma\theta}\displaystyle\int_0^{\daleth} (\daleth - \daleth')^{\theta-1}\zeta(\daleth', \widetilde{r}(\daleth'))d\daleth', \\ \widetilde{s}(\daleth) = \widetilde{s}_0 + \dfrac{1}{\Gamma\theta}\displaystyle\int_0^{\daleth} (\daleth - \daleth')^{\theta-1}\zeta(\daleth', \widetilde{s}(\daleth'))d\daleth'. \end{cases} \tag{4.3}
$$

And

$$\begin{cases} \widetilde{r}(\gimel) = & \widetilde{r}_0 \ominus_{\mathbb{A}} \dfrac{1}{\Gamma\theta}\displaystyle\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{r}(\gimel'))d\gimel', \\ \widetilde{s}(\gimel) = & \widetilde{s}_0 + \dfrac{1}{\Gamma\theta}\displaystyle\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}(2xg(\gimel',\widetilde{r}(\gimel'))+\zeta(\gimel',\widetilde{s}(\gimel')))d\gimel'. \end{cases} \tag{4.4}$$

Now, we discuss the existence and uniqueness conditions for first-order LCFFDEs in space $R^s_{F(\mathbb{A})}$.

Theorem 4.1. *Let* $|\zeta(\gimel,\widetilde{r}(\gimel))| \leq \alpha|\widetilde{r}(\gimel)|$ *where* $\alpha \geq 1$, *and* $\|\zeta(\gimel,\widetilde{s}_1(\gimel)) - \zeta(\gimel,\widetilde{s}_2(\gimel))\| \leq \|\widetilde{s}_1(\gimel) - \widetilde{s}_2(\gimel)\|$ *such that* $\left[\frac{1-e^{-\alpha\gimel}}{(\alpha)^{\theta-1}}\right] < 1$, *then Problem (1.1) with symmetric basic fuzzy number has a unique solution.*

Proof. Let us define metric

$$\begin{aligned} d(\vartheta_1(\gimel),\vartheta_2(\gimel)) &= \sup_{\gimel\in I}\left\{\|\vartheta_1(\gimel)\boxminus_{\mathbb{A}}\vartheta_2(\gimel)\|_{\psi_{\mathbb{A}}}e^{-\alpha\gimel}\right\}\ \ \alpha\geq 1,\vartheta_1,\vartheta_2\in D = C(I,R^s_{F(\mathbb{A})}), \\ &= \sup_{\gimel\in I}\left\{\|\psi_{\mathbb{A}}([\widetilde{r}_1(\gimel),\widetilde{s}_1(\gimel)]_{\equiv\mathbb{A}})\boxminus_{\mathbb{A}}\psi_{\mathbb{A}}([\widetilde{r}_2(\gimel),\widetilde{s}_2(\gimel)]_{\equiv\mathbb{A}})\|_{\psi_{\mathbb{A}}}e^{-\alpha\gimel}\right\} \\ &= \sup_{\gimel\in I}\left\{\|\psi_{\mathbb{A}}([\widetilde{r}_1(\gimel)-\widetilde{r}_2(\gimel),\widetilde{s}_1(\gimel)-\widetilde{s}_2(\gimel)]_{\equiv\mathbb{A}})\|_{\psi_{\mathbb{A}}}e^{-\alpha\gimel}\right\} \\ &= \sup_{\gimel\in I}\left\{\max\left\{\|\widetilde{r}_1(\gimel)-\widetilde{r}_2(\gimel)\|,\|\widetilde{s}_1(\gimel)-\widetilde{s}_2(\gimel)\|,\|2x(\widetilde{r}_2(\gimel)-\widetilde{r}_1(\gimel))+\widetilde{s}_1(\gimel)-\widetilde{s}_2(\gimel)\|\right\}e^{-\alpha\gimel}\right\}. \end{aligned}$$

Case 1. First, we find the condition of a unique solution to Problem (1.1) with a non-decreasing diameter on $I_0 \subseteq I$. Let the operator $T : D \to D$ be defined as $T(\vartheta(\gimel)) = \vartheta(\gimel)$, where $\vartheta(\gimel) = \psi_{\mathbb{A}}([\widetilde{r}(\gimel),\widetilde{s}(\gimel)]_{\equiv\mathbb{A}})$; therefore, by using Eq (4.3), one can have

$$T(\vartheta(\gimel)) = T([\widetilde{r}(\gimel),\widetilde{s}(\gimel)]_{\equiv\mathbb{A}}) = \psi_{\mathbb{A}}([\widetilde{r}_0+\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{r}(\gimel'))d\gimel',\widetilde{s}_0+\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{s}(\gimel'))d\gimel']_{\equiv\mathbb{A}}).$$

Now, we have to prove that T is a contraction.

$$\begin{aligned} d(T\vartheta_1(\gimel),T\vartheta_2(\gimel)) &= \sup_{\gimel\in I}\left\{\|\vartheta_1(\gimel)\boxminus_{\mathbb{A}}\vartheta_2(\gimel)\|_{\psi_{\mathbb{A}}}e^{-\alpha\gimel}\right\}\ \ \alpha\geq 1,\vartheta_1,\vartheta_2\in C(I,R^s_{F(\mathbb{A})}) \\ &= \sup_{\gimel\in I}\Bigg\{\max\Bigg\{\Bigg\|\psi_{\mathbb{A}}\Bigg(\Bigg[\Big(\widetilde{r}_0+\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{r}_1(\gimel'))d\gimel' - (\widetilde{r}_0+\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{r}_2(\gimel'))d\gimel')\Big), \\ &\quad \Big(\widetilde{s}_0+\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{s}_1(\gimel'))d\gimel' - (\widetilde{s}_0+\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{s}_2(\gimel'))d\gimel')\Big)\Bigg]_{\equiv\mathbb{A}}\Bigg)\Bigg\|_{\psi_{\mathbb{A}}}\Bigg\}e^{-\alpha\gimel}\Bigg\} \\ &= \sup_{\gimel\in I}\Bigg\{\max\Bigg\{\Bigg\|\psi_{\mathbb{A}}\Bigg(\Bigg[\Big(\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{r}_1(\gimel'))d\gimel' - \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{r}_2(\gimel'))d\gimel'\Big), \\ &\quad \Big(\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{s}_1(\gimel'))d\gimel' - \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\zeta(\gimel',\widetilde{s}_2(\gimel'))d\gimel'\Big)\Bigg]_{\equiv\mathbb{A}}\Bigg)\Bigg\|_{\psi_{\mathbb{A}}}\Bigg\}e^{-\alpha\gimel}\Bigg\} \end{aligned}$$

$$= \sup_{\gimel\in I}\Bigg\{\max\Bigg\{\|\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\Big(\zeta(\gimel',\widetilde{r}_1(\gimel'))-\zeta(\gimel',\widetilde{r}_2(\gimel'))\Big)d\gimel'\|,\|\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel-\gimel')^{\theta-1}\Big(\zeta(\gimel',\widetilde{s}_1(\gimel'))$$

$$- \zeta(\gimel', \widetilde{s}_2(\gimel'))\Big)d\gimel'\|, \|2x\Big(\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\Big(\zeta(\gimel', \widetilde{r}_2(\gimel')) - \zeta(\gimel', \widetilde{r}_1(\gimel'))\Big)d\gimel'\Big) + \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\Big(\zeta(\gimel', \widetilde{s}_1(\gimel'))$$

$$- \zeta(\gimel', \widetilde{s}_2(\gimel'))\Big)d\gimel'\|\Big\}e^{-\alpha\gimel}\Big\}$$

$$\leq \sup_{\gimel\in I}\Big\{\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\max\Big\{\|\zeta(\gimel, \widetilde{r}_1(\gimel)) - \zeta(\gimel, \widetilde{r}_2(\gimel))\|, \|\zeta(\gimel, \widetilde{s}_1(\gimel)) - \zeta(\gimel, \widetilde{s}_2(\gimel))\|, \|2x\Big(\zeta(\gimel, \widetilde{r}_2(\gimel))$$

$$- \zeta(\gimel, \widetilde{r}_1(\gimel))\Big) + \Big(\zeta(\gimel, \widetilde{s}_1(\gimel)) - \zeta(\gimel, \widetilde{s}_2(\gimel))\Big)\|\Big\}d\gimel' e^{-\alpha\gimel}\Big\}$$

$$\leq \sup_{\gimel\in I}\Big\{\alpha\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\max\Big\{\|\widetilde{r}_1(\gimel) - \widetilde{r}_2(\gimel)\|, \|\widetilde{s}_1(\gimel) - \widetilde{s}_2(\gimel)\|, \|2x\Big(\widetilde{r}_2(\gimel) - \widetilde{r}_1(\gimel)\Big) + \Big(\widetilde{s}_1(\gimel)) - \widetilde{s}_2(\gimel))\Big)\|\Big\}d\gimel' e^{-\alpha\gimel}\Big\}$$

$$\leq \sup_{\gimel\in I}\Big\{\alpha\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\psi_{\mathbb{A}}([\widetilde{r}_1(\gimel) - \widetilde{r}_2(\gimel), \widetilde{s}_1(\gimel) - \widetilde{s}_2(\gimel)]_{\equiv\mathbb{A}})\|_{\psi_{\mathbb{A}}} e^{-\alpha\gimel'}e^{\alpha\gimel'}d\gimel' e^{-\alpha\gimel}\Big\}$$

$$= \sup_{\gimel\in I}\{\frac{\alpha}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\|\vartheta_1(\gimel') - \vartheta_2(\gimel')\|e^{-\alpha\gimel'}e^{\alpha\gimel'}d\gimel' e^{-\alpha\gimel}\} = \sup_{\gimel\in I}\{\frac{\alpha}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}e^{\alpha\gimel'}d\gimel' e^{-\alpha\gimel}d(\vartheta_1(\gimel), \vartheta_2(\gimel))\}.$$

Using the two parameter Mittag-Leffler function $E_{1,\theta+1}$ as follows:

$$= \sup_{\gimel\in I}\{\Big[\gimel^{\theta}E_{1,\theta+1}(\alpha\gimel)e^{-\alpha_2\gimel}\Big]\}e^{-\alpha\gimel}\alpha d(\vartheta_1(\gimel), \vartheta_2(\gimel)).$$

Using the series expression of Mittag-Leffler function $E_{1,\theta+1}$ as follows:

$$= \sup_{\gimel\in I}\Big[\gimel^{\theta}\frac{1}{(\alpha\gimel)^{\theta}}\Big\{e^{\alpha\gimel} - \sum_{k=0}^{\theta-1}\frac{(\alpha\gimel)^k}{k!}\Big\}e^{-\alpha\gimel}\Big]\alpha d(\vartheta_1(\gimel), \vartheta_2(\gimel)) = \Big[\frac{1 - e^{-\alpha\gimel_0}}{(\alpha)^{\theta-1}}\Big]d(\vartheta_1(\gimel), \vartheta_2(\gimel)).$$

Hence, $d(T\vartheta_1(\gimel), T\vartheta_2(\gimel)) \leq kd(\vartheta_1(\gimel), \vartheta_2(\gimel))$ where $k = \Big[\frac{1-e^{-\alpha\gimel}}{(\alpha)^{\theta-1}}\Big]$. If $\alpha \geq 1$ is choose enough large that $0 < k = \Big[\frac{1-e^{-\alpha\gimel}}{(\alpha)^{\theta-1}}\Big] < 1$, then T is a contraction, and problem (1.1) has a unique solution with a non-decreasing diameter on $I_0 \subseteq I$.

Case 2. Now, we find the condition of a unique solution of the problem (1.1) with a non-increasing diameter on $I_1 \subseteq I$. Let the operator $T : D \to D$ be defined by $T(\vartheta(\gimel)) = \vartheta(\gimel)$; therefore, by using Eq (4.4) one can obtain

$$T(\vartheta(\gimel)) = \psi_{\mathbb{A}}\Big([\widetilde{r}_0 - \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel', \widetilde{r}(\gimel'))d\gimel', \widetilde{s}_0 + \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}(2xg(\gimel', \widetilde{r}(\gimel')) + \zeta(\gimel', \widetilde{s}(\gimel')))d\gimel']_{\equiv\mathbb{A}}\Big).$$

Now, we have to prove that T is a contraction.

$$d(T\vartheta_1(\gimel), T\vartheta_2(\gimel)) = \sup_{\gimel\in I}\Big\{\|\vartheta_1(\gimel) \boxminus_{\mathbb{A}} \vartheta_2(\gimel)\|_{\psi_{\mathbb{A}}}e^{-\alpha\gimel}\Big\}, \ \ \alpha \geq 1, \vartheta_1, \vartheta_2 \in C(I, R^s_{F(\mathbb{A})})$$

$$= \sup_{\gimel\in I}\Big\{\max\Big\{\Big\|\psi_{\mathbb{A}}\Big(\Big[\Big(\widetilde{r}_0 - \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel', \widetilde{r}_1(\gimel'))d\gimel' - (\widetilde{r}_0 - \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\zeta(\gimel', \widetilde{r}_2(\gimel'))d\gimel'\Big),$$

$$\Big(\widetilde{s}_0 + \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}(2xg(\gimel', \widetilde{r}_1(\gimel')) + \zeta(\gimel', \widetilde{s}_1(\gimel')))d\gimel' - (\widetilde{s}_0 + \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}(2xg(\gimel', \widetilde{r}_2(\gimel'))$$

$$+ \zeta(\gimel', \widetilde{s}_2(\gimel')))d\gimel')\Big)\Big]_{\equiv\mathbb{A}}\Big)\Big\|_{\psi_{\mathbb{A}}}\Big\}e^{-\alpha\gimel}\Big\}$$

$$= \sup_{\gimel\in I}\Big\{\max\Big\{\|\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\Big(\zeta(\gimel', \widetilde{r}_2(\gimel')) - \zeta(\gimel', \widetilde{r}_1(\gimel'))\Big)d\gimel')\|, \|\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\Big(2xg(\gimel', \widetilde{r}_1(\gimel'))$$

$$- 2x\zeta(\gimel', \widetilde{r}_2(\gimel')) + \zeta(\gimel', \widetilde{s}_1(\gimel')) - \zeta(\gimel', \widetilde{s}_2(\gimel'))\Big)d\gimel'\|, \|2x\Big(\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\Big(\zeta(\gimel', \widetilde{r}_2(\gimel')) - \zeta(\gimel', \widetilde{r}_1(\gimel'))\Big)d\gimel'\Big)$$

$$+ \frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\Big(2xg(\gimel', \widetilde{r}_1(\gimel')) - 2x\zeta(\gimel', \widetilde{r}_2(\gimel')) + \zeta(\gimel', \widetilde{s}_1(\gimel')) - \zeta(\gimel', \widetilde{s}_2(\gimel'))\Big)d\gimel'\|\Big\}e^{-\alpha\gimel}\Big\}$$

$$= \sup_{\gimel\in I}\Big\{\max\Big\{\|\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\Big(\zeta(\gimel', \widetilde{r}_2(\gimel')) - \zeta(\gimel', \widetilde{r}_1(\gimel'))\Big)d\gimel')\|, \|\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\Big(2x\Big(\zeta(\gimel', \widetilde{r}_1(\gimel'))$$

$$- \zeta(\gimel', \widetilde{r}_2(\gimel'))\Big) + \zeta(\gimel', \widetilde{s}_1(\gimel')) - \zeta(\gimel', \widetilde{s}_2(\gimel'))\Big)d\gimel'\|, \|\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\Big(2x\zeta(\gimel', \widetilde{r}_2(\gimel')) - 2x\zeta(\gimel', \widetilde{r}_1(\gimel'))$$

$$+ 2xg(\gimel', \widetilde{r}_1(\gimel')) - 2x\zeta(\gimel', \widetilde{r}_2(\gimel')) + \zeta(\gimel', \widetilde{s}_1(\gimel')) - \zeta(\gimel', \widetilde{s}_2(\gimel'))\Big)d\gimel'\|\Big\}e^{-\alpha\gimel}\Big\}$$

$$\leq \sup_{\gimel\in I}\Big\{\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\max\Big\{\|\zeta(\gimel, \widetilde{r}_1(\gimel)) - \zeta(\gimel, \widetilde{r}_2(\gimel))\|, \|\zeta(\gimel, \widetilde{s}_1(\gimel)) - \zeta(\gimel, \widetilde{s}_2(\gimel))\|, \|2x\Big(\zeta(\gimel, \widetilde{r}_1(\gimel))$$

$$- \zeta(\gimel, \widetilde{r}_2(\gimel))\Big) + \Big(\zeta(\gimel, \widetilde{s}_1(\gimel)) - \zeta(\gimel, \widetilde{s}_2(\gimel))\Big)\|\Big\}d\gimel' e^{-\alpha\gimel}\Big\}$$

$$\leq \sup_{\gimel\in I}\Big\{\alpha\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\max\Big\{\|\widetilde{r}_1(\gimel) - \widetilde{r}_2(\gimel)\|, \|\widetilde{s}_1(\gimel) - \widetilde{s}_2(\gimel)\|, \|2x\Big(\widetilde{r}_1(\gimel) - \widetilde{r}_2(\gimel)\Big) + \Big(\widetilde{s}_1(\gimel)) - \widetilde{s}_2(\gimel))\Big)\|\Big\}d\gimel' e^{-\alpha\gimel}\Big\}$$

$$\leq \sup_{\gimel\in I}\Big\{\alpha\frac{1}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\psi_{\mathbb{A}}([\widetilde{r}_1(\gimel) - \widetilde{r}_2(\gimel), \widetilde{s}_1(\gimel) - \widetilde{s}_2(\gimel)]_{\equiv\mathbb{A}})\|_{\psi_{\mathbb{A}}} e^{-\alpha\gimel'} e^{\alpha\gimel'} d\gimel' e^{-\alpha\gimel}\Big\}$$

$$= \sup_{\gimel\in I}\{\frac{\alpha}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}\|\vartheta_1(\gimel) - \vartheta_2(\gimel))\|e^{-\alpha\gimel'}e^{\alpha\gimel'}d\gimel' e^{-\alpha\gimel}\} = \sup_{\gimel\in I}\{\frac{\alpha}{\Gamma\theta}\int_0^{\gimel}(\gimel - \gimel')^{\theta-1}e^{\alpha\gimel'}d\gimel' e^{-\alpha\gimel}d(\vartheta_1(\gimel), \vartheta_2(\gimel))\}.$$

Using the two parameter Mittag-Leffler function $E_{1,\theta+1}$ as follows:

$$= \sup_{\gimel\in I}\{\Big[\gimel^{\theta}E_{1,\theta+1}(\alpha\gimel)e^{-\alpha_2\gimel}\Big]\}e^{-\alpha\gimel}d(\vartheta_1(\gimel), \vartheta_2(\gimel).$$

Using series expression of Mittag-Leffler function $E_{1,\theta+1}$ as follow:

$$= \sup_{\gimel\in I}\Big[\gimel^{\theta}\frac{1}{(\alpha\gimel)^{\theta}}\Big\{e^{\alpha\gimel} - \sum_{k=0}^{\theta-1}\frac{(\alpha\gimel)^k}{k!}\Big\}e^{-\alpha\gimel}\Big]\alpha d(\vartheta_1(\gimel), \vartheta_2(\gimel)) = \Big[\frac{1 - e^{-\alpha\gimel}}{(\alpha)^{\theta}}\Big]\alpha d(\vartheta_1(\gimel), \vartheta_2(\gimel)).$$

Hence, $d(T\vartheta_1(\gimel), T\vartheta_2(\gimel)) \leq kd(\vartheta_1(\gimel), \vartheta_2(\gimel))$ where, $k = \Big[\frac{1-e^{-\alpha\gimel}}{(\alpha)^{\theta-1}}\Big]$. If $\alpha \geq 1$ is chosen enough large that $0 < k = \Big[\frac{1-e^{-\alpha\gimel}}{(\alpha)^{\theta-1}}\Big] < 1$, then T is a contraction and Problem (1.1) has a unique solution with non-increasing diameter on $I_1 \subseteq I$. Consequently, Problem (1.1) has a unique solution. □

Example 4.2. *If the symmetric BF-number,* $\mathbb{A} = (-1; 0; 1)$, *has symmetric point* 0 *we consider the* *following FFDEs:*

$$\begin{cases} {}^{C}_{LC}D^{\frac{1}{2}}_{0^+}\vartheta(\gimel) = \dfrac{2\sqrt{\gimel-1}}{\sqrt{\pi}(2-\gimel)} \odot_{\mathbb{A}} \vartheta(\gimel) \boxminus_{\mathbb{A}} \dfrac{2(\gimel-1)^{\frac{3}{2}}}{\sqrt{\pi}(2-\gimel)}, \\ \vartheta(0) = 2\mathbb{A}. \end{cases} \tag{4.5}$$

The Eq (4.5) has a fuzzy solution $\vartheta(\gimel) = (2-\gimel)\mathbb{A} + \gimel$ *with a non-increasing diameter in* $[0, 2)$. *By choosing* $\alpha \geq 1$ *sufficiently large, the conditions of Theorem (4.1) hold for all* $\gimel \in [0, 2)$ *as* $|\zeta(\gimel, r(\gimel))| = |\frac{2\sqrt{\gimel-1}}{\sqrt{\pi}}| \leq \alpha|2-\gimel|$, *but for* $\gimel = 2$ *we cannot find any* $\alpha \geq 1$ *at which the condition holds and the Eq.(4.5) is undefined on* $\gimel = 2$. *Hence, the fuzzy solution* $\vartheta(\gimel) = (2-\gimel)\mathbb{A} + \gimel$ *is a unique solution having a non-increasing diameter in* $[0, 2)$. *Figure 3 show* $2D$ *and* $3D$ *plots of the solution. Moreover, with solutions* $\vartheta(\gimel) = (2-\gimel)\mathbb{A} + \gimel$, *the Eq (4.5) produces following LCFFDEs discussed in the example* (3.2) *of paper [28]:*

$$\begin{cases} {}^{C}_{LC}D^{\frac{1}{2}}_{1^+}\vartheta(\gimel) = \dfrac{2\sqrt{\gimel-1}}{\sqrt{\pi}}\mathbb{A} + \dfrac{2\sqrt{\gimel-1}}{\sqrt{\pi}}, \\ \vartheta(1) = \mathbb{A} + 1. \end{cases} \tag{4.6}$$

But the Eq (4.6) does not have unique solutions $\vartheta(\gimel) = (2-\gimel)\mathbb{A} + \gimel$, *because* $\vartheta(\gimel) = (\gimel+1)\mathbb{A} + \gimel$ *and all equations in the form* $\vartheta(\gimel) = r(\gimel)\mathbb{A} + \gimel$, *with* $r'(\gimel) = 1$ *and* $r'(\gimel) = -1$, *are also there solutions. Hence, Eq (4.6) does not represent a suitable form of first-order FFDEs when the BF-number is symmetric. This underscores the significance of existence theory and the structure of Eq (1.1) in handling the physical model of LCFFDEs.*

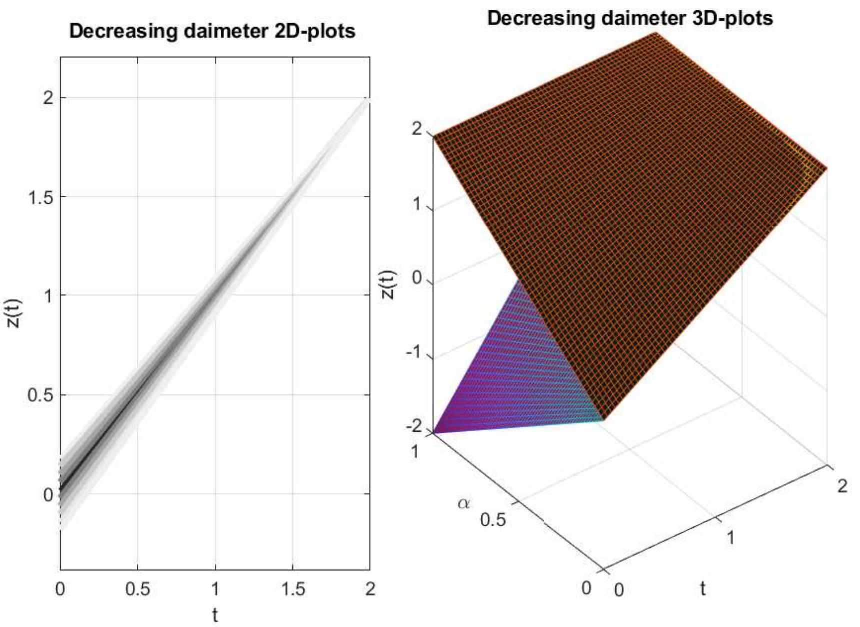

Figure 3. Show the 2D and 3D plots of the solution of example 4.2 with decreasing diameter in $[0, 2)$.

Example 4.3. *If the symmetric BF-number,* $\mathbb{A} = (-1; 0; 1)$ *has symmetric point* 0, *we consider the following FFDEs:*

$$\begin{cases} {}^{C}_{LC}D^{\frac{1}{2}}_{1^+}\vartheta(\gimel) = \dfrac{2\sqrt{\gimel-1}}{\sqrt{\pi}(\gimel+1)} \odot_{\mathbb{A}} \vartheta(\gimel) \oplus_{\mathbb{A}} \dfrac{4(\gimel-1)^{\frac{3}{2}}}{3\sqrt{\pi}}, \\ \vartheta(1) = \mathbb{A} - 1. \end{cases} \tag{4.7}$$

The Eq (4.7) has a fuzzy solution $\vartheta(\gimel) = (\gimel+1)\mathbb{A} + \gimel^2 - 1$ *with a non-decreasing diameter in* $[0,\infty)$. *By choosing* $\alpha \geq 1$ *sufficiently large conditions of Theorem (4.1) hold for all* $\gimel \in [0,\infty)$, $|\zeta(\gimel, r(\gimel))| = |\frac{2\sqrt{\gimel-1}}{\sqrt{\pi}}| \leq \alpha|\gimel+1|$, *therefore fuzzy solution of non-decreasing diameter in* $[0,\infty)$ *is unique solution. Figure 4 show* $2D$ *and* $3D$ *plots of the solution. Moreover, with solutions* $\vartheta(\gimel) = (\gimel+1)\mathbb{A} + \gimel^2 - 1$, *the Eq (4.7) produces the following LCFFDEs, discussed in the example* (4.2) *of paper [28].*

$${}^{C}_{LC}D^{\frac{1}{2}}_{1^+}\vartheta(\gimel) = \frac{2\sqrt{\gimel-1}}{\sqrt{\pi}}\mathbb{A} + \frac{8(\gimel-1)\sqrt{\gimel-1}}{\sqrt{3\pi}}. \tag{4.8}$$

But the Eq (4.8) does not have unique but solutions $\vartheta(\gimel) = (\gimel+1)\mathbb{A}+\gimel^2-\gimel$, *because* $\vartheta(\gimel) = |2-\gimel|\mathbb{A}+\gimel^2-1$ *and all equations in the form* $\vartheta(\gimel) = r(\gimel)\mathbb{A}+\gimel^2-1$, *with* $r'(\gimel) = 1$ *and* $r'(\gimel) = -1$ *are also their solutions. Therefore, Eq (4.8) is not a proper form of first order FFDEs if the BF-number is symmetric.*

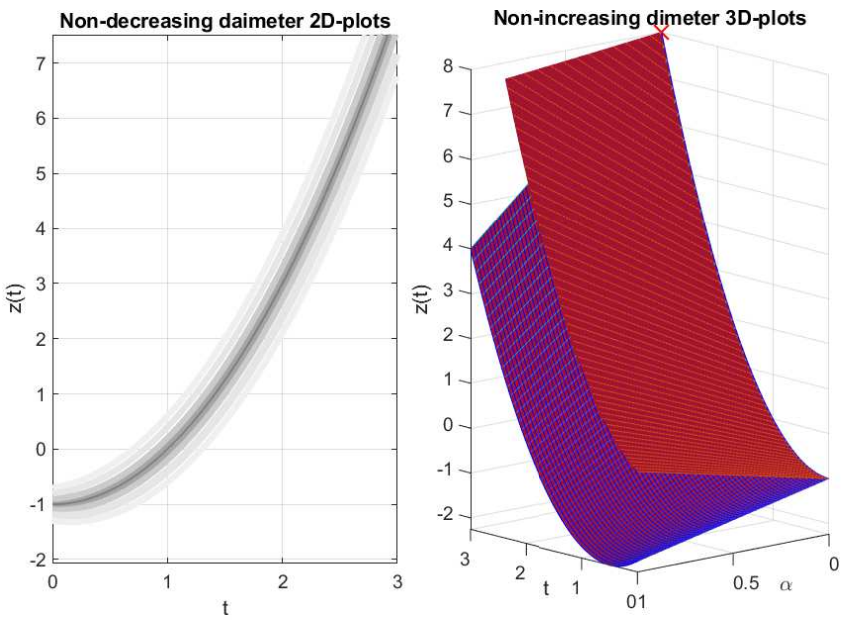

Figure 4. Show the 2D and 3D plots of the solution of example 4.3 with non-decreasing diameter in $[0,\infty)$.

Example 4.4. *If symmetric BF-number,* $\mathbb{A} = (-1; 0; 1)$ *has symmetric point* 0, *we consider the following LCFFDEs:*

$$\begin{cases} {}^{C}_{LC}D^{\frac{1}{2}}_{0^+}\vartheta(\gimel) = \dfrac{8\gimel\sqrt{\gimel}}{3\sqrt{\pi}(\gimel^2 - 4)} \odot_{\mathbb{A}} \vartheta(\gimel) \oplus_{\mathbb{A}} \dfrac{2(6-\gimel)\sqrt{\gimel}}{3(\gimel+2)\sqrt{\pi}}, \\ \quad \vartheta(0) = 4\mathbb{A} - 2. \end{cases} \tag{4.9}$$

The Eq (4.9) has a fuzzy solution $\vartheta(\gimel) = (4-\gimel^2)\mathbb{A} + \gimel - 2$ *with a non-increasing diameter in* $[0, 2)$. *By choosing* $\alpha \geq 1$ *sufficiently large, the conditions of Theorem (4.1),* $|\zeta(\gimel, r(\gimel))| = |\frac{\gimel\sqrt{\gimel}}{3\sqrt{\pi}}| \leq \alpha|4-\gimel^2|$, *holds for all* $\gimel \in [0, 2)$, *but for* $\gimel = 2$ *we cannot find any* $\alpha \geq 1$ *at which the condition holds. Hence, the fuzzy solution* $\vartheta(\gimel) = (4-\gimel^2)\mathbb{A} + \gimel - 2$ *is a unique solution having a non-increasing diameter in* $[0, 2)$. *Figure 5 show* $2D$ *and* $3D$ *plots of the solution.*

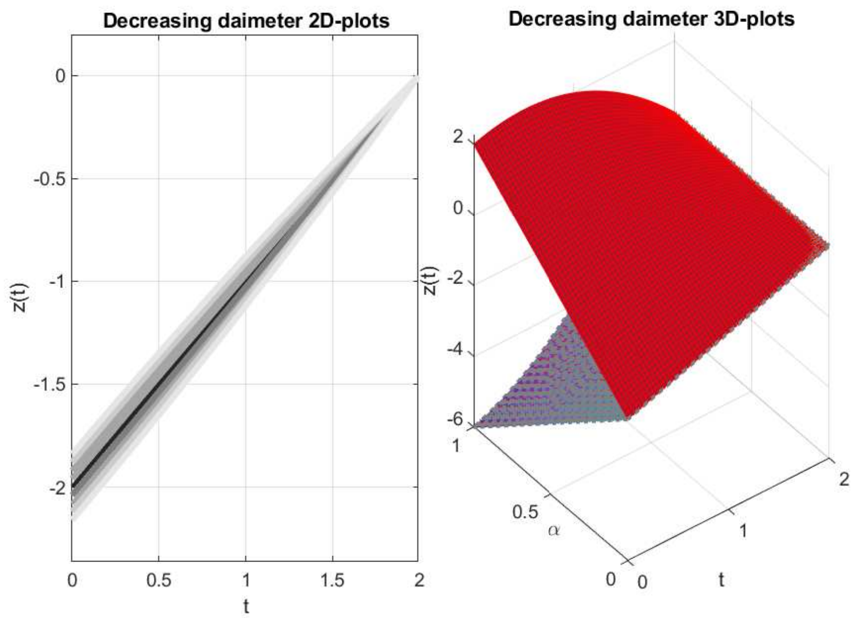

Figure 5. Show the 2D and 3D plots of the solution of example 4.4 with decreasing diameter in $[0, 2)$.

5. Conclusions

In this manuscript, we discussed conditions for the existence of unique solutions of LCFFDEs in the spaces of linear correlated fuzzy numbers of non-symmetric basic fuzzy numbers $R^n_{R(A)}$. For the useability of established results, we also discussed numerical problems. The LCFFDEs in the spaces of linear correlated fuzzy numbers of symmetric basic fuzzy numbers $R^s_{R(A)}$ have many differentiable and continuous solutions due to the extension process. In this research work, we discussed the cases of extension and the importance of the form of Eq (1.1) for the first-order LCFFDEs. This work indicates the drawbacks of taking any first-order LCFFDEs. We, also discussed conditions for the existence of unique solution of LCFFDEs in the spaces with symmetric basic fuzzy numbers, $R^s_{R(A)}$. For the utilization of established conditions for unique solutions of LCFFDEs in spaces with symmetric basic fuzzy numbers, we discussed numerical problems. In the numerical examples, we discussed the comparative analysis between our fractional problems and the fractional problems discussed in the paper [28] to show the importance of existence theory. Moreover, we provided 2D and 3D plots to enhance our findings. This study is also helpful in discussing the stability and control of LCFFDEs in future. The pair solutions of LCFFDEs in the canonical form are also interesting for future study. Moreover, the study of LCFFDEs with other fractional differentiability, like ABC-differentiability, is also interesting for research.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are thankful to Prince Sultan University for the support of this work through TAS research lab.

Funding

This research was funded by the National Science, Research, and Innovation Fund (NSRF) and King Mongkut's University of Technology North Bangkok with Contract No. KMUTNB-FF-67-B-31.

Conflicts of interest

The authors declare no conflicts of interest.

References

1. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

2. Q. Li, H. Wei, D. Hua, Stabilization of semi-markovian jumping uncertain complex-valued networks with time-varying delay: A sliding-mode control approach, *Neural Process. Lett.*, **56** (2024). https://doi.org/10.21203/rs.3.rs-3244753/v1

3. Q. Li, J. Liang, W. Gong, K. Wang, J. Wang, Nonfragile state estimation for semi-Markovian switching CVNs with general uncertain transition rates: An event-triggered scheme, *Math. Comput. Simul.*, **218** (2024), 204–222. https://doi.org/10.1016/j.matcom.2023.11.028

4. A. A. Sori, A. Ebrahimnejad, H. Motameni, Elite artificial bees' colony algorithm to solve robot's fuzzy constrained routing problem, *Comput. Intell.*, **36** (2020), 659–681. https://doi.org/10.1111/coin.12258

5. N. Jan, J. Gwak, D. Pamucar, A robust hybrid decision making model for Human-Computer interaction in the environment of Bipolar complex picture fuzzy soft Sets, *Inf. Sci.*, **645** (2023), 119163. https://doi.org/10.1016/j.ins.2023.119163
6. S. H. Nasseri, A. Ebrahimnejad, O. Gholami, Fuzzy stochastic data envelopment analysis with undesirable outputs and its application to banking industry, *Int. J. Fuzzy Syst.*, **20** (2018), 534–548. https://doi.org/10.1007/s40815-017-0367-1
7. Y. Xi, Y. Ding, Y. Cheng, J. Zhao, M. Zhou, S. Qin, Evaluation of the medical resource allocation: Evidence from China, *Healthcare*, **11** (2023), 829. https://doi.org/10.3390/healthcare11060829
8. C. Carlsson, R. Fullér, P. Majlender, Additions of completely correlated fuzzy numbers, *IEEE Int. Conf. Fuzzy Sys.*, **1** (2004), 535–539. https://doi.org/10.1109/FUZZY.2004.1375791
9. L. C. Barros, F. S. Pedro, Fuzzy differential equations with interactive derivative, *Fuzzy Sets Syst.*, **309** (2017), 64–80. https://doi.org/10.1016/j.fss.2016.04.002
10. E. Esmi, F. S. Pedro, L. C. Barros, W. Lodwick, Fréchet derivative for linearly correlated fuzzy function, *Inf. Sci.*, **435** (2018), 150–160. https://doi.org/10.1016/j.ins.2017.12.051
11. Y. H. Shen, Calculus for linearly correlated fuzzy number-valued functions, *Fuzzy Sets Syst.*, **429** (2022), 101–135. https://doi.org/10.1016/j.fss.2021.02.017
12. Y. H. Shen, A novel difference and derivative for linearly correlated fuzzy number-valued functions, *J. Intell. Fuzzy Syst.*, **42** (2022), 6027–6043. https://doi.org/10.3233/JIFS-212908
13. B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions, *Fuzzy Sets Syst.*, **230** (2013), 119–141. https://doi.org/10.1016/j.fss.2012.10.003
14. Y. Shen, First-order linear fuzzy differential equations on the space of linearly correlated fuzzy numbers, *Fuzzy Sets Syst.*, **429** (2022), 136–168. https://doi.org/10.1016/j.fss.2020.11.010
15. N. Jamal, M. Sarwar, S. Hussain, Existence criteria for the unique solution of first order linear fuzzy differential equations on the space of linearly correlated fuzzy numbers, *Fractals*, **8** (2022), 1–13. https://doi.org/10.1142/S0218348X22402216
16. N. Jamal, M. Sarwar, N. Mlaiki, A. Aloqaily, Solution of linear correlated fuzzy differential equations in the linear correlated fuzzy spaces, *AIMS Math.*, **9** (2023), 2695–2721. https://doi.org/10.3934/math.2024134
17. H. Khan, J. F. Gomez-Aguilar, T. Abdeljawad, A. Khan, Existence results and stability criteria for ABC-fuzzy-Volterra integro-differential equation, *Fractals*, **28** (2020), 2040048. https://doi.org/10.1142/S0218348X20400484
18. D. Luo, M. Tian, Q. Zhu, Some results on finite-time stability of stochastic fractional-order delay differential equations, *Chaos, Solitons Fract.*, **158** (2022), 111996. https://doi.org/10.1016/j.chaos.2022.111996
19. G. Li, Y. Zhang, Y. Guan, W. Li, Stability analysis of multi-point boundary conditions for fractional differential equation with noninstantaneous integral impulse, *Math. Biosci. Eng.*, **20** (2023), 7020–7041. https://doi.org/10.3934/mbe.2023303
20. H. Fan, J. Tang, K. Shi, Y. Zhao, Hybrid impulsive feedback control for drive-response synchronization of fractional-order multi-link memristive neural networks with multi-delays, *Fractal Fract.*, **495** (2023), 1–19. https://doi.org/10.3390/fractalfract7070495

21. H. Fan, Y. Rao, K. Shi, H. Wen, Global synchronization of fractional-order multi-delay coupled neural networks with multi-link complicated structures via hybrid impulsive control, *Mathematics*, **11** (2023), 3051. https://doi.org/10.3390/math11143051

22. L. C. Barrosa, R. C. Bassanezia, P. A. Tonelli, Fuzzy modelling in population dynamics, *Ecol. Model.*, **128** (2000), 27–33. https://doi.org/10.1016/S0304-3800(99)00223-9

23. D. L. Mon, C. H. Cheng, J. C. Lin, Evaluating weapon system using fuzzy analytic hierarchy process based on entropy weight, *Fuzzy Sets Syst.*, **62** (1994), 127–134. https://doi.org/10.1016/0165-0114(94)90052-3

24. A. Tony Thomas, S. K. Thangarasu, T. Sowmithra, Simulation and experimental validation of electro hydraulic servo system using fuzzy logic controller, *AIP Conf. Proc.*, **2336** (2021), 040021. https://doi.org/10.1063/5.0045724

25. G. Narayanan, M. S. Ali, Q. Zhu, B. Priya, G. K. Thakur, Fuzzy observer-based consensus tracking control for fractional-order multi-agent systems under cyber-attacks and its application to electronic circuits, *IEEE Trans. Netw. Sci. Eng.*, **10** (2023), 698–708. https://doi.org/10.1109/TNSE.2022.3217618

26. M. Arfan, K. Shah, A. Ullah, T. Abdeljawad, Study of fuzzy fractional order diffusion problem under the Mittag-Leffler Kernel Law, *Phys. Scr.*, **96** (2021), 074002. https://doi.org/10.1088/1402-4896/abf582

27. N. Ahmad, A. Ullah, A. Ullah, S. Ahmad, K. Shah, I. Ahmad, On analysis of the fuzzy fractional order Volterra-Fredholm integro-differential equation, *Alex. Eng. J.*, **60** (2021), 1827–1838. https://doi.org/10.1016/j.aej.2020.11.031

28. N. T. K. Son, H. T. P. Thao, T. Allahviranloo, H. V. Long, State feedback control for fractional differential equation system in the space of linearly correlated fuzzy numbers, *Fuzzy Sets Syst.*, **453** (2023), 164–191. https://doi.org/10.1016/j.fss.2022.06.022

29. B. Bede, *Mathematics of fuzzy sets and fuzzy logic*, London: Springer, 2013.

30. O. Kaleva, A note on fuzzy differential equations, *Nonlinear Anal.*, **64** (2006), 895–900. https://doi.org/10.1016/j.na.2005.01.003

31. A. Erdélyied, *Higher transcendental functions*, New York: McGraw-Hill, 1955.